LI-YORKE CHAOS IN NONAUTONOMOUS HOPF BIFURCATION PATTERNS - I.

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Abstract. We analyze the characteristics of the global attractor of a type of dissipative nonautonomous dynamical systems in terms of the Sacker and Sell spectrum of its linear part. The model gives rise to a pattern of nonautonomous Hopf bifurcation which can be understood as a generalization of the classical autonomous one. We pay special attention to the dynamics at the bifurcation point, showing the possibility of occurrence of Li-Yorke chaos in the corresponding attractor and hence of a high degree of unpredictability.

1. Introduction

The nonautonomous bifurcation theory for ordinary or partial differential equations is a relatively new, complex, and challenging subject of study for which many fundamental problems remain open. One of the initial questions is to determine what kind of objects are those whose structural variation determines the occurrence of bifurcation, as it happens with the constant and periodic solutions in the autonomous case. The use of the skew-product formalism in the analysis of the solutions of nonautonomous differential equations, highly developed during the last decades, has given several possible answers to this question: those objects we referred to can be bounded solutions, recurrent solutions, minimal sets, local attractors, or global attractors. The choice of each one of these categories defines a different research line, and all of them have shown their relevance. The works of Braaksma *et al.* [9], Alonso and Obaya [2], Johnson and Mantellini [24], Fabbri *et al.* [14], Langa *et al.* [29, 30], Rasmussen [43, 44], Núñez and Obaya [37], Pötzsche [41, 42], Anagnostopoulou and Jäger [3], Fuhrmann [16], and Caraballo *et al.* [12], develop some of these lines, providing nonautonomous transcritical, saddle-node and pitchfork bifurcation patterns. In some cases these nonautonomous phenomena admit a dynamical description analogous to that of the autonomous case. But in other cases they present some extremely complex dynamical phenomena, which cannot occur in the autonomous scenery or even in the periodic one.

The question of determining what a Hopf bifurcation means in the nonautonomous case is even more complex. Among the works devoted to this problem we can mention those of Braaksma and Broer [8] and Braaksma *et al.* [9] (who focus on the quasiperidic case and talk about bifurcation when there is a change of dimension of invariant tori), Johnson and Yi [27] (where special attention is paid to the case in which an invariant two-torus looses stability), Johnson *et al.* [23]

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(who study a two-step bifurcation pattern of Arnold type, in terms of the global change of the local and pullback attractor) Anagnostopoulou *et al.* [4] (where a Hopf bifurcation pattern for a discrete skew-product flow is analyzed, and where the bifurcation corresponds to a global change in the global attractor), and Franca *et al.* [15] (where conditions are established for nonautonomous perturbations of a classical autonomous pattern of Andronov-Hopf bifurcation ensuring the persistence of the perturbation), as well as some of the references therein.

The present work is the first of two papers devoted to the analysis of Hopf bifurcation phenomena occurring for a one-parametric family of families of nonautonomous two-dimensional systems of ODEs of the form

$$\mathbf{y}' = A^\varepsilon(\sigma(t,\omega))\,\mathbf{y} - k_\rho(|\mathbf{y}|)\,\mathbf{y}\,, \qquad \omega\in\Omega: \tag{1.1}$$

each one of the systems is defined along one of the orbits of a continuous flow $(\Omega,\sigma,\mathbb{R})$ on a compact metric space Ω, which is minimal and uniquely ergodic. Here, $A^\varepsilon\colon\Omega\to\mathbb{M}_{2\times 2}(\mathbb{R})$ is a family of continuous maps for ε in a given interval; $|\mathbf{y}|$ represents the Euclidean norm of $\mathbf{y}\in\mathbb{R}^2$; and the map k_ρ is given by

$$k_\rho\colon \mathbb{R}^+\to\mathbb{R}^+, \quad r\mapsto \begin{cases} 0 & \text{if } 0\le r\le\rho\,,\\ (r-\rho)^2 & \text{if } r\ge\rho\,, \end{cases}$$

for a fixed value of $\rho\in(0,1]$. That is, k is C^1 on $\mathbb{R}^+$, convex, increasing and unbounded, and it vanishes on $[0,\rho]$.

Frequently, this setting comes from a one-parametric system of the form

$$\mathbf{y}' = A_0^\varepsilon(t)\,\mathbf{y} - k_\rho(|\mathbf{y}|)\,\mathbf{y}\,.$$

For instance, let A_0 be an almost-periodic matrix-valued function on $\mathbb{R}$, and $A_0^\varepsilon = A_0+\varepsilon I_2$. In this case we can take Ω as the hull A_0 (that is, the closure in the compact-open topology of the set $\{A_s\mid s\in\mathbb{R}\}$ with $A_s(t)=A_0(t+s)$), define σ by time translation (that is, $\sigma(t,\omega)(s)=\omega(t+s)$), and define $A(\omega)=\omega(0)$ (so that $A(\sigma(t,\omega))=\omega(t)$) and $A^\varepsilon = A+\varepsilon I_2$. The properties of compactness of Ω, of continuity, minimality and ergodic uniqueness of the flow σ, and of continuity of the map A are proved in Sell [47]. Note that the almost-periodic situation includes as particular cases those in which A_0 is constant (with Ω given by a point) or periodic (with Ω given by a circle), which are the simplest ones, as well as those in which A_0 is quasiperiodic (with Ω given by a torus) or limit-periodic (in which case Ω can be a solenoid: see [33], and observe that in this example Ω is not a locally connected space, so that it cannot be identified with a differentiable manifold).

But we do not restrict ourselves to the almost-periodic situation, which makes our setting more general; it is known that there exist other functions for which the flow on the hull is minimal and uniquely ergodic, although they are not easy to describe. The main advantage of having a family like (1.1) instead of a single system is that the solutions of all the systems corresponding to any fixed ε allow us to define a skew-product flow $(\Omega\times\mathbb{R}^2,\tau^\varepsilon_{\mathbb{R}},\mathbb{R})$ with $(\Omega,\sigma,\mathbb{R})$ as base component. In addition, the boundedness of A^ε combined with our particular choice for k_ρ ensures the dissipativity of the family for each value of ε (as we will check in Section 4), and hence the existence of a global attractor $\mathcal{A}^\varepsilon\subset\Omega\times\mathbb{R}^2$ for $\tau^\varepsilon_{\mathbb{R}}$.

The existence of this global attractor is the starting point: our bifurcation analysis is focused on the evolution of $\mathcal{A}^\varepsilon$ as ε varies, so that a substantial change on its structure determines a bifurcation point. We put special attention in exploring the possibility of occurrence of Li-Yorke chaos at the bifurcation points, which is indeed the situation in some examples that we describe.

We will show that the occurrence of bifurcation points is determined by the variation of the Sacker and Sell spectrum Σ_{A^ε} of the systems $\mathbf{y}' = A^\varepsilon(\sigma(t,\omega))\,\mathbf{y}$, which due to the minimality of the base flow can be defined for any fixed $\omega \in \Omega$: it is the compact set of points $\lambda \in \mathbb{R}$ such that the translated system $\mathbf{y}' = (A^\varepsilon(\sigma(t,\omega)) - \lambda I_2)\,\mathbf{y}$ does not have exponential dichotomy over $\mathbb{R}$. Let $\lambda_-(\varepsilon)$ and $\lambda_+(\varepsilon)$ be the left and right edge points of Σ_{A^ε}. In this first paper we will analyze the bifurcation occurring at the value $\widetilde{\varepsilon}$ of the parameter when:

- if $\varepsilon < \widetilde{\varepsilon}$, then $\lambda_+(\varepsilon) < 0$: every point in the spectrum is strictly negative;
- $\lambda_-(\widetilde{\varepsilon}) = \lambda_+(\widetilde{\varepsilon}) = 0$: 0 is the unique point in the spectrum;
- if $\varepsilon > \widetilde{\varepsilon}$, then $\lambda_-(\varepsilon) > 0$: every point in the spectrum is strictly positive.

Note that in order to construct a family A^ε with these properties it suffices to take as starting point a matrix A with $\Sigma_A = \{0\}$ and then define $A^\varepsilon := A + \varepsilon\, I_2$ (so that $\widetilde{\varepsilon} = 0$). But many more situations may fit in our conditions. We will show that the global attractor reduces to $\Omega \times \{\mathbf{0}\}$ for $\varepsilon < \widetilde{\varepsilon}$. But, for $\varepsilon > \widetilde{\varepsilon}$, it is given by a set which is homeomorphic to a solid cylinder around $\Omega \times \{\mathbf{0}\}$; and in addition, the boundary of the attractor is the global attractor for the flow restricted to the set $\Omega \times (\mathbb{R}^2 - \{\mathbf{0}\})$ (which is invariant, since so is $\Omega \times \{\mathbf{0}\}$). We will call this pattern *nonautonomous Hopf bifurcation with zero spectrum* due to the analogies of this structure with classical autonomous Hopf bifurcation models. To understand these analogies, just think about the classical autonomous model for Hopf bifurcation $\mathbf{y}' = A^\varepsilon \mathbf{y} - |\mathbf{y}|^2 \mathbf{y}$ with $A^\varepsilon = \left[\begin{smallmatrix} \varepsilon & 1 \\ -1 & \varepsilon \end{smallmatrix}\right]$: for $\varepsilon < 0$ the global attractor of the induced flow on $\mathbb{R}^2$ reduces to the origin of coordinates, while for $\varepsilon > 0$ it is given by a closed disk centered at the origin, whose border attracts all the orbits different from the origin.

At the bifurcation point $\widetilde{\varepsilon}$, many possibilities arise. In the simplest one, the attractor is again homeomorphic to a solid cylinder. Therefore even in this case the bifurcation is discontinuous: just compare with the behavior for $\varepsilon < \widetilde{\varepsilon}$. And there are cases for which both the shape of the attractor and the dynamics on it are extremely complex, with the occurrence of Li-Yorke chaos.

Note that the specific form of $\mathcal{A}^\varepsilon$ is determined by the Sacker and Sell spectrum of the linear part, and hence by the matrix-valued function A^ε. The map k_ρ, responsible of the nonlinearity of the dynamics, plays the role of guaranteing the dissipativity, and the constant ρ determines the global size of the attractor. In addition, the fact that k_ρ vanishes on $[0,\rho]$ is fundamental to make the occurrence of Li-Yorke chaos possible, and causes the bifurcation to be discontinuous even in the simplest case.

The second paper of the series will include the analysis of a *nonautonomous two-step transcritical-Hopf bifurcation* pattern. In this case we will assume the existence of $\widetilde{\varepsilon}_1$ and $\widetilde{\varepsilon}_2$ such that:

- if $\varepsilon < \widetilde{\varepsilon}_1$, then $\lambda_+(\varepsilon) < 0$: every point in the spectrum is strictly negative;
- $\lambda_-(\widetilde{\varepsilon}_1) < 0 = \lambda_+(\widetilde{\varepsilon}_1)$: 0 is the superior of the spectrum but not its unique element;
- if $\varepsilon \in (\widetilde{\varepsilon}_1, \widetilde{\varepsilon}_2)$, then $\lambda_-(\varepsilon) < 0 < \lambda_+(\varepsilon)$: there are strictly negative and strictly positive points in the spectrum;
- $\lambda_-(\widetilde{\varepsilon}_2) = 0 < \lambda_+(\widetilde{\varepsilon}_2)$: 0 is the inferior of the spectrum but not its unique element;

– if $\varepsilon > \widetilde{\varepsilon}_2$, then $\lambda_-(\varepsilon) > 0$: every point in the spectrum is strictly positive.

The simplest example corresponding to this situation can be $A^\varepsilon := \left[\begin{smallmatrix} \varepsilon-1 & 0 \\ 0 & \varepsilon \end{smallmatrix}\right]$: here, $\widetilde{\varepsilon}_1 = 0$ and $\widetilde{\varepsilon}_2 = 1$. For more complex (and always time-dependent) choices of A^ε, we will show that Li-Yorke chaos may appear at ε_1 and/or ε_2. And we will also describe the possibility of persistence of the Li-Yorke chaos when $\Sigma_{A^\varepsilon} \subset (0,\infty)$. This last result is interesting for both patterns: for the second one we have $\Sigma_{A^\varepsilon} \subset (0,\infty)$ if $\varepsilon > \widetilde{\varepsilon}_2$; and for the pattern analyzed in this paper, we have $\Sigma_{A^\varepsilon} \subset (0,\infty)$ if ε is greater than the unique bifurcation point $\widetilde{\varepsilon}$.

It is clear that the situations analyzed in these two papers are far away from exhausting the possibilities. But they suffice to illustrate, once more, the extreme complexity of the bifurcation phenomena in the nonautonomous case: there may appear scenarios of dynamical unpredictability which are not possible in the autonomous case.

Let us sketch briefly the structure of the paper. In Section 2 we recall the basic notions and results on topological dynamics and measure theory which we will use. In the rest of this Introduction, $(\Omega,\sigma,\mathbb{R})$ will always represent a continuous flow on a compact metric space, which is assumed to be minimal and uniquely ergodic. Also in Section 2, we pay special attention to the skew-product flows induced on the bundles $\Omega\times\mathbb{R}^2$, $\Omega\times\mathbb{S}$ and $\Omega\times\mathbb{P}$ (where $\mathbb{S}$ is the unit circle in $\mathbb{R}^2$ and $\mathbb{P}$ is the real projective line) by families of two-dimensional linear systems of ODEs given by the evaluation of a continuous matrix along the orbits of the flow on Ω. Systems of this type are also the object of analysis in Section 3. We prove there that the flow given on $\Omega\times\mathbb{P}$ by a weakly elliptic family of linear systems is Li-Yorke chaotic in the case that it admits an invariant measure which is absolutely continuous with respect to the product measure on the bundle. Apart from the intrinsic interest of this result, some of the properties shown in its proof will be used in Section 6.

From this point the paper is focused on the analysis of the attractor $\mathcal{A}^\varepsilon \subset \Omega\times\mathbb{R}^2$ for a dissipative family of systems of the type (1.1). Let us fix a value of the parameter ε, and omit the superscript on A^ε and $\mathcal{A}^\varepsilon$. In Section 4 we describe this model in detail, as well as the flows induced on $\Omega\times\mathbb{R}^2$, $\Omega\times\mathbb{S}\times\mathbb{R}^+$ and $\Omega\times\mathbb{P}\times\mathbb{R}^+$. We show that they are dissipative, so that they admit global attractors, part of whose basic properties we describe to complete the section.

In Section 5 we relate the global shape and characteristics of the attractor $\mathcal{A}$ with the characteristics of the Sacker and Sell spectrum Σ_A of the family of systems $\mathbf{y}' = A(\omega{\cdot}t)\,\mathbf{y}$. We do not contemplate in this paper all the possibilities: since we are interested in the nonautonomous Hopf bifurcation with zero spectrum pattern, the analysis will be reduced to the cases $\sup\Sigma_A < 0$, $\inf\Sigma_A > 0$, and $\Sigma_A = \{0\}$. As we have already mentioned, the attractor is trivial if $\sup\Sigma_A < 0$: $\mathcal{A} = \Omega\times\{\mathbf{0}\}$. We have also mentioned that $\mathcal{A}$ takes the form of a solid cylinder around Ω with continuous boundary if $\inf\Sigma_A > 0$. More precisely, for each $\omega\in\Omega$ the section $\mathcal{A}_\omega := \{\mathbf{y}\in\mathbb{R}^2 \,|\, (\omega,\mathbf{y})\in\mathcal{A}\}$ contains and is homeomorphic to a closed disk centered at $\mathbf{0}$, and the set $\mathcal{A}_\omega$ varies continuously with respect to ω. And, in addition the boundary of the "cylinder", which is continuous and invariant, is the attractor of the flow restricted to the invariant set $\Omega\times(\mathbb{R}^2-\{\mathbf{0}\})$. The attractor $\mathcal{A}$ also takes the form of a solid cylinder with continuous boundary in the case $\Sigma_A = \{0\}$ if in addition all the solutions of all the linear systems are bounded. Therefore, even in this simplest case there is a lack of continuity in the bifurcation. We complete Section 5 by showing with some simple figures the evolution of the global attractor when $A^\varepsilon = A + \varepsilon I_2$ and A is a quasiperiodic matrix-valued function fitting in the situation just described, in order to clarify the sense of talking about a Hopf bifurcation pattern.

Finally, here we do not say too much about the general properties of $\mathcal{A}$ if $\Sigma_A = \{0\}$ and one unbounded solution exists. However, this last case is precisely the most interesting one for the purposes of the paper. The results obtained in Section 3 are a fundamental tool in Section 6, which is devoted to establish conditions ensuring the occurrence of Li-Yorke chaos, in a very strong sense, on the attractor $\mathcal{A}$. We complete the section and the paper by showing that these conditions are fulfilled in some interesting cases. For instance, when the family (1.1) is of the type

$$\mathbf{y}' = \big(\widetilde{A}(\sigma(t,\omega)) + (e(\sigma(t,\omega)) + \varepsilon)\, I_2\big)\,\mathbf{y} - k_\rho(|\mathbf{y}|)\,\mathbf{y}\,, \qquad \omega \in \Omega\,,$$

where $\widetilde{A}$ has null trace, all the solutions of all the linear systems of the family $\mathbf{y}' = \widetilde{A}(\omega{\cdot}t)\,\mathbf{y}$ are bounded, and $e\colon \Omega \to \mathbb{R}$ is a continuous function providing the following (highly complex) dynamics for the flow induced on $\Omega \times \mathbb{R}$ by the family of scalar equations $x' = e(\sigma(t,\omega))\,x$: for almost every system of the family (with respect to the unique ergodic measure) the solutions are bounded; and there are systems for which the solutions are not only unbounded but strongly oscillating at $-\infty$ and $+\infty$. There are well known examples of quasi-periodic functions $e_0\colon \mathbb{R} \to \mathbb{R}$ giving rise to a hull Ω and a map e with these characteristics, as those described by Johnson in [22] and Ortega and Tarallo in [40]. And recently Campos *et al.* [10] have proved that there exist functions $e\colon \Omega \to \mathbb{R}$ with the required properties whenever the (minimal and uniquely ergodic) flow on Ω is not periodic.

The conclusion is that the carried-on analysis provides a pattern of nonautonomous Hopf bifurcation, in which a extremely high degree of complexity is possible. This possibility is one of the strongest differences with the classical autonomous bifurcation theory.

2. Preliminaries

2.1. **Basic concepts.** We begin by recalling some basic concepts and properties of topological dynamics and measure theory, and by fixing some notation.

Let Ω be a complete metric space, and let dist_Ω be the distance on Ω. A (*real and continuous*) *flow* on Ω is given by a continuous map $\sigma\colon \mathbb{R}{\times}\Omega \to \Omega,\ (t,\omega) \mapsto \sigma(t,\omega)$ such that $\sigma_0 = \mathrm{Id}$ and $\sigma_{s+t} = \sigma_t \circ \sigma_s$ for each $s,t \in \mathbb{R}$, where $\sigma_t(\omega) := \sigma(t,\omega)$. The flow is *local* if the map σ is defined, continuous, and satisfies the previous properties on an open subset of $\mathbb{R}{\times}\Omega$ containing $\{0\}{\times}\Omega$.

Let the flow $(\Omega,\sigma,\mathbb{R})$ be defined on $\mathcal{U} \subseteq \mathbb{R}{\times}\Omega$. The set $\{\sigma_t(\omega)\,|\,(t,\omega) \in \mathcal{U}\}$ is the σ-*orbit* of the point $\omega \in \Omega$. This orbit is *globally defined* if $(t,\omega) \in \mathcal{U}$ for all $t \in \mathbb{R}$. Restricting the time to $t \geq 0$ or $t \leq 0$ provides the definition of *forward* or *backward σ-semiorbit*. A subset $\mathcal{C} \subseteq \Omega$ is σ-*invariant* if it is composed by globally defined orbits; i.e., if $\sigma_t(\mathcal{C}) := \{\sigma(t,\omega)\,|\,\omega \in \mathcal{C}\}$ is defined and agrees with $\mathcal{C}$ for every $t \in \mathbb{R}$. A σ-invariant subset $\mathcal{M} \subseteq \Omega$ is *minimal* if it is compact and does not contain properly any other compact σ-invariant set; or, equivalently, if each one of the two semiorbits of anyone of its elements is dense in it. The flow $(\Omega,\sigma,\mathbb{R})$ is *minimal* if Ω itself is minimal. If the set $\{\sigma_t(\omega_0)\,|\,t \geq 0\}$ is well-defined and relatively compact, the *omega limit set* of ω_0 is given by the points $\omega \in \Omega$ such that $\omega = \lim_{m\to\infty} \sigma_{t_m}(\omega_0)$ for some sequence $(t_m) \uparrow \infty$. This set is nonempty, compact, connected and σ-invariant. By taking sequences $(t_m) \downarrow -\infty$ we obtain the definition of the *alpha limit set* of ω_0. A global flow is *distal* if $\inf_{t\in\mathbb{R}} \mathrm{dist}_\Omega(\sigma_t(\omega_1),\sigma_t(\omega_2)) > 0$ whenever $\omega_1 \neq \omega_2$. The next definitions are less standard:

Definition 2.1. Let $(\Omega,\sigma,\mathbb{R})$ be a continuous flow on a compact metric space. Let ω_1,ω_2 be two points of Ω whose forward σ-orbits are globally defined. The points ω_1,ω_2 form a *positively distal pair* for the flow if $\liminf_{t\to\infty} \mathrm{dist}_\Omega(\sigma_t(\omega_1),\sigma_t(\omega_2)) > 0$, and a *positively asymptotic pair* if $\limsup_{t\to\infty} \mathrm{dist}_\Omega(\sigma_t(\omega_1),\sigma_t(\omega_2)) = 0$. The points ω_1,ω_2 form *Li-Yorke pair* for the flow if the pair is neither positively distal nor positively asymptotic. A set $\mathcal{S}\subseteq\Omega$ such that every pair of different points of $\mathcal{S}$ form a Li-Yorke pair is called a *scrambled set* for the flow. The flow $(\Omega,\sigma,\mathbb{R})$ is *Li-Yorke chaotic* if there exists an uncountable scrambled set.

This notion was introduced in [31] in 1975. The interested reader can find in [7] and [1] some dynamical properties associated to Li-Yorke chaos and its relation with other notions of chaotic dynamics.

