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On the asymptotic balance between electric and magnetic energies for hydromagnetic relativistic flows

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In the equations of classical magnetohydrodynamics, the displacement current is considered vanishingly small due to low plasma velocities. For velocities comparable to the speed of light, the full relativistic electromagnetic equations must be used. In the absence of gravitational forcings and with an isotropic Ohm's law, it is proved that for poloidal magnetic field and velocity and toroidal electric field, the electric and magnetic energies tend to be equivalent in average for large times. This represents a partial extension of Cowling's theorem for axisymmetric fields. © *2013 AIP Publishing LLC*. [http://dx.doi.org/10.1063/1.4812197]

I. INTRODUCTION

Magnetohydrodynamics (MHD) represents a large scale approach to the study of the evolution of conducting flows. To obtain this model, a number of drastic simplifications are performed: Electron inertia is neglected and the phase velocities within the fluid are assumed so slow that the displacement current may safely be omitted from the Maxwell equations. While this leaves out a large quantity of physics, the success of classical magnetohydrodynamics in plasma modeling in many areas of science and engineering, ranging from astrophysics to metallurgy, is unquestionable. However, when we allow the plasma velocity to be comparable to the speed of light, the MHD approximation is unsatisfactory and one must turn to the so-called General Relativistic Magnetohydrodynamics (GRMHD). This discipline starts with the studies of Lichnerowicz[1] and Anile[2] but the most useful expression of the equations, the so-called 3+1 split, was defined later. While the equations may be set in several ways (see Refs. 3 and 4 for the hydrodynamic system; Refs. 5 and 6 for the magnetohydrodynamic one), there is a general consensus about perfectly conducting fluids, but as soon as diffusivity of some kind occurs there is no universal agreement of the appropriate mathematical setting. This holds for kinetic viscosity[7,8] as well as for magnetic diffusivity, i.e., resistivity. It is not clear what terms may be ignored in a general law: Some models are extremely complex.[9,10] The simplest course is to take an isotropic conductivity as in, e.g., Refs. 11–13, although an anisotropic one, also depending on the magnetic field size, is probably closer to reality.[14,15] A simple deduction of the relevant magnetohydrodynamic equations, also taking into account possible gravitational effects, appears in Ref. 16. Those equations are valid for any metric, and in fact purely gravitational distortions of space-time may yield growth of the electromagnetic field even in the absence of flow;[17] however, we do not wish to use external forcings that could disguise the nature of electromagnetic evolution, so we take a flat or Minkowski space-time. In these conditions, the Maxwell equations may be written as

$$\frac{\partial \mathbf{E}}{\partial t} - \nabla \times \mathbf{B} = -\mathbf{J}, \tag{1}$$

$$\frac{\partial \mathbf{B}}{\partial t} + \nabla \times \mathbf{E} = \mathbf{0}, \tag{2}$$

$$\nabla \cdot \mathbf{E} = q, \tag{3}$$

$$\nabla \cdot \mathbf{B} = 0, \tag{4}$$

where $\mathbf{B}$ is the magnetic field, $\mathbf{E}$ the electric one, $\mathbf{J}$ the current density, and q the charge density. Since Maxwell's equations are in themselves special relativistic, it is logical that Eqs. (1)–(4) have the classical form. The difference starts when the flow is involved, i.e., in Ohm's law, which in the simplest isotropic case studied here may be written as

$$W(\mathbf{E} - (\mathbf{E}\cdot\mathbf{v})\mathbf{v} + \mathbf{v}\times\mathbf{B}) = \eta(\mathbf{J} - q\mathbf{v}), \tag{5}$$

where $\mathbf{v}$ is the flow velocity, $W = (1 - v^2)^{-1/2}$ the Lorentz factor, and η the (assumed constant) resistivity. Notice that we take geometric units, i.e., the speed of light c is taken as 1. Classical limits may be obtained as follows: For ideal MHD $\eta = 0$, so that the left-hand term of (5) is equal to zero. Multiplying it by $\mathbf{v}$ we obtain $\mathbf{E}\cdot\mathbf{v} = 0$, so that $\mathbf{E} = -\mathbf{v}\times\mathbf{B}$. As for the slow velocity case, $v \ll 1, W \sim 1$ and $|(\mathbf{E}\cdot\mathbf{v})\mathbf{v}| \ll |\mathbf{E}|$. For neutral plasmas ($q = 0$), this yields the classical law of Ohm: $\mathbf{E} + \mathbf{v}\times\mathbf{B} = \eta\mathbf{J}$.

