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LOCAL SUPERSYMMETRY IN SUPERGRAVITY

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Abstract

We have studied the local supersymmetry in two $D = 4$ supergravity models, with $N = 1$ and $N = 2$, given the Lagrangians [1]. We have used a simple method based on the differential of the action H, which provides an alternative systematic derivation of the gauge field variations in the first order formalism. This method may be used to find non-relativistic limits of supergravity models.

1 Introduction

There are two main groups of particles in nature, which are known as fermions and bosons. The first type has a half-integer spin whereas it is integer for the other. The spin-statistics connection tells us that they have a different statistical behaviour. Generally, the elementary particle physics decompose matter into quarks and leptons (fermions) and describe the basic forces between them in terms of exchange particles (bosons). An exception would be the Higgs boson, which is not associated to any interaction.

Now, we are going to introduce supersymmetry [2]. By definition, this concept is based on the idea of a symmetry that relates fermions and bosons. A supersymmetric theoretical model is one which exhibits this symmetry. Such a model will cause the disappearance of the distinction between matter and interaction. Supersymmetry implies the existence of a new kind of particles called 'superpartners', which are associated with their equivalent known particles, and they have the same mass but differ by 1/2 in spin. This means that the superpartners of the bosons are fermions (bosinos) and those of the fermions are bosons (sfermions). When there is supersymmetry, fermions and bosons may be organized in multiplets with same mass and different spin ('supermultiplets'), and in this way we could eliminate the distinction between matter and interaction that we mentioned before. However, superpartners have not been found yet, so that supersymmetric theories are at present only theoretical models. The fact that there is not experimental evidence of supersymmetry tells us that such models must have spontaneously broken supersymmetry, which implies that the ground state is not invariant under supersymmetric transformations and its energy cannot be zero.

Working in a relativistic field theory frame, the supersymmetric transformations are generated by quantum fermionic operators Q that interchange fermionic with bosonic states, so they change the state spin by 1/2. These operators have to obey certain anti-commutation relations. Adding anti-commutation relations we can generalise Lie algebras in what is called 'Lie superalgebras' or 'Z_2-graded Lie algebras'. Lie superalgebras are associated to Lie supergroups whose elements provide this kind of symmetry. Then, they have the property of combining spacetime and internal symmetries, which gives these theories a special interest to study them.

In spite of the lack of experimental results, there are some powerful theoretical reasons to study these theories. A first reason is the cancellation of some divergences in field theories when supersymmetry is considered. It caused some interest in the hierarchy problem of the Grand Unified Theories (GUTs), the thirteen magnitude order between the GUT mass of 10^{15} GeV/c^2 and the W boson mass. Normally, a gap of this magnitude is not stable in perturbation theory and it only can be maintained by fine-tuning repetition until high orders in the perturbation expansion. Considering supersymmetry, we can avoid the mixture in mass and the fine-tuning associated, and when the hierarchy is established, it is stabilized. But, as we said before, the symmetry must be broken, so that it should not happen at a high energy scale if we still want to solve the hierarchy problem.

The other main reason is the following. Supersymmetry started to be studied with the aim of unifying the four basic interactions: gravity, electromagnetic, weak and strong. In these sense, it was of crucial importance the Coleman-Mandula (C-M) Theorem [3]. It states that the most general symmetry algebra of the S matrix must be decomposed in a direct sum of two subalgebras, contained in it, which enclosed the spacetime and internal symmetries: $\mathcal{P} \oplus \mathcal{G}$, respectively. As a result of this, it looked impossible to join gravity (based on spacetime symmetries) with the other interactions (based on internal symmetries). However, Haag, Lopuszanski and Sohnius [4] made an extension of the C-M result, considering supersymmetry operations. They introduced the Q operators mentioned before and classified all the relevant superalgebras compatible with the C-M Theorem. Doing it, they realized that these theories solved the initial problem and could relate spacetime with internal symmetries, so that supersymmetric theories were the main candidates to unify the fundamental natural forces. Later, in the paper [5] it was argued that susy was able to make the running coupling constants for the weak, electromagnetic and strong interactions converge at high energies, something that is not possible to achieve in the Standard Model of particles and interactions.

So far, we have dealt with supersymmetric theories generally. But, as gauge theories, they can be considered globally or locally. Global supersymmetry is only called supersymmetry and it does not depend on the spacetime point. In the gauge theories of strong and electro-weak interactions, the spin of the fields cannot be greater than 1, what limits the number of supersymmetries to 4. From the Yang-Mills theories, the case with the maximum number of super-Yang-Mills supersymmtries has 4. Local supersymmetry depends on the spacetime point and these models are known as supergravity models. Supergravity always includes gravity: it contains the graviton, the theoretical [1] boson that describes gravity, whose spin is $s = 2$ and its fermionic superpartner, the gravitino, whose spin is $s = 3/2$. In these models, the spin is limited to a value of 2, so that the maximum number of supersymmetries is 8. The $D = 4$ supergravity with 8 supersymmetries is a dimensional reduction of the model in 11 dimensions. Gravity is a non-renormalizable theory, so the infinities that always arise in field theory cannot be consistently absorbed. Therefore, the only chance of making sense of a quantum theory of gravity in the perturbation theory framework is that the theory turned out to be finite. Although gravity is one-loop finite, it is divergent already at two loops [6]. The fact that infinities may cancel, when there is supersymmetry, makes supergravity less divergent, but it is still divergent at three loops when $D = 4$ [7]. This has been interpreted as being the result of ignoring high energy degrees of freedom present in a finite theory of gravity. A candidate for such a theory is superstring theory. When string theory began to be formulated, supersymmetry played an important role: introducing this symmetry in string theory gave place to superstring theories. There were five consistent superstring theories, which it seemed to be a problem, but through the discovery of three dualities that relate them, they were enclosed in one theory: M theory, the main candidate theory in the unification problem. Obviously, all these theories are still models without experimental improvement, and we have to do a huge amount of work to find their final formulations. $D = 11$ supergravity is the low-energy limit model of M theory. The superstring theories also have as a low-energy limit a supergravity model each one, but in this case in 10 dimensions.

