Nonlinear symmetries of perfectly invisible PT-regularized conformal and superconformal mechanics systems

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Abstract

We investigate how the Lax-Novikov integral in the perfectly invisible PT-regularized zero-gap quantum conformal and superconformal mechanics systems affects on their (super)-conformal symmetries. We show that the expansion of the conformal symmetry with this integral results in a nonlinearly extended generalized Shrödinger algebra. The PT-regularized superconformal mechanics systems in the phase of the unbroken exotic nonlinear $\mathcal{N} = 4$ super-Poincaré symmetry are described by nonlinearly super-extended Schrödinger algebra with the $osp(2|2)$ sub-superalgebra. In the partially broken phase, the scaling dimension of all odd integrals is indefinite, and the $osp(2|2)$ is not contained as a sub-superalgebra.

1 Introduction and summary

Conformal mechanics model was introduced and investigated by de Alfaro, Fubini and Furlan (AFF) [1] as a (0+1)-dimensional conformal field theory. It corresponds to the two-particle Calogero system [2] with eliminated center of mass degree of freedom. Supersymmetric extension of the model was considered in [3] and [4]. The geometric aspects of conformal and superconformal mechanics were investigated in [5, 6]. The many-particle generalizations of superconformal mechanics were studied in [7, 8, 9, 10, 11, 12, 13]. A revival of interest to (super)conformal mechanics was induced in connection with the AdS/CFT correspondence [14, 15, 16] when it was observed that the dynamics of a superparticle near the horizon of an extreme Reissner-Nordström black hole is described by superconformal mechanics [17, 18, 19, 20]. In the same line of the AdS/CFT correspondence, recently superconformal mechanics was employed in the study of physics underlying the confinement dynamics in QCD [21, 22]. For some further references and reviews on conformal and superconformal mechanics see [23]–[63].

The conformal mechanics model without confining potential term,

$$H_g = -\frac{d^2}{dx^2} + \frac{g}{x^2}\,, \qquad x > 0\,, \tag{1.1}$$

is related at special values of the coupling constant $g = n(n+1)$, $n = 1, 2, \ldots$, to the Korteweg-de Vries (KdV) equation: at $n = 1$ its potential is a solution of the stationary KdV equation, while for $n > 1$ it satisfies the n-th stationary equation of the KdV hierarchy. This is related to a broader picture according to which the Calogero-Moser systems govern the dynamics of the moving poles of rational solutions to the KdV equation [64, 65, 66]. The conformal mechanics (1.1) with coupling constant $g = n(n+1)$ plays also a special role in the bispectral problem [67] as well as in the Huygens' principle [68].

The potential of conformal mechanics model with the indicated special values of the coupling constant can be obtained from the potentials of the quantum reflectionless and finite-gap systems via appropriate complex shift of the argument and subsequent application of a certain limit procedure [62]. One-dimensional quantum reflectionless and finite-gap systems, in turn, are the algebro-geometric solutions to equations of the KdV hierarchy via the Lax pair representation [69, 70, 71]. Each such quantum system is characterized by a nontrivial Lax-Novikov integral of motion which is a differential operator of odd order $2n+1 \geq 3$. There exists no classical analog for this integral having a purely quantum origin and nature. It detects all the non-degenerate bound and edge-states by annihilating them, and distinguishes the doubly degenerate states inside the valence and conduction/scattering bands in the spectrum of a corresponding quantum system. In the case of reflectionless systems the integral admits a representation in the form of a Darboux-dressed momentum operator of the free quantum particle [44, 72, 73].

The Lax-Novikov integral also plays an important role in supersymmetric constructions. According to the Burchnall-Chaundy theorem [74, 75], the square of the Lax-Novikov integral of differential order $2n+1$ is equal to a (spectral) polynomial of the same order $2n+1$ in Hamiltonian operator of the corresponding quantum system. As a consequence, each non-extended ("purely bosonic") finite-gap or reflectionless quantum mechanical system is characterized by a hidden nonlinear bosonized supersymmetry [72, 73]. On the other hand, it is because of this integral that the convensional $\mathcal{N} = 2$ supersymmetry of the pairs of the Darboux-intertwined reflectionless or finite-gap quantum systems expands up to an exotic nonlinear $\mathcal{N} = 4$ supersymmetric structure which includes the matrix Lax-Novikov integral as a bosonic central charge [76, 77].

A rather natural question is therefore if the Lax-Novikov integral can be identified for the conformal mechanics model with special values of the coupling constant, and if so, how such integral could influence on the conformal and superconformal symmetries.

Some time ago it was observed that the conformal mechanics model with coupling constant $g = n(n+1)$ possesses a differential operator P_n of order $2n+1$, which commutes with the Hamitlonian H_g and satisfies a Burchnall-Chaundy relation of the form $(P_n)^2 = (H_g)^{2n+1}$ [78]. There appears an obstacle, however, that the operator P_n is *not physical* from the point of view of quantum mechanics: as it was shown in [79], acting on eigenstates of the conformal mechanics model satisfying the Dirichlet boundary condition at $x = 0$, it transforms them into formal, non-physical eigenstates of the Hamiltonian operator which satisfy Neumann boundary condition at $x = 0$.

In spite of a non-physical nature of the operator P_n from the point of view of the quantum mechanical system H_g with $g = n(n+1)$, it is the Lax-Novikov operator which underlies the above-mentioned relation of conformal mechanics to the KdV equation and its hierarchy. The Burchnall-Chaundy relation means an algebraic dependence between H_g and the formal integral P_n, and so, the presence of the latter does not influence on integrablity of one-dimensional quantum systems (1.1), which, as any other one-dimensional quantum system with conserved Hamiltonian, is (maximally) super-integrable. However, this relation has a super-algebraic nature implying that the Lax-Novikov operator is a kind of a square root operator from the odd order polynomial (monomial here) in Hamiltonian.

In a recent paper [62] it was shown that the indicated deficiency of the Lax-Novikov operator P_n can be cured by the PT-regularization of the conformal mechanics model by making a shift $x \to x + i\alpha$, where α is a nonzero real parameter, and extending x from the half-line $x > 0$ to the whole real line $x \in \mathbb{R}$. With such a shift and extended domain, the Hamiltonian $H_n(x+i\alpha) = -\frac{d^2}{dx^2} + \frac{n(n+1)}{(x+i\alpha)^2}$ is PT-symmetric satisfying the relation $[PT, H_n(x+i\alpha)] = 0$ [80, 81]. Here P is a space reflection (parity) operator, $Px = -xP$, $P^2 = 1$, and a complex conjugation operator T is defined by $Tz = \bar{z}T$, $T^2 = 1$, where $z \in \mathbb{C}$ is an arbitrary complex number [1]. The obtained quantum systems $H_n(x + i\alpha)$ are characterized by the property of the perfect invisibility : the transmission amplitude in them is not simply just a phase as it happens in the case of reflectionless systems, but exactly equals one like in the free quantum particle system. Unlike the free quantum particle, however, each such a system has a unique quadratically integrable bound state of zero energy at the very edge of the continuous part of the spectrum [2]. The corresponding systems are identified as *perfectly invisible zero-gap* systems, and they possess some other interesting properties due to their relation to the KdV hierarchy [62]. It may be noted that the parameter α is a constant with dimension of length, and in this aspect the PT-regularization of the conformal mechanics model turns out to be alternative in some sense to regularization of the conformal mechanics model considered in the original article [1], where it was realized effectively via the introduction into the Hamiltonian (1.1) of the confining harmonic oscillator potential term accompanied by a length parameter.

It was observed for the first time by Bender and Boettcher in the pioneering work [85] that Hermiticity is not a necessary condition for the reality of the spectrum of a quantum mechanical system, and that it can be substituted for the requirement of the PT-symmetry of Hamiltonian [3]. Later this type of systems was investigated in different aspects, in particular, in the context of connection between the theories of ordinary differential equations and integrable models [86], and supersymmetry [87]. The results on reality of the spectrum were extended then for non-Hermitian Hamiltonians of a more general form, for reviews see refs. [80] and [81]. The PT-regularization applied in [62] to (1.1) was employed earlier in general context of the PT-symmetry for the AFF model with confining potential term [88]. PT-symmetric multi-soliton solutions to the Korteweg-de Vries equation were discussed

[1]The peculiarity of PT-symmetry is associated with the anti-unitary character of the operator T [82].

[2]A similar picture appears in finite-gap systems where non-periodic defects can produce bound states at the very edge of the valence and conduction bands [83, 84].

[3]Reality of spectrum in some quantum mechanical systems with non-Hermitian Hamiltonian operators was observed earlier, but the reason of this phenomenon associated with the presence of PT-symmetry was established for the first time in [85], see the corresponding discussion in [80].

recently in [89], and PT-symmetric deformations of Calogero models were studied in [90, 91]. Nowadays PT-symmetry finds interesting applications in diverse areas of physics [92].

- The purpose of this article is to investigate how the Lax-Novikov integral in the PT-regularized quantum conformal and superconformal mechanics models affects on their (super)-conformal symmetries.

It is necessary to stress here that the PT-regularization $x \to x + i\alpha$ cannot be considered as a kind of a simple analytic continuation of the models (1.1) since the domain of the latter corresponds to the half-line, while the PT-regularized systems are defined on the whole real line $x \in \mathbb{R}$. As we shall see, this results in essential difference in symmetry properties of the PT-regularized systems $H_n(x + i\alpha)$ and of their supersymmetric versions in comparison with (1.1) and its superextensions. The essential difference can also be expected *a priori* if, by analogy, we compare the properties of the quantum systems described by potentials $u_1 = \frac{n(n+1)}{\sinh^2 x}$ and $u_2 = -\frac{n(n+1)}{\cosh^2 x}$. Both potentials (shifted for appropriate additive constant terms) are solutions to the corresponding stationary equations of the KdV hierarchy, and the second quantum system is related to the first one by a simple complex shift $x \to x + i\pi/2$. The first quantum system, however, is singular at $x = 0$ being defined on the real half-line $x > 0$ (or $x < 0$), its non-degenerate spectrum is continuous and has no ground state. The second system is reflectionless being defined on the whole real line, its continuous spectrum is doubly degenerate (except of the non-degenerate state with $E = 0$ at the lower edge), and has n non-degenerate negative energies corresponding to bound states, with energy of the ground state $\psi_0(x) = 1/\cosh^n x$ equal to $E_0 = -n^2$ [72, 44].

Our results can be summarized briefly as follows. We first show that the extension of the set of generators of conformal symmetry of the PT-regularized conformal mechanics model with coupling constant $g = 2$ ($n = 1$) by its Lax-Novikov integral generates three more dynamical (explicitly depending on time) integrals of motion together with a central charge. As a result, the conformal $so(2,1) \simeq sl(2,\mathbb{R})$ Lie algebra expands up to a nonlinearly extended Schrödinger algebra, in which the $so(2,1)$ generators appear quadratically in commutation relations of the four new non-trivial (including Lax-Novikov) integrals of motion. With respect to the adjoint action of the $so(2,1)$ generators, the complete set of integrals separates into one-dimensional representation corresponding to the mass central charge, while the rest of generators are eigenstates of eigenvalues $(-3/2, -1, -1/2, 0, 1/2, 1, 3/2)$ of the dilatation generator. The simplest supersymmetric extension of the PT-regularized $g = 2$ conformal mechanics model is realized via a usual construction of $\mathcal{N} = 2$ supersymmetric quantum mechanics based on superpotential $\mathcal{W}_e = -1/(x + i\alpha)$. In this case the PT-regularized conformal mechanics model is paired with the free particle, and the system is described by the exotic nonlinear $\mathcal{N} = 4$ super-Poncaré algebra which corresponds to the phase of *exact*, unbroken supersymmetry with a non-degenerate zero energy ground state. The matrix Lax-Novikov integral is generated by anticommutator of the first and second order supercharges. The extension of the set of generators of the exotic nonlinear $\mathcal{N} = 4$ super-Poncaré algebra by the matrix generators of the dilatations and special conformal symmetry transformations gives rise to the nonlinearly extended generalized super-Schrödinger algebra. The set of its nontrivial bosonic generators includes the matrix generalization of the above mentioned seven integrals of motion, the generator of a $u(1)$ R-symmetry, and two more integrals which are the momentum and the Galileo boost generator of the free particle subsystem. The set of the fermionic generators includes three pairs of dynamical integrals of motion in addition to the two pairs of the supercharges of the nonlinear $\mathcal{N}=4$ super-Poncaré sub-superalgebra. The superconformal $osp(2|2)$ symmetry is the Lie sub-superalgebra, whose expansion by any other even or odd integral of motion results in generation of the whole nonlinearly super-extended Schrödinger algebra. All the even and odd integrals form a supermultiplet with respect to the adjoint action of the generators of the $osp(2|2)$ superconformal symmetry, and the structure coefficients in the (anti)commutation relations between the rest of the even and odd integrals are linear in generators of the conformal $so(2,1)$ symmetry.