A celebrated result of T. G. Cowling (see, e.g., Ref. 18) shows that axisymmetric fields and velocities yield a decaying magnetic field. This anti-dynamo theorem was extremely influential in the early attempts to obtain a working model dynamo. We intend to study if it can be adapted to relativistic MHD.

It may be asked if there exist physical settings where Eqs. (1)–(4) represent an appropriate model. An obvious choice could be the study of accretions disks rotating around black holes, where velocities reach an appreciable fraction of the

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velocity of light. However, near the horizon the gravitational terms omitted in (1)–(4) cannot be neglected, although it is true that these terms decay at least as fast as the inverse of the radius, while Keplerian velocities decay as the inverse of the square root of the radius, so that at a moderate distance (1)–(4) are a reasonably good approximation. There exists another example where plasmas at relativistic velocities are essentially free of gravitational constraints: jets emerging either from compact objects or galaxies. In fact, some of these have been studied using special relativity (see, e.g., Ref. 19).

II. EVOLUTION OF THE MAGNETIC FLUX

Our purpose is to study Eqs. (1)–(4) when both velocity and magnetic field are axisymmetric, poloidal vectors, while the electric field is axisymmetric and toroidal (also called azimuthal). The notation simplifies somewhat if instead we take $\mathbf{v}$ and $\mathbf{B}$ as plane vectors, depending on the space variables (x,y) (plus time), while $\mathbf{E}$ is vertical and depends also on (t,x,y). Our results will be valid in the cylindrical case. With this assumption, Ohm's law (5) simplifies to

$$W(\mathbf{E}+\mathbf{v}\times\mathbf{B}) = \eta\mathbf{J}, \tag{6}$$

and a vector potential $\mathbf{A} = (0, 0, A)$ exists such that

$$\mathbf{B} = \left(\frac{\partial A}{\partial y}, -\frac{\partial A}{\partial x}, 0\right), \tag{7}$$

provided the (plane) domain Ω under consideration is simply connected. This choosing represents a Coulomb gauge $(\nabla\cdot\mathbf{A} = 0)$ such that in the lateral boundary of any three-dimensional domain with base Ω, $\Omega\times(a,b)$, we have $\mathbf{A}\cdot\mathbf{n} = 0$, whereas at the upper and lower parts (which do not exist in the periodic case, or if $(a,b) = (-\infty,\infty)$), we have $\mathbf{A}\times\mathbf{n} = \mathbf{0}$.

A is determined up to a spatial constant, which however may depend on time. We also assume that the flow is incompressible, and in fact the rest mass density ρ_0 is constant. The continuity equation (Ref. 4, p. 241) is

$$\frac{\partial}{\partial t}(\rho_0 W) + \nabla\cdot(\rho_0 W\mathbf{v}) = 0, \tag{8}$$

which in this case becomes simply

$$\nabla\cdot(W\mathbf{v}) = 0. \tag{9}$$

The uncurling of Eq. (2) yields, with our assumptions on $\mathbf{E} = (0, 0, \mathrm{E})$

$$\frac{\partial A}{\partial t} + E = F(t), \tag{10}$$

F being a function of time obtained by choosing the constant in A. If we take a fixed point $(x_0,y_0)\in\Omega$ and choose $A(t,x_0,y_0)$ at this point such that (10) holds at it with $F(t) = 0$, then the right hand side of (10) is always zero. Taking this to (1), we obtain the scalar equation for the magnetic flux

$$\frac{\partial^2\mathbf{A}}{\partial t^2} - \Delta\mathbf{A} = \mathbf{J}. \tag{11}$$

$\Delta = \nabla^2$ represents the Laplacian. Since

$$\mathbf{v}\times(\nabla\times\mathbf{A}) = (0,0,-\mathbf{v}\cdot\nabla A), \tag{12}$$

(1) may be written as the scalar equation

$$\frac{\partial^2 A}{\partial t^2} = \Delta A - \frac{W}{\eta}\left(\frac{\partial A}{\partial t} + \mathbf{v}\cdot\nabla A\right). \tag{13}$$

This is a hyperbolic equation, as it should be; parabolic ones, such as the classical MHD induction equation, are not admissible in any relativistic framework because of the instantaneous propagation inherent in them.