Although supergravity is now 40 years old, recently there has been some interest in the construction of supergravity actions based on spacetime algebras different from the Poincaré one. For instance, the AdS/CFT correspondence that relates superstring theory in the bulk with a conformal field theory on the AdS boundary, has led to the study of Galilean gravity/supergravity models [8]. For that, a way to build locally supersymmetric actions is needed.

[1]'Theoretical' means that it has not been discovered yet.

There is not a satisfactory tensor calculation in supergravity. Then, we are going to briefly introduce the main different formalisms to formulate supergravity, explaining and comparing their foundations:

- Noether formalism [9]: It is a formalism which consists in adding pieces successively starting from an action for gravity, perhaps including the gravitino kinetic term. An advantage is that it sometimes lets check the symmetries of the actions easily and it leads directly to the physical action. The main shortcomings are that it is not geometric, it makes difficult to build the actions with local supersymmetry.

- Superfields formalism [10]: It is a geometric formalism which lets build actions through the superfields of the generic form $\psi(x^\mu, \theta^\alpha)$, where x^μ are the spacetime coordinates, and θ^α are extra anticommuting coordinates labelled by the fermionic index α. Due to the anticommuting character of the θ^α one has a finite expansion $\phi(x^\mu, \theta^\alpha) = \phi^0(x^\mu) + \phi^1_\alpha(x^\mu)\theta^\alpha + \phi^2_{\alpha\beta}(x^\mu)\theta^\alpha\theta^\beta + \ldots$ where the coefficients $\phi^k_{\alpha_1 \ldots \alpha_k}(x^\mu)$ are called component fields. In general there are too many component fields in a superfield, and the non-physical ones have to be removed. This is done by imposing constraints. The problem is that there is not a general method to establish these constraints, and also in high dimensions the constraints themselves imply the superspace field equations.

- Geometric formalism [11]: It is often called Rheonomic. It is a geometric formalism supported in differential forms (the gauge one-forms of an algebra, among others), and it is expressed in components. This formalism, and the next one rely in the process called 'gauging of a Lie algebra' which, given a Lie algebra of commutation relations $[X_i, X_j] = C^k{}_{ij} X_k$, uses the gauge one-forms A^i and two-form curvatures $F^i = dA^i + \frac{1}{2} C^i{}_{jk} A^j \wedge A^k$ as building blocks for constructing the action [12]. The resulting actions are always in the first order formulation of gravity (in terms of the spin connection ω_{ab}, see Section 3). This formalism also is supported in the superspace and may be used to check the invariance of the actions but it does not let build them easily.

- In this work we present a slightly different geometric formalism [13], which is similar to the previous one but does not use the superspace. This formalism is expressed in terms of differential forms, as the previous one, and it is expressed in components too. The main difference from the other formalisms is that it focuses on the exterior differential of the action. We will see that this formalism is suitable to check the local supersymmetry for first-order supergravity actions, and we will consider two cases: $D = 4$, $N = 1$ and $D = 4$, $N = 2$. Also, when the algebra is an expansion [14] of the Poincaré superalgebra that leads to a Galilei-type superalgebra, by performing the expansion of the Poincaré supergravity, in this formalism it is almost immediate to see whether the expanded action is locally supersymmetric.

The present work is organized as follows. In Section 2 we describe the general setting of the model. Section 3 is devoted to the simpler $N = 1$ case. The $N = 2$ case, which contains many features of the higher dimensional supergravities, is described in Section 4. Section 5 contains our conclusions. After this, we include a proof of the Lemma 1 as an appendix. We also attach a document with two annexes, which include all the calculations done during this work: the calculations of $N = 1$ and $N = 2$ cases. These annexes are not needed for the proper understanding of this work, they are included for the interested reader.

2 General theoretical aspects

The purpose of this work consists in studying the local supersymmetry in supergravity. We have chosen a model in 4 dimensions, $D = 4$, and have considered two cases: $N = 1$ and $N = 2$, where N is the number of supersymmetries. At first, we are going to write some foundations in order to explain our development.