A general case of the PT-regularized conformal mechanics with coupling constant $g = n(n+1)$ is characterized by a symmetry described by a nonlinearly extended Schrödinger algebra generated by $2n+5$ integrals of motion whose scaling dimensions are $(-(n+1/2), -(n-1/2), \ldots, n-1/2, n+1/2)$, plus a central charge. The $\mathcal{N}=2$ supersymmetric extension of this PT-regularized conformal mechanics model is realized by the construction of supersymmetric quantum mechanical system given by a superpotential $\mathcal{W}_e = -n/(x+i\alpha)$. The obtained in such a way 2×2 matrix system is characterized by the exotic nonlinear $\mathcal{N}=4$ super-Poincaré type symmetry, in which the anticommutator of the supercharges, being operators of differential orders 1 and $2n$, generates the matrix Lax-Novikov integral. Extension of this nonlinear $\mathcal{N}=4$ super-Poincaré algebra by generators of dilatations and special conformal transformations results in expansion of the superalgebra up to a nonlinear super-extended Shrödinger symmetry, which contains the superconformal $osp(2|2)$ algebra as a Lie sub-superalgebra. With respect to the adjoint action of the $osp(2|2)$ generators, the rest of the $4n+2$ bosonic and $4n+2$ fermionic integrals form an irreducible represenation. The structure coefficients in (anti)-commutation relations between additional even and odd generators are polynomials of order $2n-1$ in generators of the conformal $so(2,1)$ symmetry of the system.

We also consider supersymmetric system given by the superpotential $\mathcal{W}_b = 1/(x+i\alpha_1) - 1/(x+i\alpha_2) + i/(\alpha_1-\alpha_2)$, $\mathbb{R} \ni \alpha_j \neq 0$, $j=1,2$, $\alpha_1 \neq \alpha_2$. It represents a matrix system of the Darboux-paired PT-regularized conformal mechanics models with $g=2$ characterized by different values of the shift parameters α_1 and α_2. This system is described by the spontaneously *partially broken* phase of the exotic nonlinear $\mathcal{N}=4$ super-Poncaré symmetry. The essential peculiarity of the system is that all its fermionic generators commute nontrivially with the matrix dilatation generator, neither of them has a definite scaling dimension. The superalgebra of the system represents some nonlinear super-extension of the Schrödinger symmetry which *does not* include the superconformal $osp(2|2)$ symmetry as a sub-superalgebra. Its generators transform nontrivially into those of the system given by superpotential $\mathcal{W}_e = -1/(x+i\alpha)$ in the limit when one of the PT-regularization parameters is sent to infinity.

The paper is organized as follows. In Section 2, we describe briefly the construction of the perfectly invisible PT-regularized zero-gap conformal mechanics systems together with their Lax-Novikov integrals. The nonlinearly extended Schrödinger symmetry of the PT-regularized conformal mechanics model with coupling constant $g=2$ is discussed in Section 3. Symmetries of the simplest superconformal extension of this system are investigated in Section 4. The results of the two previous sections are generalized for the case of $g =$

$n(n+1)$ in Section 5. The case of the super-extended conformal mechanics with $g=2$ in the phase of spontaneously partially broken exotic nonlinear $\mathcal{N}=4$ super-Poincaré symmetry is considered in Section 6. The concluding discussion is presented in Section 7.

2 PT-regularized conformal mechanics models

Let us re-denote the Hamiltonian operator of the conformal mechanics model (1.1) with non-negative coupling constant $g=\nu(\nu+1)$, $\nu\geq 0$, as H_ν. It admits two factorizations $H_\nu=A_\nu A_\nu^{\#}=A_{\nu+1}^{\#}A_{\nu+1}$ in terms of the first order differential operators $A_\nu=x^\nu\frac{d}{dx}x^{-\nu}=\frac{d}{dx}-\frac{\nu}{x}$, $A_\nu^{\#}=A_\nu^\dagger=-\frac{d}{dx}-\frac{\nu}{x}$. The A_ν and $A_\nu^{\#}$ intertwine the Hamiltonian operators with different in one values of the index: $A_\nu H_{\nu-1}=H_\nu A_\nu$, $A_\nu^{\#}H_\nu=H_{\nu-1}A_\nu^{\#}$. The system H_ν is defined on the half-axis $x>0$, and the case with $\nu=0$ can be considered as a limit case $\nu\to 0$ corresponding to the quantum free particle on the half-line $x>0$ subject to the Dirichlet boundary condition $\psi(0)=0$ [62]. We denote the Hamiltonian operator of such a limit system by H_0^+. The case of integer values of the parameter $\nu=n$ is special in the sense that the conformal mechanics model Hamiltonian H_n can be intertwined with that of H_0^+, $\mathcal{A}_n^{\#}H_n=H_0^+\mathcal{A}_n^{\#}$, $\mathcal{A}_nH_0^+=H_n\mathcal{A}_n$, by the generators of the Darboux-Crum transformation given by the n-th order differential operators $\mathcal{A}_n=A_nA_{n-1}\ldots A_1$ and $\mathcal{A}_n^{\#}=\mathcal{A}_n^\dagger$. In correspondence with this, the eigenstates $\psi_k^{(0)}(x)=\sin kx$ of $H_0^\dagger$ are mapped into eigenstates of H_n, $\psi_k^{(n)}(x)=\mathcal{A}_n\psi_k^{(0)}(x)$, $H_n\psi_k^{(n)}=k^2\psi_k^{(n)}$. Unlike the quantum free particle system $H_0=-\frac{d^2}{dx^2}$, $x\in\mathbb{R}$, the half-free particle system H_0^+ is not translation invariant and $P_0=-i\frac{d}{dx}$ *is not* its physical operator (observable) since acting on the states $\psi_k^{(0)}(x)$ it transforms them into the wave functions satisfying the Neumann boundary condition $\psi'(0)=0$ instead of the Dirichlet boundary condition. As a result, instead of the algebra of Schrödinger symmetry of the free particle on a whole line given by the Hamiltonian H_0, the conformal mechanics systems H_n, including the case H_0^+, are described only by the algebra $sl(2,\mathbb{R})$ of its conformal symmetry subgroup.

By virtue of the intertwining relations between H_n and H_0^+, each of the systems H_n possesses a *formal* integral of motion $P_n=\mathcal{A}_nP_0\mathcal{A}_n^{\#}$, $[P_n,H_n]=0$, which, similarly to the operator P_0 for H_0^+, 'conflicts' with the Dirichlet boundary condition, and so, *is not* a physical operator. This 'deficiency' can be removed by the PT-symmetric regularization of conformal mechanics systems H_n via a purely imaginary shift of the argument, $x\to x+i\alpha$, $\mathbb{R}\ni\alpha\neq 0$, accompanied by extension of x from the half-line to the whole real line. As it was shown in [62], the resulting Hamiltonian

$$H_n^\alpha=-\frac{d^2}{dx^2}+\frac{n(n+1)}{(x+i\alpha)^2}\,,\qquad x\in\mathbb{R}\,,\tag{2.1}$$

describes a perfectly invisible zero-gap PT-symmetric system, which is characterized by a *purely real* spectrum in conformity with general properties of such class of the systems [80, 81]. The peculiarity of the system (2.1), however, is that its transmission amplitude is equal to one for all values of energy $E>0$, and it has one bound state of zero energy given by a square-integrable on $\mathbb{R}$ wave function $\psi_0^{(n)}(x)=1/(x+i\alpha)^n$ generated from the eigenstate $\psi_0^{(0)}=1$ of zero energy of the free particle system: $\psi_0^{(n)}=\mathcal{A}_n^\alpha\psi_0^{(0)}$, where $\mathcal{A}_n^\alpha=A_n^\alpha A_{n-1}^\alpha\ldots A_1^\alpha$,

$A_l^\alpha = \frac{d}{dx} - \frac{l}{x+i\alpha}$. The Lax-Novikov integral

$$P_n^\alpha = \mathcal{A}_n^\alpha P_0 \mathcal{A}_n^{\alpha\#}\,, \tag{2.2}$$

where $\mathcal{A}_n^{\alpha\#} = A_1^{\alpha\#}\ldots A_n^{\alpha\#}$, $A_l^{\alpha\#} = -\frac{d}{dx} - \frac{l}{x+i\alpha}$, detects the ground state $\psi_0^{(n)}(x)$ by annihilating it, and distinguishes the deformed plane wave eigenstates $\psi^{\pm k} = \mathcal{A}_n^\alpha e^{\pm ikx}$ with $E = k^2 > 0$ in the continuous part of the spectrum, $P_n^\alpha \psi^{\pm k} = \pm k^{2n+1}\psi^{\pm k}$.

We pass over now to investigation of the effect of the presence of the Lax-Novikov integrals on symmetries of the PT-regularized conformal mechanics systems H_n^α and of their $\mathcal{N}=2$ super-extended versions. To this aim we first consider the case of the simplest system H_1^α, and then we study its supersymmetric extension.

3 Symmetries of the H_1^α system

The generator of Galilean transformations $G_0 = x - 2tP_0 = x + 2it\frac{d}{dx}$ of the free quantum particle $H_0 = -\frac{d^2}{dx^2}$, $x \in \mathbb{R}$, depends explicitly on the time parameter t, and satisfies Heisenberg equation of motion of the form $i\frac{d}{dt}G_0 = i\frac{\partial G_0}{\partial t} - [H_0, G_0] = 0$. In correspondence with this property, G_0 is identified as a dynamical integral of motion. Two other dynamical integrals of motion of H_0 are generators of dilatations, $D_0 = \frac{1}{4}\{G_0, P_0\} = -\frac{i}{2}\left(x\frac{d}{dx} + \frac{1}{2}\right) - tH_0$, and special conformal transformations, $K_0 = (G_0)^2 = x^2 - 8tD_0 - 4t^2H_0$. The integrals G_0, D_0 and K_0 are not translationally-invariant, $G_0(x+\tau) = G_0(x) + \tau$, $\mathcal{D}_0(x+\tau) = \mathcal{D}_0(x) + \frac{1}{2}\tau P_0$, $K_0(x+\tau) = K_0(x) + \frac{1}{2}\tau G_0(x) + \frac{1}{4}\tau^2$. The set of integrals H_0, P_0, G_0, D_0 and K_0 and the unit operator $\mathbb{I}$ generate the Schrödinger algebra

$$[D_0, H_0] = iH_0\,, \qquad [D_0, K_0] = -iK_0\,, \qquad [K_0, H_0] = 8iD_0\,, \tag{3.1}$$

$$[D_0, P_0] = \tfrac{i}{2}P_0\,, \qquad [D_0, G_0] = -\tfrac{i}{2}G_0\,, \tag{3.2}$$

$$[H_0, G_0] = -2iP_0\,, \qquad [H_0, P_0] = 0\,, \tag{3.3}$$

$$[K_0, P_0] = 2iG_0\,, \qquad [K_0, G_0] = 0\,, \tag{3.4}$$

$$[G_0, P_0] = i\,\mathbb{I}\,. \tag{3.5}$$

This Lie algebra describes symmetry of the free particle. It is the semi-direct sum of the conformal algebra $sl(2,\mathbb{R})$ generated by H_0, D_0 and K_0, and of the one-dimensional Heisenberg algebra generated by P_0, G_0 and $\mathbb{I}$. The identity operator $\mathbb{I}$ (in the chosen units $\hbar = 1$, $m = 1/2$) is the mass central element of the algebra. According to Eqs. (3.2), (3.3) and (3.4), the generators of translations, P_0, and Galileo transformations, G_0, form a doublet under the adjoint action of the generators H_0, K_0 and D_0 of the conformal $sl(2,\mathbb{R})$ symmetry. The algebra (3.1)–(3.5) is characterized by the automorphism corresponding to a spatial reflection, $\rho_1 : P_0 \to -P_0$, $G_0 \to -G_0$, $H_0 \to H_0$, $K_0 \to K_0$, $D_0 \to D_0$, $\mathbb{I} \to \mathbb{I}$. It also has another automoprphism

$$\rho_2: \; H_0 \to K_0\,, \quad K_0 \to H_0\,, \quad D_0 \to -D_0\,, \quad P_0 \to -G_0\,, \quad G_0 \to P_0\,, \quad \mathbb{I} \to \mathbb{I}\,, \tag{3.6}$$

which at $t = 0$ corresponds to a unitary (canonical) transformation $x \to -i\frac{d}{dx}$, $-i\frac{d}{dx} \to -x$.