Let m be a Borel measure on Ω; i.e., a regular measure defined on the Borel sets. The measure m is *σ-invariant* if $m(\sigma_t(\mathcal{B})) = m(\mathcal{B})$ for every Borel subset $\mathcal{B}\subseteq\Omega$ and every $t\in\mathbb{R}$. Suppose that m is finite and normalized; i.e., that $m(\Omega)=1$. Then it is *σ-ergodic* if it is σ-invariant and, in addition, $m(\mathcal{B})=0$ or $m(\mathcal{B})=1$ for every σ-invariant subset $\mathcal{B}\subseteq\Omega$. If Ω is compact, the continuous flow $(\Omega,\sigma,\mathbb{R})$ admits at least an ergodic measure. The flow is *uniquely ergodic* if it admits just a normalized invariant measure, in which case this measure is ergodic.

Let $(\Omega,\sigma,\mathbb{R})$ be a global flow on a compact metric space, and let $\mathbb{Y}$ be a complete metric space. Let dist_Ω and $\mathrm{dist}_\mathbb{Y}$ be the distances on Ω and $\mathbb{Y}$. Then the map $\mathrm{dist}_{\Omega\times\mathbb{Y}}((\omega_1,y_1),(\omega_2,y_2)) := \mathrm{dist}_\Omega(\omega_1,\omega_2)+\mathrm{dist}_\mathbb{Y}(y_1,y_2)$ defines a distance on $\Omega\times\mathbb{Y}$, and we have a new complete metric space. In what follows, this product space is understood as a bundle over Ω. The sets Ω and $\mathbb{Y}$ will be referred to as the *base* and the *fiber* of the bundle. A *skew-product flow on* $\Omega\times\mathbb{Y}$ *projecting onto* $(\Omega,\sigma,\mathbb{R})$ is a (local or global) flow given by a continuous map τ of the form

$$\tau\colon \mathcal{U}\subseteq\mathbb{R}\times\Omega\times\mathbb{Y}\to\Omega\times\mathbb{Y},\quad (\omega,y)\mapsto(\omega{\cdot}t,\tau_2(t,\omega,y))\,.$$

The flow $(\Omega,\sigma,\mathbb{R})$ is the *base flow* of $(\Omega\times\mathbb{Y},\tau,\mathbb{R})$. Note that the *fiber component* τ_2 of τ satisfies $\tau_2(s+t,\omega,y)=\tau_2(s,\omega{\cdot}t,\tau_2(t,\omega,y))$ whenever the right-hand term is defined. If $\mathbb{Y}$ is a vector space, a global skew-product flow is *linear* if the map $\mathbb{Y}\to\mathbb{Y}$, $y\mapsto\tau_2(t,\omega,y)$ is linear for all $(t,\omega)\in\mathbb{R}\times\Omega$. A measurable map $\alpha\colon\Omega\to\mathbb{Y}$ is an *equilibrium for* τ if $\tau_2(t,\omega,\alpha(\omega))=\alpha(\omega{\cdot}t)$ for all $t\in\mathbb{R}$ and $\omega\in\Omega$. A set $\mathcal{K}\subset\Omega\times\mathbb{Y}$ is a *copy of the base for* τ if it is the graph of a continuous equilibrium.

Definition 2.2. Let $(\Omega\times\mathbb{Y},\tau,\mathbb{R})$ be a skew-product flow over a minimal and uniquely ergodic base $(\Omega,\sigma,\mathbb{R})$, and let $\mathcal{K}\subseteq\Omega\times\mathbb{Y}$ be a τ-invariant compact set. Then the restricted flow $(\mathcal{K},\tau,\mathbb{R})$ is *Li-Yorke fiber-chaotic in measure* if there exists a set $\Omega_0\subseteq\Omega$ with full measure such that $\mathcal{K}$ contains an uncountable scrambled set of Li-Yorke pairs with first component ω for each $\omega\in\Omega_0$.

Remark 2.3. It is clear that, in the case of skew-product flow $(\Omega\times\mathbb{Y},\tau,\mathbb{R})$, a pair of points (ω,y_1), (ω,y_2) (with common first component) form: a positively distal pair if and only if $\liminf_{t\to\infty}\mathrm{dist}_\mathbb{Y}(\tau_2(t,\omega,y_1),\tau_2(t,\omega,y_2))>0$; a positively asymptotic pair if and only if $\limsup_{t\to\infty}\mathrm{dist}_\mathbb{Y}(\tau_2(t,\omega,y_1),\tau_2(t,\omega,y_2))=0$; and a Li-Yorke pair if these two conditions fail.

Note also that the notion of Li-Yorke fiber-chaos in measure, much more exigent than that of Li-Yorke chaos, makes only sense in the setting of skew-product flows.

The same happens with the notion of residually Li-Yorke chaotic flow, previously analyzed in [6] and [19]. Li-Yorke chaos for nonautonomous dynamical systems is also the object of analysis in [11] and [12].

The *Hausdorff semidistance* from $\mathcal{C}_1$ to $\mathcal{C}_2$, where $\mathcal{C}_1, \mathcal{C}_2 \subset \Omega \times \mathbb{Y}$, is

$$\mathrm{dist}(\mathcal{C}_1, \mathcal{C}_2) := \sup_{(\omega_1, y_1) \in \mathcal{C}_1} \left(\inf_{(\omega_2, y_2) \in \mathcal{C}_2} \big(\mathrm{dist}_{\Omega \times \mathbb{Y}}((\omega_1, y_1), (\omega_2, y_2)) \big) \right).$$

Definition 2.4. A set $\mathcal{B} \subset \Omega \times \mathbb{Y}$ is said *to attract a set* $\mathcal{C} \subseteq \Omega$ *under* τ if $\tau_t(\mathcal{C})$ is defined for all $t \geq 0$ and, in addition, $\lim_{t\to\infty} \mathrm{dist}(\tau_t(\mathcal{C}), \mathcal{B}) = 0$. The flow τ is *bounded dissipative* if there exists a bounded set $\mathcal{B}$ attracting all the bounded subsets of $\Omega \times \mathbb{Y}$ under τ. And a set $\mathcal{A} \subset \Omega \times \mathbb{Y}$ is a *global attractor* for τ if it is compact, τ-invariant, and it attracts every bounded subset of $\Omega \times \mathbb{Y}$ under τ.

As usual, given a subset $\mathcal{C} \subseteq \Omega \times \mathbb{Y}$, we will represent its sections over the base elements by $\mathcal{C}_\omega := \{y \in \mathbb{Y} \mid (\omega, y) \in \mathcal{C}\}$. Finally, given a normalized Borel measure on m_Ω on Ω and a regular measure $m_\mathbb{Y}$ on $\mathbb{Y}$, we represent by $m_\Omega \times m_\mathbb{Y}$ the product measure on $\Omega \times \mathbb{Y}$. A measure m on $\Omega \times \mathbb{Y}$ *projects onto* m_Ω if $m(\mathcal{B} \times \mathbb{Y}) = m_\Omega(\mathcal{B})$ for any Borel set $\mathcal{B} \subseteq \Omega$. If this is the case and m is τ-invariant, then m_Ω is σ-invariant.

2.2. **The flows induced by a linear family.** As usual, we identify the unit circle $\mathbb{S}$ of $\mathbb{R}^2$ and the one-dimensional real projective line $\mathbb{P}$ with the quotient spaces $\mathbb{R}/(2\pi\mathbb{Z})$ and $\mathbb{R}/(\pi\mathbb{Z})$, respectively. In this way, the map

$$\mathfrak{p} \colon \mathbb{S} \to \mathbb{P}, \quad \theta \mapsto \theta (\mathrm{mod}\, \pi)$$

defines a projection of $\mathbb{S}$ onto $\mathbb{P}$. Note that $\mathbb{S}$ can be understood as a 2-cover of $\mathbb{P}$: if $\theta \in \mathbb{P}$, then $\mathfrak{p}^{-1}(\theta) = \{\theta, \theta + \pi\} \subset \mathbb{S}$. The map $\mathfrak{p}$ will be frequently used.

Let $(\Omega, \sigma, \mathbb{R})$ be a minimal flow on a compact metric space. (The ergodic uniqueness is not required by now.) Given four continuous functions $a, b, c, d \colon \Omega \to \mathbb{R}$, we consider the family of nonautonomous two-dimensional linear systems of ODEs

$$\mathbf{y}' = \begin{bmatrix} a(\omega{\cdot}t) & b(\omega{\cdot}t) \\ c(\omega{\cdot}t) & d(\omega{\cdot}t) \end{bmatrix} \mathbf{y} \tag{2.1}$$

for $\omega \in \Omega$, with $\mathbf{y} \in \mathbb{R}^2$. We call $A := \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]$. We will use the notation $(2.1)_\omega$ to refer to the system of this family corresponding to the point ω, and will proceed in an analogous way for the rest of the equations appearing in the paper. And we represent by $\mathbf{y}_l(t, \omega, \mathbf{y}_0)$ the (globally defined) solution of the system $(2.1)_\omega$ with initial data $\mathbf{y}_l(0, \omega, \mathbf{y}_0) = \mathbf{y}_0$: the subindex l makes reference to the linearity of the systems. Then the map

$$\tau_{l,\mathbb{R}} \colon \mathbb{R} \times \Omega \times \mathbb{R}^2 \to \Omega \times \mathbb{R}^2, \quad (t, \omega, \mathbf{y}_0) \mapsto (\omega{\cdot}t, \mathbf{y}_l(t, \omega, \mathbf{y}_0))$$

defines a linear skew-product flow with base $(\Omega, \sigma, \mathbb{R})$. It is possible to write

$$\mathbf{y}_l(t, \omega, \mathbf{y}_0) = \begin{bmatrix} r_l(t, \omega, \theta, r_0) \sin(\widehat{\theta}(t, \omega, \theta)) \\ r_l(t, \omega, \theta, r_0) \cos(\widehat{\theta}(t, \omega, \theta)) \end{bmatrix} \quad \text{for } \mathbf{y}_0 = \begin{bmatrix} r_0 \sin\theta \\ r_0 \cos\theta \end{bmatrix}: \tag{2.2}$$

here $t \mapsto \widehat{\theta}(t, \omega, \theta)$ is the solution of the equation

$$\theta' = f(\omega{\cdot}t, \theta) \tag{2.3}$$

for

$$f(\omega, \theta) := -c(\omega) \sin^2\theta + b(\omega) \cos^2\theta + (a(\omega) - d(\omega)) \sin\theta \cos\theta \tag{2.4}$$

with initial data $\widehat\theta(0,\omega,\theta)=\theta$, which we understand as an element of $\mathbb{S}$; and the map $t\mapsto r_l(t,\omega,\theta,r_0)$ solves

$$r'=r\,g(\omega{\cdot}t,\widehat\theta(t,\omega,\theta)) \tag{2.5}$$

with $r_l(0,\omega,\theta,r_0)=r_0$, for

$$g(\omega,\theta):=a(\omega)\sin^2\theta+d(\omega)\cos^2\theta+(b(\omega)+c(\omega))\sin\theta\cos\theta\,. \tag{2.6}$$

Note that the map

$$\widehat\sigma\colon \mathbb{R}\times\Omega\times\mathbb{S}\to\Omega\times\mathbb{S}\,,\quad (t,\omega,\theta)\mapsto(\omega{\cdot}t,\widehat\theta(t,\omega,\theta)) \tag{2.7}$$

defines a skew-product flow on $\Omega\times\mathbb{S}$. Note also that $f(\omega,\theta)=f(\omega,\theta+\pi)$ and $g(\omega,\theta)=g(\omega,\theta+\pi)$: we can define them either on $\Omega\times\mathbb{P}$ or on $\Omega\times\mathbb{S}$. Consequently,

$$\widehat\theta(t,\omega,\theta+\pi)=\widehat\theta(t,\omega,\theta)+\pi \quad\text{and}\quad r_l(t,\omega,\theta,r_0)=r_l(t,\omega,\theta+\pi,r_0)\,.$$

In particular, we can understand the solutions of (2.3) as elements $\mathbb{P}$: let us write $\widetilde\theta(t,\omega,\theta)=\mathfrak{p}(\widehat\theta(t,\omega,\theta))=\widehat\theta(t,\omega,\theta)(\mathrm{mod}\,\pi)$ for $(t,\omega,\theta)\in\mathbb{R}\times\Omega\times\mathbb{P}$, and note that

$$\widetilde\sigma\colon \mathbb{R}\times\Omega\times\mathbb{P}\to\Omega\times\mathbb{P}\,,\quad (t,\omega,\theta)\mapsto(\omega{\cdot}t,\widetilde\theta(t,\omega,\theta)) \tag{2.8}$$

defines a global skew-product flow on $\Omega\times\mathbb{P}$. We say that $(\Omega\times\mathbb{S},\widehat\sigma,\mathbb{R})$ *projects onto* $(\Omega\times\mathbb{P},\widetilde\sigma,\mathbb{R})$. Note also that $\mathfrak{p}(\widehat\theta(t,\omega,\theta))=\widetilde\theta(t,\omega,\mathfrak{p}(\theta))$ for $(t,\omega,\theta)\in\mathbb{R}\times\Omega\times\mathbb{S}$.

Let $U(t,\omega)$ be the fundamental matrix solution of $(2.1)_\omega$ with $U(0,\omega)=I_2$, so that $\mathbf{y}_l(\omega,\omega,\mathbf{y}_0)=U(t,\omega)\,\mathbf{y}_0$. For further purposes we recall that

$$\det U(t,\omega_0)=\exp\Big(\int_0^t \operatorname{tr}A(\omega_0{\cdot}s)\,ds\Big)\,. \tag{2.9}$$

Definition 2.5. The family (2.1) has *exponential dichotomy over* Ω if there exist constants $c\ge 1$ and $\gamma>0$ and a splitting $\Omega\times\mathbb{R}^2=F^+\oplus F^-$ of the bundle into the Whitney sum of two closed subbundles such that

- F^+ and F^- are invariant under the flow $(\Omega\times\mathbb{R}^2,\tau_{l,\mathbb{R}},\mathbb{R})$,
- $|\,U(t,\omega)\,\mathbf{y}_0|\le c\,e^{-\gamma t}|\mathbf{y}_0|$ for every $t\ge 0$ and $(\omega,\mathbf{y}_0)\in F^+$,
- $|\,U(t,\omega)\,\mathbf{y}_0|\le c\,e^{\gamma t}|\mathbf{y}_0|$ for every $t\le 0$ and $(\omega,\mathbf{y}_0)\in F^-$.

Remarks 2.6. 1. Since the base flow $(\Omega,\sigma,\mathbb{R})$ is minimal, the exponential dichotomy of the family (2.1) over Ω is equivalent to the exponential dichotomy of any of its systems over $\mathbb{R}$: see e.g. Theorem 2 and Section 3 of [45].

2. The family (2.1) has exponential dichotomy over Ω if and only if no one of its systems has a nontrivial bounded solution: see e.g. Theorem 1.61 of [25].

Definition 2.7. The *Sacker and Sell spectrum* or *dynamical spectrum* of the linear family (2.1) is the set Σ_A of $\lambda\in\mathbb{R}$ such that the family $\mathbf{y}'=(A(\omega{\cdot}t)-\lambda I_2)\,\mathbf{y}$ does not have exponential dichotomy over Ω.

Now we assume also that the base flow $(\Omega,\sigma,\mathbb{R})$ is uniquely ergodic, and represent by m_Ω the unique σ-invariant (ergodic) measure. The next theorem summarizes part of the information provided by the Oseledets theorem (see Section 2 of [26]) and the Sacker and Sell spectral theorem (see Theorem 6 of [46]).

Theorem 2.8. *One of the following situations holds.*

Case 1. $\Sigma_A=[\gamma_1,\gamma_2]$ *with* $\gamma_1<\gamma_2$. *In this case there exists a* σ*-invariant subset* $\Omega_0\subset\Omega$ *with* $m_\Omega(\Omega_0)=1$ *such that, for all* $\omega\in\Omega_0$, o.1) *there exist two one-dimensional vector spaces* $W^{\gamma_1}_\omega$ *and* $W^{\gamma_2}_\omega$ *such that* $\lim_{|t|\to\infty}(1/t)\ln|U(t,\omega)\,\mathbf{y}_0| = \gamma_j$ *for* $\mathbf{y}_0\in W^{\gamma_j}_\omega-\{\mathbf{0}\}$.

o.2) $\mathbb{R}^2 = W^{\gamma_1}_\omega \oplus W^{\gamma_2}_\omega$.

In particular, if $\omega\in\Omega_0$ *and* $\mathbf{y}^j = \begin{bmatrix} y_1^j \\ y_2^j\end{bmatrix} \in W^{\gamma_j}_\omega - \{\mathbf{0}\}$*, and we call* $\theta^{\gamma_j}(\omega) := \tan^{-1}(y_1^j/y_2^j)\in\mathbb{P}$*, then* $\theta^{\gamma_j}(\omega)$ *is uniquely determined for* $j=1,2$*. In addition,*

o.3) *the maps* $\Omega_0\to\mathbb{P}\,,\ \omega\mapsto\theta^{\gamma_j}(\omega)$ *are measurable and satisfy* $\widetilde\theta(t,\omega,\theta^{\gamma_j}(\omega)) = \theta^{\gamma_j}(\omega{\cdot}t)$ *for all* $t\in\mathbb{R}$ *and* $\omega\in\Omega_0$*, and for* $j=1,2$*.*

o.4) $\lim_{|t|\to\infty}(1/t)\ln\det U(t,\omega) = \gamma_1+\gamma_2$ *for all* $\omega\in\Omega$.

Consequently, the measurable subbundles

$$W^{\gamma_j} := \{(\omega,\mathbf{y})\in\Omega\times\mathbb{R}^2 \mid \omega\in\Omega_0 \text{ and } \mathbf{y}\in W^{\gamma_j}_\omega\} \tag{2.10}$$

are $\tau_{l,\mathbb{R}}$*-invariant for* $j=1,2$.

Case 2. $\Sigma_A=\{\gamma_1\}\cup\{\gamma_2\}$. *Then all the assertions in* Case 1 *hold for* $\Omega_0=\Omega$*. In addition:* W^{γ_1} *and* W^{γ_2} *are closed subbundles;* θ^{γ_1} *and* θ^{γ_2} *are continuous maps;* $\Omega\times\mathbb{R}^2 = W^{\gamma_1}\oplus W^{\gamma_2}$ *as Whitney sum;* $\lim_{t\to-\infty}\operatorname{dist}_{\mathbb{P}}(\widetilde\theta(t,\omega,\theta),\theta^{\gamma_1}(\omega{\cdot}t))=0$ *if* $\theta\neq\theta^{\gamma_2}(\omega)$*; and* $\lim_{t\to\infty}\operatorname{dist}_{\mathbb{P}}(\widetilde\theta(t,\omega,\theta),\theta^{\gamma_2}(\omega{\cdot}t))=0$ *if* $\theta\neq\theta^{\gamma_1}(\omega)$.

Case 3. $\Sigma_A=\{\gamma\}$. *In this case, for all* $\omega\in\Omega$, $\lim_{|t|\to\infty}(1/t)\ln|\,U(t,\omega)\,\mathbf{y}_0|=\gamma$ *for* $\mathbf{y}_0\in\mathbb{R}^2-\{\mathbf{0}\}$*, and* $\lim_{|t|\to\infty}(1/t)\ln\det U(t,\omega)=2\gamma$.

Definition 2.9. In cases 1 and 2, the sets W^{γ_j} defined by (2.10) are the *Oseledets subbundles* of the family of linear systems (2.1), and the numbers γ_1 and γ_2 are its *Lyapunov exponents*. In case 3, the value γ is the unique *Lyapunov exponent*.

Remark 2.10. In the case that $\Sigma_A\subset(-\infty,0)$, $F^+=\Omega\times\mathbb{R}^2$; and, if $\Sigma_A\subset(0,\infty)$, then $F^-=\Omega\times\mathbb{R}^2$. These assertions follow easily from the fact that Σ_A contains all the Lyapunov exponents of the family (see e.g. Theorem 2.3 of [26]) and from the casuistic described in Theorem 2.8.

3. Li-Yorke chaos for weakly elliptic linear systems

As explained in the Introduction, this section provides conditions on a certain type of linear systems which ensure the occurrence of Li-Yorke chaos for the corresponding projective flow. This is done in Theorem 3.4. As a matter of fact, we will show that for almost every point $\omega\in\Omega$, the scrambled set of points of the form (ω,θ) for the flow $(\Omega\times\mathbb{P},\widetilde\sigma,\mathbb{R})$ contains all the points of $\{\omega\}\times\mathbb{P}$ excepting at most one. Apart from the independent interest of this result, its proof includes some arguments which will be essential in the proof of our main result, in Section 6.

For the rest of the paper, $(\Omega,\sigma,\mathbb{R})$ will be a minimal and uniquely ergodic flow on a compact metric space. The unique σ-ergodic measure on Ω will be denoted by m_Ω; $l_{\mathbb{R}^n}$ will be the Lebesgue measure on $\mathbb{R}^n$; and $l_{\mathbb{S}}$ and $l_{\mathbb{P}}$ will denote the normalized Lebesgue measures on $\mathbb{S}$ and $\mathbb{P}$.

We consider a family of linear (Hamiltonian) systems with zero trace,

$$\mathbf{y}' = \widetilde A(\omega{\cdot}t)\,\mathbf{y} \tag{3.1}$$

for $\omega\in\Omega$, where $\widetilde A = \left[\begin{smallmatrix}\widetilde a & b\\ c & -\widetilde a\end{smallmatrix}\right]$. (The choice of the names for the coefficients is due to the fact that b and c will later agree with those of (2.1).) The angular equation is

$$\theta' = f(\omega{\cdot}t,\theta) \tag{3.2}$$

for

$$f(\omega,\theta) := -c(\omega)\sin^2\theta + b(\omega)\cos^2\theta + 2\,\widetilde a(\omega)\sin\theta\cos\theta\,. \tag{3.3}$$

As in the previous section, we will represent by $\widehat\theta(t,\omega,\theta)$ and $\widetilde\theta(t,\omega,\theta)$ the solutions on $\mathbb{S}$ and $\mathbb{P}$ of the equation with initial datum θ, and by $(\Omega{\times}\mathbb{S},\widehat\sigma,\mathbb{R})$ and $(\Omega{\times}\mathbb{P},\widetilde\sigma,\mathbb{R})$ the corresponding flows, given by the expressions (2.7) and (2.8).

Definition 3.1. The family (3.1) *is in the weakly elliptic case* if its Sacker and Sell spectrum is $\Sigma_{\widetilde A}=\{0\}$ and at least one of its systems has an unbounded solution.