We start writing the Lagrangian, B_N ($N = 1, 2$) [2], which describes the model chosen, in terms of differential forms, the gauge fields and their associated curvatures, as Julia and Silva did in their paper [1]. Given B, in order to write an action, we have to realize the gauge curvatures on the spacetime $\mathcal{M}$, $A^i = A^i_\mu dx^\mu$, $F^i = F^i_{\mu\nu} dx^\mu \wedge dx^\nu$. Inserting this in the expression of B, we obtain a realization of B on $\mathcal{M}$ given by $B = B_{\mu_1 \dots \mu_D} dx^{\mu_1} \wedge \dots \wedge dx^{\mu_D}$, and the action is given by

$$I[A] = \int_{\mathcal{M}} B = \int_{\mathcal{M}} B_{\mu_1 \dots \mu_D} dx^{\mu_1} \wedge \dots \wedge dx^{\mu_D} = D! \int_{\mathcal{M}} B_{0 \dots D-1} dx^0 \wedge \dots \wedge dx^{D-1} = D! \int_{\mathcal{M}} B_{\mu_1 \dots \mu_D} d^D x \ . \tag{1}$$

Our procedure is based on calculating the differential of the Lagrangian, defined by $H_N = dB_N$. Once we have done it, we can easily derive from H_N the motion equations by calculating an interior derivation of the expression with respect to the curvatures. The next step is writing the equation obtained from the derivation with respect to the gravitino field, denoted by ψ. We must express the last relation in terms of the field equations, obtained before. The final step consists in studying the symmetries from the last result. The original gauge symmetry transformations which are derived from the algebra of the model, are not symmetries of the action for $D > 3$, so the actual variations that lead to a symmetry are in general *modified* gauge transformations.

In order to study the symmetries of the action of our model we may use the following lemma (see the appendix for a proof):

Lemma 1. *Let A^i, $F^i = dA^i + \frac{1}{2} C^i{}_{jk} A^j \wedge A^k$ be the gauge one-forms and curvature two-forms for a Lie algebra $\mathcal{G}$, $i, j, k = 1 \dots dim\, \mathcal{G}$, and let G^I denote a set of zero-forms. Assume that the D-form B is an element of the exterior algebra generated by A^i, F^i, G^I, dG^I so that in particular $\int_{\mathcal{M}^D} B$ defines an action as described above. We define the interior derivations (resp. antiderivations) I_{F^i}, (resp. I_{A^i}, I_{dG^I}) by giving the only non-zero cases,*

$$I_{F^i} F^j = \delta_i^j \ , \quad I_{A^i} A^j = \delta_i^j \ , \quad I_{dG^I} dG^J = \delta_I^J \ . \tag{2}$$

Then,

1. *The field equations corresponding to the action are given by*

$$I_{F^i} H = 0 \ , \quad I_{dG^I} H = 0 \ . \tag{3}$$

2. *The gauge variation of the action is given by*

$$\delta_{gauge} \int_{\mathcal{M}^D} B = \int_{\mathcal{M}^D} \alpha^i \quad I_{A^i} H \ . \tag{4}$$

[2]When we write B_N (equivalently H_N), we want to remark that we are focusing in our two cases. Whereas, if we write B (equivalently H), we are writing expressions for general Lagrangians (or their differentials).

3. *If for a certain value of i there exist on $\mathcal{M}$ one-forms $X^j{}_i$ and zero-forms $Y^J{}_i$ on M^D such that*

$$I_{A^i}H = X^j{}_i \wedge I_{F^j}H + Y^J{}_i I_{dG^J}H \tag{5}$$

where $H = dB$, then the action is invariant under the variations

$$\begin{aligned} \delta A^k &= \delta^k_i d\alpha^i - C^k{}_{ij}\alpha^i A^j - X^k{}_i\alpha^i \quad \textit{no sum in } i \\ \delta G^J &= -Y^J{}_i\alpha^i \quad \textit{no sum in } i\ . \end{aligned} \tag{6}$$

In particular, if $I_{A^i}H = 0$, then the action is invariant under the gauge transformation with parameter α^i.

As a particular case, if there are no 0-forms, i.e., $G^I = 0$, and the action is exclusively expressed in terms of curvatures, it is a Chern-Simons action. This is the case of gravity [15] and supergravity [16] in three dimensions.

In the next two sections we are going to particularize what we have just described, for the mentioned cases: $N = 1$ and $N = 2$, $D = 4$ supergravities. Before doing it, we establish our conventions: we take the metric signature $\eta^{ab} = \{-,+,+,+\}$ and the Levi-Civita tensor $\epsilon_{0123} = 1$. The matrices γ^a are the Dirac matrices which satisfy the Clifford algebra $\{\gamma^a, \gamma^b\} = 2\eta^{ab}$ and $\gamma^5 := \gamma^0\gamma^1\gamma^2\gamma^3$, with $(\gamma^5)^2 = I$ (I is the identity matrix). The constant $4\kappa^2 = 16\pi G$ where G is the Newton constant.