Let us look now for extension and generalization of the Schrödinger Lie algebra (3.1)–(3.5) for the case of the PT-regularized conformal mechanics model H_1^α. For this we first identify formally the integrals for the conformal mechanics model H_1 by Darboux-dressing the integrals of H_0 and considering their commutation relations. Only generators of the conformal $sl(2,\mathbb{R})$ symmetry identified in such a way will be true integrals of motion of H_1, while other formal integrals will be non-physical: acting on physical eigentstates of H_1 satisfying the Dirichlet boundary condition $\psi(0) = 0$, they produce non-physical states satisfying the Neumann boundary condition $(\frac{d}{dx}\psi)(0) = 0$. The subsequent shift $x \to x + i\alpha$ (accompanied by extension $x > 0 \to x \in \mathbb{R}$ and omission of boundary condition for wave functions at $x = 0$) transforms then all the true and formal integrals of H_1 into the true integrals of motion of the PT-regularized conformal mechanics system H_1^α. The key point also is that here the substitution $x \to x + i\alpha$, $\frac{d}{dx} \to \frac{d}{dx}$ changes the operators but does not touch the form of all the corresponding nonlinear algebraic relations they satisfy.

Similarly to a formal (non-physical) integral P_1 obtained via the Darboux-dressing of the free particle momentum, we identify the analog of G_0 to be a differential operator

$$G_1(x) \equiv A_1(x)G_0(x)A_1^\#(x) = x\frac{d}{dx}A_1^\#(x) - 2tP_1(x)\,. \tag{3.7}$$

Here the argument in integrals $P_1(x)$ and $G_1(x)$ is shown to stress that they are obtained from the corresponding integrals of the free particle H_0 via Darboux-dressing by the intertwining operators $A_1(x) = \frac{d}{dx} - \frac{1}{x}$ and $A_1^\#(x) = -\frac{d}{dx} - \frac{1}{x}$. The analog of $D_0(x)$ for the system $H_1(x) = A_1(x)A_1^\#(x)$ is

$$D_1(x) = -\tfrac{i}{2}\left(x\tfrac{d}{dx} + \tfrac{1}{2}\right) - tH_1(x)\,. \tag{3.8}$$

It can be extracted from the Darboux-dressed form of $D_0(x)$ by using the relations $A_1(x)D_0(x)A_1^\#(x) = \left(D_1(x) - \frac{i}{2}\right)H_1(x) = H_1(x)\left(D_1(x) + \frac{i}{2}\right)$, Analogously, from the relations $A_1(x)K_0(x)A_1^\#(x) = K_1(x)H_1(x) - 4iD_1(x) - 1 = H_1(x)K_1(x) + 4iD_1(x) - 1$ we extract a dynamical integral

$$K_1(x) = x^2 - 8tD_1(x) - 4t^2H_1(x)\,. \tag{3.9}$$

The operators D_1, K_1 and H_1 generate the $sl(2,\mathbb{R})$ algebra of the form (3.1):

$$[D_1, H_1] = iH_1\,, \qquad [D_1, K_1] = -iK_1\,, \qquad [K_1, H_1] = 8iD_1\,. \tag{3.10}$$

The commutation relations

$$[H_1, G_1] = -2iP_1\,, \qquad [H_1, P_1] = 0\,,$$

are a direct analog of (3.3), while relations (3.2) and the first relation from (3.4) are replaced by

$$[D_1, P_1] = \tfrac{3}{2}iP_1\,, \qquad [D_1, G_1] = \tfrac{i}{2}G_1\,, \tag{3.11}$$

$$[K_1, P_1] = 6iG_1\,. \tag{3.12}$$

Instead of (3.5) we have a nonlinear commutation relation

$$[G_1, P_1] = 3i(H_1)^2\,. \tag{3.13}$$

The zero commutator $[K_0, G_0] = 0$ is changed for nonzero one,

$$[K_1, G_1] = -4iV_1\,, \tag{3.14}$$

where

$$V_1(x) = ix^2 A_1^{\#}(x) - 4tG_1(x) - 4t^2 P_1(x) \tag{3.15}$$

is identified as a new explicitly depending on time formal integral of motion for the system H_1. It has the following commutation relations with other integrals:

$$[V_1, H_1] = 4iG_1\,, \qquad [V_1, D_1] = \tfrac{i}{2}V_1\,, \qquad [V_1, K_1] = 2iR_1\,, \tag{3.16}$$

$$[V_1, P_1] = 12iH_1D_1 - 6H_1\,, \qquad [V_1, G_1] = 12i(D_1)^2 + \tfrac{3}{4}i\,\mathbb{I}\,. \tag{3.17}$$

Here the operator

$$R_1(x) = x^3 - 6tV_1(x) - 12t^2 G_1(x) - 8t^3 P_1 \tag{3.18}$$

has to be identified as yet another new formal dynamical integral of motion of H_1. Its commutation relations with the rest of the integrals are

$$[R_1, H_1] = 6iV_1\,, \qquad [R_1, D_1] = \tfrac{3}{2}iR_1\,, \quad [R_1, K_1] = 0\,, \tag{3.19}$$

$$[R_1, P_1] = 36i\,D_1^2 + \tfrac{21}{4}i\,\mathbb{I}\,, \quad [R_1, G_1] = 12i\,D_1K_1 - 6K_1\,, \quad [R_1, V_1] = 3i\,K_1^2\,. \tag{3.20}$$

It may be noted here that the commutation relations of the form similar to (3.13) are satisfied by ladder operators in rationally deformed harmonic oscillator systems as well as in rationally deformed conformal mechanics model of de Alfaro-Fubini-Furlan with the included confining harmonic potential term [93, 94].

The explicitly depending on time *formal* integrals of H_1 have been identified via the Darboux-dressing of the corresponding integrals of the free particle with additional step of subsequent 'extraction' in the case of the integrals D_1 and K_1. The dynamical integrals also can be obtained via the "time-dressing" by the evolution operator $U_1(t) = \exp(iH_1t)$. For this we note that the dynamical integral $X_1(t) = U_1^{-1}(t)xU_1(t)$ is given by an infinite series in t and $\frac{d}{dx}$, and so, is a *nonlocal* in x operator. This is essentially different from the free particle case where $U_0(t) = \exp(iH_0t)$ and $U_0^{-1}(t)xU_0(t) = G_0$ is the local operator G_0. However, the time dressing of the operators x^2, $\frac{1}{4}\{x, -i\frac{d}{dx}\}$, $x\frac{d}{dx}A_1^{\#}$, $ix^2A_1^{\#}(x)$ and x^3 generates the local operators to be exactly the dynamical integrals of motion $K_1(x)$, $D_1(x)$, $G_1(x)$, $V_1(x)$ and $R_1(x)$, respectively.

The integrals P_1 and G_1, being Darboux-dressed free particle's generators of translations, P_0, and Galileo transformations, G_0, are the third order differential operators [4]. According to (3.2) and (3.11), the scaling dimensions $-1/2$ and $+1/2$ of P_0 and G_0 given by relation $i[D_0, F] = s_F F$, are changed here for the scaling dimensions $-3/2$ and $-1/2$ of P_1 and G_1 given by analogous relation with D_0 changed for D_1. Coherently with this, the central charge in commutation relation (3.5) is changed in (3.13) for the operator $(H_1)^2$ having the scaling dimension -2. In addition, two new formal dynamical integrals V_1 and R_1 of the scaling dimensions $+1/2$ and $+3/2$ are generated via the commutation of G_1 with generator of the special conformal transformations K_1 having the scaling dimension $+1$. As a result, instead

[4] The not-depending explicitly on t term in (3.7) is the second order differential operator.

of the Lie algebraic structure of the Schrödinger symmetry of the free particle we obtain the nonlinear algebra in which the commutators (3.13), (3.17) and (3.20) are quadratic in generators of the $sl(2,\mathbb{R})$ Lie subalgebra (3.10). Instead of the doublet of integrals (G_0, P_0) under the adjoint action of the operators H_0, K_0 and D_0 in the case of the free particle, here we have the quartet (R_1, V_1, G_1, P_1) under the adjoint action of the $sl(2,\mathbb{R})$ generators H_1, K_1 and D_1.

The described nonlinear algebra is characterized by the automorphism corresponding to a spatial reflection ρ_1. It also has the automorphism ρ_2,

$$\rho_2: \; H_1 \to K_1 \to H_1\,, \; D_1 \to -D_1\,, \; P_1 \to -R_1\,, \; R_1 \to P_1\,, \; V_1 \to -G_1\,, \; G_1 \to V_1\,. \tag{3.21}$$

Till the moment we discussed the integrals of the system H_1. Except the generators H_1, D_1 and K_1 of the $sl(2,\mathbb{R})$ symmetry, the rest of them are formal integrals being non-physical operators. By shifting the argument $x \to x + i\alpha$ and extending $x > 0$ for $x \in \mathbb{R}$, we obtain the corresponding set of the true integrals H_1^α, D_1^α, K_1^α, P_1^α, G_1^α, V_1^α and R_1^α of the PT-regularized conformal mechanics system H_1^α. They satisfy the nonlinearly extended Schrödinger algebra of the same described form.

4 Symmetries of the $\mathcal{N} = 2$ super-extended H_1^α system

Consider now the extended system described by the diagonal matrix Hamiltonian operator

$$\mathcal{H}^\alpha = \text{diag}\,(H_1^\alpha,\; H_0) \tag{4.1}$$

composed from the PT-regularized conformal mechanics Hamiltonian H_1^α and the free particle Hamiltonian H_0. To simplify notations, below we omit the upper index α in the Hamiltonian and matrix integrals of the system (4.1). The system (4.1) is described by the superpotential $\mathcal{W}_e = -1/(x + i\alpha)$, $\mathcal{H} = -\frac{d^2}{dx^2} + \mathcal{W}_e^2 - \mathcal{W}_e'\sigma_3$, $\mathcal{W}_e' = \frac{d}{dx}\mathcal{W}_e$, and is characterized by the supercharges

$$\mathcal{Q}_1 = \begin{pmatrix} 0 & A_1^\alpha \\ A_1^{\alpha\#} & 0 \end{pmatrix}, \qquad \mathcal{Q}_2 = i\sigma_3 \mathcal{Q}_1\,, \tag{4.2}$$

which are the matrix first order differential operators, where $A_1^\alpha = \frac{d}{dx} + \mathcal{W}_e(x)$, $A_1^{\alpha\#} = -\frac{d}{dx} + \mathcal{W}_e(x)$. The peculiarity of the system (4.1) is that besides the generators of the $\mathcal{N} = 2$ supersymmetry $\mathcal{Q}_a$, $a = 1, 2$, it also possesses two more supercharges

$$\mathcal{S}_1 = \begin{pmatrix} 0 & -iA_1^\alpha P_0 \\ iP_0 A_1^{\alpha\#} & 0 \end{pmatrix}, \qquad \mathcal{S}_2 = i\sigma_3 \mathcal{S}_1\,. \tag{4.3}$$

The appearance of additional supercharges (4.3) to be matrix differential operators of the second order, as it clear from their structure, is explained by existence of the momentum integral P_0 in the free particle system. As a consequence, the Hamiltonian operator H_1^α can be intertwined with the free particle Hamiltonian H_0 not only by the first order operators A_1^α and $A_1^{\alpha\#}$ but also by the second order operators from which supercharges (4.3) are composed.