We will describe in the next remarks two already classical procedures which will be used several times in what follows, and fix the corresponding notation.

Remark 3.2. Let $C\colon\Omega\to\mathbb{M}_{2\times 2}(\mathbb{R})$ be continuous, with $\det C\equiv 1$, and such that $C'\colon\Omega\to\mathbb{M}_{2\times 2}(\mathbb{R})$ with $C'(\omega):=(d/dt)\,C(\omega{\cdot}t)|_{t=0}$ is a well-defined and continuous function. Let us consider the change of variables $(\omega,\mathbf{y})\mapsto(\omega,\mathbf{z})$ given by $\mathbf{z}=C(\omega)\,\mathbf{y}$. This change of variables takes the system $\mathbf{y}'=\widetilde A(\omega{\cdot}t)\,\mathbf{y}$ to $\mathbf{z}'=\widetilde A^*(\omega{\cdot}t)\,\mathbf{z}$ with $\widetilde A^*:=(C'+CA)\,C^{-1}$. It is easy to check that $\operatorname{tr}\widetilde A^*=0$. In addition,

- **p1.** $\mathbf{y}'=\widetilde A(\omega{\cdot}t)\,\mathbf{y}$ is in the weakly elliptic case if and only if $\mathbf{z}'=\widetilde A^*(\omega{\cdot}t)\,\mathbf{z}$ is, as immediately deduced from $\mathbf{z}(t,\omega,\mathbf{z}_0)=C(\omega{\cdot}t)\,\mathbf{y}(t,\omega,C^{-1}(\omega)\,\mathbf{z}_0)$ together with the boundedness of C and C^{-1}.
- **p2.** The linear and continuous change of variables induces a homeomorphism $\Phi\colon\Omega{\times}\mathbb{P}\to\Omega{\times}\mathbb{P}$ with $\Phi(\omega,\theta)=(\omega,\phi_\omega(\theta))$ and $\widetilde\sigma^*(t,\Phi(\omega,\theta))=\Phi(\widetilde\sigma(t,\omega,\theta))$. It follows from here that two points $(\omega,\theta_1),(\omega,\theta_2)$ are a Li-Yorke pair for $(\Omega{\times}\mathbb{P},\widetilde\sigma,\mathbb{R})$ if and only if the points $(\omega,\phi_\omega(\theta_1)),(\omega,\phi_\omega(\theta_1))$ are a Li-Yorke pair for $(\Omega{\times}\mathbb{P},\widetilde\sigma^*,\mathbb{R})$. The same argument shows that the property is also true for positively distal pairs instead of Li-Yorke pairs.
- **p3.** If the flow $(\Omega{\times}\mathbb{P},\widetilde\sigma,\mathbb{R})$ admits an invariant measure μ which is absolutely continuous with respect to $m_\Omega{\times}l_{\mathbb{P}}$, then so does the flow $(\Omega{\times}\mathbb{P},\widetilde\sigma^*,\mathbb{R})$ defined from the family $\mathbf{z}'=\widetilde A^*(\omega{\cdot}t)\,\mathbf{z}$. To prove it, we use the fact that, for all $\omega\in\Omega$, the homeomorphism $\phi_\omega\colon\mathbb{P}\to\mathbb{P}$ takes measures of $\mathbb{P}$ which are absolutely continuous with respect to $l_{\mathbb{P}}$ to measures of the same type. (In turn, this property can be deduced from the fact that $\mathbf{z}\mapsto C(\omega)\,\mathbf{z}$ preserves the Lebesghe measure in $\mathbb{R}^2$, since $\det C(\omega)=1$.) This assertion combined with Fubini's theorem ensures that the measure μ^* defined over the Borel sets by $\mu^*(\mathcal{B}):=\mu(\Phi^{-1}(\mathcal{B}))$ (which is $\widetilde\sigma^*$-invariant) is absolutely continuous with respect to $m_\Omega{\times}l_{\mathbb{P}}$. (The maps Φ and ϕ_ω are defined in **p2**.)

Remark 3.3. Let us consider the flow $(\Omega\times\mathbb{S},\widehat\sigma,\mathbb{R})$. We take a $\widehat\sigma$-minimal set $\mathcal{M}\subseteq\Omega\times\mathbb{S}$. For each $(\omega,\theta^1)\in\mathcal{M}$, we define $\widetilde a^1(\omega,\theta^1):=\widetilde a(\omega)$, $b^1(\omega,\theta^1):=b(\omega)$ and $c^1(\omega,\theta^1):=c(\omega)$, and consider the family of linear systems

$$\mathbf{y}'=\begin{bmatrix}\widetilde a^1(\widehat\theta(t,\omega,\theta^1)) & b^1(\widehat\theta(t,\omega,\theta^1))\\ c^1(\widehat\theta(t,\omega,\theta^1)) & -\widetilde a^1(\widehat\theta(t,\omega,\theta^1))\end{bmatrix}\mathbf{y} \tag{3.4}$$

for $(\omega,\theta^1)\in\mathcal{M}$. Note that the angular equation

$$\theta':=-c^1(\widehat\theta(t,\omega,\theta^1))\sin^2\theta+b^1(\widehat\theta(t,\omega,\theta^1))\cos^2\theta+2\,\widetilde a^1(\widehat\theta(t,\omega,\theta^1))\sin\theta\cos\theta \tag{3.5}$$

corresponding to (ω,θ^1) agrees with $(3.2)_\omega$. Therefore, the two skew-product flows with base $\mathcal{M}$ defined by the family (3.5) are

$$\begin{aligned}&\widehat\vartheta_{\mathcal{M}}\colon\mathbb{R}{\times}\mathcal{M}{\times}\,\mathbb{S}\to\mathcal{M}\times\mathbb{S}\,,\quad (t,\omega,\theta^1,\theta)\mapsto(\widehat\theta(t,\omega,\theta^1),\widehat\theta(t,\omega,\theta))\,,\\ &\widetilde\vartheta_{\mathcal{M}}\colon\mathbb{R}{\times}\mathcal{M}{\times}\,\mathbb{P}\to\mathcal{M}\times\mathbb{P}\,,\quad (t,\omega,\theta^1,\theta)\mapsto(\widehat\theta(t,\omega,\theta^1),\widetilde\theta(t,\omega,\theta))\,.\end{aligned} \tag{3.6}$$

We list some properties relating (3.1) to (3.4), later required.

p4. If the family (3.1) is in the weakly elliptic case, then so (3.4) is: 0 is its unique Lyapunov exponent, and there exists an unbounded solution.

p5. Let us take $(\omega,\theta^1)\in\mathcal{M}$ and $\theta_1,\theta_2\in\mathbb{P}$. Then the points $(\omega,\theta_1),(\omega,\theta_2)$ are a Li-Yorke pair for $(\Omega\times\mathbb{P},\widetilde\sigma,\mathbb{R})$ if and only if the points $(\omega,\theta^1,\theta_1),(\omega,\theta^1,\theta_2)$ are a Li-Yorke pair for $(\mathcal{M}\times\mathbb{P},\widetilde\vartheta_{\mathcal{M}},\mathbb{R})$. This fact follows immediately from the fact that the fiber component of $\widetilde\vartheta_{\mathcal{M}}(t,\omega,\theta^1,\theta)$ agrees with that of $\widetilde\sigma(t,\omega,\theta)$. And the property is also true for positively distal pairs.

p6. Let us assume that $(\Omega\times\mathbb{P},\widetilde\sigma,\mathbb{R})$ admits an invariant measure m which is absolutely continuous with respect to $m_\Omega\times l_{\mathbb{P}}$, and let $m_{\mathcal{M}}$ be a $\widehat\sigma$-ergodic measure on $\mathcal{M}$, which projects onto m_Ω. Let $q\in L^1(\Omega\times\mathbb{P},m_\Omega\times l_{\mathbb{P}})$ be the density function of m, and let f be defined by (3.3). Proposition 2.2 of [39] ensures that there exists a measurable function $p\colon\Omega\times\mathbb{P}\to\mathbb{R}^+$ with $p(\omega,\theta)=q(\omega,\theta)$ for $(m_\Omega\times l_{\mathbb{P}})$-a.a. $(\omega,\theta)\in\Omega\times\mathbb{P}$ such that

$$p(\widetilde\sigma(l,\omega,\theta))=p(\omega,\theta)\exp\Big(-\int_0^l\frac{\partial f}{\partial\theta}\,(\widetilde\sigma(s,\omega,\theta))\,ds\Big)\tag{3.7}$$

for all $(\omega,\theta)\in\Omega\times\mathbb{P}$ and $l\in\mathbb{R}$. Let us define $p^1(\omega,\theta^1,\theta):=p(\omega,\theta)$ for all $(\omega,\theta^1,\theta)\in\mathcal{M}\times\mathbb{P}$. It is easy to check that the nonnegative function p^1 belongs to $L^1(\mathcal{M}\times\mathbb{P},m_{\mathcal{M}}\times l_{\mathbb{P}})$ and that it satisfies the equation (3.7) corresponding to $(\mathcal{M}\times\mathbb{P},\widetilde\vartheta_{\mathcal{M}},\mathbb{R})$ for all $(\omega,\theta^1,\theta)\in\mathcal{M}\times\mathbb{P}$ and $l\in\mathbb{R}$. A new application of Proposition 2.2 of [39] shows that $(\mathcal{M}\times\mathbb{P},\widetilde\vartheta_{\mathcal{M}},\mathbb{R})$ admits an invariant measure which is absolutely continuous with respect to $m_{\mathcal{M}}\times l_{\mathbb{P}}$.

Note also that the set $\mathcal{M}^1:=\{(\omega,\theta^1,\theta^1)\,|\,(\omega,\theta^1)\in\mathcal{M}\}$ is a copy of the base for the flow $(\mathcal{M}\times\mathbb{S},\widehat\vartheta_{\mathcal{M}},\mathbb{R})$. The last property is the main achievement of this procedure: the existence of this copy of the base (which may not be the case for (3.1)), will allow us to define linear and continuous changes of variables taking (3.4) to families of systems whose corresponding dynamics are easier to describe; and from this description we will be able to derive the required conclusions for the initial family.

Theorem 3.4. *Let us assume that the family* (3.1) *is in the weakly elliptic case. Let us assume also that the flow* $(\Omega\times\mathbb{P},\widetilde\sigma,\mathbb{R})$ *admits an invariant measure which is absolutely continuous with respect to* $m_\Omega\times l_{\mathbb{P}}$. *Then there exists a* σ*-invariant subset* $\Omega_0\subseteq\Omega$ *with* $m_\Omega(\Omega_0)=1$ *such that for every* $\omega\in\Omega_0$ *there exists a subset* $\mathcal{P}_\omega\subseteq\mathbb{P}$ *with the next properties:* $\mathbb{P}-\mathcal{P}_\omega$ *contains at most one element; and for every pair of different points* $\theta_1,\theta_2\in\mathcal{P}_\omega$*, the points* $(\omega,\theta_1),(\omega,\theta_2)$ *form a Li-Yorke pair for* $(\Omega\times\mathbb{P},\widetilde\sigma,\mathbb{R})$. *Hence, the flow* $(\Omega\times\mathbb{P},\widetilde\sigma,\mathbb{R})$ *is Li-Yorke fiber-chaotic in measure.*

Proof. The proof is carried-out in two steps: the first one contains auxiliary results for the second one, which proves the statements.

Step 1. We will begin by assuming that the family (3.1) is triangular,

$$\mathbf{y}'=\begin{bmatrix}\widetilde a(\omega{\cdot}t) & 0\\ c(\omega{\cdot}t) & -\widetilde a(\omega{\cdot}t)\end{bmatrix}\mathbf{y}\,.\tag{3.8}$$

Later on we will assume that the flow $(\Omega\times\mathbb{P},\widetilde\sigma,\mathbb{R})$ either does not contain a positively distal pair, or it has two minimal sets.

The angular equation for (3.8) is $\theta'=2\,\widetilde a(\omega{\cdot}t)\sin\theta\cos\theta-c(\omega{\cdot}t)\sin^2\theta$, so that the compact set $\{(\omega,0)\}\subset\Omega\times\mathbb{P}$ is $\widetilde\sigma$-invariant. Therefore it concentrates a $\widetilde\sigma$-invariant measure, which together with the assumed existence of a $\widetilde\sigma$-invariant measure absolutely continuous with respect to $m_\Omega\times l_{\mathbb{P}}$ allows us to apply Proposition 3.3 of [35] in order to conclude that: there exist a σ-invariant set Ω_1 with $m_\Omega(\Omega_1)=1$ and measurable functions $m_a\colon\Omega_1\to\mathbb{R}$ and $\varphi_0\colon\Omega_1\to\mathbb{P}$ (with $(d/dt)\,m_a(\omega{\cdot}t)=-\widetilde a(\omega{\cdot}t)\,m_a(\omega{\cdot}t)$ and such that $t\mapsto\varphi_0(\omega{\cdot}t)$ satisfies the angular equation, in both cases for all $\omega\in\Omega_1$) such that $\Omega\times\mathbb{P}$ decomposes into the disjoint union of the measurable $\widetilde\sigma$-invariant sets $\mathcal{S}_\eta:=\{(\omega,\varphi_\eta(\omega))\,|\,\omega\in\Omega_1\}$ for $\eta\in(-\infty,\infty]$, where $\varphi_\eta(\omega):=\operatorname{arccot}(\eta\,m_a^2(\omega)+\cot\varphi_0(\omega))$. (These sets are the *ergodic* 1*-sheets*, using the language of [35]; see also [17].) Let $\mathcal{K}\subseteq\Omega_1$ be a compact set with $m_\Omega(\mathcal{K})>0$ such that the restrictions of m_a and φ_0 to $\mathcal{K}$ are continuous. Let $\Omega_0^*\subseteq\Omega$ be the set of points ω for which there exists a sequence $(t_n)\uparrow\infty$ with $\omega{\cdot}t_n\in\mathcal{K}$. Birkhoff's ergodic theorem ensures that $m_\Omega(\Omega_0^*)=1$. Now we fix $\omega\in\Omega_0^*$ and choose a sequence $(t_n)\uparrow\infty$ such that $\omega{\cdot}t_n\in\mathcal{K}$ for all $n\ge 0$. We take two different points $\theta_1,\theta_2\in\mathbb{P}$ and write them as $\theta_i=\varphi_{\eta_i}(\omega)$, so that $\eta_1\ne\eta_2$. Then $\operatorname{dist}_{\mathbb{P}}(\widetilde\theta(t_n,\omega,\theta_1),\widetilde\theta(t_n,\omega,\theta_2))\ge\inf_{\widetilde\omega\in\mathcal{K}}\operatorname{dist}_{\mathbb{P}}(\varphi_{\eta_2}(\widetilde\omega),\varphi_{\eta_2}(\widetilde\omega))>0$ if $n\ge 0$. In particular, for $\omega\in\Omega_0^*$ and $\theta_1\ne\theta_2$,

$$\limsup_{t\to\infty}\operatorname{dist}_{\mathbb{P}}(\widetilde\theta(t,\omega,\theta_1),\widetilde\theta(t,\omega,\theta_2))>0\,. \tag{3.9}$$

Now we consider two different situations. The first one is simple: the flow $(\Omega\times\mathbb{P},\widetilde\sigma,\mathbb{R})$ does not admit a positively distal pair. Then (3.9) ensures that the pair (ω,θ_1), (ω,θ_2) is of Li-Yorke type whenever $\omega\in\Omega_0^*$ and $\theta_1,\theta_2\in\mathbb{P}$ with $\theta_1\ne\theta_2$. To complete the proof in this case, we define $\Omega_0:=\Omega_0^*$ and $\mathcal{P}_\omega:=\mathbb{P}$ for each $\omega\in\Omega_0$.

The second case we consider is that the flow $(\Omega\times\mathbb{P},\widetilde\sigma,\mathbb{R})$ admits two different minimal sets. The proof of Proposition 4.4 in [21] (which does not require the almost-periodicity of the base flow, there assumed) shows that these minimal sets are the unique ones: otherwise all the solutions of the systems of the family (3.8) would be bounded, which is not the case. We already know that one of these minimal sets is $\{(\omega,0)\,|\,\omega\in\Omega\}$. Therefore there exists $\delta>0$ such that any (ω,θ^1) in the second minimal set satisfies $\delta\le\theta^1\le\pi-\delta$. Let us take a $\widehat\sigma$-minimal set $\mathcal{M}\subset\Omega\times\mathbb{S}$ projecting onto the second minimal set of $\Omega\times\mathbb{P}$, and perform the procedure described in Remark 3.3, taking now (3.8) as starting point. Note that the obtained system (3.4) is now also triangular: $b^1\equiv 0$. In addition, the set $\mathcal{M}^1:=\{(\omega,\theta^1,\theta^1)\,|\,(\omega,\theta^1)\in\mathcal{M}\}$ is a copy of the base for the flow $(\mathcal{M}\times\mathbb{S},\widehat\vartheta_{\mathcal{M}},\mathbb{R})$, and it is contained either in $\Omega\times\mathbb{S}\times[\delta,\pi-\delta]$ or in $[\pi+\delta,2\pi-\delta]$. A straightforward computation shows that the bounded and continuous change of variables on $\mathcal{M}\times\mathbb{R}^2$ given by $(\omega,\theta^1,\mathbf{y})\mapsto(\omega,\theta^1,\mathbf{w})$ for $\mathbf{w}:=\left[\begin{smallmatrix}1&0\\-\cot\theta^1&1\end{smallmatrix}\right]\mathbf{y}$ takes the (now triangular) family (3.4) to a new family of the form

$$\mathbf{w}'=\begin{bmatrix}\widehat a^{\,1}(\widehat\theta(t,\omega,\theta^1)) & 0\\ 0 & -\widehat a^{\,1}(\widehat\theta(t,\omega,\theta^1))\end{bmatrix}\mathbf{w}\,. \tag{3.10}$$

Let $m_{\mathcal{M}}$ be a $\widehat\sigma$-ergodic measure concentrated on $\mathcal{M}$. Property **p1** in Remark 3.2 combined with property **p4** in Remark 3.3 ensures that the family (3.10) is also in the weakly elliptic case; and properties **p3** and **p6** ensure that the new flow $(\mathcal{M}\times\mathbb{P},\widetilde\vartheta^*_{\mathcal{M}},\mathbb{R})$ given by the family of angular equations $\theta'=2\,\widehat a^{\,1}(\widehat\theta(t,\omega,\theta^1))\sin\theta\cos\theta$ (corresponding to (3.10)), admits an invariant measure absolutely continuous with respect to $m_{\mathcal{M}}\times l_{\mathbb{P}}$. Clearly, the set $\{(\omega,\theta^1,0)\,|\,(\omega,\theta^1)\in\mathcal{M}\}\subset\mathcal{M}\times\mathbb{P}$ is $\widetilde\vartheta^*_{\mathcal{M}}$-minimal (as $\{(\omega,\theta^1,\pi/2)\,|\,(\omega,\theta^1)\in\mathcal{M}\}$). Proposition 3.3 of [35] allows us to ensure that there exists a $\widehat\sigma$-invariant subset $\mathcal{M}_1$ of $\mathcal{M}$ with $m_{\mathcal{M}}(\mathcal{M}_1)=1$ and a measurable function $m_a^*\colon \mathcal{M}_1 \to \mathbb{R}^+$ with $(d/dt)\, m_a^*(\widehat{\theta}(t,\omega,\theta^1)) = -\widehat{a}(\widehat{\theta}(t,\omega,\theta^1))\, m_a^*(\widehat{\theta}(t,\omega,\theta^1))$ for all $(\omega,\theta^1) \in \mathcal{M}_1$ such that $\mathcal{M}\times\mathbb{P}$ decomposes into the disjoint union of the measurable $\widetilde{\vartheta}^*_{\mathcal{M}}$-invariant sets (ergodic 1-sheets) $\mathcal{S}^*_\eta := \{(\omega,\theta^1,\varphi^*_\eta(\omega,\theta^1)) \,|\, (\omega,\theta^1) \in \mathcal{M}_1\}$ for $\eta \in (-\infty,\infty]$, where $\varphi^*_\eta(\omega,\theta^1) := \operatorname{arccot}(\eta\,(m_a^*)^2(\omega,\theta^1))$.

Let us call Ω_0 to the intersection of the projection of $\mathcal{M}_1$ onto Ω with the set Ω_0^* for which (3.9) holds, and note that there Ω_0 is σ-invariant with $m_\Omega(\Omega_0) = 1$. We fix $\omega_0 \in \Omega_0$ and choose $\theta_0^1 \in \mathbb{S}$ with $(\omega_0,\theta_0^1) \in \mathcal{M}_1$. Given two different points $\theta_1,\theta_2 \in \mathbb{P}$, we write $\theta_i = \varphi^*_{\eta_i}(\omega_0,\theta_0^1)$ for $i = 1,2$, so that $\eta_1,\eta_2 \in (-\infty,\infty]$ and $\eta_1 \neq \eta_2$. Note that the fiber component of $\widetilde{\vartheta}^*_{\mathcal{M}}$ satisfies $\widetilde{\theta}^*_{\mathcal{M}}(t,\omega_0,\theta_0^1,\theta_i) = \varphi^*_{\eta_i}(\widehat{\sigma}(t,\omega_0,\theta_0^1))$. Note also that there cannot exist $\kappa_1,\kappa_2 \in \mathbb{R}$ such that $0 < \kappa_1 \le m_a^*(\widehat{\sigma}(t,\omega_0,\theta_0^1)) \le \kappa_2 < \infty$ for all $t \ge 0$: otherwise all the solutions of all the systems of the family (3.4) would be bounded (see e.g. Proposition A.1 in [28]), which is not the case. Assume that there exists a sequence $(s_n) \uparrow \infty$ such that $\lim_{n\to\infty} m_a^*(\widehat{\sigma}(s_n,\omega_0,\theta_0^1)) = 0$. Then $\lim_{n\to\infty} \widetilde{\theta}^*_{\mathcal{M}}(s_n,\omega_0,\theta_0^1,\theta_i) = \pi/2$ if $\theta_i \neq 0$ (since $\eta_i \neq \infty$). In this case we set $\mathcal{P}^*_{\omega_0} := \mathbb{P} - \{0\}$. If the previous property does not hold, then $\lim_{n\to\infty} m_a^*(\widehat{\sigma}(s_n,\omega_0,\theta_0^1)) = \infty$ for a sequence $(s_n) \uparrow \infty$, which in turn ensures that $\lim_{n\to\infty} \widetilde{\theta}^*_{\mathcal{M}}(s_n,\omega_0,\theta_0^1,\theta_i) = 0$ if $\theta_i \neq \pi/2$ (since $\eta_i \neq 0$). In this case we take $\mathcal{P}^*_{\omega_0} := \mathbb{P} - \{\pi/2\}$. In both cases we conclude that $\liminf_{t\to\infty} \operatorname{dist}_{\mathbb{P}}(\widetilde{\theta}^*_{\mathcal{M}}(t,\omega_0,\theta_0^1,\theta_1), \widetilde{\theta}^*_{\mathcal{M}}(t,\omega_0,\theta_0^1,\theta_2)) = 0$ if $\theta_1,\theta_2 \in \mathcal{P}^*_{\omega_0}$.