3 $N = 1$ case

The 4-form Lagrangian of $D = 4$, $N = 1$ supergravity, in the first order formulation reads:

$$B_1 = -\frac{1}{8\kappa^2}\epsilon_{abcd}R^{ab} \wedge e^c \wedge e^d + \frac{i}{2}\bar{\psi} \wedge \gamma^5\gamma_a e^a \wedge \rho \tag{7}$$

It depends on the 1-form vierbein e^a, the associated 1-form spin-connection ω^{ab}, the spinor-1-form the gravitino ψ and the 2-form curvatures R^{ab} and ρ. The 1-form gauge fields and its associated 2-form curvatures are related through the algebra equations:

$$\begin{cases} R^{ab} = d\omega^{ab} + \omega^a{}_c \wedge \omega^{cb} \\ T^a = de^a + \omega^a{}_b \wedge e^b - \dfrac{i\kappa^2}{2}\bar{\psi}\gamma^a \wedge \psi \\ \rho \equiv D\psi = d\psi + \dfrac{1}{4}\gamma_{ab}\omega^{ab} \wedge \psi \end{cases} \tag{8}$$

The Riemann curvature R^{ab} is associated to ω^{ab}, the torsion T^a to e^a and ρ to ψ. We also have that γ_{ab} are antisymmetrized gamma matrices with two indices, $\gamma_{ab} \equiv \gamma_{[a}\gamma_{b]} = \frac{1}{2}(\gamma_a\gamma_b - \gamma_b\gamma_a)$. If we set the curvatures in (8) equal to zero, we obtain the Maurer-Cartan equations, *i.e.* the dual version, of the superPoincaré algebra.

Now, we derive the expression of $H_1 = dB_1$, where d is the exterior derivative. Note that we do not realize the abstract for B_1 on spacetime yet, otherwise this differential would be equal to zero. In order to compute the exterior differential, we may use that B_1 is a Lorentz scalar form, so the differential coincides with the Lorentz covariant differential, so we just have to compute the covariant exterior differential of B_1. So, $H_1 = DB_1$. The result reads:

$$H_1 = \frac{1}{4\kappa^2}\epsilon_{abcd}R^{ab} \wedge e^c \wedge T^d + \frac{i}{2}\bar{\rho} \wedge \gamma^5\gamma_a e^a \wedge \rho - \frac{i}{2}\bar{\psi} \wedge \gamma^5\gamma_a T^a \wedge \rho \tag{9}$$

In order to obtain this expression, we have made use of the following facts:

- $DR^{ab} = 0$ by the Bianchi identities.
- From the second equation of (8) we write: $De^a = T^a + \frac{i\kappa^2}{2}\bar{\psi}\gamma^a \wedge \psi$.
- It follows that $D\rho = \frac{1}{4}\gamma_{ab}R^{ab} \wedge \psi$.
- The quadratic terms in ψ are cancelled between them.
- The cubic term in ψ vanishes by the Fierz rearrangement $\bar{\psi}\gamma_a \wedge \bar{\psi}\gamma^a \wedge \psi = 0$.

Then, it is easy to read the motion equations associated with each 1-form from (9):

$$\begin{aligned} I_{R^{ab}}H_1 &= \frac{1}{2\kappa^2}\epsilon_{abcd}e^c \wedge T^d = 0 \\ I_{T^d}H_1 &= \frac{1}{4\kappa^2}\epsilon_{abcd}R^{ab} \wedge e^c - \frac{i}{2}\bar{\psi} \wedge \gamma^5\gamma_d\rho = 0 \\ I_{\bar{\rho}}H_1 &= i\gamma^5\gamma_a e^a \wedge \rho - \frac{i}{2}\gamma^5\gamma_a T^a \wedge \psi = 0 \end{aligned} \tag{10}$$

In the last one, we could equally have used ρ.

From the first equation, we can obtain an important result: torsion vanishes, $T^a = 0$. Also, the vanishing of the torsion allows us to solve for ω_{ab} in terms of the e^a_μ, its inverse and ψ_μ, leaving as independent fields the e^a_μ corresponding to the gravity field, and ψ_μ, *i.e.*, the gravitino field. Inserting the expression for ω_{ab} in the action leads to the second order formalism. The second equation gives us the vacuum Einstein field equations, $\frac{1}{\kappa^2}[R^{sb}{}_{db} - \frac{1}{2}\eta^s{}_d R^{uv}{}_{uv}] = 0$, where $R^{ab}{}_{cd}$ are the Riemann curvature components. And the last one, the fermionic equation, lets us derive the Rarita-Schwinger (R-S) equation, $\gamma^{\nu\alpha\sigma}\rho_{\nu\alpha} = 0$, where $\rho_{\nu\alpha}$ are the components of ρ and $\gamma^{\nu\alpha\sigma}$ are the antisymmetrized products of three gamma matrices.

Finally, we have to study the symmetries of the action. To do it, we are going to compute the interior derivation with respect to ψ in (9) in order to show that

$$I_{\bar{\psi}}H_1 = -\frac{i}{2}\gamma^5\gamma_a T^a \wedge \rho \cong 0 \tag{11}$$

The last symbol means that the relation vanishes when we impose it the field equations, *i.e.*, if we take the result $T^a = 0$.