These four supercharges satisfy the relations

$$[\mathcal{H},\mathcal{Q}_a]=0\,,\qquad [\mathcal{H},\mathcal{S}_a]=0\,, \tag{4.4}$$

$$\{\mathcal{Q}_a,\mathcal{Q}_b\}=2\delta_{ab}\mathcal{H}\,,\qquad \{\mathcal{S}_a,\mathcal{S}_b\}=2\delta_{ab}\mathcal{H}^2\,, \tag{4.5}$$

$$\{\mathcal{Q}_a,\mathcal{S}_b\}=2\epsilon_{ab}\mathcal{L}\,, \tag{4.6}$$

$$[\mathcal{L},\mathcal{H}]=[\mathcal{L},\mathcal{Q}_a]=[\mathcal{L},\mathcal{S}_a]=0\,, \tag{4.7}$$

where

$$\mathcal{L}=\mathrm{diag}\,(P_1^\alpha,\,H_0P_0) \tag{4.8}$$

is the bosonic integral of motion composed from the Lax-Novikov integral P_1^α of the subsystem H_1^α and the momentum operator P_0 of the free particle subsystem H_0. The integrals $\mathcal{H}$, $\mathcal{Q}_a$, $\mathcal{S}_a$ and $\mathcal{L}$ generate the exotic nonlinear $\mathcal{N}=4$ superalgebra of the system (4.1), in which the operator $\mathcal{L}$ plays a role of the bosonic central charge. System (4.1) has the unique bound state of zero energy given by a square-integrable on $\mathbb{R}$ wave function of the form $\Psi^t=((x+i\alpha)^{-1},0)^t$, which is annihilated by all the superacharges $\mathcal{Q}_a$ and $\mathcal{S}_a$ as well as by the Lax-Novikov integral $\mathcal{L}$.

We note here that the extension of the $\mathcal{N}=2$ supersymmetry up to the exotic nonlinear $\mathcal{N}=4$ supersymmetry in general case of the quantum reflectionless and finite-gap systems also is based on existence of the two pairs of intertwiners to be differential operators of the even and odd orders. In finite-gap systems, however, no analog of the free particle system with its proper integral of motion P_0 does appear. In another way the extension can be related to a presentation of the Lax-Novikov integral in such systems in the form of the product of two non-singular operators of the even and odd differential orders. For the detailed discussion of these aspects see refs. [44, 76, 77, 83, 84, 95], where also an important phenomenon of reduction of higher order intertwining operators to intertwining operators of a lower order is discussed. We consider some example of the reduction below in Section 6. Because of such a reduction mechanism, each finite-gap or reflectionless system, including the PT-regularized conformal mechanics systems (2.1), is characterized by a corresponding unique irreducible Lax-Novikov integral: reducible intertwining operators generate the same Lax-Novikov integral up to multiplication by a polynomial in the system's Hamiltonian.

The Hamiltonian (4.1) together with the matrix dynamical bosonic integrals $\mathcal{D}=\mathrm{diag}\,(D_1^\alpha,D_0^\alpha)$ and $\mathcal{K}=\mathrm{diag}\,(K_1^\alpha,K_0^\alpha)$ generates the conformal algebra $sl(2,\mathbb{R})$,

$$[\mathcal{D},\mathcal{H}]=i\mathcal{H}\,,\qquad [\mathcal{D},\mathcal{K}]=-i\mathcal{K}\,,\qquad [\mathcal{K},\mathcal{H}]=8i\mathcal{D}\,, \tag{4.9}$$

where we use the notation $D_0^\alpha=D_0(x+i\alpha)$, $K_0^\alpha=K_0(x+i\alpha)$. Extending this set of the bosonic integrals with the supercharge operators $\mathcal{Q}_a$ and $\mathcal{S}_a$, and taking all the (anti)commutators of these operators and the new integrals generated in this procedure, we obtain a nonlinear superalgebra which corresponds to some nonlinear extension of the super-Schrödinger algebra. It is generated by the set of the bosonic integrals $\mathcal{H}$, $\mathcal{D}$, $\mathcal{K}$, $\mathcal{L}$, $\mathcal{G}$, $\mathcal{V}$, $\mathcal{R}$, $\mathcal{P}_-$, $\mathcal{G}_-$, $\Sigma=\sigma_3$, $\mathcal{I}=\mathrm{diag}\,(1,1)$, and by the fermionic integrals $\mathcal{Q}_a$, $\mathcal{S}_a$, and λ_a, μ_a and κ_a, $a=1,2$, where

$$\mathcal{G}=\mathrm{diag}\,\left(G_1^\alpha,\,\tfrac{1}{2}\{G_0^\alpha,H_0\}\right)\,, \tag{4.10}$$

$$\mathcal{V}=i(x+i\alpha)^2A_1^{\alpha\#}\mathcal{I}-4t\mathcal{G}-4t^2\mathcal{L}\,, \tag{4.11}$$

$$\mathcal{R} = (x+i\alpha)^3\mathcal{I} - 6t\mathcal{V} - 12t^2\mathcal{G} - 8t^3\mathcal{L}\,, \tag{4.12}$$

$$\mathcal{P}_- = \tfrac{1}{2}(1-\sigma_3)P_0\,, \qquad \mathcal{G}_- = \tfrac{1}{2}(1-\sigma_3)G_0^\alpha\,, \tag{4.13}$$

$$\lambda_1 = \begin{pmatrix} 0 & i(x+i\alpha) \\ -i(x+i\alpha) & 0 \end{pmatrix} - 2t\mathcal{Q}_1\,, \qquad \lambda_2 = i\sigma_3\lambda_1\,, \tag{4.14}$$

$$\mu_1 = \begin{pmatrix} 0 & (x+i\alpha)P_0 \\ P_0(x+i\alpha) & 0 \end{pmatrix} - 2t\mathcal{S}_1\,, \qquad \mu_2 = i\sigma_3\mu_1\,, \tag{4.15}$$

$$\kappa_1 = \begin{pmatrix} 0 & (x+i\alpha)^2 \\ (x+i\alpha)^2 & 0 \end{pmatrix} - 4t\mu_1 - 4t^2\mathcal{S}_1\,, \qquad \kappa_2 = i\sigma_3\kappa_1\,. \tag{4.16}$$

The dynamical integrals represent the corresponding time-independent operators "dressed" by the matrix evolution operator $\mathcal{U}(t) = \exp(i\mathcal{H}t)$: $F(t) = \mathcal{U}^{-1}(t)F(0)\mathcal{U}(t)$. Particularly, the dynamical integral $\mathcal{G}$ corresponds to a dressed form of the diagonal matrix operator $g = \mathrm{diag}\,\Big((x+i\alpha)\frac{d}{dx}A_1^{\alpha\#},\, (x+i\alpha)A_1^{\alpha\#}\frac{d}{dx}\Big)$.

All the integrals are eigenstates of the operator $\mathcal{D}$ in the sense of its adjoint action:

$$[\mathcal{D},\mathcal{I}] = [\mathcal{D},\mathcal{D}] = [\mathcal{D},\Sigma] = [\mathcal{D},\mu_a] = 0\,, \tag{4.17}$$

$$[\mathcal{D},\mathcal{H}] = i\mathcal{H}\,, \quad [\mathcal{D},\mathcal{K}] = -i\mathcal{K}\,, \tag{4.18}$$

$$[\mathcal{D},\mathcal{L}] = \tfrac{3}{2}i\mathcal{L}\,, \qquad [\mathcal{D},\mathcal{R}] = -\tfrac{3}{2}i\mathcal{R}\,, \tag{4.19}$$

$$[\mathcal{D},\mathcal{G}] = \tfrac{i}{2}\mathcal{G}\,, \quad [\mathcal{D},\mathcal{V}] = -\tfrac{i}{2}\mathcal{V}\,, \quad [\mathcal{D},\mathcal{P}_-] = \tfrac{i}{2}\mathcal{P}_-\,, \quad [\mathcal{D},\mathcal{G}_-] = -\tfrac{i}{2}\mathcal{G}_-\,, \tag{4.20}$$

$$[\mathcal{D},\mathcal{Q}_a] = \tfrac{i}{2}\mathcal{Q}_a\,, \quad [\mathcal{D},\mathcal{S}_a] = i\mathcal{S}_a\,, \quad [\mathcal{D},\lambda_a] = -\tfrac{i}{2}\lambda_a \quad [\mathcal{D},\kappa_a] = -i\kappa_a\,. \tag{4.21}$$

The complete set of other (anti)-commutation relations is:

$$[\mathcal{H},\mathcal{K}] = -8i\mathcal{D}\,, \qquad [\mathcal{H},\mathcal{D}] = -i\mathcal{H}\,, \tag{4.22}$$

$$[\mathcal{H},\mathcal{R}] = -6i\mathcal{V}\,, \quad [\mathcal{H},\mathcal{V}] = -4i\mathcal{G}\,, \quad [\mathcal{H},\mathcal{G}] = -2i\mathcal{L}\,, \quad [\mathcal{H},\mathcal{L}] = 0\,, \tag{4.23}$$

$$[\mathcal{H},\mathcal{G}_-] = -2i\mathcal{P}_-\,, \qquad [\mathcal{H},\mathcal{P}_-] = 0\,, \tag{4.24}$$

$$[\mathcal{H},\mathcal{Q}_a] = 0\,, \qquad [\mathcal{H},\mathcal{S}_a] = 0\,, \tag{4.25}$$

$$[\mathcal{H},\kappa_a] = -4i\mu_a\,, \qquad [\mathcal{H},\mu_a] = -2i\mathcal{S}_a\,, \qquad [\mathcal{H},\lambda_a] = -2i\mathcal{Q}_a\,, \tag{4.26}$$

$$[\mathcal{K},\mathcal{L}] = 6i\mathcal{G}\,, \quad [\mathcal{K},\mathcal{G}] = 4i\mathcal{V}\,, \quad [\mathcal{K},\mathcal{V}] = 2i\mathcal{R}\,, \quad [\mathcal{K},\mathcal{R}] = 0\,, \tag{4.27}$$

$$[\mathcal{K},\mathcal{P}_-] = 2i\mathcal{G}_-\,, \qquad [\mathcal{K},\mathcal{G}_-] = 0\,, \tag{4.28}$$

$$[\mathcal{K},\mathcal{Q}_a] = 2i\lambda_a\,, \quad [\mathcal{K},\lambda_a] = 0\,, \quad [\mathcal{K},\mathcal{S}_a] = 4i\mu_a\,, \quad [\mathcal{K},\mu_a] = 2i\kappa_a\,, \quad [\mathcal{K},\kappa_a] = 0\,, \tag{4.29}$$

$$[\mathcal{L},\mathcal{G}] = -3i\mathcal{H}^2\,, \qquad [\mathcal{L},\mathcal{P}_-] = 0\,, \qquad [\mathcal{L},\mathcal{G}_-] = -3i\Pi_-\mathcal{H}\,, \tag{4.30}$$

$$[\mathcal{L},\mathcal{V}] = -3\left(1+\Upsilon^+\right)\mathcal{H}\,, \qquad [\mathcal{L},\mathcal{R}] = -i\left(36\mathcal{D}^2 + \tfrac{9}{2}\Sigma + \tfrac{3}{4}\mathcal{I}\right), \tag{4.31}$$

$$[\mathcal{L},\mathcal{Q}_a] = 0\,, \qquad [\mathcal{L},\mathcal{S}_a] = 0\,, \tag{4.32}$$

$$[\mathcal{L},\lambda_a] = 3i\epsilon_{ab}\mathcal{S}_b\,, \quad [\mathcal{L},\mu_a] = 3i\mathcal{H}\epsilon_{ab}\mathcal{Q}_b\,, \quad [\mathcal{L},\kappa_a] = 3\big(\Upsilon^+\epsilon_{ab} + i\delta_{ab}\big)\mathcal{Q}_b\,, \tag{4.33}$$