The performed change of variables induces a flow homeomorphism from $(\mathcal{M}\times\mathbb{P},\widetilde{\vartheta}_{\mathcal{M}},\mathbb{R})$ to $(\mathcal{M}\times\mathbb{P},\widetilde{\vartheta}^*_{\mathcal{M}},\mathbb{R})$, given by $\Phi^*(\omega,\theta^1,\theta) := (\omega, \operatorname{arccot}(\cot\theta - \cot\theta^1))$. Let us define $\mathcal{P}_{\omega_0} := \{\theta \in \mathbb{P} \,|\; \Phi^*(\omega,\theta^1,\theta) \in \mathcal{P}^*_{\omega_0}\}$. Then $\mathbb{P} - \mathcal{P}_{\omega_0}$ contains one element; and, since the fiber component of $\widetilde{\vartheta}_{\mathcal{M}}$ is $\widetilde{\theta}$ (see (3.6)), we have $\liminf_{t\to\infty} \operatorname{dist}_{\mathbb{P}}(\widetilde{\theta}(t,\omega_0,\theta_1), \widetilde{\theta}(t,\omega_0,\theta_2)) = 0$ if $\theta_1,\theta_2 \in \mathcal{P}_{\omega_0}$. Combining this property with (3.9) we conclude that for all $\omega_0 \in \Omega_0$ and all $\theta_1,\theta_2 \in \mathcal{P}_{\omega_0}$ with $\theta_1 \neq \theta_2$, the points $(\omega_0,\theta_1),(\omega_0,\theta_2)$ form a Li-Yorke pair for $(\Omega\times\mathbb{P},\widetilde{\sigma},\mathbb{R})$. This completes the proof in this case, and STEP 1. We point out that the ergodic uniqueness of $(\Omega,\sigma,\mathbb{R})$ has not been required, which will be fundamental in STEP 2.

STEP 2. We will now prove the statement in the general case. The idea is: to reformulate the family in a new base on which we can find a continuous change of variables taking it to a new family with triangular form for which the hypotheses remain true and which fits in one of the two situations analyzed in STEP 1; to apply the results of that step; and to show that the conclusions for the new base suffice to our purposes.

We follow again the procedure described in Remark 3.3: we fix a $\widehat{\sigma}$-minimal set $\mathcal{M} \subseteq \Omega\times\mathbb{S}$ and a $\widehat{\sigma}$-ergodic measure $m_{\mathcal{M}}$ concentrated on $\mathcal{M}$, and consider the family of systems (3.4) and the flows $(\mathcal{M}\times\mathbb{S},\widehat{\vartheta}_{\mathcal{M}},\mathbb{R})$ and $(\mathcal{M}\times\mathbb{P},\widetilde{\vartheta}_{\mathcal{M}},\mathbb{R})$. (For the moment being $\mathcal{M}$ is any set with these properties; later on we will need to be more precise with its choice.) Recall that $\mathcal{M}^1 := \{(\omega,\theta^1,\theta^1) \,|\; (\omega,\theta^1) \in \mathcal{M}\}$ is a copy of the base for $\widehat{\vartheta}_{\mathcal{M}}$. Let us now consider the change of variables given on $\mathcal{M}\times\mathbb{R}^2$ by $(\omega,\theta^1,\mathbf{y}) \mapsto (\omega,\theta^1,\mathbf{z})$ with $\mathbf{z} := \left[\begin{smallmatrix} \cos\theta^1 & -\sin\theta^1 \\ \sin\theta^1 & \cos\theta^1 \end{smallmatrix}\right]\mathbf{y}$, which induces the rotation $(\omega,\theta^1,\theta) \mapsto (\omega,\theta^1,\theta-\theta^1)$ on $\mathcal{M}\times\mathbb{S}$. Clearly, the minimal set $\mathcal{M}^1$ is taken to $\{(\omega,\theta^1,0) \,|\, (\omega,\theta^1) \in \mathcal{M}\}$, which ensures that the solutions of (3.1) are taken to those of a new family of the form

$$\mathbf{z}' = \begin{bmatrix} \widehat{a}^{\,1}(\widehat{\theta}(t,\omega,\theta^1)) & 0 \\ \widehat{c}^{\,1}(\widehat{\theta}(t,\omega,\theta^1)) & -\widehat{a}^{\,1}(\widehat{\theta}(t,\omega,\theta^1)) \end{bmatrix} \mathbf{z}\,: \tag{3.11}$$

the coefficient $\widehat{b}^1$ is 0 since the function 0 solves the corresponding equation (3.5); and the trace of the new matrix is zero, as explained in Remark 3.2.

Let us now consider the new flow $(\mathcal{M}\times\mathbb{P},\widetilde{\vartheta}^*_{\mathcal{M}},\mathbb{R})$ given by the family of angular equations $\theta' = -\widehat{c}^{\,1}(\widehat{\theta}(t,\omega,\theta^1))\sin^2\theta + 2\,\widehat{a}^{\,1}(\widehat{\theta}(t,\omega,\theta^1))\sin\theta\cos\theta$ (corresponding to (3.11)). As in STEP 1, we can ensure that the family (3.11) is in the weakly elliptic case, and that the flow $(\mathcal{M}\times\mathbb{P},\widetilde{\vartheta}^*_{\mathcal{M}},\mathbb{R})$ admits an invariant measure which is absolutely continuous with respect to $m_{\mathcal{M}}\times l_{\mathbb{P}}$. In other words, the family (3.11) satisfies the hypotheses of the theorem, and it is triangular. We will see that a suitable choice if $\mathcal{M}$ makes it fit in one of the two situations analyzed in STEP 1.

Assume first that the flow $(\Omega\times\mathbb{P},\widehat{\sigma},\mathbb{R})$ does not admit a positively distal pair. Then, according to the last assertion in **p5** (in Remark 3.3) and **p2** (in Remark 3.2), the flow $(\mathcal{M}\times\mathbb{P},\widetilde{\vartheta}^*_{\mathcal{M}},\mathbb{R})$ does not admit a positively distal pair, and hence it fits in the first situation of STEP 1 (no matter the choice of $\mathcal{M}$).

Now let us assume that the flow $(\Omega\times\mathbb{P},\widehat{\sigma},\mathbb{R})$ admits a positively distal pair. We will check that $\mathcal{M}$ can be chosen in such a way that $(\mathcal{M}\times\mathbb{P},\widetilde{\vartheta}_{\mathcal{M}},\mathbb{R})$ admits two different minimal sets. Hence so does $(\mathcal{M}\times\mathbb{P},\widetilde{\vartheta}^*_{\mathcal{M}},\mathbb{R})$, and consequently this flow fits in the second situation analyzed in STEP 1.

Let the points $(\omega,\theta_1),(\omega,\theta_2)$ form a positively distal pair for $(\Omega\times\mathbb{P},\widetilde{\sigma},\mathbb{R})$. Let us take a $\widetilde{\sigma}$-minimal set $\widetilde{\mathcal{M}}\subseteq\Omega\times\mathbb{P}$ contained in the omega limit set of (ω,θ_1), and a point $(\omega_*,\theta^1_*)\in\widetilde{\mathcal{M}}$. Then we can choose a sequence $(t_n)\uparrow\infty$ and a point $(\omega_*,\theta^2_*)\in\Omega\times\mathbb{P}$ such that $(\omega_*,\theta^i_*)=\lim_{n\to\infty}\widetilde{\theta}(t_n,\omega,\theta_i)$ for $i=1,2$. Is is clear that $(\omega_*,\theta^1_*),(\omega_*,\theta^2_*)$ form a positively distal pair. Let us take a $\widehat{\sigma}$-minimal set $\mathcal{M}\subseteq\Omega\times\mathbb{S}$ projecting onto $\widetilde{\mathcal{M}}$ such that $(\omega_*,\theta^1_*)\in\mathcal{M}$, consider the flow $(\mathcal{M}\times\mathbb{P},\widetilde{\vartheta}_{\mathcal{M}},\mathbb{R})$ defined by (3.6), and note that $(\omega_*,\theta^1_*,\theta^1_*),(\omega_*,\theta^1_*,\theta^2_*)$ form a positively distal pair for this flow. Note also that $(\omega_*,\theta^1_*,\theta^1_*)$ belongs to the minimal set $\mathcal{M}^1:=\{(\omega,\theta^1,\mathfrak{p}(\theta^1))\,|\,(\omega,\theta^1)\in\mathcal{M}\}$, which is a copy of the base. Now we take a minimal set $\mathcal{M}^2$ contained in the omega limit set of $(\omega_*,\theta^1_*,\theta^2_*)$ for the flow $\widetilde{\vartheta}_{\mathcal{M}}$. It is easy to deduce from $\inf_{t\ge0}\operatorname{dist}_{\mathbb{P}}(\widetilde{\theta}(t,\omega_*,\theta^1_*),\widetilde{\theta}(t,\omega_*,\theta^2_*))>0$ that $(\omega_*,\theta^1_*,\theta^1_*)\notin\mathcal{M}^2$, so that $\mathcal{M}^1\neq\mathcal{M}^2$. This proves our assertion.

In both cases, we have checked in STEP 1 that there exists a $\widehat{\sigma}$-invariant subset $\mathcal{M}_0\subseteq\mathcal{M}$ with $m_{\mathcal{M}}(\mathcal{M}_0)=1$ such that for every $(\omega,\theta^1)\in\mathcal{M}_0$ there exists a subset $\mathcal{P}_{(\omega,\theta^1)}\subseteq\mathbb{P}$ with the next properties: $\mathbb{P}-\mathcal{P}_{(\omega,\theta^1)}$ contains at most one element; and for every pair of different points $\theta_1,\theta_2\in\mathcal{P}_{(\omega,\theta^1)}$, the points $(\omega,\theta^1,\theta_1),(\omega,\theta^1,\theta_2)$ form a Li-Yorke pair for $(\mathcal{M}\times\mathbb{P},\widetilde{\vartheta}^*,\mathbb{R})$. We define Ω_0 as the projection of $\mathcal{M}_0$ on Ω and note that it is σ-invariant. In addition, since $m_{\mathcal{M}}$ projects onto m_Ω, we have $m_\Omega(\Omega_0)=1$. Given $\omega\in\Omega_0$ we look for $(\omega,\theta^1)\in\mathcal{M}_0$ and define $\mathcal{P}_\omega:=\mathcal{P}_{(\omega,\theta^1)}$. Finally, we use the information provided by **p2** and **p5** to conclude that Ω_0 and $\{\mathcal{P}_\omega\,|\,\omega\in\Omega_0\}$ satisfy all the assertions of the theorem. The proof is complete. □

Remark 3.5. Let us consider the family (3.11) appearing in STEP 2 of the previous proof, and a $\widehat{\sigma}$-ergodic measure $m_{\mathcal{M}}$ concentrated on $\mathcal{M}$. As seen at the beginning of the proof of STEP 1, the associated flow $(\mathcal{M}\times\mathbb{P},\widetilde{\vartheta}^*_{\mathcal{M}},\mathbb{R})$ decomposes into ergodic

1-sheets: we can write $\mathcal{M}\times\mathbb{P}=\bigcup_{\eta\in(-\infty,\infty]}\mathcal{S}^*_\eta$, where $\mathcal{S}^*_\eta$ is a $\widetilde{\vartheta}^*$-invariant set of the form $\{(\omega,\theta^1,\varphi^*_\eta(\omega,\theta^1))\mid(\omega,\theta^1)\in\mathcal{M}\}$ for a measurable map $\varphi^*_\eta\colon\mathcal{M}\to\mathbb{P}$. And, in addition, due to the expression of φ^*_η (namely, $\varphi^*_\eta(\omega,\theta^1)=\operatorname{arccot}(\eta\, m_a^2(\omega,\theta^1)+\cot\varphi_0(\omega,\theta^1))$ for certain measurable functions m_a and φ_0), Lusin's theorem provides a compact set $\mathcal{K}\subseteq\mathcal{M}$ with measure $m_{\mathcal{M}}(\mathcal{K})>1/2$ such that the restrictions of all the maps φ^*_η to $\mathcal{K}$ are continuous. On the other hand, the map $\mathcal{M}\times\mathbb{P}\to\mathcal{M}\times\mathbb{P}$, $(\omega,\theta^1,\theta)\mapsto(\omega,\theta^1,\theta+\theta^1)$ takes $(\mathcal{M}\times\mathbb{P},\widetilde{\vartheta}^*_{\mathcal{M}},\mathbb{R})$ to $(\mathcal{M}\times\mathbb{P},\widetilde{\vartheta}_{\mathcal{M}},\mathbb{R})$. We define now $\varphi_\eta(\omega,\theta^1):=\varphi^*_\eta(\omega,\theta^1)+\theta^1$ and note: that the restrictions of all the maps φ_η to $\mathcal{K}$ are continuous; and that the associated flow $(\mathcal{M}\times\mathbb{P},\widetilde{\vartheta}_{\mathcal{M}},\mathbb{R})$ decomposes into the ergodic 1-sheets $\mathcal{S}_\eta:=\{(\omega,\theta^1,\varphi_\eta(\omega,\theta^1))\mid(\omega,\theta^1)\in\mathcal{M}\}$ for $\eta\in(-\infty,\infty]$. This information will be used in the proof of Theorem 6.8.

4. The boundaries of the global attractors for the flows induced by a family of dissipative systems

Recall that $(\Omega,\sigma,\mathbb{R})$ is a continuous global flow on a compact metric space, minimal and uniquely ergodic, that m_Ω is its only ergodic measure, and that $l^2_{\mathbb{R}}$, $l_{\mathbb{S}}$ and $l_{\mathbb{P}}$ represent the Lebesgue measures on $\mathbb{R}^2$, $\mathbb{S}$ and $\mathbb{P}$ (normalized in the last two cases). In the rest of the paper we will work with a particular type of family of nonlinear systems defined along the σ-orbits, which we now describe. Given a real value $\rho\in(0,1]$ and the C^1-map

$$k_\rho\colon\mathbb{R}^+\to\mathbb{R}^+,\quad r\mapsto\begin{cases}0 & \text{if } 0\le r\le\rho\,,\\ (r-\rho)^2 & \text{if } r\ge\rho\,,\end{cases}\tag{4.1}$$

we consider the family of nonautonomous two-dimensional systems of ODEs

$$\mathbf{y}'=A(\omega{\cdot}t)\,\mathbf{y}-k_\rho(|\mathbf{y}|)\,\mathbf{y}\tag{4.2}$$

for $\omega\in\Omega$, whose linear part

$$\mathbf{y}'=A(\omega{\cdot}t)\,\mathbf{y}\tag{4.3}$$

agrees with (2.1). A discrete version of this continuous model has been studied in [4]. The main difference in our approach is that we put the focus on the unpredictability of the dynamics on the attractor at the bifurcation point.

We will use the notation established in Section 2 for the flows $(\Omega\times\mathbb{R}^2,\tau_{l,\mathbb{R}},\mathbb{R})$, $(\Omega{\times}\mathbb{S},\widehat{\sigma},\mathbb{R})$ and $(\Omega{\times}\mathbb{P},\widetilde{\sigma},\mathbb{R})$ induced by (4.3). The value of ρ will be fixed throughout the paper, so that we will not include it in the notation. The family (4.2) also induces a (now local) skew-product flow with base $(\Omega,\sigma,\mathbb{R})$ on $\Omega\times\mathbb{R}^2$, defined by

$$\tau_{\mathbb{R}}\colon\mathcal{U}\subseteq\mathbb{R}\times\Omega\times\mathbb{R}^2\to\Omega\times\mathbb{R}^2,\quad(t,\omega,\mathbf{y}_0)\mapsto(\omega{\cdot}t,\mathbf{y}(t,\omega,\mathbf{y}_0))\,,\tag{4.4}$$

where $\mathbf{y}(t,\omega,\mathbf{y}_0)$ represents the solution of the system $(4.2)_\omega$ with initial data $\mathbf{y}(0,\omega,\mathbf{y}_0)=\mathbf{y}_0$. Note that $\mathbf{y}(t,\omega,-\mathbf{y}_0)=-\mathbf{y}(t,\omega,\mathbf{y}_0)$. Note also that (4.2) and (4.3) agree as long as $\mathbf{y}$ belongs to the Euclidean closed disk centered at the origin and with radius ρ; but not outside this disk, where (4.2) is no longer linear. In particular, $\mathbf{y}(t,\omega,\mathbf{y}_0)$ may not be globally defined; and if $|\mathbf{y}_0|\le\rho$ then $\mathbf{y}(t,\omega,\mathbf{y}_0)=\mathbf{y}_l(t,\omega,\mathbf{y}_0)$ at the interval of points containing 0 at which $|\mathbf{y}(t,\omega,\mathbf{y}_0)|\le\rho$.

Now we take coordinates $y_1=r\sin\theta$ and $y_2=r\cos\theta$ and obtain the equations

$$\begin{aligned}\theta'&=f(\omega{\cdot}t,\theta)\,,\\ r'&=r\,\big(g(\omega{\cdot}t,\theta)-k_\rho(r)\big),\end{aligned}\tag{4.5}$$

where $f\colon \Omega\times\mathbb{S}\to\mathbb{R}$ and $g\colon\Omega\times\mathbb{S}\to\mathbb{R}$ are given by (2.4) and (2.6). Recall that $f(\omega,\theta)=f(\omega,\theta+\pi)$ and $g(\omega,\theta)=g(\omega,\theta+\pi)$. Observe that the family of equations given by the first line in (4.5) depends neither on ρ nor on r. In fact, it coincides with the family (2.3), so that its solutions define the global flows $(\Omega\times\mathbb{S},\widehat\sigma,\mathbb{R})$ and $(\Omega\times\mathbb{P},\widetilde\sigma,\mathbb{R})$ given by (2.7) and (2.8). Note also that the r component of the solution of $(4.5)_\omega$ with initial data (θ,r_0) agrees with the solution of the equation

$$r'=r\left(g(\omega{\cdot}t,\widehat\theta(t,\omega,\theta))-k_\rho(r)\right) \tag{4.6}$$

with initial data r_0. We can consider this family varying either on $\Omega\times\mathbb{S}$ or on $\Omega\times\mathbb{P}$, since $g(\omega{\cdot}t,\widehat\theta(t,\omega,\theta))=g(\omega{\cdot}t,\widetilde\theta(t,\omega,\theta))$. Let $r(t,\omega,\theta,r_0)$ be the solution of $(4.6)_{(\omega,\theta)}$ with $r(t,\omega,\theta,r_0)=r_0\ge 0$, and note that if $r_0\le\rho$ then $r(t,\omega,\theta,r_0)=r_l(t,\omega,\theta,r_0)$ at the interval containing 0 at which $r(t,\omega,\theta,r_0)\le\rho$. We can define two new local skew-product flows with bases $(\Omega\times\mathbb{S},\widehat\sigma,\mathbb{R})$ and $(\Omega\times\mathbb{P},\widetilde\sigma,\mathbb{R})$, given by

$$\widehat\tau\colon \widehat{\mathcal{V}}\subseteq\mathbb{R}\times\Omega\times\mathbb{S}\times\mathbb{R}^+\to\Omega\times\mathbb{S}\times\mathbb{R}^+,\quad (t,\omega,\theta,r_0)\mapsto(\omega{\cdot}t,\widehat\theta(t,\omega,\theta),r(t,\omega,\theta,r_0))$$

and

$$\widetilde\tau\colon \widetilde{\mathcal{V}}\subseteq\mathbb{R}\times\Omega\times\mathbb{P}\times\mathbb{R}^+\to\Omega\times\mathbb{P}\times\mathbb{R}^+,\quad (t,\omega,\theta,r_0)\mapsto(\omega{\cdot}t,\widetilde\theta(t,\omega,\theta),r(t,\omega,\theta,r_0))\,.$$

Recall that

$$\mathfrak{p}(\widehat\theta(t,\omega,\theta))=\widetilde\theta(t,\omega,\mathfrak{p}(\theta))\quad\text{and}\quad r(t,\omega,\theta,r_0)=r(t,\omega,\mathfrak{p}(\theta),r_0)\,. \tag{4.7}$$

It is clear that

$$\mathbf{y}(t,\omega,\mathbf{y}_0)=\begin{bmatrix} r(t,\omega,\theta,r_0)\sin(\widehat\theta(t,\omega,\theta))\\ r(t,\omega,\theta,r_0)\cos(\widehat\theta(t,\omega,\theta))\end{bmatrix}\quad\text{if}\ \ \mathbf{y}_0=\begin{bmatrix} r_0\sin\theta\\ r_0\cos\theta\end{bmatrix}.$$

Therefore the flows $\tau_{\mathbb{R}}$ and $\widehat\tau$ are closely related. As a matter of fact, they can be identified outside the (respectively invariant) sets $\Omega\times\{\mathbf{0}\}\subset\Omega\times\mathbb{R}^2$ and $\Omega\times\mathbb{S}\times\{0\}\subset\Omega\times\mathbb{S}\times\mathbb{R}^+$. In addition, $(\Omega\times\mathbb{S}\times\mathbb{R}^+,\widehat\tau,\mathbb{R})$ projects onto $(\Omega\times\mathbb{P}\times\mathbb{R}^+,\widetilde\tau,\mathbb{R})$.