Now, we have to write (11) in terms of $I_{R^{ab}}H_1 \equiv E(\omega^{ab})$. So, we have that

$$I_{\bar{\psi}}H_1 = -\frac{i}{2}\gamma^5\gamma_a T^a \wedge \rho = \frac{1}{2}X^{ab} \wedge E(\omega^{ab}) \tag{12}$$

Where X^{ab} is the 1-form that we have to compute in order to study the invariance under variations, as we said in the lemma. After an easy calculation, we obtain,

$$X^{ab} = -i\kappa^2\epsilon^{abcd}\gamma^5\gamma_c e^u \wedge \rho_{ud} - \frac{i\kappa^2}{2}\epsilon^{abcd}\gamma^5\gamma^u e_u \wedge \rho_{cd} \tag{13}$$

We already have all the ingredients to work out the field variations. According to the lemma, they are:

$$\begin{cases} \delta\omega^{ab} = -\bar{\epsilon}X^{ab} \\ \delta e^a = i\kappa^2\bar{\epsilon}\gamma^a\psi \\ \delta\psi = d\epsilon + \dfrac{1}{4}\gamma_{ab}\omega^{ab}\epsilon, \end{cases} \tag{14}$$

With ϵ the parameter of the gauge transformations. The difference respect to the original variations is that $\delta\omega^{ab} = 0$ in the gauge case. With these results we conclude our work in the $N = 1$ case.

4 $N = 2$ case

The Lagrangian of $N = 2$ supergravity depends on the 1-form e^a, the associated 1-form $so(1,3)$ spin-connection ω^{ab}, two spinor-1-forms the gravitinos ψ^A ($A = 1,2$ is the internal global $SO(2)$ R-symmetry [3] index) and one Abelian 1-form connection, the Maxwell field A. The need for the field A is apparent if we remember that the number of bosonic degrees of freedom equals the number of fermionic degrees of freedom in each supermultiplet due to supersymmetry. The action also contains auxiliary zero-forms F^{ab} needed to write, in the first order formalism, the kinetic term for the field A. Since now we have two gravitinos, then the graviton degrees of freedom are not enough to match the fermionic ones. We write the 4-form Lagrangian as,

$$B_2 = -\frac{1}{8\kappa^2}\epsilon_{abcd}R^{ab}\wedge e^c\wedge e^d + \frac{i}{2}\bar{\psi}_A\wedge\gamma^5\gamma_a e^a\wedge\rho^A - \frac{1}{2}F\wedge_* F + _*F\wedge dA - _*F\wedge a - b\wedge dA + \frac{1}{2}a\wedge b. \tag{15}$$

Where

$$a = \frac{i\kappa}{2}\varepsilon^A{}_B\bar{\psi}_A\wedge\psi^B, \quad b = \frac{i\kappa}{2}\varepsilon^A{}_B\bar{\psi}_A\gamma^5\wedge\psi^B \tag{16}$$

The symbol $*$ refers to the standard Hodge-duality and $\varepsilon^A{}_B$ is the antisymmetric $SO(2)$-invariant tensor, which satisfies $\varepsilon^1{}_2 = -\varepsilon^2{}_1 = 1$ and $\varepsilon^A{}_C\varepsilon^C{}_B = -\delta^A{}_B$. In relation to the field strength F, we express it and its duality in components:

$$F = F_{ab}\wedge e^a\wedge e^b, \quad _*F = \frac{1}{2}\epsilon_{abcd}F^{ab}\wedge e^c\wedge e^d \tag{17}$$

As we did in the previous section, we give the algebra of our model whose equations relate the curvatures and the fields:

$$\begin{cases} R^{ab} = d\omega^{ab} + \omega^a{}_c\wedge\omega^{cb} \\ T^a = de^a + \omega^a{}_b\wedge e^b - \dfrac{i\kappa^2}{2}\bar{\psi}_A\gamma^a\wedge\psi^A \\ \rho^A = d\psi^A + \dfrac{1}{4}\gamma_{ab}\omega^{ab}\wedge\psi^A \\ \mathcal{F} = dA - a \end{cases} \tag{18}$$

With $\mathcal{F}$ the associated curvature to A, and the 'curvature' of the zero-form F^{ab} is simply DF^{ab}. Again, setting the curvatures equal to zero, we obtain a superalgebra that includes internal bosonic generator as well as the Poincaré ones. This is an example of the non-trivial algebras allowed by the Haag-Lopuszanski-Sohnius Theorem.

[3] A R-symmetry is a proper symmetry of supersymmetry operators.

Proceeding as in the previous case, we calculate the differential of the action, $H_2 = DB_2$. The process is more tedious now, and we have to do it more carefully. We use the following facts:

- In a similar way as before: $DR^{ab} = 0$, $De^a = T^a + \frac{i\kappa^2}{2}\bar{\psi}_A\gamma^a \wedge \psi^A$ and $D\rho^A = \frac{1}{4}\gamma_{ab}R^{ab} \wedge \psi^A$.
- The Fierz identity is now, $\gamma_a\psi^A \wedge (\bar{\psi}_B\gamma^a \wedge \psi^B) + \varepsilon^A{}_B\psi^B \wedge (\varepsilon^C{}_D\bar{\psi}_C \wedge \psi^D) + \varepsilon^A{}_B\gamma^5\psi^B \wedge (\varepsilon^C{}_D\bar{\psi}_C\gamma^5 \wedge \psi^D) = 0$, which is more complicated.
- We take control of the terms that depend on T^a and dA. To do it, we use the fact that $T^a = 0$, according to the motion equations which we will compute later, and we express the terms which are proportional to dA in the next way: $(dA - F - a) + (F + a)$, expression which relates A and F. So, these terms that are of our interest are those that are proportional to T^a and $(dA - F - a)$.