$$[\mathcal{G},\mathcal{V}] = 3i\mathcal{H}\mathcal{K} - 12\mathcal{D} + \tfrac{i}{2}(7\Sigma - \mathcal{I})\,, \qquad [\mathcal{G},\mathcal{R}] = 3(1+\Upsilon^-)\mathcal{K}\,, \tag{4.34}$$
$$[\mathcal{G},\mathcal{P}_-] = i\Pi_-\mathcal{H}\,, \qquad [\mathcal{G},\mathcal{G}_-] = -4i\Pi_-\mathcal{D}\,, \tag{4.35}$$
$$[\mathcal{G},\mathcal{Q}_a] = i\epsilon_{ab}\mathcal{S}_b\,, \qquad [\mathcal{G},\mathcal{S}_a] = -2i\mathcal{H}\epsilon_{ab}\mathcal{Q}_b\,, \tag{4.36}$$
$$[\mathcal{G},\lambda_a] = -2i\epsilon_{ab}\mu_b\,, \tag{4.37}$$
$$[\mathcal{G},\mu_a] = \tfrac{1}{2}\Upsilon^+\epsilon_{ab}\mathcal{Q}_b\,, \qquad [\mathcal{G},\kappa_a] = -\big(2\Upsilon^-\epsilon_{ab} + i\delta_{ab}\big)\lambda_b, \tag{4.38}$$

$$[\mathcal{V},\mathcal{R}] = -3i\mathcal{K}^2\,, \qquad [\mathcal{V},\mathcal{P}_-] = 4i\Pi_-\mathcal{D}\,, \qquad [\mathcal{V},\mathcal{G}_-] = -i\Pi_-\mathcal{K}\,, \tag{4.39}$$
$$[\mathcal{V},\mathcal{Q}_a] = 2i\epsilon_{ab}\mu_b\,, \qquad [\mathcal{V},\mathcal{S}_a] = -\big(2\Upsilon^+\epsilon_{ab} + i\delta_{ab}\big)\mathcal{Q}_b\,, \tag{4.40}$$
$$[\mathcal{V},\lambda_a] = -i\epsilon_{ab}\kappa_b\,, \quad [\mathcal{V},\mu_a] = \tfrac{1}{2}\Upsilon^-\epsilon_{ab}\lambda_b\,, \quad [\mathcal{V},\kappa_a] = 4i\mathcal{K}\epsilon_{ab}\lambda_b\,, \tag{4.41}$$

$$[\mathcal{R},\mathcal{P}_-] = 3i\Pi_-\mathcal{K}\,, \qquad [\mathcal{R},\mathcal{G}_-] = 0\,, \tag{4.42}$$
$$[\mathcal{R},\mathcal{Q}_a] = -3i\epsilon_{ab}\kappa_b\,, \qquad [\mathcal{R},\mathcal{S}_a] = 3\big(\Upsilon^-\epsilon_{ab} + i\delta_{ab}\big)\lambda_b\,, \tag{4.43}$$
$$[\mathcal{R},\lambda_a] = 0\,, \qquad [\mathcal{R},\mu_a] = -3i\mathcal{K}\epsilon_{ab}\lambda_b\,, \qquad [\mathcal{R},\kappa_a] = 0\,, \tag{4.44}$$

$$[\mathcal{P}_-,\mathcal{G}_-] = -\tfrac{i}{2}(\mathcal{I} - \Sigma)\,, \qquad [\mathcal{P}_-,\mathcal{Q}_a] = -i\mathcal{S}_a\,, \qquad [\mathcal{P}_-,\mathcal{S}_a] = i\mathcal{H}\mathcal{Q}_a\,, \tag{4.45}$$
$$[\mathcal{P}_-,\lambda_a] = -i\mu_a\,, \tag{4.46}$$
$$[\mathcal{P}_-,\mu_a] = \tfrac{1}{2}\Upsilon^+\mathcal{Q}_a\,, \qquad [\mathcal{P}_-,\kappa_a] = \left(-\tfrac{1}{2}\Upsilon^-\delta_{ab} + i\epsilon_{ab}\right)\lambda_b\,, \tag{4.47}$$

$$[\mathcal{G}_-,\mathcal{Q}_a] = -i\mu_a\,, \qquad [\mathcal{G}_-,\mathcal{S}_a] = -\mathcal{Q}_a + i\mathcal{H}\lambda_a\,, \tag{4.48}$$
$$[\mathcal{G}_-,\lambda_a] = -i\kappa_a\,, \qquad [\mathcal{G}_-,\mu_a] = -\tfrac{1}{2}\Upsilon^-\lambda_a\,, \qquad [\mathcal{G}_-,\kappa_a] = i\mathcal{K}\lambda_a\,, \tag{4.49}$$

$$[\Sigma,\mathcal{B}] = 0\,, \qquad \mathcal{B} = \Gamma,\mathcal{H},\mathcal{L},\mathcal{D},\mathcal{K},\mathcal{G},\mathcal{V},\mathcal{R},\mathcal{P}_-,\mathcal{G}_-\,, \tag{4.50}$$
$$[\Sigma,\mathcal{F}_a] = 2i\epsilon_{ab}\mathcal{F}_b\,, \qquad \mathcal{F}_a = \mathcal{Q}_a,\mathcal{S}_a,\lambda_a,\mu_a,\kappa_a\,, \tag{4.51}$$

$$\{\mathcal{Q}_a,\mathcal{Q}_b\} = 2\delta_{ab}\mathcal{H}\,, \qquad \{\mathcal{Q}_a,\mathcal{S}_b\} = 2\epsilon_{ab}\mathcal{L}\,, \tag{4.52}$$
$$\{\mathcal{Q}_a,\lambda_b\} = 4\delta_{ab}\mathcal{D} + 2\epsilon_{ab}\big(\mathcal{I} - \tfrac{1}{2}\Sigma\big)\,, \tag{4.53}$$
$$\{\mathcal{Q}_a,\mu_b\} = -2\delta_{ab}\mathcal{P}_- + 2\epsilon_{ab}\mathcal{G}\,, \qquad \{\mathcal{Q}_a,\kappa_b\} = -4\delta_{ab}\mathcal{G}_- + 2\epsilon_{ab}\mathcal{V}\,, \tag{4.54}$$

$$\{\mathcal{S}_a,\mathcal{S}_b\} = 2\delta_{ab}\mathcal{H}^2\,, \tag{4.55}$$
$$\{\mathcal{S}_a,\lambda_b\} = 4\delta_{ab}\mathcal{P}_- - 2\epsilon_{ab}\mathcal{G}\,, \qquad \{\mathcal{S}_a,\mu_b\} = \big(-i(1+\Upsilon^+)\delta_{ab} + (1-2\Sigma)\epsilon_{ab}\big)\mathcal{H}\,, \tag{4.56}$$
$$\{\mathcal{S}_a,\kappa_b\} = \big(\tfrac{1}{2}\mathcal{H}\mathcal{K} - \mathcal{I}\big)\,\delta_{ab} + \big(2i\delta_{ab} + 4(1-2\Sigma)\epsilon_{ab}\big)\mathcal{D}\,, \tag{4.57}$$

$$\{\lambda_a,\lambda_b\} = 2\delta_{ab}\mathcal{K}\,, \qquad \{\lambda_a,\mu_b\} = 2\epsilon_{ab}\mathcal{V} + 2\delta_{ab}\mathcal{G}_-\,, \qquad \{\lambda_a,\kappa_b\} = 2\epsilon_{ab}\mathcal{R}\,, \tag{4.58}$$
$$\{\mu_a,\mu_b\} = \tfrac{1}{2}\Upsilon^+\Upsilon^-\delta_{ab}\,, \qquad \{\mu_a,\kappa_b\} = \big(i(1+\Upsilon^-)\delta_{ab} + (1-2\Sigma)\epsilon_{ab}\big)\mathcal{K}\,, \tag{4.59}$$
$$\{\kappa_a,\kappa_b\} = 2\delta_{ab}\mathcal{K}^2\,. \tag{4.60}$$

We use here a notation $\Pi_- = \frac{1}{2}(1-\Sigma)$ and $\Upsilon^\pm = 1 \pm 4i\mathcal{D}$ for operator-valued coefficients appearing in some (anti)-commutation relations.

The superalgebra (4.17)–(4.60) represents a nonlinear generalization of the super-Schrödinger symmetry with ten even and ten odd generators plus a bosonic central charge $\mathcal{I}$. It is characterized by the quadratic Casimir operator $C_0 = \{\mathcal{K}, \mathcal{H}\} - 8\mathcal{D}^2 + 2\left(\mathcal{I} - \frac{1}{2}\Sigma\right)^2 - 3\mathcal{I}^2$, which takes zero value $C_0 = 0$ for the system (4.1). The set of operators $\mathcal{H}$, $\mathcal{K}$, $\mathcal{D}$, $(\mathcal{I} - \frac{1}{2}\Sigma)$, $\mathcal{Q}_a$ and λ_a generates the Lie sub-superalgebra $osp(2|2)$ of superconformal symmetry of the system. The extension of the indicated set of operators by the Lax integral $\mathcal{L}$ or by any other integral not appearing in the list produces all the rest of the generators and expands the superalgebra $osp(2|2)$ up to the described nonlinear (quadratic) superalgebra. As an example, one such a chain of the (anti)commutation relations can be presented schematically as

$$\mathcal{L} \xrightarrow{\mathcal{K}} \mathcal{G} \xrightarrow{\mathcal{K}} \mathcal{V} \xrightarrow{\mathcal{K}} \mathcal{R}\,, \qquad \mathcal{L} \xrightarrow{\lambda_a} \mathcal{S}_a \xrightarrow{\lambda_a} \mathcal{P}_- \xrightarrow{\lambda_a} \mu_a \xrightarrow{\lambda_a} \mathcal{G}_- \xrightarrow{\lambda_a} \kappa_a\,, \qquad \mathcal{G}_- \xrightarrow{\mathcal{P}_-} \mathcal{I}, \Sigma\,,$$

where $\mathcal{L} \xrightarrow{\mathcal{K}} \mathcal{G}$ corresponds to the first commutation relation in (4.27), etc.

The peculiarity of the obtained nonlinear superalegbra is that the (anti)-commutators of the generators of the $osp(2|2)$ sub-superalgebra with any other generator is linear in generators; at the same time, in the (anti)-commutators of the rest of the integrals, the generators $\mathcal{H}$, $\mathcal{K}$, $\mathcal{D}$ and Σ of the $sl(2,\mathbb{R}) \oplus u(1)$ sub-algebra appear as operator-valued coefficients.

The superalgebra has the automorphism corresponding to a spatial reflection ρ_1, under which the integrals $\mathcal{L}$, $\mathcal{G}$, $\mathcal{V}$, $\mathcal{R}$, $\mathcal{P}_-$, $\mathcal{G}_-$, $\mathcal{Q}_a$, and λ_a are odd (change the sign), while the rest of generators is even. Another set of transformations

$$\begin{gathered}\rho_2:\ \mathcal{H} \to \mathcal{K}\,, \quad \mathcal{K} \to \mathcal{H}\,, \quad \mathcal{D} \to -\mathcal{D}\,, \quad \mathcal{R} \to \mathcal{L}\,, \quad \mathcal{L} \to -\mathcal{R}\,,\\ \mathcal{V} \to -\mathcal{G}\,, \quad \mathcal{G} \to \mathcal{V}\,, \quad \mathcal{G}_- \to \mathcal{P}_-\,, \quad \mathcal{P}_- \to -\mathcal{G}_-\,, \quad \Sigma \to \Sigma\,, \quad \mathcal{I} \to \mathcal{I}\,,\\ \mathcal{Q}_a \to -\lambda_a\,, \quad \lambda_a \to \mathcal{Q}_a\,, \quad \mathcal{S}_a \to \kappa_a\,, \quad \kappa_a \to \mathcal{S}_a\,, \quad \mu_a \to -\mu_a\,,\end{gathered}$$

corresponds to automorphism of the superalgebra that unifies and generalizes relations (3.6) and (3.21) for the superextended system (4.1).

5 Symmetries of H_n^α and of its super-extended version

In this section we generalize the analysis of the previous two sections for the case of the PT-regularized conformal mechanics system H_n^α and its super-extended version.