Remark 4.1. For further purposes, we point out that the skew-product semiflow $\widetilde\tau$ is *concave*; that is, its fiber component satisfies

$$r(t,\omega,\theta,\eta\,r_1+(1-\eta)\,r_2)\ge\eta\,r(t,\omega,\theta,r_1)+(1-\eta)\,r(t,\omega,\theta,r_2)$$

for all $\eta\in[0,1]$, $(\omega,\theta)\in\Omega\times\mathbb{P}$, $r_1,r_2\in\mathbb{R}^+$, and all the values of $t\ge 0$ such that all the involved terms are defined. This assertion follows from the fact that $k_\rho(\eta r_1+(1-\eta)\,r_2)\le\eta\,k_\rho(r_1)+(1-\eta)\,k_\rho(r_2)$ for all $\eta\in[0,1]$ and $r_1,r_2\in\mathbb{R}^+$ combined with a standard argument of comparison of solutions. And, of course, it is also *monotone*: $0\le r(t,\omega,\theta,r_1)\le r(t,\omega,\theta,r_2)$ whenever $0\le r_1\le r_2$. The monotonicity will be often used without further reference.

Since the function g of (4.6) (given by (2.6)) is bounded, the definition (4.1) of k_ρ shows that, if we fix any $\delta>0$, we can find a r_ρ such that

$$g(\omega,\theta)-k_\rho(r)<-\delta\quad\text{ for all } (\omega,\theta)\in\Omega\times\mathbb{S}\ \text{ and }\ r\ge r_\rho\,. \tag{4.8}$$

This constant r_ρ plays a fundamental role in the statement of the next result: it will bound the zones of the phase spaces in which the attractors lie. Note also that the minimal choice of r_ρ satisfying (4.8) increases as ρ increases. This is the only point in which the choice of a particular $\rho\in(0,1]$ has influence.

Theorem 4.2. *Let the constant $r_\rho>0$ satisfy* (4.8).

(i) *The flow* $(\Omega\times\mathbb{R}^2,\tau_{\mathbb{R}},\mathbb{R})$ *is bounded dissipative, and it has a global attractor* $\mathcal{A}\subset\Omega\times\mathbb{R}^2$ *which contains* $\Omega\times\{\mathbf{0}\}$. *In addition,*

$$\begin{aligned}\mathcal{A}&=\{(\omega,\mathbf{y}_0)\in\Omega\times\mathbb{R}^2|\ \sup_{t\in\mathbb{R}}|\mathbf{y}(t,\omega,\mathbf{y}_0)|<\infty\}\\&=\{(\omega,\mathbf{y}_0)\in\Omega\times\mathbb{R}^2|\ \sup_{t\in\mathbb{R}^-}|\mathbf{y}(t,\omega,\mathbf{y}_0)|<\infty\}\,;\end{aligned}$$

$\mathcal{A}\subseteq\Omega\times\{\mathbf{y}\in\mathbb{R}^2|\ |\mathbf{y}|\le r_\rho\}$; *and* $(\omega,\mathbf{y}_0)\in\mathcal{A}$ *if and only if* $(\omega,-\mathbf{y}_0)\in\mathcal{A}$.

(ii) *The flow* $(\Omega\times\mathbb{S}\times\mathbb{R}^+,\widehat{\tau},\mathbb{R})$ *is bounded dissipative, and it has a global attractor* $\widehat{\mathcal{B}}\subset\Omega\times\mathbb{S}\times\mathbb{R}^+$ *which contains* $\Omega\times\mathbb{S}\times\{0\}$. *In addition,*

$$\begin{aligned}\widehat{\mathcal{B}}&=\{(\omega,\theta,r_0)\in\Omega\times\mathbb{S}\times\mathbb{R}^+|\ \sup_{t\in\mathbb{R}}r(t,\omega,\theta,r_0)<\infty\}\\&=\{(\omega,\theta,r_0)\in\Omega\times\mathbb{S}\times\mathbb{R}^+|\ \sup_{t\in\mathbb{R}^-}r(t,\omega,\theta,r_0)<\infty\}\,;\end{aligned}$$

$\widehat{\mathcal{B}}\subseteq\Omega\times\mathbb{S}\times[0,r_\rho]$; *and* $(\omega,\theta)\in\widehat{\mathcal{B}}$ *for* $\theta\in[0,\pi)$ *if and only if* $(\omega,\theta+\pi)\in\widehat{\mathcal{B}}$.

(iii) *The set*

$$\widetilde{\mathcal{B}}:=\{(\omega,\mathfrak{p}(\theta),r_0)\,|\,(\omega,\theta,r_0)\in\widehat{\mathcal{B}}\}\subseteq\Omega\times\mathbb{P}\times[0,r_\rho]$$

is a global attractor for the bounded dissipative flow $(\Omega\times\mathbb{P}\times\mathbb{R}^+,\widetilde{\tau},\mathbb{R})$.

(iv) *If* $r_0>0$, *then* $\big(\omega,\left[\begin{smallmatrix}r_0\sin\theta\\ r_0\cos\theta\end{smallmatrix}\right]\big)\in\mathcal{A}$ *if and only if* $(\omega,\theta,r_0)\in\widehat{\mathcal{B}}$. *In addition, the dynamics of* $\widehat{\tau}$ *on* $\widehat{\mathcal{B}}-(\Omega\times\mathbb{S}\times\{0\})$ *can be recovered from that of* $\tau_{\mathbb{R}}$ *on* $\mathcal{A}-(\Omega\times\{\mathbf{0}\})$, *and the converse is also true.*

Proof. It is easy to deduce from (4.8) that the maximal interval of definition of $\mathbf{y}(t,\omega,\mathbf{y}_0)$ contains $[0,\infty)$ for all $(\omega,\mathbf{y}_0)\in\Omega\times\mathbb{R}^2$, and that the set $\mathcal{C}_\rho:=\Omega\times\{\mathbf{y}\in\mathbb{R}^2\,|\ |\mathbf{y}|\le r_\rho\}$ attracts any bounded set under $\tau_{\mathbb{R}}$. Or, equivalently, that $r(t,\omega,\theta,r_0)$ is defined at least on $[0,\infty)$ for all $(\omega,\theta,r_0)\in\Omega\times\mathbb{S}\times\mathbb{R}^+$ and that the set $\widehat{\mathcal{C}}_\rho:=\Omega\times\mathbb{S}\times[0,r_\rho]$ attracts any bounded set under $\widehat{\tau}$. Therefore both flows are bounded dissipative, and the classical theory ensures that the sets $\mathcal{A}$ and $\widehat{\mathcal{B}}$ of (i) and (ii) are the respective global attractors: see e.g. Section 2.4 of [18] and Section 1.2 of [13]. Relation (4.8) also guarantees that any globally bounded solution $\mathbf{y}(t,\omega,\mathbf{y}_0)$ of (4.2) satisfies $|\mathbf{y}(t,\omega,\mathbf{y}_0)|\le r_\rho$ for all $t\in\mathbb{R}$: if $|\mathbf{y}_0|>r_\rho$, then (4.8) would force $|\mathbf{y}(t,\omega,\mathbf{y}_0)|$ to tend to ∞ as t tends to the left edge of the maximal interval of definition. so that it is not bounded; therefore $|\mathbf{y}_0|\le r_\rho$, and hence we can use again (4.8) to prove the assertion. Consequently, $\mathcal{A}\subseteq\mathcal{C}_\rho$ and $\widehat{\mathcal{B}}\subseteq\widehat{\mathcal{C}}_\rho$. And obviously $\mathbf{y}(t,\omega,\mathbf{y}_0)$ is globally bounded if and only if $\mathbf{y}(t,\omega,-\mathbf{y}_0)=-\mathbf{y}(t,\omega,\mathbf{y}_0)$ is globally bounded, which completes the proof of (i) and (ii). The assertions in (iii) follows from (ii) and (4.7). Finally, the properties sated in (iv) follow from (i) and (ii) and from the relation between $\tau_{\mathbb{R}}$ and $\widehat{\tau}$ explained before. □

Theorem 4.3. *Let* $\mathcal{A}\subset\Omega\times\mathbb{R}^2$, $\widehat{\mathcal{B}}\subset\Omega\times\mathbb{S}\times\mathbb{R}^+$ *and* $\widetilde{\mathcal{B}}\subset\Omega\times\mathbb{P}\times\mathbb{R}^+$ *be the global attractors of the flows* $(\Omega\times\mathbb{R}^2,\tau_{\mathbb{R}},\mathbb{R})$, $(\Omega\times\mathbb{S}\times\mathbb{R}^+,\widehat{\tau},\mathbb{R})$ *and* $(\Omega\times\mathbb{P}\times\mathbb{R}^+,\widetilde{\tau},\mathbb{R})$, *and let* r_ρ *satisfy* (4.8). *Then, there exists an upper semicontinuous map*

$$\widetilde{\beta}\colon\Omega\times\mathbb{P}\to[0,r_\rho]$$

such that

(i) *the attractors are given by*

$$\mathcal{A} = \bigcup_{\omega\in\Omega} \{(\omega,\mathbf{y}) \mid \mathbf{y} = \left[\begin{smallmatrix} r\sin\theta \\ r\cos\theta \end{smallmatrix}\right] \;\textit{ for } \theta\in\mathbb{S} \textit{ and } r\in[0,\widetilde{\beta}(\omega,\mathfrak{p}(\theta))]\}\,,$$

$$\widehat{\mathcal{B}} = \bigcup_{(\omega,\theta)\in\Omega\times\mathbb{S}} \big(\{(\omega,\theta)\}\times[0,\widetilde{\beta}(\omega,\mathfrak{p}(\theta))]\big)\,,$$

$$\widetilde{\mathcal{B}} = \bigcup_{(\omega,\theta)\in\Omega\times\mathbb{P}} \big(\{(\omega,\theta)\}\times[0,\widetilde{\beta}(\omega,\theta))]\big)\,.$$

(ii) $\widetilde{\beta}(\widetilde{\sigma}(t,\omega,\theta)) = r(t,\omega,\theta,\widetilde{\beta}(\omega,\theta))$ *for all* $(t,\omega,\theta)\in\mathbb{R}\times\Omega\times\mathbb{P}$*; that is,* $\widetilde{\beta}$ *is an equilibrium for* $\widetilde{\sigma}$*.*

(iii) *If there exists* $\beta_0>0$ *such that* $\widetilde{\beta}(\omega,\theta)\ge\beta_0$ *for all* $(\omega,\theta)\in\Omega\times\mathbb{P}$*, then* $\widetilde{\beta}$ *defines a uniformly stable equilibrium: for each* $\varepsilon>0$ *there exists* $\delta(\varepsilon)>0$ *such that if* $(\omega,\theta)\in\Omega\times\mathbb{P}$ *and* $0\le|\widetilde{\beta}(\omega,\theta)-r_0|\le\delta(\varepsilon)$ *then*

$$|\widetilde{\beta}(\widetilde{\sigma}(t,\omega,\theta)) - r(t,\omega,\theta,r_0)| \le \varepsilon \quad \textit{for all } t\ge 0\,. \tag{4.9}$$

Proof. Recall that the flow $(\Omega\times\mathbb{P}\times\mathbb{R}^+,\widetilde{\tau},\mathbb{R})$ is induced by the family of equations $r' = r\,\big(g(\omega{\cdot}t,\widetilde{\theta}(t,\omega,\theta)) - k_\rho(r)\big)$, which agree with (4.6). Relation (4.8) ensures that $0 > r_\rho\,\big(g(\widehat{\theta}(t,\omega,\theta)) - k_\rho(r_\rho)\big)$, so that the function $r(t)\equiv r_\rho$ is an upper solution for all these equations. In addition, the set $\{r(t,\omega,\theta,r_\rho)\mid t\ge 0,\,(\omega,\theta)\in\Omega\times\mathbb{P}\}$ is bounded, since $(\Omega\times\mathbb{P}\times\mathbb{R}^+,\widetilde{\tau},\mathbb{R})$ is bounded dissipative. In these conditions, Theorem 3.6 of [34] (see also its proof, which does not require the minimality of the base flow, in our case $(\Omega\times\mathbb{P},\widetilde{\sigma},\mathbb{R})$) shows that

$$\widetilde{\beta}(\omega,\theta) := \lim_{t\to\infty} r(t,\widetilde{\sigma}(-t,\omega,\theta),r_\rho)$$

defines an upper semicontinuous function satisfying $\widetilde{\beta}(\omega,\theta)\le r_\rho$ and property (ii). In addition, if $r(t,\omega_0,\theta_0,r_0)$ is globally bounded, then Theorem 4.2(i) ensures that $r(-t,\omega_0,\theta_0,r_0)\le r_\rho$ for all $t\ge 0$, so that $r_0 = r(t,\widetilde{\sigma}(-t,\omega_0,\theta_0),r(-t,\omega_0,\theta_0,r_0)) \le r(t,\widetilde{\sigma}(-t,\omega_0,\theta_0),r_\rho)$ and hence $r_0\le\widetilde{\beta}(\omega_0,\theta_0)$. And conversely, since (ii) holds, any solution $r(t,\omega_0,\theta_0,r_0)$ with $r_0\le\widetilde{\beta}(\omega_0,\theta_0)$ is globally bounded. This and the descriptions of $\mathcal{A}$, $\widehat{\mathcal{B}}$ and $\widetilde{\mathcal{B}}$ made in Theorem 4.2 complete the proof of (i) and (ii).

Let us prove (iii). Since $k_\rho(\eta\,r)$ is smaller than $\eta\,k_\rho(r)$ for $r\ge 0$ and $\eta\in[0,1]$ and greater for $r\ge 0$ and $\eta\ge 1$, a standard argument of comparison of solutions and property (ii) ensure that

$$\eta\,\widetilde{\beta}(\widetilde{\sigma}(t,\omega,\theta)) \le r(t,\omega,\eta\,\widetilde{\beta}(\omega,\theta)) \quad \text{for all } t\ge 0 \text{ and } (\omega,\theta)\in\Omega\times\mathbb{P} \text{ if } \eta\in[0,1]\,,$$

$$\eta\,\widetilde{\beta}(\widetilde{\sigma}(t,\omega,\theta)) \ge r(t,\omega,\eta\,\widetilde{\beta}(\omega,\theta)) \quad \text{for all } t\ge 0 \text{ and } (\omega,\theta)\in\Omega\times\mathbb{P} \text{ if } \eta\ge 1\,.$$

Given $\varepsilon>0$ we take $\delta(\varepsilon)=\varepsilon\,\beta_0/r_\rho$, where β_0 satisfies the assumption in (iii). Then, if $|\widetilde{\beta}(\omega,\theta)-r_0|\le\delta(\varepsilon)$ for a point $(\omega,\theta)\in\Omega\times\mathbb{P}$, we have $(1-\varepsilon/r_\rho)\,\widetilde{\beta}(\omega,\theta)\le r_0\le(1+\varepsilon/r_\rho)\,\widetilde{\beta}(\omega,\theta)$. Combining this fact, the previous properties, and (ii), we get $(1-\varepsilon/r_\rho)\,\widetilde{\beta}(\widetilde{\sigma}(t,\omega,\theta))\le r(t,\omega,\theta,r_0)\le(1+\varepsilon/r_\rho)\,\widetilde{\beta}(\widetilde{\sigma}(t,\omega,\theta))$ for all $t\ge 0$, which together with $0\le\widetilde{\beta}(\widetilde{\sigma}(t,\omega,\theta))\le r_\rho$ yields (4.9). □

It is clear that the properties of the semicontinuous map $\widetilde{\beta}$ of Theorem 4.3 determine the shapes of the three global attractors. Let us see that $\widetilde{\beta}(\omega,\theta)>0$ if and only if the solutions $r_l(t,\omega,\theta,r_0)$ of $(2.5)_{(\omega,\theta)}$ with initial data $r_0>0$ are bounded. By linearity, it is enough to consider the solutions $r_l(t,\omega,\theta,1)$.

Proposition 4.4. *Let us fix $(\omega,\theta)\in\Omega\times\mathbb{P}$, and let $\widetilde\beta$ be defined in Theorem 4.3. Then $\widetilde\beta(\omega,\theta)>0$ if and only if $\sup_{t\le 0} r_l(t,\omega,\theta,1)<\infty$.*

Proof. Recall that $\rho\in(0,1]$ is fixed from the beginning, and that $\widetilde\beta$ takes values in $[0,r_\rho]$ with r_ρ satisfying (4.8). We fix $(\omega,\theta)\in\Omega\times\mathbb{P}$ and assume that $\beta_0:=\widetilde\beta(\omega,\theta)>0$. Since $k_\rho\ge 0$, a standard arguments of comparison of solutions for scalar ODEs applied to (2.5) and (4.6), and Theorem 4.3(ii), guarantee

$$\beta_0\, r_l(t,\omega,\theta,1)=r_l(t,\omega,\theta,\beta_0)\le r(t,\omega,\theta,\beta_0)=\widetilde\beta(\widetilde\sigma(t,\omega,\theta))\le r_\rho \quad \text{for all } t\le 0\,.$$

Therefore $\sup_{t\le0} r_l(t,\omega,\theta,1)<\infty$, as asserted.

Assume now that $\kappa:=\sup_{t\le0} r_l(t,\omega,\theta,1)<\infty$, so that $\sup_{t\le0} r_l(t,\omega,\theta,\rho/\kappa)\le\rho$. Hence $r(t,\omega,\theta,\rho/\kappa)=r_l(t,\omega,\theta,\rho/\kappa)\le\kappa$ for all $t\le0$, which together with the dissipativity of the flows ensures that $r(t,\omega,\theta,\rho/\kappa)$ is globally defined and bounded. Theorem 4.2(ii)&(iii) ensure that $(\omega,\theta,\rho/\kappa)\in\widetilde{\mathcal{B}}$, and hence Theorem 4.3(ii) guarantees that $\widetilde\beta(\omega,\theta)\ge\rho/\kappa$. This completes the proof. □

The previous result shows that the sets

$$(\Omega\times\mathbb{P})^+:=\{(\omega,\theta)\in\Omega\times\mathbb{P}\mid \widetilde\beta(\omega,\theta)>0\}\,,$$
$$(\Omega\times\mathbb{P})^0:=\{(\omega,\theta)\in\Omega\times\mathbb{P}\mid \widetilde\beta(\omega,\theta)=0\}$$

can be characterized in terms of the existence of bounded solutions on $\mathbb{R}^-$ for the family of equations (2.5); or, equivalently, of the family of systems (4.3). Note also that both sets are $\widetilde\sigma$-invariant, as Theorem 4.3(ii) guarantees.

Remarks 4.5. 1. Since two different points in $\mathbb{P}$ determine two linearly independent solutions of $(4.3)_\omega$, Proposition 4.4 shows that there are three possibilities for the sections $(\Omega\times\mathbb{P})^+_\omega$ and $(\Omega\times\mathbb{P})^0_\omega$: $(\Omega\times\mathbb{P})^+_\omega=\mathbb{P}$ and $(\Omega\times\mathbb{P})^0_\omega$ is empty; $(\Omega\times\mathbb{P})^+_\omega$ is empty and $(\Omega\times\mathbb{P})^0_\omega=\mathbb{P}$; and $(\Omega\times\mathbb{P})^+_\omega=\{\theta_0\}$ and $(\Omega\times\mathbb{P})^0_\omega=\mathbb{P}-\{\theta_0\}$. In addition, if a point ω is in one of these cases, the same happens with all the points $\omega{\cdot}t$ of its σ-orbit: for all $\theta_0\in\mathbb{P}$ and all $t\in\mathbb{R}$ there exists a unique $\theta_{-t}\in\mathbb{P}$ with $(\omega{\cdot}t,\theta_0)=\widetilde\sigma(t,\omega,\theta_{-t})$, namely $\theta_{-t}:=\widetilde\theta(-t,\omega{\cdot}t,\theta_0)$.

2. The upper semicontinuity of $\widetilde\beta$ ensures that it is continuous at every point of a residual set $\mathcal{C}\subseteq\Omega\times\mathbb{P}$. In addition, since $\widetilde\beta\ge0$, $(\Omega\times\mathbb{P})^0\subseteq\mathcal{C}$. Assume now that $(\Omega\times\mathbb{P})^0$ contains a set $\mathcal{D}$ which is dense in $\Omega\times\mathbb{P}$. It is easy to deduce that $\widetilde\beta$ vanishes at any point of $\mathcal{C}$. So that, in this case, the set $(\Omega\times\mathbb{P})^0$ is residual in $\Omega\times\mathbb{P}$.

We also define the set

$$\Omega^+:=\{\omega\in\Omega\mid (\Omega\times\mathbb{P})^+_\omega=\mathbb{P}\}=\{\omega\in\Omega\mid \widetilde\beta(\omega,\theta)>0 \text{ for all } \theta\in\mathbb{P}\}\,, \tag{4.10}$$

which is $\widetilde\sigma$-invariant: see Remark 4.5.1. Proposition 4.4 shows that $\Omega^+=\Omega$ if and only if all the solutions of all the systems (4.3) are bounded on $\mathbb{R}^-$. Let us check that, if this is not the case, then Ω^+ is of the first Baire category.

Proposition 4.6. *Suppose that $\Omega^+\ne\Omega$. Then $\widetilde\beta$ vanishes exactly at the residual $\widetilde\sigma$-invariant set of points of $\Omega\times\mathbb{P}$ at which it is continuous. In particular, the set Ω^+ is of the first Baire category.*

Proof. We will check below that $\widetilde\beta$ vanishes at a dense set of points of $\Omega\times\mathbb{P}$, which according to Remark 4.5.2 ensures that the $\widetilde\sigma$-invariant set $(\Omega\times\mathbb{P})^0$ is the residual set of continuity points of $\widetilde\beta$. Since $(\Omega\times\mathbb{P})^0$ projects onto $\Omega-\Omega^+$, this set is also residual (see Proposition 3.1 of [49]), which proves the second assertion.

Let us take $\omega_0 \notin \Omega^+$ and note that $(\Omega\times\mathbb{P})^0_{\omega_0} \supseteq \mathbb{P} - \{\theta_0\}$ for a point $\theta_0 \in \mathbb{P}$ (see Remark 4.5.1). It follows easily from the minimality of $(\Omega,\sigma,\mathbb{R})$ that the set $\mathcal{D} := \{\widetilde{\sigma}(t,\omega_0,\theta) \mid t \in \mathbb{R}\,, \theta \neq \theta_0\}$ is dense in $\Omega\times\mathbb{P}$. Finally, the $\widetilde{\sigma}$-invariance of $(\Omega\times\mathbb{P})^0$ ensures that $\widetilde{\beta}$ vanishes at the points of $\mathcal{D}$. □

We finish this section by relating the measure of Ω^+ to those of the attractors.