To obtain the evolution of the flux energy, we multiply (13) by A and integrate in Ω. In order to do this, we use the following identities:

$$\frac{\partial^2 A}{\partial t^2}\cdot A = \frac{1}{2}\frac{\partial^2 A^2}{\partial t^2} - \left(\frac{\partial A}{\partial t}\right)^2 = \frac{1}{2}\frac{\partial^2 A^2}{\partial t^2} - E^2, \tag{14}$$

$$\Delta A\cdot A = \nabla\cdot(A\nabla A) - |\nabla A|^2 = \nabla\cdot(A\nabla A) - B^2, \tag{15}$$

$$(W\mathbf{v}\cdot\nabla A)A = \frac{1}{2}\nabla\cdot(W\mathbf{v}\nabla A^2). \tag{16}$$

Using the divergence theorem, we obtain the main equation

$$\begin{aligned}&\frac{d^2}{dt^2}\int_\Omega A^2\,dV + \frac{1}{\eta}\int_\Omega W\frac{\partial A^2}{\partial t}\,dV\\&\quad = 2\int_\Omega (E^2 - B^2)\,dV + \int_{\partial\Omega}\frac{\partial A^2}{\partial n}d\sigma - \frac{1}{\eta}\int_{\partial\Omega}A^2 W\mathbf{v}\cdot\mathbf{n}\,d\sigma.\end{aligned} \tag{17}$$

It will be convenient to write (17) in the form

$$\begin{aligned}&\frac{d^2}{dt^2}\int_\Omega A^2\,dV + \frac{1}{\eta}\frac{d}{dt}\int_\Omega WA^2\,dV - \frac{1}{\eta}\int_\Omega A^2\frac{\partial W}{\partial t}\,dV\\&\quad = 2\int_\Omega (E^2 - B^2)\,dV + \int_{\partial\Omega}\frac{\partial A^2}{\partial n}d\sigma - \frac{1}{\eta}\int_{\partial\Omega}A^2 W\mathbf{v}\cdot\mathbf{n}\,d\sigma.\end{aligned} \tag{18}$$

Notice that $E^2 - B^2$ is a Maxwell-Lorentz invariant, and independent of the observer. Its value will determine the evolution of the quadratic mean of the magnetic flux. Integrating in time (18) once, we obtain

$$\begin{aligned}&\frac{d}{dt}\int_\Omega A^2\,dV + \frac{1}{\eta}\int_\Omega WA^2\,dV - \frac{1}{\eta}\int_0^t dr\int_\Omega A^2\frac{\partial W}{\partial t}\,dV\\&\quad = 2\int_0^t dr\int_\Omega (E^2 - B^2)\,dV + \int_0^t dr\int_{\partial\Omega}\left(\frac{\partial A^2}{\partial n} - \frac{1}{\eta}A^2 W\mathbf{v}\cdot\mathbf{n}\right)d\sigma\\&\qquad + \left(\frac{1}{\eta}\int_\Omega WA^2 + \frac{\partial A^2}{\partial t}dV\right)_{t=0}.\end{aligned} \tag{19}$$

We will denote the right hand side of (19) by $f(t)$. It has three terms: the Maxwell-Lorentz invariant, the boundary integral, and a constant due to the initial conditions. The boundary integral itself is the sum of two terms: The first one, involving the normal derivative of A^2, measures the diffusive input (or output) of flux density through the boundary. The second one involves the incoming or outgoing flux transported by the flow; it vanishes, e.g., if the velocity is parallel to the boundary, so that Ω is closed for the flow. These boundary integrals may be dragged throughout the subsequent analysis, but they tend to obscure the role played by the difference between the electric and the magnetic energies. Therefore, we will assume that Ω is closed by the flow and there is no input of magnetic flux. Then (19) becomes

$$\frac{d}{dt}\int_\Omega A^2\,dV+\frac{1}{\eta}\int_\Omega WA^2\,dV-\frac{1}{\eta}\int_0^t dr\int_\Omega A^2\frac{\partial W}{\partial t}\,dV = 2\int_0^t dr\int_\Omega (E^2-B^2)\,dV+\left(\frac{1}{\eta}\int_\Omega WA^2+\frac{\partial A^2}{\partial t}\,dV\right)_{t=0}. \tag{20}$$