We have the necessary tips to work out the final expression of H_2. After a long computation (see the correspondent annex) and simplifying:

$$
\begin{aligned}
H_2 =& \frac{1}{4\kappa^2}\epsilon_{abcd}R^{ab} \wedge e^c \wedge T^d + \frac{i}{2}\bar{\rho}_A \wedge \gamma^5\gamma_a e^a \wedge \rho^A - \frac{i}{2}\bar{\psi}_A \wedge \gamma^5\gamma_a T^a \wedge \rho^A \\
&+ \frac{1}{4}\epsilon_{abcd}DF^{ab} \wedge e^c \wedge e^d \wedge F_{uv} \wedge e^u \wedge e^v - \frac{1}{4}\epsilon_{abcd}F^{ab} \wedge e^c \wedge e^d \wedge DF_{uv} \wedge e^u \wedge e^v \\
&+ \frac{1}{2}\epsilon_{abcd}F^{ab} \wedge e^c \wedge e^d \wedge F_{uv} \wedge e^u \wedge T^v - \frac{i\kappa^2}{4}\epsilon_{abcd}F^{ab} \wedge e^c \wedge \bar{\psi}_A\gamma^d \wedge \psi^A \wedge F_{uv} \wedge e^u \wedge e^v \\
&+ \frac{i\kappa^2}{4}\epsilon_{abcd}F^{ab} \wedge e^c \wedge e^d \wedge F_{uv} \wedge e^u \wedge \bar{\psi}_A\gamma^v \wedge \psi^A - \epsilon_{abcd}F^{ab} \wedge e^c \wedge T^d \wedge (dA - a - \frac{F}{2}) \\
&+ \frac{i\kappa}{2}\epsilon_{abcd}F^{ab} \wedge e^c \wedge e^d \wedge \varepsilon^A{}_B\bar{\psi}_A \wedge \rho^B + i\kappa\varepsilon^A{}_B\bar{\psi}_A\gamma^5 \wedge \rho^B \wedge F_{uv} \wedge e^u \wedge e^v \\
&+ \frac{1}{2}\epsilon_{abcd}DF^{ab} \wedge e^c \wedge e^d \wedge (dA - F - a) - \frac{i\kappa^2}{2}\epsilon_{abcd}F^{ab} \wedge e^c \wedge \bar{\psi}_A\gamma^d \wedge \psi^A \wedge (dA - F - a) \\
&+ i\kappa\varepsilon^A{}_B\bar{\psi}_A\gamma^5 \wedge \rho^B \wedge (dA - F - a)
\end{aligned}
\tag{19}
$$

As we can see, the first three terms are the same as in the $N = 1$ case. The rest are new, so that it is easy to note the difficulty raised in this case.

It is easy to read from (19) the motion equations. A simple calculation gives:

$$
\begin{aligned}
I_{R^{ab}}H_2 =& \frac{1}{2\kappa^2}\epsilon_{abcd}e^c \wedge T^d = 0 \\
I_{T^d}H_2 =& \frac{1}{4\kappa^2}\epsilon_{abcd}R^{ab} \wedge e^c - \frac{i}{2}\bar{\psi}_A\gamma^5\gamma_d \wedge \rho^A + \frac{1}{2}\epsilon_{abcd}F^{ab} \wedge e^c \wedge e^d \wedge F_{uv} \wedge e^u \\
&- \epsilon_{abcd}F^{ab} \wedge e^c \wedge (dA - a - \frac{1}{2}F) = 0 \\
I_{\bar{\rho}_A}H_2 =& i\gamma^5\gamma_a e^a \wedge \rho^A - \frac{i}{2}\gamma^5\gamma_a T^a \wedge \psi^A - i\kappa({}_*F + \gamma^5 F) \wedge \varepsilon^A{}_B\psi^B \\
&- i\kappa\varepsilon^A{}_B\psi^B\gamma^5 \wedge (dA - F - a) = 0 \\
I_{DF^{ab}}H_2 =& \frac{1}{2}\epsilon_{abcd}e^c \wedge e^d \wedge (dA - a - F) = 0 \\
I_{\mathcal{F}}H_2 =& D({}_*F - b) = 0
\end{aligned}
\tag{20}
$$

The first one tells us, as before, that the torsion vanishes. This is why it is important to take account of the terms depending on T^a in H_2. According to the importance of $(dA - F - a)$ we can check the equation of F^{ab}.