System (2.1) has the dynamical integrals of motion

$$D_n^\alpha = -\tfrac{i}{2}\left((x+i\alpha)\tfrac{d}{dx} + \tfrac{1}{2}\right) - tH_n^\alpha\,, \qquad K_n^\alpha = (x+i\alpha)^2 - 8tD_n^\alpha - 4t^2 H_n^\alpha\,, \tag{5.1}$$

which together with Hamiltonian H_n^α generate the conformal $sl(2,\mathbb{R})$ symmetry. It also has the Lax-Novikov integral of motion P_n^α, $[P_n^\alpha, H_n^\alpha] = 0$, being the Darboux-dressed momentum integral P_0 of the free particle. Its scale dimension is $-(n+\frac{1}{2})$, $i[D_n^\alpha, P_n^\alpha] = -(n+\frac{1}{2})P_n^\alpha$. Repeated commutation of P_n^α with K_n^α, $[P_n^\alpha, K_n^\alpha]$, $[[P_n^\alpha, K_n^\alpha], K_n^\alpha]$, ..., generates in addition to D_n^α and K_n^α, the $2n$ dynamical integrals of the half-integer scale dimensions $-(n-\frac{1}{2}), \ldots, (n+\frac{1}{2})$. These are the operators which together with the integral P_n^α can be presented in the form

$$\mathcal{X}_{n,\ell}^\alpha = U_n^{-1}(t) X_{n,\ell}^\alpha U_n(t) = X_{n,\ell}^\alpha + \ldots\,, \qquad \ell = 0, 1, \ldots, 2n+1\,, \tag{5.2}$$

where $U_n(t) = \exp(itH_n^\alpha)$, the ellipsis corresponds to a polynomial of order ℓ in t,

$$X_{n,\ell}^\alpha = (x+i\alpha)^\ell \mathcal{A}_{n,\ell}^\alpha\,, \tag{5.3}$$

$$\mathcal{A}_{n,\ell}^\alpha = A_{\ell-n}^\diamondsuit A_{\ell+1-n}^\diamondsuit \ldots A_n^\diamondsuit\,, \quad \ell = 0,1,\ldots,2n\,, \qquad \mathcal{A}_{n,2n+1}^\alpha = 1\,, \tag{5.4}$$

and we denote $A_{-k}^\diamondsuit = A_k^\alpha$ for $k>0$, $A_0^\diamondsuit = \frac{d}{dx}$, and $A_k^\diamondsuit = A_k^{\alpha\#}$ for $k>0$. In particular cases of $\ell=0$ and $\ell = 2n+1$ the integrals (5.2) are reduced to $\mathcal{X}_{n,0} = iP_n^\alpha$ and $\mathcal{X}_{n,2n+1} = U_n^{-1}(t)(x+i\alpha)^{2n+1}U_n(t)$. The integrals (5.2) form a $2(n+1)$-dimensional representation with respect to the adjoint action of the $sl(2,\mathbb{R})$ generators,

$$[D_n^\alpha, \mathcal{X}_{n,\ell}^\alpha] = i(n-\ell+\tfrac{1}{2})\mathcal{X}_{n,\ell}^\alpha\,, \tag{5.5}$$

$$[H_n^\alpha, \mathcal{X}_{n,\ell}^\alpha] = 2\ell\,\mathcal{X}_{n,\ell-1}^\alpha\,, \qquad [K_n^\alpha, \mathcal{X}_{n,\ell}^\alpha] = 2(2n+1-\ell)\mathcal{X}_{n,\ell+1}^\alpha\,. \tag{5.6}$$

The commutation relations of $\mathcal{X}_{n,\ell}^\alpha$ between themselves are reduced to the form

$$[\mathcal{X}_{n,\ell}^\alpha, \mathcal{X}_{n,\ell'}^\alpha] = \mathcal{P}_{\ell,\ell'}^{(2n)}(H_n^\alpha, K_n^\alpha, D_n^\alpha)\,, \tag{5.7}$$

where $\mathcal{P}_{\ell,\ell'}^{(2n)}$ are some polynomials of order $2n$ in the $sl(2,\mathbb{R})$ generators.

The supersymmetric system given by the superpotential $\mathcal{W}_e = -n/(x+i\alpha)$ is described by the matrix Hamiltonian

$$\mathcal{H}_n = \mathrm{diag}\,(H_n^\alpha,\, H_{n-1}^\alpha)\,. \tag{5.8}$$

System (5.8) is characterized by nonlinear extension of the Schrödinger superalgebra, which in this case is generated by $4n+6$ bosonic integrals and the same number of fermionic integrals, and by the bosonic central charge $\mathcal{I}$. The set of bosonic generators is given by $\mathcal{H}_n$, $\mathcal{D}_n = \mathrm{diag}\,(D_n^\alpha, D_{n-1}^\alpha)$, $\mathcal{K}_n = \mathrm{diag}\,(K_n^\alpha, K_{n-1}^\alpha)$, Σ, $\mathcal{X}_{n,\ell}^+ = \mathcal{X}_{n,\ell}^\alpha \Pi_+$, $\ell = 0,\ldots,2n+1$, and $\mathcal{X}_{n,k}^- = \mathcal{X}_{n-1,k}^\alpha \Pi_-$, $k=0,\ldots,2n-1$, where $\Pi_\pm = \frac{1}{2}(1\pm\sigma_3)$. The set of fermionic generators is $\mathcal{Q}_{n,a}$, $\lambda_{n,a}$, $\mathcal{S}_{n,a}$, and $\mu_{n,k,a}$, $k=0,\ldots,2n-1$. The explicit form of these generators with index $a=1$ is given by

$$\mathcal{Q}_{n,1} = \begin{pmatrix} 0 & A_n^- \\ A_n^\# & 0 \end{pmatrix}, \qquad \lambda_{n,1} = \begin{pmatrix} 0 & i(x+i\alpha) \\ -i(x+i\alpha) & 0 \end{pmatrix} - 2t\mathcal{Q}_{n,1}\,, \tag{5.9}$$

$$\mathcal{S}_{n,1} = \begin{pmatrix} 0 & -iA_n^\alpha P_{n-1}^\alpha \\ iP_{n-1}^\alpha A_n^{\alpha\#} & 0 \end{pmatrix}, \tag{5.10}$$

$$\mu_{n,k,1} = \mathcal{U}_n^{-1}(t) \begin{pmatrix} 0 & (x+i\alpha)X_{n-1,k}^\alpha \\ (-1)^{k+1}X_{n-1,k}^\alpha(x+i\alpha) & 0 \end{pmatrix} \mathcal{U}_n(t)\,. \tag{5.11}$$

The generators with index $a=2$ are obtained by multiplication with $i\sigma_3$, $\mathcal{Q}_{n,2} = i\sigma_3\mathcal{Q}_{n,1}$, etc., and here $\mathcal{U}_n(t) = \exp(i\mathcal{H}_n t)$ is the evolution operator. The operators $\mu_{n,k,a}$ with $k=2n-1$ and $k=n-1$ correspond, respectively, to the fermionic dynamical integrals (4.16) and (4.15) in the case of $n=1$.

The operators $\mathcal{H}_n$, $\mathcal{K}_n$, $\mathcal{D}_n$, $(\mathcal{I}-\frac{1}{2}\Sigma)$, $\mathcal{Q}_{n,a}$ and $\lambda_{n,a}$ are generators of the Lie sub-superalgebra $osp(2|2)$ of superconformal symmetry of the system (5.8). Extension of this set by any other integral (different from the central charge $\mathcal{I}$) results in expansion of the $osp(2|2)$ up to the whole nonlinearly extended super-Schrödinger algebra. As in the $n=1$ case, the (anti)-commutators of all the integrals with the $osp(2|2)$ generators are linear in the generators of the extended super-Schrödinger algebra. Particularly, the operators $\mathcal{X}^+_{n,\ell}$ and $\mathcal{X}^-_{n,k}$ have the scaling dimensions $-(n-\ell+\frac{1}{2})$ and $-(n-k-\frac{1}{2})$, respectively, given by the adjoint action of $i\mathcal{D}_n$, while the pairs of fermionic operators $S_{n,a}$ and $\mu_{n,k,a}$, $a=1,2$, have the scaling dimensions $-n$ and $(k+1-n)$. In the (anti)-commutators of additional generators with generators of the $osp(2|2)$ superconformal symmetry, the structure coefficients are certain polynomials of order $2n-1$ in generators $\mathcal{H}$, $\mathcal{K}$, $\mathcal{D}$ and Σ of the $sl(2,\mathbb{R})\oplus u(1)$ sub-algebra. The Hamiltonian $\mathcal{H}_n$, the supercharges $\mathcal{Q}_{n,a}$ and $\mathcal{S}_{n,a}$, and the bosonic Lax-Novikov matrix integral taken in the form $\mathcal{L}_n=\mathcal{X}^+_{n,0}+\mathcal{H}_n\mathcal{X}^-_{n,0}=\text{diag}\,(P^\alpha_n,H^\alpha_{n-1}P^\alpha_{n-1})$ generate the exotic nonlinear $\mathcal{N}=4$ super-Poincaré algebra of the form (4.4), (4.5), (4.6), (4.7) with the unique difference in the anti-commutation relation between the higher order supercharges $\mathcal{S}_{n,a}:\ \{\mathcal{S}_{n,a},\mathcal{S}_{n,b}\}=2\delta_{ab}(\mathcal{H}_n)^{2n}$. As in the particular case of $n=1$, the exotic nonlinear $\mathcal{N}=4$ supersymmetry of the system (5.8) is unbroken: its unique ground state of zero energy $\Psi^t=((x+i\alpha)^{-n},0)^t$ is annihilated by all the supercharges $\mathcal{Q}_{n,a}$ and $\mathcal{S}_{n,a}$ as well as by the Lax-Novikov integral [62]. Here, the operator $\mathcal{G}_n=\mathcal{X}^+_{n,1}+\frac{1}{2}\{\mathcal{H}_n,\mathcal{X}^-_{n,1}\}=\text{diag}\,(\mathcal{X}^\alpha_{n,1},\frac{1}{2}\{\mathcal{H}_{n-1},\mathcal{X}^\alpha_{n-1,1}\})$ is the analog of the integral (4.10) of the case $n=1$.

6 Spontaneoulsy broken phase of the exotic SUSY

We study here the case of the PT-regularized $g=2$ superconformal mechanics model in the phase of the partially broken exotic nonlinear $\mathcal{N}=4$ super-Poincaré symmetry. The symmetry of the system we investigate is described by the same number of bosonic and fermionic generators as in the system from Section 4, but the structure of nonlinear superalgebra they generate is essentially different from that of the model (4.1). Nevertheless, its symmetry is shown may be related to the nonlinearly extended super-Schrödinger symmetry of the system (4.1) in the limit of the transition to the phase of the unbroken exotic nonlinear $\mathcal{N}=4$ supersymmetry.