We will denote by D and k the following values:

$$D(t)=2\int_0^t dr\int_\Omega (E^2-B^2)\,dV, \tag{21}$$

$$k=\left(\frac{1}{\eta}\int_\Omega WA^2+\frac{\partial A^2}{\partial t}\,dV\right)_{t=0}, \tag{22}$$

so that $f(t)=k+D(t)$.

We close this section by asking if this simple geometry could represent some realistic physical setting. There exist cases of jets created by a Schwarzschild black hole where velocity is mostly radial and the magnetic field dipolar, i.e., both of them poloidal.[20] However, in most cases the magnetic field in a jet is helical in the large scale, so that our model does not apply. The reason of our choice is none other than simplicity: It is one of the few instances where the evolution of electric and magnetic fields is governed by a single scalar equation, and it is eminently reasonable to start the study of a complex problem by considering a tractable example.

III. EVOLUTION OF THE MAGNETIC FLUX WITH MAGNETIC ENERGY LARGER THAN THE ELECTRIC ONE

Take W_0 such that $W\ge W_0$ for all points in Ω and all times. (Recall that always $W\ge 1$, but conceivably 1 is not the best, i.e., largest constant.) Also let γ be another constant such that $\partial W/\partial t\le\gamma$; then (20) implies

$$\frac{d}{dt}\int_\Omega A^2\,dV+\frac{W_0}{\eta}\int_\Omega A^2\,dV-\frac{\gamma}{\eta}\int_0^t ds\int_\Omega A^2\,dV=h(t)\le f(t). \tag{23}$$

Let

$$x(t)=\int_\Omega A^2(t)\,dV. \tag{24}$$

(23) may be written as

$$x'(t)+\frac{W_0}{\eta}x(t)-\frac{\gamma}{\eta}\int_0^t x(r)\,dr=h(t)\le f(t). \tag{25}$$

This integro-differential equation is easy to solve, e.g., by using the Laplace transform. Take first the worst case scenario $\gamma>0$, and consider the polynomial

$$P(s)=s^2+\frac{W_0}{\eta}s-\frac{\gamma}{\eta}, \tag{26}$$

whose roots are

$$s_1=-\frac{W_0}{2\eta}-\frac{1}{2}\sqrt{\frac{W_0^2}{\eta^2}+\frac{4\gamma}{\eta}}<0,\qquad s_2=-\frac{W_0}{2\eta}+\frac{1}{2}\sqrt{\frac{W_0^2}{\eta^2}+\frac{4\gamma}{\eta}}>0. \tag{27}$$

Then

$$\frac{1}{P(z)}=\frac{-s_1}{s_2-s_1}\frac{1}{z-s_1}+\frac{s_2}{s_2-s_1}\frac{1}{z-s_2}, \tag{28}$$

so that the solution to (25) is

$$x(t)=\frac{s_2e^{s_2t}-s_1e^{s_1t}}{s_2-s_1}x(0)+\frac{1}{s_2-s_1}\int_0^t\left[s_2e^{s_2(t-r)}-s_1e^{s_1(t-r)}\right]h(r)\,dr. \tag{29}$$

Notice that all the coefficients $s_2,\ -s_1, s_2-s_1$ are positive. Since $h\le f$

$$x(t)\le\frac{s_2e^{s_2t}-s_1e^{s_1t}}{s_2-s_1}x(0)+\frac{1}{s_2-s_1}\int_0^t\left[s_2e^{s_2(t-r)}-s_1e^{s_1(t-r)}\right]f(r)\,dr. \tag{30}$$