At this point, we only have to study the symmetries from the $I_{\bar{\psi}_A} H_2$ equation. The expression is:

$$\begin{aligned} I_{\bar{\psi}_A} H = &- \frac{i}{2}\gamma^5\gamma_a T^a \wedge \rho^A - \frac{i\kappa^2}{2}\epsilon_{abcd} F^{ab} \wedge e^c\gamma^d \wedge \psi^A \wedge F + i\kappa^2{}_*F \wedge F_{uv} \wedge e^u\gamma^v \wedge \psi^A \\ &+ i\kappa({}_*F + \gamma^5 F) \wedge \varepsilon^A{}_B\rho^B - i\kappa^2\epsilon_{abcd}F^{ab} \wedge e^c\gamma^d \wedge \psi^A \wedge (dA - F - a) \\ &+ i\kappa\varepsilon^A{}_B\gamma^5\rho^B \wedge (dA - F - a) \cong 0 \end{aligned} \tag{21}$$

It vanishes when we impose the motion equations, as we wanted to show. Indeed, first we note that we can take $T^a = 0$ and $(dA - F - a) = 0$ on-shell since they are obtained from some field equations. The remaining terms are cancelled through the fermionic equation, as we now show. Taking in this relation $T^a = 0$ and $(dA - F - a) = 0$, it is easy to prove that

$$M\gamma^5\gamma_a e^a = ({}_*F + \gamma^5 F) \tag{22}$$

With M a matrix which takes the value:

$$M = \frac{1}{2}\not{e} \wedge \not{F} \tag{23}$$

Where the slashed notation is defined as: $\not{e} = \gamma_a e^a$ and $\not{F} = \gamma_{ab}F^{ab}$. Following this recipe we can check what we said, $I_{\bar{\psi}_A} H_2 \cong 0$.

The last step consists in writing this equation in terms of $I_{R^{ab}} H_2 \equiv E(\omega^{ab})$, $I_{DF^{ab}} H_2 \equiv E(F^{ab})$ and $I_{\bar{\rho}_A} H_2 \equiv E(\bar{\psi}_A)$ in order to compute the field variations to study the symmetries according to the lemma, which is our task. So that, we have to express (21) as,

$$I_{\bar{\psi}_A} H_2 = \frac{1}{2} X^{ab,A} \wedge E(\omega^{ab}) + \frac{1}{2} Y^{ab,A} E(F^{ab}) + Z^A{}_B \wedge E(\bar{\psi}_B) = 0 \tag{24}$$

It is a vanishing relation because it directly depends on motion equations. $X^{ab,A}$, $Y^{ab,A}$ and $Z^A{}_B$ are, respectively, the 1-form, the 0-form and the 1-form matrix, which let us to work out the gauge field transformations that we are looking for. To compute them, we have done it term by term. It is easy to relate (21) with the fermionic equation through (22), which gives the value of $Z^A{}_B$. $X^{ab,A}$ is obtained from those terms which depend on T^a, and those that depend on $(dA - F - a)$ give the value of $Y^{ab,A}$. This is possible because, as we have mentioned before, $T^a = 0$ and $(dA - F - a) = 0$ are equivalent to the ω_{ab} and F_{ab} field equations respectively.

A tedious, but simple, systematic calculation leads us to:

$$\begin{cases} X^{ab,A} = -i\kappa^2\epsilon^{abcd}\gamma^5\gamma_c e^u \wedge (\rho^A{}_{ud} - \kappa(M\psi)^A{}_{ud}) - \dfrac{i\kappa^2}{2}\epsilon^{abcd}\gamma^5\gamma^u e_u \wedge (\rho^A{}_{cd} - \kappa(M\psi)^A{}_{cd}) \\ Y^{ab,A} = -i\kappa\epsilon^{abcd}\varepsilon^A{}_B\gamma^5(\rho^B{}_{cd} + \kappa(M\psi)^B{}_{cd}) - 2i\kappa^2 F^{ab}\gamma_c \wedge \psi^{c,A} + 4i\kappa^2 F^{[ac}\gamma^{b]} \wedge \psi^A_c \\ Z^A{}_B = \kappa M \end{cases} \tag{25}$$

Finally, using the results of the Lemma presented in Sec. 2, we immediately obtain the searched variations:

$$\begin{cases} \delta\omega^{ab} = -\bar{\epsilon}_A X^{ab,A} \\ \delta e^a = i\kappa^2 \bar{\epsilon}_A \gamma^a \psi^A \\ \delta\psi^A = d\epsilon^A + \dfrac{1}{4}\gamma_{ab}\omega^{ab}\epsilon^A - Z^A{}_B \epsilon^B \\ \delta A = i\kappa \varepsilon^A{}_B \bar{\epsilon}_A \psi^B \\ \delta F^{ab} = -\bar{\epsilon}_A Y^{ab,A} \end{cases} \tag{26}$$

With ϵ^A (equivalently $\bar{\epsilon}_A$) the local supersymmetry parameter. Again, the variations include new terms beyond the gauge variations obtained directly from the Poincaré Lie superalgebra.

5 Conclusions

We have shown how to derive the supersymmetry transformation rules of $D = 4$, $N = 1$ and $N = 2$ supergravities using a method that exploits the exterior differential of the Lagrangian form, when it is given by an element of the exterior algebra generated by the gauge fields and curvatures of a superalgebra (in our case, superPoincaré), plus some additional zero-forms and their exterior differentials in the $N = 2$ case. We have not considered higher-dimensional supergravities such as $D = 11$ supergravity but nearly all their features are present already in the $D = 4$, $N = 2$ case.