Proposition 4.7. *The attractor $\widetilde{\mathcal{B}}$ of the flow $(\Omega\times\mathbb{P}\times\mathbb{R}^+,\widetilde{\tau},\mathbb{R})$ has positive measure $m_\Omega\times l_\mathbb{P}\times l_\mathbb{R}$ on $\Omega\times\mathbb{P}\times\mathbb{R}^+$ if and only if $m_\Omega(\Omega^+)=1$, where Ω^+ is the σ-invariant set defined by* (4.10).

Proof. It is clear that the measure of $\widetilde{\mathcal{B}}$ is given by $\int_{\Omega\times\mathbb{P}} \widetilde{\beta}(\omega,\theta)\, d(m_\Omega\times l_\mathbb{P})$, which agrees with $\int_{(\Omega\times\mathbb{P})^+} \widetilde{\beta}(\omega,\theta)\, d(m_\Omega\times l_\mathbb{P})$; so that this measure is positive if and only if $(m_\Omega\times l_\mathbb{P})((\Omega\times\mathbb{P})^+) > 0$. In turn, $(m_\Omega\times l_\mathbb{P})((\Omega\times\mathbb{P})^+) = \int_\Omega l_\mathbb{P}((\Omega\times\mathbb{P})^+_\omega)\, dm_\Omega$, so that it is positive if and only if the set of points ω with $l_\mathbb{P}((\Omega\times\mathbb{P})^+_\omega) > 0$ has positive measure m_Ω. It follows from Remark 4.5.1 that this set agrees with Ω^+. And, since Ω^+ is σ-invariant and m_Ω is σ-ergodic, $m_\Omega(\Omega^+) > 0$ is equivalent to $m_\Omega(\Omega^+) = 1$. □

Remark 4.8. Note that $(m_\Omega\times l_\mathbb{P}\times l_\mathbb{R})(\widetilde{\mathcal{B}}) > 0$ is equivalent to $(m_\Omega\times l_\mathbb{S}\times l_\mathbb{R})(\widehat{\mathcal{B}}) > 0$ and to $(m_\Omega\times l_{\mathbb{R}^2})(\mathcal{A}) > 0$: see Theorems 4.2 and 4.3.

5. The shape of the global attractors in terms of the Sacker and Sell spectrum of the associated linear system

We continue in this section with the analysis of the shape and properties of the global attractor $\mathcal{A}$ (and hence of $\widehat{\mathcal{B}}$ and $\widetilde{\mathcal{B}}$) associated to the family of systems (4.2) described at the beginning of Section 4. As explained in the Introduction, we will relate these properties to the characteristics of the Sacker and Sell spectrum Σ_A of the linear family (4.3) (or of any one of its systems: see Definitions 2.5 and 2.7, and Remark 2.6.1). We have also explained there that in this paper we will consider just three cases: $\Sigma_A \subset (-\infty,0)$, $\Sigma_A \subset (0,\infty)$, and $\Sigma_A = \{0\}$, and that this casuistic can be understood as a bifurcation pattern.

Recall that the boundaries of the attractors are determined by the function $\widetilde{\beta}\colon \Omega\times\mathbb{P} \to [0,r_\rho]$ of Theorem 4.3.

A. The case $\boldsymbol{\Sigma_A \subset (-\infty,0)}$. This first case is the simplest one. We will check that the attractors $\mathcal{A}$, $\widehat{\mathcal{B}}$ and $\widetilde{\mathcal{B}}$ are trivial. Note that the conclusion is irrespective of the fact that Σ_A reduces to a point, it is a nondegenerate interval, or it is composed by two negative points, which are the three possibilities: see Theorem 2.8.

Theorem 5.1. *Suppose that $\Sigma_A \subset (-\infty,0)$. Then $\mathcal{A} = \{(\omega,\mathbf{0}) \mid \omega \in \Omega\}$, $\widehat{\mathcal{B}} = \{(\omega,\theta,0) \mid (\omega,\theta) \in \Omega\times\mathbb{S}\}$, and $\widetilde{\mathcal{B}} = \{(\omega,\theta,0) \mid (\omega,\theta) \in \Omega\times\mathbb{P}\}$.*

Proof. It is enough to prove the assertion for $\widetilde{\mathcal{B}}$: see Theorem 4.2. Note that the family (4.3) has exponential dichotomy with $F^+ = \Omega\times\mathbb{R}^2$: see Remark 2.10. This and Remark 2.6.2 ensure that $\limsup_{t\to-\infty} r_l(t,\omega,\theta,r_0) = \infty$ for all $(\omega,\theta) \in \Omega\times\mathbb{P}$ and $r_0 > 0$. Hence, Proposition 4.4 proves the assertion. □

B. The case $\boldsymbol{\Sigma_A \subset (0,\infty)}$. Now we will show that, in the three cases of $\Sigma_A \subset (0,\infty)$ (see Theorem 2.8), the map $\widetilde{\beta}$ is continuous and strictly positive on $\Omega\times\mathbb{P}$, which means that $\mathcal{A}$ can be identified with a “solid cylinder” which has Ω as axis.

Theorem 5.2. *Suppose that* $\Sigma_A \subset (0,\infty)$. *Then,*

(i) *the map* $\widetilde{\beta}\colon \Omega\times\mathbb{P}\to\mathbb{R}^+$ *of Theorem 4.3 is continuous and strictly positive.*

(ii) *The compact set* $\{(\omega,\theta,\widetilde{\beta}(\omega,\theta))\mid(\omega,\theta)\in\Omega\times\mathbb{P}\}$ *is the global attractor for the flow* $\widetilde{\tau}$ *restricted to* $\Omega\times\mathbb{P}\times(0,\infty)$. *More precisely,*

$$\lim_{t\to\infty}(\widetilde{\beta}(\widetilde{\sigma}(t,\omega,\theta))-r(t,\omega,\theta,r_0))=0 \quad \textit{for all } (\omega,\theta)\in\Omega\times\mathbb{P} \textit{ and } r_0>0\,.$$

(iii) *The compact set* $\left\{\left(\omega,\begin{bmatrix}\widetilde{\beta}(\omega,\theta)\sin\theta\\ \widetilde{\beta}(\omega,\theta)\cos\theta\end{bmatrix}\right)\mid(\omega,\theta)\in\Omega\times\mathbb{P}\right\}$ *is the global attractor for the flow* $\tau_{\mathbb{R}}$ *restricted to* $\Omega\times(\mathbb{R}^2-\{\mathbf{0}\})$.

Proof. (i) The property $\Sigma_A\subset(0,\infty)$ ensures that the family of linear systems (4.3) has exponential dichotomy, with $F^-=\Omega\times\mathbb{R}^2$: see Remark 2.10. More precisely, there exist $c\ge 1$ and $\gamma>0$ such that

$$r_l(t,\omega,\theta,r_0)\le c\,e^{\gamma\,t}r_0 \quad \text{for all } (\omega,\theta)\in\Omega\times\mathbb{P},\ r_0\ge 0, \text{ and } t\le 0\,.$$

We take $\delta_\rho=\rho/c>0$. Then $r_l(t,\omega,\theta,r_0)\le\rho$ for $t\le 0$ if $r_0\le\delta_\rho$, so that

$$r(t,\omega,\theta,r_0)=r_l(t,\omega,\theta,r_0)\le c\,e^{\gamma\,t}r_0 \quad \text{for all } (\omega,\theta)\in\Omega\times\mathbb{P} \text{ and } t\le 0\,.$$

We will deduce from this fact that there exists $t_\rho>0$ such that

$$r(t,\omega,\theta,\delta_\rho)\ge\delta_\rho \quad \text{for all } (\omega,\theta)\in\Omega\times\mathbb{P} \text{ and } t\ge t_\rho\,. \tag{5.1}$$

Assume for contradiction the existence of sequences $(t_n)\uparrow\infty$ in $\mathbb{R}^+$ and $((\omega_n,\theta_n))$ in $\Omega\times\mathbb{P}$ such that $r(t_n,\omega_n,\theta_n,\delta_\rho)=:r_n<\delta_\rho$. Then, on the one hand,

$$r(-t_n,\widetilde{\sigma}(t_n,\omega_n,\theta_n)),\delta_\rho)>r(-t_n,\widetilde{\sigma}(t_n,\omega_n,\theta_n)),r_n)=\delta_\rho$$

for all $n\in\mathbb{N}$; and, on the other hand, $0<r(-t_n,\widetilde{\sigma}(t_n,\omega_n,\theta_n)),\delta_\rho)\le c\,e^{-\gamma\,t_n}\delta_\rho$, which tends to 0 as n increases. This contradiction proves (5.1). And the same argument proves that, given any $r_0>0$, there exists $t_{r_0,\rho}$ such that

$$r(t,\omega,\theta,r_0)\ge\delta_\rho \quad \text{for all } (\omega,\theta)\in\Omega\times\mathbb{P} \text{ and } t\ge t_{r_0,\rho}\,. \tag{5.2}$$

A fundamental consequence derives from (5.1). Let us define the sequence of continuous functions (α_n) by

$$\alpha_n\colon\Omega\times\mathbb{P}\to\mathbb{R}^+,\quad(\omega,\theta)\mapsto r(nt_\rho,\widetilde{\sigma}(-nt_\rho,\omega,\theta),\delta_\rho)\,.$$

Note that all these functions are bounded from below by δ_ρ and from above by ρ. We can reason as in the proof of Theorem 3.6 of [34] in order to prove: that

$$r(t_\rho,\omega,\theta,\alpha_n(\omega,\theta))\ge\alpha_n(\widetilde{\sigma}(t_\rho,\omega,\theta)) \quad \text{for all } (\omega,\theta)\in\Omega\times\mathbb{P} \text{ and } n\ge 1\,;$$

that the sequence $(\alpha_n(\omega,\theta))_n$ increases with n for all $(\omega,\theta)\in\Omega\times\mathbb{P}$; that the limit

$$\alpha(\omega,\theta):=\lim_{n\to\infty}\alpha_n(\omega,\theta)$$

satisfies

$$\alpha(\widetilde{\sigma}(mt_\rho,\omega,\theta))=r_l(mt_\rho,\omega,\theta,\alpha(\omega,\theta)) \quad \text{for all } m\in\mathbb{N} \text{ and } (\omega,\theta)\in\Omega\times\mathbb{P}\,; \tag{5.3}$$

and that α is a lower semicontinuous map: if $(\omega,\theta)=\lim_{n\to\infty}(\omega_n,\theta_n)$, then $\alpha(\omega,\theta)\le\liminf_{n\to\infty}\alpha(\omega_n,\theta_n)$. Note also that $\delta_\rho\le\alpha\le r_\rho$.

On the other hand, $\widetilde{\beta}(\omega,\theta)=\lim_{n\to\infty}r(nt_\rho,\widetilde{\sigma}(-nt_\rho,\omega,\theta),r_\rho)$ (see the proof of Theorem 4.3), so that $\widetilde{\beta}\ge\alpha\ge\delta_\rho>0$. We will check that, as a matter of fact, they agree, which together with the semicontinuity properties of $\widetilde{\beta}$ and α shows the continuity of $\widetilde{\beta}$ and completes the proof.

We will make this in two steps. In the first one we assume that $(\Omega\times\mathbb{P},\widetilde{\sigma},\mathbb{R})$ is minimal. Let (ω_0,θ_0) be a continuity point of $\widetilde{\beta}$. Theorem 3.6 of [34] proves that

$$\begin{aligned}\mathcal{K} :&= \text{closure}_{\Omega\times\mathbb{P}\times\mathbb{R}^+}\{\widetilde{\tau}(t,\omega_0,\theta_0,\widetilde{\beta}(\omega_0,\theta_0))\mid t\ge 0\}\\ &= \text{closure}_{\Omega\times\mathbb{P}\times\mathbb{R}^+}\{(\widetilde{\sigma}(t,\omega_0,\theta_0),\widetilde{\beta}(\widetilde{\sigma}(t,\omega_0,\theta_0)))\mid t\ge 0\}\end{aligned}$$

is a $\widetilde{\tau}$-minimal set. The upper semicontinuity of $\widetilde{\beta}$ ensures that $r_0\ge r\ge\delta_\rho$ for all $(\omega,\theta,r)\in\mathcal{K}$. In the words of [38], the minimal $\mathcal{K}$ is strongly above the equilibrium 0. In addition, (5.2) also ensures that any $\widetilde{\tau}$-minimal different from that given by 0 is contained in $\{(\omega,\theta,r)\mid r\ge\delta_\rho\}\subset\Omega\times\mathbb{P}\times\mathbb{R}^+$. Recall that the flow $\widetilde{\tau}$ is monotone and concave: see Remark 4.1. In these conditions, Theorem 3.8 of [38] ensures that $\mathcal{K}$ is a uniformly exponentially stable copy of the base which attracts all the forward semiorbits starting above zero. That is, we can write $\mathcal{K}=\{(\omega,\theta,c(\omega,\theta))\mid(\omega,\theta)\in\Omega\times\mathbb{P}\}$ for a continuous map $c\colon\Omega\times\mathbb{P}\to\mathbb{R}^+$; and given $\varepsilon>0$ there exists $\eta(\varepsilon)>0$ such that if $r_0>0$ and $|r_0-c(\omega,\theta)|\le\eta(\varepsilon)$, then $|r(t,\omega,\theta,r_0)-c(\widetilde{\sigma}(t,\omega,\theta))|\le\varepsilon$ for all $t\ge 0$. It is very easy to deduce from the definition of $\mathcal{K}$ that c and $\widetilde{\beta}$ agree at the continuity points of $\widetilde{\beta}$. Now we will check that $\widetilde{\beta}$ and c agree everywhere: we fix $(\omega,\theta)\in\Omega\times\mathbb{P}$ and any $\varepsilon>0$; choose a sequence $(s_n)\downarrow-\infty$ such that $(\omega_0,\theta_0)=\lim_{n\to\infty}\widetilde{\sigma}(s_n,\omega,\theta)$; deduce from $\widetilde{\beta}(\omega_0,\theta_0)=c(\omega_0,\theta_0)$ that there exists s_n with $|\widetilde{\beta}(\widetilde{\sigma}(s_n,\omega,\theta))-c(\widetilde{\sigma}(s_n,\omega,\theta))|\le\eta(\varepsilon)$; and conclude that

$$|\widetilde{\beta}(\omega,\theta)-c(\omega,\theta)|=|r(-s_n,\omega{\cdot}s_n,\widetilde{\beta}(\widetilde{\sigma}(s_n,\omega,\theta)))-c(\widetilde{\sigma}(-s_n,\widetilde{\sigma}(s_n,\omega,\theta)))|\le\varepsilon\,,$$

which means that $\widetilde{\beta}(\omega,\theta)=c(\omega,\theta)$. Note that this proves that $\widetilde{\beta}$ is continuous in the minimal case; but our goal is to prove that it agrees with α, which will be required in the general case.

The continuity and positiveness of $\widetilde{\beta}$ allows us to find $\eta\in(0,1)$ such that $0<\eta\widetilde{\beta}\le\delta_\rho$, so that $\eta\widetilde{\beta}\le\alpha$. In addition, as said in the proof of Theorem 4.3, $\eta\,\widetilde{\beta}(\widetilde{\sigma}(t,\omega,\theta))\le r(t,\omega,\eta\,\widetilde{\beta}(\omega,\theta))$ for all $t\ge 0$ and $(\omega,\theta)\in\Omega\times\mathbb{P}$. In these conditions, Theorem 3.6 of [34] and its proof ensure that the map $\gamma(\omega,\theta):=\lim_{t\to\infty}r(t,\widetilde{\sigma}(-t,\omega,\theta),\eta\beta(\widetilde{\sigma}(-t,\omega,\theta)))$ defines a new equilibrium above 0. We can repeat all the procedure before performed with $\widetilde{\beta}$ in order to conclude that γ is continuous. But Theorem 3.8 of [38] ensures that $\widetilde{\beta}$ is the only strictly positive continuous equilibrium. That is, $\gamma=\widetilde{\beta}$. Consequently,

$$\begin{aligned}\beta(\omega,\theta)&=\lim_{n\to\infty}r(nt_\rho,\widetilde{\sigma}(-nt_\rho,\omega,\theta),\eta\beta(\widetilde{\sigma}(-nt_\rho,\omega,\theta)))\\ &\le\lim_{n\to\infty}r(nt_\rho,\widetilde{\sigma}(-nt_\rho,\omega,\theta),\alpha(\widetilde{\sigma}(-nt_\rho,\omega,\theta)))=\alpha(\omega,\theta)\,,\end{aligned}$$

which completes the proof in this case.

Let us now consider the general case. We already know that α and $\widetilde{\beta}$ agree over each $\widetilde{\sigma}$-minimal subset $\mathcal{M}\subset\Omega\times\mathbb{P}$. We fix $(\omega,\theta)\in\Omega\times\mathbb{P}$, take a minimal subset $\mathcal{M}$ contained in its alpha limit set for $\widetilde{\sigma}$, choose a point $(\bar{\omega},\bar{\theta})$ in $\mathcal{M}$, take a sequence $(s_n)\downarrow-\infty$ such that $(\bar{\omega},\bar{\theta})=\lim_{n\to\infty}\widetilde{\sigma}(s_n,\omega,\theta)$, and assume without restriction that $s_n=m_nt_\rho-\mu_n$ with $-m_n\in\mathbb{N}$ and $\mu_n\in[0,t_\rho)$ and that there exists $\widetilde{\mu}:=\lim_{n\to\infty}\mu_n\in[0,t_\rho]$. The continuity of $\widetilde{\sigma}$ allows us to ensure that $\lim_{n\to\infty}\widetilde{\sigma}(m_nt_\rho,\omega,\theta)=\widetilde{\sigma}(\widetilde{\mu},\bar{\omega},\bar{\theta})=:(\widetilde{\omega},\widetilde{\theta})\in\mathcal{M}$. In particular, $\alpha(\widetilde{\omega},\widetilde{\theta})=\widetilde{\beta}(\widetilde{\omega},\widetilde{\theta})$, Now we fix $\varepsilon>0$; take $\delta(\varepsilon)$ satisfying (4.9); assume without restriction that the sequences $(\alpha(\widetilde\sigma(m_nt_\rho,\omega,\theta)))$ and $(\widetilde\beta(\widetilde\sigma(m_nt_\rho,\omega,\theta)))$ converge; use the inequalities

$$0=\widetilde\beta(\widetilde\omega,\widetilde\theta)-\alpha(\widetilde\omega,\widetilde\theta)\ge\lim_{n\to\infty}(\widetilde\beta(\widetilde\sigma(m_nt_\rho,\omega,\theta))-\alpha(\widetilde\sigma(m_nt_\rho,\omega,\theta))\ge0$$

in order to find n_* such that

$$0\le\widetilde\beta(\widetilde\sigma(m_{n_*}t_\rho,\omega,\theta))-\alpha(\widetilde\sigma(m_{n_*}t_\rho,\omega,\theta))\le\delta(\varepsilon)\,;$$

and combine Theorem 4.3(ii), (5.3) and (4.9) to deduce that

$$\begin{aligned}0\le\widetilde\beta(\omega,\theta)-\alpha(\omega,\theta)&=r(-m_{n_*}t_\rho,\widetilde\sigma(m_{n_*}t_\rho,\omega,\theta),\widetilde\beta(\widetilde\sigma(m_{n_*}t_\rho,\omega,\theta)))\\&\quad-r(-m_{n_*}t_\rho,\widetilde\sigma(m_{n_*}t_\rho,\omega,\theta),\alpha(\widetilde\sigma(m_{n_*}t_\rho,\omega,\theta)))\le\varepsilon\,.\end{aligned}$$

This completes the proof of the equality also in the general case.

(ii) Let us take $(\omega,\theta,r_0)\in\Omega\times\mathbb{P}\times(0,\infty)$. Recall that there exists t_0 such that $r(t,\omega,\theta,r_0)\ge\delta_\rho>0$ for all $(\omega,\theta)\in\Omega\times\mathbb{P}$ and all $t\ge t_0$: see (5.2). We look for a $\widetilde\sigma$-minimal set $\mathcal{M}\subseteq\Omega\times\mathbb{P}$ contained in the omega limit set of (ω,θ), and a sequence $(t_n)\uparrow\infty$ such that there exists $\lim_{n\to\infty}(\widetilde\sigma(t_n,\omega,\theta),r(t_n,\omega,\theta,r_0)) =: (\widetilde\omega,\widetilde\theta,\widetilde r)\in\mathcal{M}\times\mathbb{R}^+$. In particular, $\widetilde r\ge\delta_\rho>0$. We fix $\varepsilon>0$ and take $\delta(\varepsilon)$ satisfying (4.9). The convergence to $\widetilde\beta$ in the minimal case explained in the previous step allows us to take $\widetilde s>0$ with $|r(\widetilde s,\widetilde\omega,\widetilde\theta,\widetilde r)-\widetilde\beta(\widetilde\sigma(\widetilde s,\widetilde\omega,\widetilde\theta))|\le\delta(\varepsilon)/3$. And we also look for t_m such that $|r(t_m+\widetilde s,\omega,\theta,r_0)-r(\widetilde s,\widetilde\omega,\widetilde\theta,\widetilde r)|\le\delta(\varepsilon)/3$ and $|\widetilde\beta(\widetilde\sigma(\widetilde s+t_m,\omega,\theta))-\widetilde\beta(\widetilde\sigma(\widetilde s,\widetilde\omega,\widetilde\theta))|\le\delta(\varepsilon)/3$. These inequalities and (4.9) ensure that $|r(t,\omega,\theta,r_0)-\widetilde\beta(\widetilde\sigma(t,\omega,\theta))|\le\varepsilon$ for all $t\ge\widetilde s+t_m$, which proves (ii).

(iii) This last assertion is an immediate consequence of (ii) and Theorem 4.2. □

C. The case $\Sigma_A=\{0\}$. In this case the family (4.3) does not admit exponential dichotomy, so that at least one of its systems admits a nontrivial bounded solution. We begin by considering the simplest situation (which is also the least interesting for the purposes of this paper): when all the solutions are bounded, the attractor $\mathcal{A}$ is homeomorphic to a solid cylinder with continuous boundary. This is proved in Theorem 5.6. Before formulating it, we explain some properties used in its proof and also in Section 6.

Definition 5.3. A continuous function $e\colon\Omega\to\mathbb{R}$ *admits a continuous primitive* if there exists a continuous function $h_e\colon\Omega\to\mathbb{R}$ such that $h_e(\omega{\cdot}t)-h_e(\omega)=\int_0^t e(\omega{\cdot}s)\,ds$ for all $\omega\in\Omega$ and $t\in\mathbb{R}$.