If we integrate by parts the last term, we obtain

$$x(t)\le\frac{s_2e^{s_2t}-s_1e^{s_1t}}{s_2-s_1}x(0)+\frac{e^{s_2t}-e^{s_1t}}{s_2-s_1}f(0)+\frac{1}{s_2-s_1}\int_0^t\left[e^{s_2(t-r)}-e^{s_1(t-r)}\right]f'(r)\,dr. \tag{31}$$

We have

$$f(0)=k,\quad f'(t)=D'(t)=2\int_\Omega (E^2-B^2)(t)\,dt. \tag{32}$$

If we assume that this amount is bounded in time (which certainly occurs if the total energy is bounded), also the integral

$$\frac{1}{s_2-s_1}\int_0^t e^{s_1(t-r)}f'(r)\,dr \tag{33}$$

is bounded. Since $e^{s_1t}\to 0$ as $t\to\infty$, all the terms in (31) multiplied by e^{s_1t} are bounded in time. The remaining quantity is

$$\frac{e^{s_2 t}}{s_2 - s_1}\left[s_2 x(0) + k + \int_0^t e^{-s_2 r} D'(r)\,dr\right]. \tag{34}$$

Since $e^{s_2 t} \to \infty$ as $t \to \infty$, if the term between brackets becomes less than a fixed negative constant $-\delta$ for all $t \geq t_0$, we find

$$x(t) \leq \text{constant} - \frac{\delta e^{s_2 t}}{s_2 - s_1} \to -\infty, \tag{35}$$

whereas $x(t) \geq 0$. This contradiction occurs if

$$s_2 \int_\Omega A(0)^2\,dV + k + \int_0^\infty e^{-s_2 r}\,dr \int_\Omega (E^2 - B^2)(r)\,dV < 0. \tag{36}$$

This shows that the magnetic energy cannot be consistently larger than the electric one. The precise meaning of "consistently larger" is somewhat diffuse, but some examples are clear. Thus, if

$$\int_\Omega (E^2 - B^2)\,dV < -\rho < 0, \tag{37}$$

and

$$s_2 \int_\Omega A(0)^2\,dV + k - \frac{\rho}{s_2} \geq 0, \tag{38}$$

(36) holds. This certainly occurs if ρ is large enough or s_2 small enough (the latter condition meaning that γ is near enough to zero). In fact, a contradiction may be found even if the difference between electric and magnetic energies tends to zero. Explicitly, assume

$$\int_\Omega (E^2 - B^2)\,dV < -\rho t^{-\alpha} < 0, \tag{39}$$

with $\alpha < 1$. Then (36) becomes

$$s_2 \int_\Omega A(0)^2\,dV + k - \rho \frac{\Gamma(1-\alpha)}{s_2^{1-\alpha}} < 0, \tag{40}$$

which again occurs if ρ is large enough or s_2 small enough.

When $\gamma \leq 0$ (i.e., the plasma velocity does not grow), the previous estimates may be improved. We simply ignore the (positive) term involving γ in (25), and find

$$x'(t) + \frac{W_0}{\eta} x(t) \leq f(t), \tag{41}$$

i.e.

$$x(t) \leq e^{-(W_0/\eta)t} x(0) + e^{-(W_0/\eta)t} \int_0^t e^{(W_0/\eta)s} f(s)\,ds. \tag{42}$$

Since $e^{-(W_0/\eta)t} \to 0$ as $t \to \infty$, if $f(t) \leq -\rho < 0$ for $t \geq t_0$ we find that $x(t)$ becomes eventually negative, which is absurd. This in particular occurs if

$$f'(t) = D'(t) = 2\int_\Omega (E^2 - B^2)(t)\,dV < 0, \tag{43}$$

no matter how small, i.e., if the magnetic energy always exceeds the electric one, even if the difference tends to disappear.

We may therefore conclude that the magnetic energy cannot exceed always the electric one, although the precise meaning of this statement depends on the initial conditions and the evolution of the Lorentz factor.