Another reason for considering $N = 2$ is the possible application of our construction to Galilean supergravities when the method of Lie algebra expansions is used. Starting from Poincaré supergravity we may perform an expansion of both $I_{\bar{\psi}}H$ and the field equation forms $I_{R^{ab}}H$, $I_{DF^{ab}}H$, $I_{\bar{\rho}}H$ to see if there is still a connection of the type (5) after the expansion. $N = 2$ is necessary to have a Galilean supersymmetry as the expanded algebra (see, for instance, [17]). However, it has recently been shown [18] that $N = 1$ superPoincaré is enough to obtain p-brane Galilean supergravities for $p = 1$ (string) and $p = 2$ (membrane), $p = 0$ being the particle case which requires $N = 2$. Our work provides a simple proof of the local supersymmetry of these models. It would be interesting to apply this method to the ultrarelativistic supergravity models associated to the Carroll algebra (see for instance [19]).

Finally, we now comment on another possible application of our results. We have seen that given a D-form Lagrangian B constructed out of the gauge forms and curvatures of a Lie superalgebra, only when $I_{A^a}H = 0$ for all A^a, the action is a Chern-Simons one. This provides a connection between CS actions and non-CS ones that may be used to check the conjectured connection between $D = 11$ supergravity and a CS theory [20].

A Appendix: Proof of Lemma 1

1. Let us compute the generic variation δI given δA^a, δG^I that vanishes in $\partial\mathcal{M}$ (or goes to zero at infinity) so that we can discard total differentials when integrating by parts:

$$\begin{aligned} \delta \int_{\mathcal{M}} B &= \int_{\mathcal{M}} \Big(\delta A^i \wedge I_{A^i}B + \delta F^i \wedge I_{F^i}B + \delta G^I I_{G^I}B + \delta(dG^I) \wedge I_{dG^I}B\Big) \\ &= \int_{\mathcal{M}} \Big(\delta A^i \wedge I_{A^i}B + d(\delta A^i) \wedge I_{F^i}B + C^i{}_{jk}\delta A^j \wedge A^k \wedge I_{F^i}B + \delta G^I I_{G^I}B + \delta(dG^I) \wedge I_{dG^I}B\Big) \\ &= \int_{\mathcal{M}} \Big\{\delta A^i \wedge \Big(I_{A^i}B + C^j{}_{ik}A^k \wedge I_{F^j}B + dI_{F^j}B\Big) + \delta G^I \left(I_{G^I}B - dI_{dG^I}B\right)\Big\} \ . \end{aligned} \tag{27}$$

Now, we note the identities

$$[d, I_{F^i}] = -I_{A^i} - C^j{}_{ik} A^k \wedge I_{F^j} \tag{28}$$

and

$$\{d, I_{dG^I}\} = I_{G^I} \ . \tag{29}$$

Then,

$$\begin{aligned} dI_{F^i} B &= i_{F^i} H - I_{A^i} B - C^j{}_{ik} A^k \wedge I_{F^j} B \\ dI_{dG^I} B &= -I_{dG^I} H + I_{G^I} B \ . \end{aligned} \tag{30}$$

Inserting this in (27) we obtain, as stated,

$$\delta \int_{\mathcal{M}} B = \int_{\mathcal{M}} \left(\delta A^i \wedge I_{F^i} H + \delta G^I I_{dG^I} H \right) \ . \tag{31}$$

2. Let us particularize (31) to the case when δA^i is the gauge variation of parameters α^i,

$$\delta_{gauge} A^i = d\alpha^i - C^i{}_{jk} \alpha^j A^k \ , \quad \delta_{gauge} G^I = 0 \ . \tag{32}$$

Then, also omitting the boundary terms,

$$\delta_{gauge} \int_{\mathcal{M}} B = \int_{\mathcal{M}} \alpha^i \left(-dI_{F^i} H - C^j{}_{ik} A^k \wedge I_{F^j} H \right) \ . \tag{33}$$

Again, using equation (28) acting on H,

$$-\, dI_{F^i} H = -I_{F^i} dH + I_{A^i} H + C^j{}_{ik} A^k \wedge I_{F^j} H \ , \tag{34}$$

we obtain

$$\delta_{gauge} \int_{\mathcal{M}} B = \int_{\mathcal{M}} \alpha^i I_{A^i} H \ . \tag{35}$$

3. Let us now assume that

$$I_{A^i} H = \ X^j{}_i \wedge i_{F^j} H + Y^J{}_i I_{dG^J} H \ . \tag{36}$$

Then, under the modified variations

$$\begin{aligned} \delta' A^i &= \delta_{sugra} A^i - X^i{}_j \alpha^j \\ \delta' G^I &= -Y^I{}_i \alpha^i \ , \end{aligned} \tag{37}$$

we have taken into account (35) and (31),

$$\begin{aligned} \delta' \int_{\mathcal{M}} B &= \delta_{gauge} \int_{\mathcal{M}} B + \int_{\mathcal{M}} \left(-X^i{}_j \wedge \alpha^j I_{F^i} H - Y^I{}_i \alpha^i I_{dG^I} H \right) \\ &= \int_{\mathcal{M}} \alpha^i \left(I_{A^i} H - X^j{}_i \wedge I_{F^j} H - Y^I{}_i I_{dG^I} H \right) = 0 \ , \end{aligned} \tag{38}$$

where in the last equality we have used (36).

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