The system we consider is generated via the $\mathcal{N}=2$ supersymmetric quantum mechanics construction based on the superpotential $\mathcal{W}_b(x)=1/\xi_1-1/\xi_2+i\delta^{-1}$, where $\xi_j=x+i\alpha_j$, $j=1,2$, $\mathbb{R}\ni\alpha_j\neq 0$, $\delta=\alpha_1-\alpha_2\neq 0$. In the limit when the parameter α_1 (or α_2) is sent to infinity, the superpotential transforms into the superpotential $\mathcal{W}_e(x)$ (or $-\mathcal{W}_e(x)$) of the system (4.1). We have $\mathcal{W}_b^2-\mathcal{W}_b'=2/\xi_1^2-\delta^{-2}$, $\mathcal{W}_b^2+\mathcal{W}_b'=2/\xi_2^2-\delta^{-2}$, that allows us to introduce the first order differential operators $A_b=\frac{d}{dx}+\mathcal{W}_b(x)$ and $A_b^\#=-\frac{d}{dx}+\mathcal{W}_b(x)$ satisfying the factorization, $A_b^\#A_b=H_1^{\alpha_1}-\delta^{-2}$, $A_bA_b^\#=H_1^{\alpha_2}-\delta^{-2}$, and the intertwining, $A_bH_1^{\alpha_1}=H_1^{\alpha_2}A_b$, $A_b^\#H_1^{\alpha_2}=H_1^{\alpha_1}A_b^\#$, relations, where $H_1^{\alpha_j}=-\frac{d^2}{dx^2}+2/\xi_j^2$. Define now the extended system described by the matrix Hamiltonian

$$\mathcal{H}=\text{diag}\,(H_1^{\alpha_2},\,H_1^{\alpha_1})\,. \tag{6.1}$$

Operators

$$\mathcal{Q}_1=\begin{pmatrix}0 & A_b\\ A_b^\# & 0\end{pmatrix},\qquad \mathcal{Q}_2=i\sigma_3\mathcal{Q}_1\,, \tag{6.2}$$

are the supercharges of the system (6.1). They satisfy the relations

$$[\mathcal{H},\mathcal{Q}_a]=0,\qquad \{\mathcal{Q}_a,\mathcal{Q}_b\}=2\delta_{ab}(\mathcal{H}-\delta^{-2})\,. \tag{6.3}$$

The partner Hamiltonians can also be intertwined by the second order operators $A_1^{\alpha_2}A_1^{\alpha_1\#}$ and $A_1^{\alpha_1}A_1^{\alpha_2\#}$ via the Hamiltonian of the intermediate ('virtual' here) free particle system: $A_1^{\alpha_2}A_1^{\alpha_1\#}H_1^{\alpha_1} = A_1^{\alpha_2}H_0A_1^{\alpha_1\#} = H_1^{\alpha_2}A_1^{\alpha_2}A_1^{\alpha_1\#}$, $A_1^{\alpha_1}A_1^{\alpha_2\#}H_1^{\alpha_2} = A_1^{\alpha_1}H_0A_1^{\alpha_2\#} = H_1^{\alpha_1}A_1^{\alpha_1}A_1^{\alpha_2\#}$. Hence, system (6.1) is characterized additionally by the second order supercharges

$$\mathcal{S}_1 = \begin{pmatrix} 0 & A_1^{\alpha_2}A_1^{\alpha_1\#} \\ A_1^{\alpha_1}A_1^{\alpha_2\#} & 0 \end{pmatrix}, \qquad \mathcal{S}_2 = i\sigma_3\mathcal{S}_1\,, \tag{6.4}$$

$$[\mathcal{H}, \mathcal{S}_a] = 0, \qquad \{\mathcal{S}_a, \mathcal{S}_b\} = 2\delta_{ab}\mathcal{H}^2\,. \tag{6.5}$$

We note here that the first order intertwining operators from which supercharges (6.2) are composed can be obtained by employing the reduction of intertwining operators mentioned in Section 4. Indeed, the Hamiltonians $H_1^{\alpha_1}$ and $H_1^{\alpha_2}$ can be intertwined by the third order differential operator $A_1^{\alpha_2}P_0A_1^{\alpha_1\#}$ by using the chain of equalities $A_1^{\alpha_2}P_0A_1^{\alpha_1\#}H_1^{\alpha_1} = A^{\alpha_2}P_0H_0A_1^{\alpha_1\#} = A_1^{\alpha_2}H_0P_0A_1^{\alpha_1\#} = H_1^{\alpha_2}A_1^{\alpha_2}P_0A_1^{\alpha_1\#}$. However, using equation (4.18) from ref. [62], or by a direct computation, one finds the relation $A_1^{\alpha_2}P_0A_1^{\alpha_1\#} = H_1^{\alpha_2}A_b - i\delta^{-1}A_1^{\alpha_2}A_1^{\alpha_1\#}$ that shows that the indicated third order intertwining operator reduces to the intertwining operators appearing in the right-upper corners of the supercharges $\mathcal{Q}_1$ and $\mathcal{S}_1$ multiplied, respectively, by the second order operator $H_1^{\alpha_2}$ and the constant $-i\delta^{-1}$.

The anti-commutator of the first and second order supercharges generates the matrix Lax-Novikov integral $\mathcal{L}_1 = \mathrm{diag}\,(P_1^{\alpha_2}, P_1^{\alpha_1})$,

$$\{\mathcal{Q}_a, \mathcal{S}_b\} = 2\left(\epsilon_{ab}\mathcal{L}_1 + i\delta_{ab}\delta^{-1}\mathcal{H}\right). \tag{6.6}$$

This operator satisfies relations $[\mathcal{L}_1, \mathcal{H}] = [\mathcal{L}_1, \mathcal{Q}_a] = [\mathcal{L}_1, \mathcal{S}_a] = 0$, and plays the role of the central charge of the exotic nonlinear $\mathcal{N} = 4$ supersymmetry generated by $\mathcal{H}$, $\mathcal{Q}_a$, $\mathcal{S}_a$ and $\mathcal{L}_1$. Unlike (4.1), system (6.1) is in the phase of the partially broken exotic nonlinear $\mathcal{N} = 4$ supersymmetry. Its two zero energy eigenstates $\Psi_{\alpha_2}^t = ((x + i\alpha_2)^{-1}, 0)^t$ and $\Psi_{\alpha_1}^t = (0, (x + i\alpha_1)^{-1})^t$ are zero modes of both second order supercharges $\mathcal{S}_a$, but neither of them is annihilated by the first order supercharges $\mathcal{Q}_a$ [62].

System (6.1) is also characterized by the dynamical integrals $\mathcal{D} = \mathrm{diag}\,(D_1^{\alpha_2}, D_1^{\alpha_1})$ and $\mathcal{K} = \mathrm{diag}\,(K_1^{\alpha_2}, K_1^{\alpha_1})$. Commuting the Lax-Novikov integral repeatedly with the dynamical integral $\mathcal{K}$, we generate three more bosonic dynamical integrals of motion,

$$[\mathcal{L}_1, \mathcal{K}] = -6i\mathcal{G}_1\,, \qquad [\mathcal{G}_1, \mathcal{K}] = -4i\mathcal{V}\,, \qquad [\mathcal{V}, \mathcal{K}] = -2i\mathcal{R}\,, \qquad [\mathcal{R}, \mathcal{K}] = 0\,, \tag{6.7}$$

$$\mathcal{G}_1 = (\Xi\mathcal{H} + \mathcal{A}^\#) - 2t\mathcal{L}_1\,, \qquad \mathcal{V} = i\Xi^2\mathcal{A}^\# - 4t\mathcal{G}_1 - 4t^2\mathcal{L}_1\,, \tag{6.8}$$

$$\mathcal{R} = \Xi^3 - 6t\mathcal{V} - 12t^2\mathcal{G}_1 - 8t^3\mathcal{L}_1\,, \tag{6.9}$$

where $\Xi = \mathrm{diag}\,(\xi_2, \xi_1)$, $\mathcal{A}^\# = \mathrm{diag}\,(A_1^{\alpha_2\#}, A_1^{\alpha_1\#})$. All these bosonic integrals have the same scaling dimensions as their analogs in the system from Section 3. The peculiarity of the system (6.1), however, is that its supercharges $\mathcal{Q}_a$ and $\mathcal{S}_a$ are not eigenstates of the dilatation generator under its adjoint action:

$$[\mathcal{D}, \mathcal{Q}_a] = \tfrac{1}{2}\delta\mathcal{S}_a\,, \qquad [\mathcal{D}, \mathcal{S}_a] = \tfrac{3}{2}i\mathcal{S}_a - \tfrac{1}{2}\delta\mathcal{H}\mathcal{Q}_a\,. \tag{6.10}$$

Coherently with this, the way in which the odd dynamical integrals for the system (6.1) are generated is also different in comparison with that for the system from Section 4. The commutation of the dynamical integral $\mathcal{K}$ with the first and second order supercharges $\mathcal{Q}_a$ and $\mathcal{S}_a$ generates the unique pair of the odd dynamical integrals μ_a:

$$[\mathcal{K},\mathcal{Q}_a]=\delta^2\mathcal{Q}_a+2\delta\mu_a\,,\qquad [\mathcal{K},\mathcal{S}_a]=\delta^2\mathcal{S}_a+3i\delta\mathcal{Q}_a-4\delta\mathcal{D}\mathcal{Q}_a+6i\mu_a\,, \tag{6.11}$$

$$\mu_1=i\Xi\sigma_1\mathcal{A}^{\#}-2t\mathcal{S}_1\,,\qquad \mu_2=i\sigma_3\mu_1\,. \tag{6.12}$$

Commutation of $\mathcal{K}$ with μ_a produces a new pair of fermionic integrals Γ_a,

$$[\mathcal{K},\mu_a]=4i\Gamma_a+\delta^2\mu_a-2\delta\mathcal{K}\mathcal{Q}_a\,, \tag{6.13}$$

$$\Gamma_1=(\Xi^2\sigma_1-\delta\Xi\sigma_2)-(4\mu_1+2\delta Q_1)t-4t^2S_1\,,\qquad \Gamma_2=i\sigma_3\Gamma_1\,. \tag{6.14}$$

The time-independent term in (6.14) is presented equivalently in the form $\xi_1\xi_2\sigma_1$. Finally, commutation of $\mathcal{K}$ with Γ_a generates dynamical odd integrals Ω_a,

$$[\mathcal{K},\Gamma_a]=\Omega_a\,, \tag{6.15}$$

$$\begin{aligned}\Omega_1=&(3\delta^2\Xi^2\sigma_1+(2\delta\Xi^3-\delta^3\Xi)\sigma_2)+t\left(-2\delta^3\mathcal{Q}_1-12\delta^2\mu_1+12\delta\mathcal{K}\mathcal{Q}_1-12i\Gamma_1\right)\\&+t^2\left(-12\delta^2\mathcal{S}_1+48\delta\mathcal{D}Q_1-24i\mu_1\right)+t^3\left(-16i\mathcal{S}_1+16\delta\mathcal{H}\mathcal{Q}_1\right)\,,\end{aligned} \tag{6.16}$$

$\Omega_2=i\sigma_3\Omega_1$. The commutation relation

$$[\mathcal{K},\Omega_a]=\delta^2\left(4\Omega_a-2\delta^2\Gamma_a-\tfrac{1}{2}\mathcal{K}\Gamma_a\right) \tag{6.17}$$

then signals that the process of generation of odd dynamical integrals terminates. Commutation relations

$$[\mathcal{D},\mu_a]=\tfrac{i}{2}\mu_a-\delta\left(\mathcal{D}-\tfrac{i}{4}\right)\mathcal{Q}_a\,,\qquad [\mathcal{D},\Gamma_a]=-\tfrac{i}{2}\Gamma_a-\tfrac{1}{2}\delta\mathcal{K}\mathcal{Q}_a+\tfrac{1}{2}\delta^2\mu_a\,, \tag{6.18}$$

$$[\mathcal{D},\Omega_a]=-\tfrac{3}{2}i\Omega_a-\tfrac{3}{2}\delta^3\mathcal{K}\mathcal{Q}_a+2i\delta^2\Gamma_a+\delta^2\left(\tfrac{1}{2}\delta^2-\mathcal{K}\right)\mu_a \tag{6.19}$$

show that, like $\mathcal{Q}_a$ and $\mathcal{S}_a$, neither of the odd dynamical integrals is eigenstate under the adjoint action of the dilation generator. The commutation relations of the odd dynamical integrals with the Hamiltonian operator are

$$[\mathcal{H},\mu_a]=-2iS_a\,,\qquad [\mathcal{H},\Gamma_a]=-2i(2\mu_a+\delta Q_a)\,, \tag{6.20}$$

$$[\mathcal{H},\Omega_a]=12\Gamma_a-i\delta^2(10\mu_a+3\delta Q_a)+4\delta^2\mathcal{D}(\mu_a+\delta Q_a)+12i\mathcal{K}Q_a\,. \tag{6.21}$$

The anti-commutation relations of dynamical integrals μ_a with μ_b, $\mathcal{Q}_b$ and $\mathcal{S}_b$ are

$$\{\mu_a,\mu_b\}=8\delta_{ab}\left(\mathcal{D}^2-4i\mathcal{D}+2i\delta\mathcal{G}_2-9\mathcal{I}\right)\,, \tag{6.22}$$

$$\{\mathcal{Q}_a,\mu_b\}=\delta_{ab}\left(-\delta\mathcal{H}+4i\delta^{-1}\mathcal{D}+\delta^{-1}\mathcal{I}\right)+\epsilon_{ab}\left(2\mathcal{G}_1+i\delta\Sigma\mathcal{H}\right)\,, \tag{6.23}$$

$$\{\mathcal{S}_a,\mu_b\}=\delta_{ab}\left(-3i\mathcal{H}+4\mathcal{D}\mathcal{H}+i\delta\mathcal{L}_2\right)+\epsilon_{ab}\left(-2\Sigma\mathcal{H}+\delta\mathcal{L}_1\right)\,, \tag{6.24}$$

where $\mathcal{L}_2=\sigma_3\mathcal{L}_1$ and $\mathcal{G}_2=\sigma_3\mathcal{G}_1$ are two additional bosonic integrals. Instead of them, one can take their linear combinations with $\mathcal{L}_1$ and $\mathcal{G}_1$ to obtain $\mathcal{L}_-=\Pi_-\mathcal{L}_1$ and $\mathcal{G}_-=\Pi_-\mathcal{G}_1$, which could be considered as analogs of the integrals (4.13) of the system (4.1). Finally, the system under consideration is described by ten bosonic integrals $\mathcal{H}$, $\mathcal{D}$, $\mathcal{K}$, Σ, $\mathcal{L}_1$, $\mathcal{G}_1$, $\mathcal{V}$, $\mathcal{R}$, $\mathcal{L}_2$ and $\mathcal{G}_2$ (or by $\mathcal{L}_-$ and $\mathcal{G}_-$ instead of the two last integrals), the same number of fermionic integrals $\mathcal{Q}_a$, $\mathcal{S}_a$, μ_a, Γ_a and Ω_a, and by the central charge $\mathcal{I}$.