IV. EVOLUTION OF THE MAGNETIC FLUX WITH ELECTRIC ENERGY LARGER THAN THE MAGNETIC ONE

The second case is not a mirror image of the first one. Assume now that there exists constants W_0', γ_0' such that

$$W \leq W_0', \quad \frac{\partial W}{\partial t} \geq \gamma'. \tag{44}$$

The first hypothesis means that the fluid velocity is always less than a fixed fraction of the speed of light, whereas the second implies that the plasma velocity cannot decay too rapidly. γ' may be negative, but in this case we take W_0' large enough for the following condition to hold:

$$\text{If } \gamma' \text{ is negative, then } \quad |\gamma'| < \frac{(W_0')^2}{4\eta}. \tag{45}$$

Then the analogous equation to (23) holds, but now the sign of the inequality is reversed

$$\frac{d}{dt}\int_\Omega A^2\,dV + \frac{W_0'}{\eta}\int_\Omega A^2\,dV - \frac{\gamma'}{\eta}\int_0^t ds \int_\Omega A^2\,dV = h(t) \geq f(t). \tag{46}$$

With the same notation as in (24), we get

$$x'(t) + \frac{W_0'}{\eta} x(t) - \frac{\gamma'}{\eta}\int_0^t x(r)\,dr = h(t) \geq f(t). \tag{47}$$

We take now the roots of the polynomial analogous to the one in (26)

$$s_1' = -\frac{W_0'}{2\eta} - \frac{1}{2}\sqrt{\frac{(W_0')^2}{\eta^2} + \frac{4\gamma'}{\eta}}, \qquad s_2' = -\frac{W_0'}{2\eta} + \frac{1}{2}\sqrt{\frac{(W_0')^2}{\eta^2} + \frac{4\gamma'}{\eta}}. \tag{48}$$

Let us consider the worst case scenario $\gamma' < 0$. With our assumption (44), we have $s_1' < s_2' < 0$. The solution to (47) may be written

$$x(t) = \frac{s_2' e^{s_2' t} - s_1' e^{s_1' t}}{s_2' - s_1'} x(0) + \frac{1}{s_2' - s_1'}\int_0^t [s_2' e^{s_2'(t-r)} - s_1' e^{s_1'(t-r)}] h(r)\,dr. \tag{49}$$

It is convenient to take positive coefficients. Since $|s_i'| = -s_i'$,

$$x(t) = \frac{|s_1'|e^{-|s_1'|t} - |s_2'|e^{-|s_2'|t}}{|s_2' - s_1'|}x(0) + \frac{1}{|s_2' - s_1'|}\int_0^t [|s_1'|e^{-|s_1'|(t-r)} - |s_2'|e^{-|s_2'|(t-r)}]h(r)\,dr. \tag{50}$$

Recall that $h(r) \geq f(r) = k + D(r)$, so that $h(r) - k \geq D(r) \geq 0$. Also

$$\frac{1}{|s_2' - s_1'|}\int_0^t [|s_1'|e^{-|s_1'|(t-r)} - |s_2'|e^{-|s_2'|(t-r)}]k\,dr = \frac{k}{|s_2' - s_1'|}(e^{-|s_2'|t} - e^{-|s_1'|t}). \tag{51}$$

Let

$$\epsilon(t) = \frac{|s_1'|e^{-|s_1'|t} - |s_2'|e^{-|s_2'|t}}{|s_2' - s_1'|}x(0) + \frac{k}{|s_2' - s_1'|}(e^{-|s_2'|t} - e^{-|s_1'|t}), \tag{52}$$

which tends exponentially to zero. Then

$$x(t) = \epsilon(t) + \frac{|s_1'|}{|s_2' - s_1'|}\int_0^t e^{-|s_1'|(t-r)}(h(r) - k)\,dr - \frac{|s_2'|}{|s_2' - s_1'|}\int_0^t e^{-|s_2'|(t-r)}(h(r) - k)\,dr. \tag{53}$$

Since the exponent $-|s_1'|(t-r)$ is smaller than $-|s_2'|(t-r)$, and $h(r) - k \geq 0$,

$$x(t) \geq \epsilon(t) + \frac{|s_1'| - |s_2'|}{|s_2' - s_1'|}\int_0^t e^{-|s_2'|(t-r)}(h(r) - k)\,dr \geq \epsilon(t) + \frac{|s_1'| - |s_2'|}{|s_2' - s_1'|}\int_0^t e^{-|s_2'|(t-r)}D(r)\,dr. \tag{54}$$