Remark 5.4. If e admits a continuous primitive, then $\sup_{(t,\omega)\in\mathbb{R}\times\Omega}\left|\int_0^t e(\omega{\cdot}s)\,ds\right|<\infty$, and Birkhoff's ergodic theorem ensures that $\int_\Omega e(\omega)\,dm_\Omega=0$. It is well-known that if $(\Omega,\sigma,\mathbb{R})$ is minimal (as in our case) then e admits a continuous primitive if and only if there exists $\omega_0\in\Omega$ with $\sup_{t\ge0}\left|\int_0^t e(\omega_0{\cdot}s)\,ds\right|<\infty$ or with $\sup_{t\le0}\left|\int_0^t e(\omega_0{\cdot}s)\,ds\right|<\infty$: see e.g. Proposition A.1 in [28].

Now we rewrite the matrix $A=\left[\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right]$ of the families (4.3) and (4.2) as

$$A(\omega)=e(\omega)\,I_2+\widetilde A(\omega)\quad\text{for }\;e(\omega):=(1/2)\operatorname{tr}A(\omega)\,,\tag{5.4}$$

so that $\widetilde A(\omega)=\left[\begin{smallmatrix}\widetilde a&b\\c&-\widetilde a\end{smallmatrix}\right]$. We consider the family of linear systems with zero trace

$$\mathbf{y}'=\widetilde A(\omega{\cdot}t)\,\mathbf{y}\tag{5.5}$$

for $\omega\in\Omega$. The notation established in Section 3 will be used.

Remarks 5.5. Recall that we assume $\Sigma_A = \{0\}$. Let e be given by (5.4).

1. The flows $(\Omega\times\mathbb{S},\widehat{\sigma},\mathbb{R})$ and $(\Omega\times\mathbb{P},\widetilde{\sigma},\mathbb{R})$ induced by (5.5) agree with those induced by the initial linear family $\mathbf{y}' = A(\omega{\cdot}t)\,\mathbf{y}$: they are defined by (2.7) and (2.8), since the function f appearing in the angular equations (2.3) and (3.2) is the same (which is due to $a-d=2\widetilde{a}$).

2. Recall that $r_l(t,\omega,\theta,1)$ is the solution of the linear equation $(2.5)_{(\omega,\theta)}$ with $r_l(0,\omega,\theta,1)=1$, and let $\widetilde{r}_l(t,\omega,\theta,1)$ be the solution with $\widetilde{r}_l(0,\omega,\theta,1)=1$ of the r-equation for (5.5), namely

$$r' = r\left(g(\omega{\cdot}t,\widetilde{\theta}(t,\omega,\theta)) - e(\omega{\cdot}t)\right) = r\left(-\frac{1}{2}\,\frac{\partial f}{\partial\theta}(\omega{\cdot}t,\widetilde{\theta}(t,\omega,\theta))\right).$$

It is very easy to check that $r_l(t,\omega,\theta,1)=\widetilde{r}_l(t,\omega,\theta,1)\exp\left(\int_0^t e(\omega{\cdot}s)\,ds\right)$ and that $\widetilde{r}_l(t,\omega,\theta,1)=\exp\left(\int_0^t(-1/2)(\partial f/\partial\theta)(\widetilde{\sigma}(s,\omega,\theta))\,ds\right)$.

3. Since $\Sigma_A=\{0\}$, Theorem 2.8 ensures that $\lim_{t\to\infty}(1/t)\ln\det U(t,\omega)=0$ for all $\omega\in\Omega$, and this, relation (2.9) and Birkhoff's ergodic theorem yield $\int_\Omega e(\omega)\,dm_\Omega=0$. This property, the previous remark and Birkhoff's ergodic theorem yield $\lim_{t\to\infty}(1/t)\ln r_l(t,\omega,\theta,1)=\lim_{t\to\infty}(1/t)\ln\widetilde{r}_l(t,\omega,\theta,1)$; and this and Theorem 2.8 ensure that the Sacker and Sell spectrum of (5.5) is also $\{0\}$.

4. According to (2.9),

$$\det U(t,\omega_0)=\exp\left(\int_0^t \operatorname{tr}A(\omega_0{\cdot}s)\,ds\right)=\exp\left(\int_0^t 2e(\omega_0{\cdot}s)\,ds\right). \tag{5.6}$$

Assume that all the solutions of (4.3) are bounded. Then (5.6) combined with Theorem A.2 of [28] ensures that e has a continuous primitive, which together with Remark 5.5.2 ensures that all the solutions of (5.5) are also bounded.

Theorem 5.6. *Suppose that $\Sigma_A=\{0\}$ and that all the solutions of all the linear systems (4.3) are bounded. Then the map $\widetilde{\beta}\colon\Omega\times\mathbb{P}\to\mathbb{R}^+$ of Theorem 4.3 is continuous and strictly positive.*

Proof. Proposition 4.4 ensures that $\widetilde{\beta}(\omega,\theta)>0$ for all $(\omega,\theta)\in\Omega\times\mathbb{P}$. Let us assume for the moment being that the flow $(\Omega\times\mathbb{P},\widetilde{\sigma},\mathbb{R})$ is minimal. Then, since all the solutions of the family of equations $r'=r\,g(\widetilde{\sigma}(t,\omega,\theta))$ are bounded, there exists a continuous function $h\colon\Omega\times\mathbb{P}\to(0,\infty)$ such that $(d/dt)h(\widetilde{\sigma}(t,\omega,\theta))=h(\omega,\theta)\,g(\widetilde{\sigma}(t,\omega,\theta))$: see e.g. Proposition A.1 in [28]. Let us call $h_\eta:=\eta\,h$ for $\eta\ge 0$. We look for $\eta>\eta_0$ such that $h_\eta(\omega,\theta)\le\rho$ for all $(\omega,\theta)\in\Omega\times\mathbb{P}$, where ρ is the constant in (4.1). Then $(d/dt)\,h_\eta(\widetilde{\sigma}(t,\omega,\theta))=h_\eta(\omega,\theta)\left(g(\widetilde{\sigma}(t,\omega,\theta))-k_\rho(h_\eta(\widetilde{\sigma}(t,\omega,\theta)))\right)$, so that h_η determines a copy of the base for $(\Omega\times\mathbb{P}\times\mathbb{R}^+,\widetilde{\tau},\mathbb{R})$ which is strongly above 0. In addition, this flow is monotone and concave: see Remark 4.1. In these conditions, Theorem 3.8(v) of [38] ensures that $\widetilde{\beta}$ is continuous.

We will now check that, in the minimal case, $\widetilde{\beta}=h_{\eta_0}$, where η_0 is determined by $\sup_{\omega,\theta\in\Omega\times\mathbb{P}}h_{\eta_0}(\omega,\theta)=\rho$. We already know that $h_{\eta_0}\le\widetilde{\beta}$. Let us choose $\eta>0$ and check that $\beta<h_\eta$, from where the assertion follows. For this η, $(d/dt)\,h_\eta(\widetilde{\sigma}(t,\omega,\theta))\ge h_\eta(\omega,\theta)\left(g(\widetilde{\sigma}(t,\omega,\theta))-k_\rho(h_\eta(\widetilde{\sigma}(t,\omega,\theta)))\right)$, and the inequality is strict at least at a point $(\omega,\theta)\in\Omega\times\mathbb{P}$. Now we combine Propositions 4.4 and 4.3 and Theorem 3.6 of [34] in order to deduce that there exists a $\delta>0$ and an equilibrium $\gamma\le h_\eta-\delta$ such that any point (ω,θ,r) with $\gamma(\omega,\theta)<r<h_\eta(\omega,\theta)$ does not belong to any copy of the base. A new application of Theorem 3.8(v) of

[38] ensures that $\beta < h_\eta$, as asserted. It follows easily from $\widetilde{\beta} = h_{\eta_0}$ that

$$\sup_{t\in\mathbb{R}} \widetilde{\beta}(\widetilde{\sigma}(t,\omega,\theta)) = \rho \quad \text{for all } (\omega,\theta) \in \Omega\times\mathbb{P} \quad \text{if } \Omega\times\mathbb{P} \text{ is } \widetilde{\sigma}\text{-minimal}\,. \tag{5.7}$$

Let us now suppose that $\Omega\times\mathbb{P}$ is not $\widetilde{\sigma}$-minimal. Then the assumed boundedness of the solutions of the linear family ensures that $\Omega\times\mathbb{P}$ is the union of an uncountable family of $\widetilde{\sigma}$-minimal sets, each one of them being an m-cover of Ω for a common $m \geq 1$. This is proved in the proof of Theorem 3.1 of [39], since according to Remark 5.5.4 all the solutions of (5.5) are bounded. Note that $\widetilde{\beta}$ is continuous over each minimal set. We define $\alpha(\omega,\theta) := \sup_{t\in\mathbb{R}} r_l(t,\omega,\theta,1)$ and deduce from Theorem 4.3(ii), the decomposition of $\Omega\times\mathbb{P}$ into minimal sets and (5.7) that $\alpha(\omega,\theta) = (1/\widetilde{\beta}(\omega,\theta))\,\sup_{t\in\mathbb{R}} \widetilde{\beta}(\widetilde{\sigma}(t,\omega,\theta)) = \rho/\widetilde{\beta}(\omega,\theta)$. Consequently, α is also continuous over each minimal set, and the global continuity of $\widetilde{\beta}$ will be guaranteed once we have checked that α is continuous on $\Omega\times\mathbb{P}$.

Note now that

$$\begin{aligned}|\alpha(\omega,\theta_1) - \alpha(\omega,\theta_2)| &= \left| \sup_{t\in\mathbb{R}} \left| U(t,\omega) \left[\begin{smallmatrix}\sin\theta_1\\ \cos\theta_1\end{smallmatrix}\right]\right| - \sup_{t\in\mathbb{R}} \left| U(t,\omega) \left[\begin{smallmatrix}\sin\theta_2\\ \cos\theta_2\end{smallmatrix}\right]\right| \right| \\ &\leq \sup_{t\in\mathbb{R}} \left| U(t,\omega)\Big(\left[\begin{smallmatrix}\sin\theta_1\\ \cos\theta_1\end{smallmatrix}\right] - \left[\begin{smallmatrix}\sin\theta_2\\ \cos\theta_2\end{smallmatrix}\right]\Big)\right| \leq c \left| \left[\begin{smallmatrix}\sin\theta_1\\ \cos\theta_1\end{smallmatrix}\right] - \left[\begin{smallmatrix}\sin\theta_2\\ \cos\theta_2\end{smallmatrix}\right]\right| \leq \widetilde{c}\, \mathrm{dist}_{\mathbb{P}}(\theta_1,\theta_2)\end{aligned}$$

for all $\omega \in \Omega$ and $\theta_1,\theta_2 \in \mathbb{P}$, where $c := \sup_{(t,\omega)\in\mathbb{R}\times\Omega} |U(t,\omega)| < \infty$. (As usual, $|U|$ represents the Euclidean matrix operator norm.) Let us take a sequence $((\omega_n,\theta_n))$ in $\Omega\times\mathbb{P}$ with limit (ω_0,θ_0). Let $\mathcal{M}^0$ be the minimal set containing (ω_0,θ_0). Then there exists a sequence (ω_n,φ_n) in $\mathcal{M}^0$ with limit (ω_0,θ_0). This assertion can be proved combining the continuity of the map $\omega \mapsto \mathcal{M}^0_\omega$ in the Hausdorff topology of the set of compact subsets of $\mathbb{P}$ (see e.g. Theorem 3.3 of [36]) with the fact that $\mathcal{M}^0_\omega$ always contains m elements. In particular, $\lim_{n\to\infty} \mathrm{dist}_{\mathbb{P}}(\theta_n,\varphi_n) = 0$. The previous bound ensures that $\lim_{n\to\infty} |\alpha(\omega_n,\varphi_n) - \alpha(\omega_n,\theta_n)| = 0$, and the continuity of α on $\mathcal{M}^0$ yields $\lim_{n\to\infty} |\alpha(\omega_0,\theta_0) - \alpha(\omega_n,\varphi_n)| = 0$. Hence, $\lim_{n\to\infty} |\alpha(\omega_0,\theta_0) - \alpha(\omega_n,\theta_n)| = 0$, which completes the proof. □

Apart from the situation described in Theorem 5.6, the casuistic for $\Sigma_A = \{0\}$ is large. One of the less trivial cases involves the occurrence of Li-Yorke chaos, which occurs under some additional conditions. We will explain this assertion, main goal of the paper, in Section 6.

5.1. **A simple quasiperiodic example.** With the aim of clarifying the ideas, we will apply the previous results to the nonautonomous system

$$\mathbf{y}' = \begin{bmatrix} \varepsilon & \cos t + \sin(\sqrt{2}t) \\ -\cos t - \sin(\sqrt{2}t) & \varepsilon \end{bmatrix} \mathbf{y} - k_{0.5}(|\mathbf{y}|)\,\mathbf{y}\,, \tag{5.8}$$

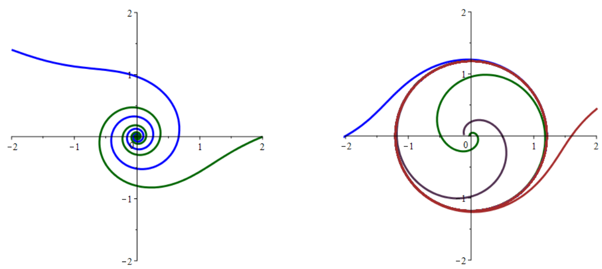
showing the analogies with

$$\mathbf{y}' = \begin{bmatrix} \varepsilon & 1 \\ -1 & \varepsilon \end{bmatrix} \mathbf{y} - k_{0.5}(|\mathbf{y}|)\,\mathbf{y}\,. \tag{5.9}$$

This second system provides a classical pattern of (autonomous) Hopf bifurcation. Some simple figures will contribute to the explanation.

Let us begin with (5.9). Its solutions define a flow on $\mathbb{R}^2$, which can be understood as a disjoint union of orbits: the projections of the graphics of the solutions. For $\varepsilon < 0$ all the orbits tend (always as t increases) to the origin $\mathbf{0} \in \mathbb{R}^2$. Therefore, this point constitutes the global attractor. For $\varepsilon > 0$ there appears another "special" orbit: the circle of radius $0.5+\sqrt{\varepsilon}$ centered at the origin. It corresponds to the periodic solutions of the system, and has the property that it attracts any other orbit excepting the fixed point provided by the origin. Therefore the global attractor is the closed disk, and its border is the attractor for the flow restricted to the invariant set $\mathbb{R}^2-\{\mathbf{0}\}$. Figure 1 shows this behaviour.

Figure 1. Orbits on $\mathbb{R}^2$, autonomous case (5.9). Left: $\varepsilon=-0.15$. Right: $\varepsilon=0.5$.



Let us go now with the nonautonomous example, for which the unperturbed linear system (that corresponding to $\varepsilon=0$) is given by the quasiperiodic matrix-valued function $A_0(t)=\left[\begin{smallmatrix}0 & \cos t+\sin(\sqrt{2}t)\\ -\cos t-\sin(\sqrt{2}t) & 0\end{smallmatrix}\right]$. The hull construction described in the Introduction provides the hull $\mathbb{T}^2$ (the two-dimensional torus $\mathbb{T}^2$), a Kronecker flow on it, and the family of systems

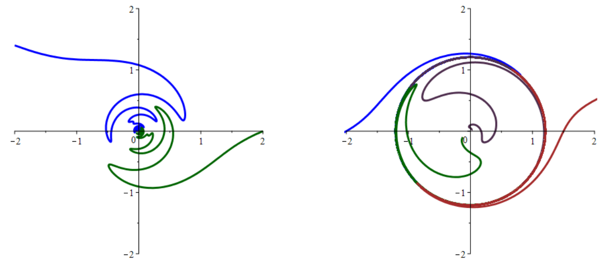
$$\mathbf{y}'=\begin{bmatrix}\varepsilon & f(\theta_1+t,\theta_2+\sqrt{2}t)\\ -f(\theta_1+t,\theta_2+\sqrt{2}t) & \varepsilon\end{bmatrix}\mathbf{y}-k_{0.5}(|\mathbf{y}|)\,\mathbf{y}$$

for $(\theta_1,\theta_2)\in\mathbb{T}^2$, where $f(\theta_1,\theta_2):=\cos(\theta_1)+\sin(\theta_2)$.

Since the base flow is minimal, we can determine the Sacker and Sell spectrum of the linear family by considering just the initial system (which is given by $(\theta_1,\theta_2)=(0,0)$). Assume that $\varepsilon=0$. It is easy to check that all the solutions of the corresponding linear system are bounded, which ensures that the Sacker and Sell spectrum is $\{0\}$. And from here it follows easily that the the Sacker and Sell spectrum of the system corresponding to ε is $\{\varepsilon\}$. Therefore, if $\varepsilon<0$, we are in the situation described by Theorem 5.1, and the global attractor is given by $\mathbb{T}^2\times\{\mathbf{0}\}$. And, if $\varepsilon>0$, the situation fits in that of Theorem 5.2, and hence the global attractor is homeomorphic to a solid cylinder around $\mathbb{T}^2$. Well, in this case it is indeed a cylinder: $\mathbb{T}^2\times\{\mathbf{y}\in\mathbb{R}^2\mid|\mathbf{y}|\le 0.5+\sqrt{\varepsilon}\}$, as one can see by writting the system in polar coordinates. We have depicted in Figure 2 the projections of the graphics of some solutions for $\varepsilon<0$ and $\varepsilon>0$. Now they are not orbits of a flow, but the analogies with the autonomous case are clear. For $\varepsilon<0$ all the projections approach the origin, which is the section of the global attractor corresponding to $(\theta_1,\theta_2)=(0,0)$. And, for $\varepsilon>0$, there appears an "invariant" circle, and the projection of the graphic of any non zero solution approaches to part of this circle.

There is an important difference among the autonomous case and the quasiperiodic one that we have chosen: the rotation number is 1 for the first and 0 for the second. But this fact causes no change in the analysis of the structure of the global

FIGURE 2. Graphic projections, quasiperiodic case (5.8). Left: $\varepsilon = -0.15$. Right: $\varepsilon = 0.5$.

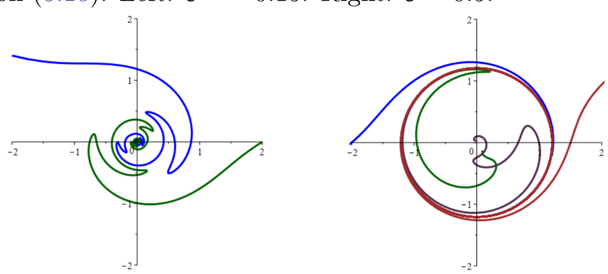


attractor: Figure 3, similar to Figure 2, depicts the projections of the graphics of the solutions of

$$\mathbf{y}' = \begin{bmatrix} \varepsilon & 0.5 + \cos t + \sin(\sqrt{2}t) \\ -0.5 - \cos t - \sin(\sqrt{2}t) & \varepsilon \end{bmatrix} \mathbf{y} - k_{0.5}(|\mathbf{y}|)\,\mathbf{y}\,, \tag{5.10}$$

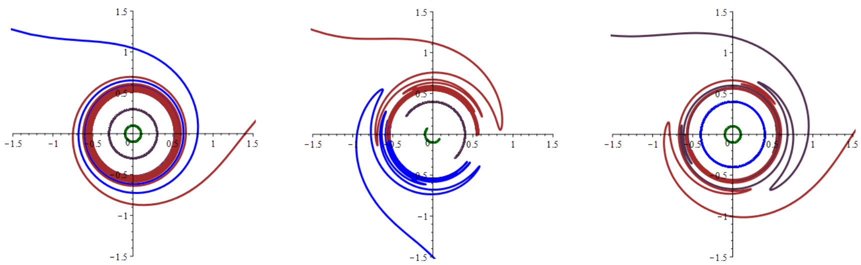
for which the rotation number is 0.5.

FIGURE 3. Graphic projections, quasiperiodic case with positive rotation (5.10). Left: $\varepsilon = -0.15$. Right: $\varepsilon = 0.5$.



Note finally that, for $\varepsilon = 0$, in the autonomous case (5.9), the global attractor is the disk of radius 0.5, so that there is a strong discontinuity with the situation for $\varepsilon < 0$; but also with the situation for $\varepsilon > 0$, since now the border of the disk is no longer the attractor outside the origin: there is a periodic orbit of radius r for any $r \in (0, 0.5)$. Naturally, this is due to the fact that $k_{0.5}$ vanishes identically on $[0, 0.5]$. And, for the nonautonomous cases (5.8) and (5.10), the situation is similar: according to Theorem 5.6 and to the expression of the systems in polar coordinates, we know that in both cases the global attractor is given for $\varepsilon = 0$ by the solid cylinder $\mathbb{T}^2 \times \{\mathbf{y} \in \mathbb{R}^2 \mid |\mathbf{y}| \le 0.5\}$. Therefore, it is again completely different from that corresponding to $\varepsilon < 0$; and also to that appearing for $\varepsilon > 0$ in the sense that now the attractor is composed by infinitely many compact invariant sets: the cylinders $\mathbb{T}^2 \times \{\mathbf{y} \in \mathbb{R}^2 \mid |\mathbf{y}| \le r\}$ for any $r \in [0, 0.5]$. This is the "lack of continuity even in the simplest situation" that we referred to in the Introduction. Figure 4 shows the projections of the graphics of some solutions of (5.9), (5.8) and (5.10) for $\varepsilon = 0$.

Figure 4. Graphic projections at the bifurcation point $\varepsilon = 0$, case of bounded orbits. Left: autonomous case (5.9). Center: quasiperiodic case with null rotation (5.8) Right: quasiperiodic case with positive rotation (5.10).



6. Occurrence of Li-Yorke chaos in the case of Sacker and Sell spectrum $\{0\}$

We continue working with the family (4.2) under the conditions described at the beginning of Section 4, and with the flows $(\Omega\times\mathbb{S}, \widehat{\sigma}, \mathbb{R})$, $(\Omega\times\mathbb{P}, \widetilde{\sigma}, \mathbb{R})$ and $(\Omega\times\mathbb{R}^2, \tau_{\mathbb{R}}, \mathbb{R})$ respectively defined by (2.7), (2.8) and (4.4). We will assume in all this section that $\Sigma_A = \{0\}$. In addition, we will also assume from the beginning that at least one of the systems of the family (4.3) has an unbounded solution: the alternative has already been analyzed in Theorem 5.6. In other words, the set Ω^+ defined by (4.10) is not the whole Ω. And we will add one more condition:

Hypotheses 6.1. The Sacker and Sell spectrum of the linear family (4.3) is $\Sigma_A = \{0\}$, and the set Ω^+ defined by (4.10) satisfies $\Omega^+ \neq \Omega$ and $m_\Omega(\Omega^+) = 1$.