Essential peculiarity here is that the PT-regularization parameter α of the dimension of length does not appear in the superalgebra of the system (4.1), while in the (anti)-commutation relations for the system (6.1) there appears the parameter $\delta = \alpha_1 - \alpha_2$.

We will not write down the missing (anti)-commutation relations of the integrals, but examine the relation between the systems (4.1) and (6.1) in the light of their nonlinearly supersymmetrically extended and deformed conformal symmetries. First of all, we observe that the nonlinear extensions of the conformal $sl(2,\mathbb{R})$ symmetry generated by their sets of the bosonic integrals are very similar. The only difference appears in commutation relations of the corresponding pairs of the operators ($\mathcal{P}_-$, $\mathcal{G}_-$) and ($\mathcal{L}_-$, $\mathcal{G}_-$). More essential difference is that system (4.1) contains the algebra of the superconformal $osp(2|2)$ symmetry as a sub-superalgebra, whereas system (6.1) does not have such a sub-superalgebra. This is reflected, particularly, in the peculiarity of the latter system encoded in relations (6.10), (6.18) and (6.19), cf. (4.21). Since in the limit $\alpha_1 \to \infty$ with identification $\alpha_2 = \alpha$ the superpotential $\mathcal{W}_b(x)$ of the system (6.1) transforms into superpotential $\mathcal{W}_e(x)$ of the system (4.1), and the Hamiltonian (6.1) transforms into (4.1), the essential difference between two systems at first glance may seem to be rather surprising. To clarify this point, let us see what happens in the indicated limit with other operators. We have $A_b \to A_1^{\alpha}$, $A_b^{\#} \to A_1^{\alpha\#}$, $A_1^{\alpha_1} \to \frac{d}{dx}$ and $A_1^{\alpha_1\#} \to -\frac{d}{dx}$. As a result, we find that the generators $\mathcal{Q}_a$, $\mathcal{S}_a$ and $\mathcal{L}_1$ of the system (6.1) transform into the corresponding integrals $\mathcal{Q}_a$, $\mathcal{S}_a$ and $\mathcal{L}$ of the system (4.1), while $\mathcal{L}_-$ transforms into the integral $\mathcal{P}_-$ multiplied with the Hamiltonian (4.1). The limit applied to other integrals is more delicate, however. The dilatation generator of the system (6.1) with $\alpha_2 = \alpha$ is represented equivalently in the form $\mathcal{D} = \mathcal{D}(x+i\alpha) + \frac{i}{2}(\alpha_1 - \alpha)\mathcal{P}_-$, where by $\mathcal{D}(x+i\alpha)$ we indicate the dilatation generator for the system (4.1). So, the integral $\mathcal{D}$ transforms into $\mathcal{D}(x+i\alpha)$ if the limit $\alpha_1 \to \infty$ is accompanied by a 'renormalization' which consists in omitting the integral $\frac{i}{2}\mathcal{P}_-$ multiplied with the factor $\delta = (\alpha_1 - \alpha) \to \infty$. A similar picture is valid for relation between other bosonic dynamical integrals of both systems in the indicated limit. Particularly, we have $\mathcal{K} = \mathcal{K}(x+i\alpha) + 2i\delta\mathcal{J}_-(x+i\alpha) - \delta^2\Pi_-$, and this integral transforms into $\mathcal{K}(x+i\alpha)$ by making the same kind of 'renormalization'. For fermionic dynamical integral μ_1 the relation $\mu_1 = \mu_1(x+\alpha) - \delta\Pi_-\mathcal{Q}_1$ is valid when $\alpha_1 \to \infty$, and, as a result, a 'renormalized' form of the integrals (6.12) will correspond to the dynamical integrals (4.15) of the system (4.1). For the integral Γ_1 with $\alpha_1 \to \infty$ we have $\Gamma_1 = \kappa_1(x+i\alpha) - i\delta\lambda_2(x+i\alpha)$, and the 'renormalized' form of the integrals (6.14) gives us the integrals (4.16) for the system (4.1). This also means that the limit $\alpha_1 \to \infty$ applied to the rescaled integrals $\delta^{-1}\Gamma_a$ reproduces the dynamical integrals (5.10). Analogously, the appropriately rescaled integrals (6.16) in the limit $\alpha_1 \to \infty$ reproduce the integrals (5.10): $\delta^{-3}\Omega_a \to \lambda_a(x+i\alpha)$.

All this shows that in the limit when one of the two parameters α_1 or α_2 is sent to infinity, the system (6.1) in the phase of the partially broken exotic nonlinear $\mathcal{N} = 4$ super-Poincaré symmetry transforms into the system (4.1) in the unbroken phase, and all the integrals of the latter system can be reproduced from those of the former system. The relation between the integrals, however, is rather non-trivial, that is behind the essential difference between the nonlinear super-extended versions of the conformal symmetry generated in two cases.

7 Discussion and outlook

The AFF model [1] is obtained from the model (1.1) by adding into the Hamiltonian operator the confining harmonic potential term x^2 that effectively results in introduction of another boundary condition $\psi(+\infty) = 0$ at another edge of the interval $x \in (0, +\infty)$ in addition to the Dirichlet boundary condition $\psi(0) = 0$. This procedure does not change the symmetry: the system described by the "regularized" AFF Hamiltonian has the same conformal symmetry as the initial system (1.1), but it changes radically the spectrum. Instead of continuous non-degenerate spectrum of the system (1.1) with $E > 0$, the AFF model has the equidistant discrete spectrum which, up to a constant shift, coincides with the spectrum of the half-harmonic oscillator [94]. This is not surprising as at particular values of the coupling constant $g = n(n + 1)$, the AFF model is generated from the half-harmonic oscillator by the appropriate nonsingular on the half-line Darboux-Crum transformation [94]. From this point of view the AFF model with confining harmonic potential term is rather a rational deformation of the half-harmonic oscillator than the deformation (by adding the harmonic term) of the two-particle Calogero system (1.1) with the omitted center of mass coordinate. Nevertheless, in this way the "regularization" recipe accepted in [1] solves the problem of the absence of the ground state in the initial system (1.1). One could not restrict the values of x to the half-line in the system (1.1) as well as in the "regularized" AFF model and consider both systems on the whole real line $x \in \mathbb{R}$. Potential with the inverse squared term is not penetrable at $x = 0$, and the probability flux between regions $x < 0$ and $x > 0$ will be equal to zero, i.e. in this case we effectively will have two copies of the system with doubly degenerate either continuous spectrum with $E > 0$ in the case of (1.1) or the discrete spectrum in the AFF model [1]. Besides the conformal symmetry, system (1.1) has another very important peculiarity. As we noted, it's potential with $g = n(n+1)$ is an algebro-geometric solution to the stationary KdV equation or the higher equation of its hierarchy, which is characterized by the existence of the differential operator of order $2n + 1$, related with a higher order Novikov equation [96]. This higher order Lax-Novikov operator commutes with the Hamiltonian operator (1.1) [71]. In this picture, a free particle on the whole real line can be treated as a zero-gap ($n = 0$) system for which the corresponding first order Lax-Novikov differential operator is just the momentum integral $-i\frac{d}{dx}$. But unlike the free particle case with $x \in \mathbb{R}$, the Lax-Novikov differential operator of order $2n + 1$ for the quantum system (1.1) with $g = n(n + 1)$ is not a true integral of motion since it takes out the wave functions from the domain of the Hamiltonian operator (1.1), as it also happens for the operator $-i\frac{d}{dx}$ for the free particle on the half-line. The PT-regularization of the system (1.1) we apply, $x \to x + i\alpha$, $\mathbb{R} \ni \alpha \neq 0$, $x > 0 \;\to x \in \mathbb{R}$, allows us to transform the Lax-Novikov operator of the algebro-geometric method of solution of the KdV equation and higher equations of its hierarchy, into the true integral for the quantum system $H^\alpha_n = -\frac{d^2}{dx^2} + \frac{n(n+1)}{(x+i\alpha)^2}$, which turns out to be a Darboux-dressed momentum of the free particle on the whole line. As a result, the symmetry of the PT-regularized system is described by nonlinearly extended generalized Shrödinger algebra which includes conformal $sl(2, \mathbb{R})$ algebra as the subalgebra, and there appears a non-degenerate bound state $\psi_0 = 1/(x + i\alpha)^n$ of zero energy at the very edge of its doubly degenerate continuous spectrum, which is nothing else as the Darboux-Crum transformed zero energy free particle's state $\psi_0 = 1$. As we showed, such extension of symmetry also has very profound consequences for supersymmetric PT-regularized conformal mechanics systems in comparison with the superconformal $osp(2|2)$ symmetry of the superextended version of the system (1.1). Particularly, in the PT-regularized superconformal mechanics system in the partially broken phase of the exotic nonlinear $\mathcal{N}=4$ super-Poincaré symmetry we considered in the previous section, the $osp(2|2)$ is not contained at all as a sub-superalgebra in the corresponding nonlinearly super-extended Schrödinger algebra.

As it was shown in [62], the systems we considered here are the simplest representatives of a broader class of the systems which can be generated by applying the chains of Darboux transformations to the quantum free particle. The interesting question is what happens with the conformal symmmetry for more complicated systems from the indicated class, particularly, in the systems related to solutions of the equations of the KdV hierarchy which reveal the properties of the extreme waves [62].

Since conformal mechanics (1.1) with special values of the coupling constant $g = n(n+1)$ plays a special role in the Huygens' principle [68], the interesting question that deserves a separate investigation corresponds to the treatment of the considered systems as (1+1)-dimensional field theories from the perspective of the associated time-dependent Schrödinger equation. An a priori complication which may appear in this direction is that due to the higher-derivative nature of the Lax-Novikov operators, it is impossible, at least directly, to associate them with Noether integrals of the corresponding field systems. The indicated field-theoretical generalization is also interesting bearing in mind that the considered PT-symmetric quantum mechanical systems are also related to the singular kinks arising as traveling waves in the Liouville and $SU(3)$ conformal Toda systems [62].

The obtained results could be generalized for the case of the AFF conformal mechanics model with the confining harmonic potential term. The essential difference in such a case is that the corresponding systems do not have Lax-Novikov integrals. However, in the case of rational deformations of such systems with special values of the coupling constant $g = n(n+1)$, instead of the Lax-Novikov integrals they are characterized by higher-derivative spectrum-generating ladder operators [93, 94]. As a result, in that case also there appear nonlinear extensions of the (super)-Schrödinger symmetry of the structure similar to that investigated here [97].

It would be interesting to generalize our results for the case of appropriately PT-regularized many-particle superconformal mechanics [7]–[13]. There, additional integrals also can be constructed in the form of higher-order Lax-Novikov type integrals via the Darboux-dressing procedure [98]. It is necessary to stress that the formal Lax-Novikov type integrals in n-particle Calogero-Moser systems with $n > 2$ also have a pure quantum nature, and they are not related to (maximal) super-integrability of the systems which takes place already at the classical level [99, 100, 101, 102]. Similarly to the present case of $n = 2$, the square of the indicated formal integrals also reduces there to a polynomial in the Liouville charges [98].

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