Recalling the definition of $D(r)$ in (21), we may see that the last term tends to infinity in several important cases. Certainly if

$$2\int_\Omega (E^2 - B^2)\,dV \geq \rho > 0, \tag{55}$$

because in this case $D(r) \geq \rho r$; even if

$$2\int_\Omega (E^2 - B^2)\,dV \geq \rho t^{-\alpha}, \quad \rho > 0,\ 0 \leq \alpha < 1. \tag{56}$$

Then, $D(r) \geq \rho t^{-\alpha+1}/(-\alpha + 1)$. This implies, according to (54)

$$x(t) \geq \epsilon(t) + \rho\frac{|s_1'| - |s_2'|}{|s_2' - s_1'|}\frac{t^{1-\alpha}}{1-\alpha} - \rho\frac{|s_1'| - |s_2'|}{|s_2' - s_1'|}\int_0^t e^{-|s_1'|(t-r)}r^{-\alpha}\,dr, \tag{57}$$

and all the terms are bounded except for $t^{1-\alpha}$, which tends to ∞.

The reasoning is simplified when $\gamma' \geq 0$, although not as much as in the previous instance. In that case we may omit the corresponding term in (47), obtaining

$$x'(t) + \frac{W_0'}{\eta}x(t) = h(t) \geq f(t), \tag{58}$$

so that

$$x(t) \geq e^{-(W_0'/\eta)t}x(0) + e^{-(W_0'/\eta)t}\int_0^t e^{(W_0'/\eta)s}f(s)\,ds = e^{-(W_0'/\eta)t}x(0) + \frac{\eta}{W_0'}k(1 - e^{-(W_0'/\eta)t}) + e^{-(W_0'/\eta)t}\int_0^t e^{(W_0'/\eta)r}D(r)\,dr. \tag{59}$$

This tends to infinity if D tends to infinity fast enough, certainly in the conditions of (56).

The difference with the previous case is that there is no obvious contradiction as before, where we obtained $x(t) < 0$; instead we obtain $x(t) \to \infty$, i.e.

$$\int_\Omega A^2(t)\,dV \to \infty. \tag{60}$$

Recall that A was chosen so that

$$\frac{\partial A}{\partial t}(t, x_0, y_0) + E(t, x_0, y_0) = 0. \tag{61}$$

Therefore, if we assume that $\int_0^t E(s, x_0, y_0)\,ds$ is bounded in time, either by the electric field decreasing to zero fast enough there, or oscillating between positive and negative values in a way such that the previous integral remains bounded, then also $A(t, x_0, y_0)$ remains bounded. In these conditions, the growth of the magnetic flux implies the one of the magnetic energies. This follows from a result of Deny and Lions;[21] see also Ref. 22, pp. 49-50. For Ω bounded and Lipschitz (i.e., rather smooth), there exists a constant $c(\Omega)$ such that

$$||A(t)||_{L^2(\Omega)} \leq c(\Omega)(||\nabla A(t)||_{L^2(\Omega)} + |A(t, x_0, y_0)|). \tag{62}$$

(In fact this is a particular case of a more general theorem.) Since $|\nabla A| = |\mathbf{B}|$, the result follows. With our assumption that the electric energy exceeds the magnetic one, this would imply that both tend to infinity, and *a fortiori* the total energy (kinetic plus electromagnetic) does the same. General considerations state that this demands an external forcing of some sort. In the absence of such forcing, we must conclude that with reasonable estimates on the Lorentz factor and the assumption that the electric field remains oscillating at some fixed point of the domain, the electric energy cannot be consistently larger than the magnetic one.

V. COWLING'S THEOREM AS THE CLASSICAL LIMIT

It is surprisingly hard to obtain the theorem of Cowling as the limit of these estimates for small velocity. This is because the displacement current in (1) is not multiplied by any small parameter which we may make tend conveniently to zero. In dimensional units, (1) should be written as

$$\epsilon \frac{\partial \mathbf{E}}{\partial t} - \frac{1}{\mu_0} \nabla \times \mathbf{B} = -\mathbf{J}. \tag{63}$$

The formal MHD limit may be reached by making ϵ tend to zero; it must be clear, however, that the dielectric constant cannot be lower than the vacuum one and the reason why the displacement current is neglected is the assumed low frequency of the plasma motions. Nevertheless, it is useful to admit $\epsilon \to 0$ to study the mathematics of this singular limit process. Keeping ϵ and taking for simplicity $\mu_0 = 1$, for low velocities we may assume $W = 1$. Then Eq. (13) becomes

$$\epsilon \frac{\partial^2 A}{\partial t^2} = \Delta A - \frac{1}{\eta}\left(\frac{\partial A}{\partial t} + \mathbf{v} \cdot \nabla A\right). \tag{64}$$

For $\epsilon = 0$, this becomes the MHD induction equation in terms of the magnetic potential. The scaled version of (18) is

$$\epsilon \frac{d^2}{dt^2}\int_\Omega A^2\,dV + \frac{1}{\eta}\frac{d}{dt}\int_\Omega A^2\,dV = 2\int_\Omega (\epsilon E^2 - B^2)\,dV + \int_{\partial\Omega} \frac{\partial A^2}{\partial n}\,d\sigma - \frac{1}{\eta}\int_{\partial\Omega} A^2 \mathbf{v}\cdot\mathbf{n}\,d\sigma. \tag{65}$$

If we ignore the boundary terms as before and integrate (65) one in time, with our previous notation

$$\epsilon x'(t) + \frac{1}{\eta}x(t) = \epsilon x'(0) + \frac{1}{\eta}x(0) + \int_0^t dr \int_\Omega (\epsilon E^2 - B^2)(r)\,dV, \tag{66}$$

whose solution is

$$x(t) = x(0) + \eta\epsilon x'(0)(1 - e^{-t/(\eta\epsilon)}) + \eta\int_0^t (1 - e^{-(t-r)/(\eta\epsilon)})\int_\Omega (\epsilon E^2 - B^2)(r)\,dV. \tag{67}$$

The second term is of order ϵ. The last term, for small epsilon, is dominated by

$$-\eta\int_0^t (1 - e^{-(t-r)/(\eta\epsilon)})\int_\Omega B^2(r)\,dV, \tag{68}$$

which is negative (and moreover depends mostly on the values of B^2 near the time t). We conclude that $x(t)$ decays for small ϵ, and continues decaying until $B^2 = 0$ and therefore A = const. This is Cowling's theorem for a plane (or poloidal) magnetic configuration.

VI. CONCLUSIONS

The assumptions of classical magnetohydrodynamics fail when the plasma velocity is comparable to the speed of light; this fact occurs in several astrophysical situations, such as accretions disks around black holes and galactic jets. In this case, the relativistic electromagnetic evolution equations must be used. While the Maxwell component of these equations is well established, Ohm's law remains a difficult subject, with several proposals in the literature. The simplest one involves a constant resistivity, isotropic and independent of the size of the magnetic field. While range of this law is limited, it may be useful to study how classical results translate to relativistic ones. To avoid the distortions caused by gravitational deformations of space-time, we consider the equations in a Minkowski metric and study if Cowling's theorem (axisymmetric velocities and fields yield a decaying magnetic field) may be generalized. We consider the simple case where the velocity and magnetic field are poloidal while the electric field is toroidal, all of them axisymmetric. With reasonable assumptions on the size and evolution of the Lorentz factor, we find that the magnetic energy cannot be always larger than the electric one, even if the difference tends to zero as an inverse power of time; if we assume also that the electric field oscillates around zero at some fixed point of the domain, we reach the analogous conclusion that the electric energy cannot be consistently larger than the magnetic one. This equipartion result is far less precise than Cowling's theorem, as it allows in principle for both energies to oscillate around each other, but on the other hand it allows for a wider range of physical situations where it is certain that the approximations of classical MHD do not hold. It is also shown that if we approach formally the classical MHD equation by multiplying the displacement current by a small parameter, the same results hold with the magnetic energy divided by this parameter. Thus, it becomes dominant and we recover the decay of magnetic field intensity as predicted by Cowling's theorem.

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