We will make some remarks on these hypotheses at the end of this section. For now, note that $\Omega^+ \neq \Omega$ ensures that Ω^+ is a set of the first Baire category (see Proposition 4.6); and recall that $m_\Omega(\Omega^+) = 1$ is equivalent to say that the attractor $\mathcal{A}$ or the flow $(\Omega\times\mathbb{R}^2, \tau_{\mathbb{R}}, \mathbb{R})$ (see Theorem 4.2) has positive measure in $\Omega\times\mathbb{R}^2$ (see Proposition 4.7 and Remark 4.8). Our main results, Theorem 6.8 and Corollary 6.9, substitute $\Omega^+ \neq \Omega$ by a stronger condition in order to guarantee that the flow $(\Omega\times\mathbb{R}^2, \tau_{\mathbb{R}}, \mathbb{R})$ is Li-Yorke chaotic; or, more precisely, that its restriction to the attractor $\mathcal{A}$ is Li-Yorke chaotic in a quite strong sense.

To prepare the way, we need some new concepts and related properties. The concept of continuous primitive is explained in Definition 5.3.

Definition 6.2. We call $\mathcal{R}(\Omega)$ to the set of continuous functions $e\colon \Omega \to \mathbb{R}$ with $\int_\Omega e(\omega)\, dm_\Omega = 0$ which do not admit a continuous primitive and such that $\sup_{t\le 0} \int_0^t e(\omega{\cdot}s)\, ds < \infty$ for m_Ω-a.e. $\omega \in \Omega$.

The next two results are basically proved in [22], [28] and [11]. We include many details of the proofs for the reader's convenience.

Proposition 6.3. *Let $e\colon \Omega \to \mathbb{R}$ be a continuous function with $\int_\Omega e(\omega)\, dm_\Omega = 0$. Then $\sup_{t\le 0} \int_0^t e(\omega{\cdot}s)\, ds < \infty$ for m_Ω-a.e. $\omega \in \Omega$ if and only if $\sup_{t\ge 0} \int_0^t e(\omega{\cdot}s)\, ds < \infty$ for m_Ω-almost every $\omega \in \Omega$.*

Proof. Let us assume that $\sup_{t\le 0} \int_0^t e(\omega{\cdot}s)\, ds < \infty$ for all ω in a set Ω_0 with $m_\Omega(\Omega_0) = 1$, which is clearly σ-invariant. Property $\int_\Omega e(\omega)\, dm_\Omega = 0$ combined with the classical recurrence result of [48] and with Fubini's theorem provides a σ-invariant set Ω_r with $m_\Omega(\Omega_r)=1$ such that for all $\omega\in\Omega_r$ there exists $(l_n)\downarrow-\infty$ with $\lim_{n\to\infty}\int_{l_n}^0 e(\omega{\cdot}s)\,ds=0$. Let us define $\Omega^e=\Omega_0\cap\Omega_r$, take $\omega_0\in\Omega^e$, and check that $\sup_{t\ge0}\int_0^t e(\omega_0{\cdot}s)\,ds<\infty$. We reason by contradiction assuming the existence of a sequence $(t_n)\uparrow\infty$ with $\lim_{n\to\infty}\int_0^{t_n} e(\omega_0{\cdot}s)\,ds=\infty$. Since $\omega_0{\cdot}t_n\in\Omega_r$ for all $n\in\mathbb{N}$, we can find $l_n<0$ such that $1/n>\big|\int_{l_n-t_n}^0 e(\omega_0{\cdot}(t_n+s))\,ds\big|=\big|\int_{l_n}^{t_n} e(\omega_0{\cdot}s)\,ds\big|$. But then $\lim_{n\to\infty}\int_{l_n}^0 e(\omega_0{\cdot}s)\,ds=-\lim_{n\to\infty}\int_0^{t_n} e(\omega_0{\cdot}s)\,ds=-\infty$, which in turn ensures that $\sup_{t\le0}\int_0^t e(\omega_0{\cdot}s)\,ds=\infty$. This is the sought-for contradiction. The proof of the converse assertion is analogous. □

Proposition 6.4. *Let $e\colon\Omega\to\mathbb{R}$ be a continuous function with $\int_\Omega e(\omega)\,dm_0=0$. The following assertions are equivalent:*

(1) $e\in\mathcal{R}(\Omega)$.
(2) *There exists a σ-invariant set $\Omega^e\subseteq\Omega$ with $m_\Omega(\Omega^e)=1$ such that, if $\omega\in\Omega^e$, then $\sup_{t\in\mathbb{R}}\int_0^t e(\omega{\cdot}s)\,ds<\infty$, $\inf_{t\le0}\int_0^t e(\omega{\cdot}s)\,ds=-\infty$, and $\inf_{t\ge0}\int_0^t e(\omega{\cdot}s)\,ds=-\infty$.*
(3) *There exist a measurable function $h_e\colon\Omega\to(-\infty,0]$ and a σ-invariant set $\Omega^e\subseteq\Omega$ with $m_\Omega(\Omega^e)=1$ such that, if $\omega\in\Omega^e$, then $h_e(\omega{\cdot}t)-h_e(\omega)=\int_0^t e(\omega{\cdot}s)\,ds$ for all $t\in\mathbb{R}$, $\sup_{t\in\mathbb{R}}h_e(\omega{\cdot}t)<\infty$, $\inf_{t\le0}h_e(\omega{\cdot}t)=-\infty$, and $\inf_{t\ge0}h_e(\omega{\cdot}t)=-\infty$.*

In addition, in this case, for every ω in a residual subset of Ω there exist four sequences (t_n^i) with $(t_n^i)\uparrow\infty$ for $i=1,3$ and $(t_n^i)\downarrow-\infty$ for $i=2,4$ such that

$$\lim_{n\to\infty}\int_0^{t_n^i} e(\omega{\cdot}s)\,ds=-\infty \text{ for } i=1,2 \quad\text{and}\quad \lim_{n\to\infty}\int_0^{t_n^i} e(\omega{\cdot}s)\,ds=\infty \text{ for } i=3,4\,.$$

Proof. We begin by assuming that $e\in\mathcal{R}(\Omega)$. Then Proposition 6.3 ensures that $\sup_{t\in\mathbb{R}}\int_0^t e(\omega{\cdot}s)\,ds<\infty$ for all ω in a set of full measure Ω^e, which is clearly σ-invariant. Hence $\inf_{t\le0}\int_0^t e(\omega{\cdot}s)\,ds=\inf_{t\ge0}\int_0^t e(\omega{\cdot}s)\,ds=-\infty$ for all $\omega\in\Omega^e$, since otherwise e would admit a continuous primitive: see Remark 5.4. This completes the proof of (1)⇒(2). The converse implication is almost obvious: see the first assertion in Remark 5.4.

Let us now assume that (2) holds, define $h(\omega):=-\sup_{t\in\mathbb{R}}\int_0^t e(\omega{\cdot}s)\,ds$ for all $\omega\in\Omega$, and note that $-\infty\le h(\omega)\le0$ for all $\omega\in\Omega$ and $-\infty<h(\omega)\le0$ for all $\omega\in\Omega^e$. It is not hard to check that $h(\omega{\cdot}t)-h(\omega)=\int_0^t e(\omega{\cdot}s)\,ds$. We modify h is order to take the value 0 on $\Omega-\Omega^e$ and obtain the function h_e of (3). Conversely, it is obvious that (3) implies (2).

A detailed proof of the last assertion can be found in Theorem A.2 of [28]. □

The relation between the linear families (4.3) and (5.5), determined by (5.4) and explained in Remarks 5.5, will be fundamental in what follows. Let us show the effect of Hypotheses 6.1 on the function e of (5.4).

Proposition 6.5. *Suppose that Hypotheses 6.1 hold. Then, the map $e=(1/2)\operatorname{tr}A$ belongs to the set $\mathcal{R}(\Omega)$ of Definition 6.2.*

Proof. As explained in Remark 5.5.3, Hypotheses 6.1 ensure that $\int_\Omega e(\omega)\,dm_\Omega=0$. Let us fix $\omega_0\in\Omega^+$. Proposition 4.4 ensures that all the solutions of the linear system $(4.3)_{\omega_0}$ are bounded on $\mathbb{R}^-$. This and (5.6) yield $\sup_{t\le0}\int_0^t e(\omega_0{\cdot}s)\,ds<\infty$.

Consequently, either $e\colon \Omega\to\mathbb{R}$ admits a continuous primitive, or it belongs to $\mathcal{R}(\Omega)$. We will check that $\inf_{t\le 0}\det U(t,\omega_0)=0$, which together with (5.6) ensures that $\inf_{t\le 0}\int_0^t e(\omega_0{\cdot}s)\,ds=-\infty$. This precludes the existence of a continuous primitive (see Remark 5.4), so that $e\in\mathcal{R}(\Omega)$.

We assume that $\inf_{t\le 0}\det U(t,\omega_0)>0$. Then $\sup_{s\le 0}|U(s,\omega_0)^{-1}|<\infty$. Hence, $\sup_{t\le 0,s\le 0}|U(t,\omega_0{\cdot}s)|=\sup_{t\le 0,s\le 0}|U(t+s,\omega_0)\,U(s,\omega_0)^{-1}|<\infty$. Therefore, since $\{\omega_0{\cdot}s\mid s\le 0\}$ is dense in Ω, $\sup_{t\le 0,\omega\in\Omega}|U(t,\omega)|<\infty$. This means that all the solutions of all the systems (4.3) are bounded on $\mathbb{R}^-$ and hence that $\Omega^+=\Omega$ (see once more Proposition 4.4). But this contradicts one of the Hypotheses 6.1. □

Out next purpose is to show that Hypotheses 6.1 impose on the family (5.5) one of the conditions of Theorem 3.4, part of whose proof will be fundamental later.

Theorem 6.6. *Suppose that Hypotheses 6.1 holds. Then the flow* $(\Omega\times\mathbb{P},\widetilde\sigma,\mathbb{R})$ *admits an invariant measure which is equivalent to* $m_\Omega\times l_\mathbb{P}$.

Proof. Let f be defined by (3.3). According to Remark 5.5.1 and Proposition 2.2 of [39], it is enough to look for a function $p\colon\Omega\times\mathbb{P}\to\mathbb{R}$ in $L^1(\Omega\times\mathbb{P},m_\Omega\times l_\mathbb{P})$ with $p(\omega,\theta)>0$ for $(m_\Omega\times l_\mathbb{P})$-a.a. (ω,θ) and such that, for all $(\omega,\theta)\in\Omega\times\mathbb{P}$ and $l\in\mathbb{R}$,

$$p(\widetilde\sigma(l,\omega,\theta))=p(\omega,\theta)\exp\Big(-\int_0^l\frac{\partial f}{\partial\theta}\,(\widetilde\sigma(s,\omega,\theta))\,ds\Big)\,: \tag{6.1}$$

if so, p is the density function of a $\widetilde\sigma$-invariant measure. Let us define

$$p_t(\omega,\theta):=\frac1t\int_{-t}^0\Big(\exp\int_0^s\frac{\partial f}{\partial\theta}\,(\widetilde\sigma(r,\omega,\theta))\,dr\Big)\,ds\qquad\text{and}\qquad p(\omega,\theta):=\liminf_{t\to\infty}p_t(\omega,\theta)$$

for $(\omega,\theta)\in\Omega{\times}\mathbb{P}$. Our goal is to proof that p satisfies the three mentioned conditions. We will do it in three steps.

Step 1. We will check that $p\in L^1(\Omega\times\mathbb{P},m_\Omega\times l_\mathbb{P})$. More precisely, that

$$\int_{\mathcal{C}}p_t(\omega,\theta)\,d(m_\Omega\times l_\mathbb{P})=\frac1t\int_{-t}^0(m_\Omega\times l_\mathbb{P})(\widetilde\sigma_s(\mathcal{C}))\,ds \tag{6.2}$$

for any measurable set $\mathcal{C}\subseteq\Omega\times\mathbb{P}$ and $t\ge 0$. This yields $\int_{\Omega\times\mathbb{P}}p_t(\omega,\theta)\,d(m_\Omega\times l_\mathbb{P})=1$, and then, as a consequence of Fatou's lemma, we obtain

$$0\le\int_{\Omega\times\mathbb{P}}p(\omega,\theta)\,d(m_\Omega\times l_\mathbb{P})\le\liminf_{t\to\infty}\int_{\Omega\times\mathbb{P}}p_t(\omega,\theta)\,d(m_\Omega\times l_\mathbb{P})=1\,,$$

so that $p\in L^1(\Omega\times\mathbb{P})$. Let us take a measurable set $\mathcal{C}\subseteq\Omega\times\mathbb{P}$. The definition of p_t, Fubini's theorem, and the equality $\chi_{\mathcal{C}}(\omega,\theta)=\chi_{\widetilde\sigma_l(\mathcal{C})}(\widetilde\sigma(l,\omega,\theta))$ for all $l\in\mathbb{R}$ yield

$$\begin{aligned}&\int_{\Omega\times\mathbb{P}}p_t(\omega,\theta)\,\chi_{\mathcal{C}}(\omega,\theta)\,d(m_\Omega\times l_\mathbb{P})\\&\quad=\frac1t\int_{-t}^0\int_\Omega\int_\mathbb{P}\Big(\exp\int_0^s\frac{\partial f}{\partial\theta}\,(\widetilde\sigma(l,\omega,\theta))\,dl\Big)\,\chi_{\widetilde\sigma_s(\mathcal{C})}(\widetilde\sigma(s,\omega,\theta))\,dl_\mathbb{P}\,dm_\Omega\,ds\,.\end{aligned} \tag{6.3}$$

It is easy to deduce from $(d/dt)\,\widetilde\theta(t,\omega,\theta)=f(\widetilde\sigma(t,\omega,\theta))=f((\omega{\cdot}t,\widetilde\theta(t,\omega,\theta))$ and $\widetilde\theta(0,\omega,\theta)=\theta$ that $(d/d\theta)\,\widetilde\theta(s,\omega,\theta)=\exp\big(\int_0^s(\partial f/\partial\theta)\,(\widetilde\sigma(l,\omega,\theta))\,dl\big)$. In turn, this implies that

$$\int_\mathbb{P}\exp\Big(\int_0^s\frac{\partial f}{\partial\theta}\,(\widetilde\sigma(l,\omega,\theta))\,dl\Big)\,\chi_{\widetilde\sigma_s(\mathcal{C})}(\widetilde\sigma(s,\omega,\theta))\,dl_\mathbb{P}=\int_\mathbb{P}\chi_{\widetilde\sigma_s(\mathcal{C})}(\omega{\cdot}s,\theta)\,dl_\mathbb{P}\,.$$

The σ-invariance of m ensures that

$$\int_\Omega \int_{\mathbb{P}} \chi_{\widetilde{\sigma}_s(\mathcal{C})}(\omega{\cdot}s,\theta)\,dl_{\mathbb{P}}\,dm_\Omega = \int_\Omega \int_{\mathbb{P}} \chi_{\widetilde{\sigma}_s(\mathcal{C})}(\omega,\theta)\,dl_{\mathbb{P}}\,dm_\Omega = (m_\Omega{\times}\,l_{\mathbb{P}})(\widetilde{\sigma}_s(\mathcal{C}))\,,$$

and this equality and (6.3) show (6.2). The first step is complete.

STEP 2. We will now check that $p(\omega,\theta)>0$ for $(m_\Omega{\times}\,l_{\mathbb{P}})$-almost all $(\omega,\theta)\in\Omega\times\mathbb{P}$. We use the notation established in Remark 5.5.2 and the information that it provides in order to write

$$p_t(\omega,\theta)=\frac{1}{t}\int_{-t}^0 \frac{1}{\widetilde{r}_l^{\,2}(s,\omega,\theta,1)}\,ds=\frac{1}{t}\int_{-t}^0\frac{1}{r_l^2(s,\omega,\theta,1)}\exp\Big(2\int_0^s e(\omega{\cdot}r)\,dr\Big)\,ds\,.$$

We write $\exp\big(2\int_0^s e(\omega{\cdot}r)\,dr\big)=H_e(\omega{\cdot}s)/H_e(\omega)$, for $H_e(\omega)=\exp(2h_e(\omega))$, where h_e is the map provided by Propositions 6.5 and 6.4. Note that $H_e(\omega)\in(0,1]$. Birkhoff's ergodic theorem ensures that the set of points ω_0 of Ω^+ for which there exists $\lim_{t\to\infty}(1/t)\int_{-t}^0 H_e(\omega_0{\cdot}s)\,ds=\int_\Omega H_e(\omega)\,dm_\Omega$ has measure 1. We fix ω_0 in this set and $\theta_0\in\mathbb{P}$. Then there exists $c_{(\omega_0,\theta_0)}\geq 1$ such that $r_l^2(t,\omega_0,\theta_0,1)\leq c_{(\omega_0,\theta_0)}$ for all $t\leq 0$: see Proposition 4.4. Altogether, we have

$$p_t(\omega_0,\theta_0)\geq\frac{1}{c_{(\omega_0,\theta_0)}\,H_e(\omega_0)}\,\frac{1}{t}\int_{-t}^0 H_e(\omega_0{\cdot}s)\,ds\,,$$

so that

$$p(\omega_0,\theta_0)\geq\frac{1}{c_{(\omega_0,\theta_0)}\,H_e(\omega_0)}\int_\Omega H_e(\omega)\,dm_\Omega>0\,.$$

STEP 3. We will finally check that p satisfies (6.1). First, we will check that

$$\begin{aligned} p_t(\widetilde{\sigma}(l,\omega,\theta))\cdot\exp\Big(\int_0^l\frac{\partial f}{\partial\theta}(\widetilde{\sigma}(r,\omega,\theta))\,dr\Big)\\ =\frac{t-l}{t}\,p_{t-l}(\omega,\theta)+\frac{1}{t}\int_0^l\exp\Big(\int_0^s\frac{\partial f}{\partial\theta}(\widetilde{\sigma}(r,\omega,\theta))\,dr\Big)\,ds \end{aligned}\tag{6.4}$$

for $0<l<t$. We call I to the left-hand term in the previous expression. Then,

$$\begin{aligned} I:&=\frac{1}{t}\int_{-t}^0\exp\Big(\int_0^s\frac{\partial f}{\partial\theta}(\widetilde{\sigma}(l+r,\omega,\theta))\,dr\Big)\,ds\cdot\exp\Big(\int_0^l\frac{\partial f}{\partial\theta}(\widetilde{\sigma}(r,\omega,\theta))\,dr\Big)\\ &=\frac{1}{t}\int_{-t}^0\exp\Big(\int_0^{s+l}\frac{\partial f}{\partial\theta}(\widetilde{\sigma}(r,\omega,\theta))\,dr\Big)\,ds\\ &=\frac{t-l}{t-l}\,\frac{1}{t}\int_{-t+l}^l\exp\Big(\int_0^s\frac{\partial f}{\partial\theta}(\widetilde{\sigma}(r,\omega,\theta))\,dr\Big)\,ds\\ &=\frac{t-l}{t}\,p_{t-l}(\omega,\theta)+\frac{1}{t}\int_0^l\exp\Big(\int_0^s\frac{\partial f}{\partial\theta}(\widetilde{\sigma}(r,\omega,\theta))\,dr\Big)\,ds\,. \end{aligned}$$

We take $\liminf$ in (6.4) in order to get (6.1) for $l>0$. From here we deduce that

$$\begin{aligned} p(\widetilde{\sigma}(-l,\omega,\theta))&=p(\omega,\theta)\,\exp\Big(\int_0^l\frac{\partial f}{\partial\theta}\,(\widetilde{\sigma}(s-l,\omega,\theta))\,ds\Big)\\ &=p(\omega,\theta)\,\exp\Big(-\int_0^{-l}\frac{\partial f}{\partial\theta}\,(\widetilde{\sigma}(s,\omega,\theta))\,ds\Big) \end{aligned}$$

if $l>0$, which completes the proof of (6.1), and hence that of the theorem. □

Some more preliminary work is required before formulating and proving the main result. Recall that the attractor $\widetilde{\mathcal{B}}$ for $(\Omega\times\mathbb{P}\times\mathbb{R},\widetilde{\tau},\mathbb{R})$ can be written in terms of the map $\widetilde{\beta}$ as $\widetilde{\mathcal{B}}=\bigcup_{(\omega,\theta)\in\Omega\times\mathbb{P}}\big(\{(\omega,\theta)\}\times[0,\widetilde{\beta}(\omega,\theta)]\big)$: see Theorem 4.3(i). And recall that $\rho\in(0,1]$ is the constant appearing in (4.1). We define

$$
\begin{aligned}
(\Omega\times\mathbb{P})_l^+ &:= \{(\omega,\theta)\in\Omega\times\mathbb{P}\,|\,0<\sup_{t\in\mathbb{R}}\widetilde{\beta}(\widetilde{\sigma}(t,\omega,\theta))\le\rho\}\subseteq(\Omega\times\mathbb{P})^+,\\
\Omega^* &:= \{\omega\in\Omega\,|\,l_{\mathbb{P}}(((\Omega\times\mathbb{P})_l^+)_\omega)=1\}\,,\\
\widetilde{\mathcal{B}}_l &:= \text{closure}_{\Omega\times\mathbb{P}\times[0,\rho]}\{(\omega,\theta,r)\in\widetilde{\mathcal{B}}\,|\,(\omega,\theta)\in(\Omega\times\mathbb{P})_l^+\}\,,\\
\mathcal{A}_l &:= \big\{\big(\omega,\left[\begin{smallmatrix} r\sin\theta\\ r\cos\theta\end{smallmatrix}\right]\big)\,|\,(\omega,\mathfrak{p}(\theta),r)\in\widetilde{\mathcal{B}}_l\big\}\,.
\end{aligned}
\tag{6.5}
$$

Note that $\widetilde{\mathcal{B}}_l$ and $\mathcal{A}_l$ are compact subsets of the global attractors $\widetilde{\mathcal{B}}$ (for the flow $(\Omega\times\mathbb{P}\times\mathbb{R}^+,\widetilde{\tau},\mathbb{R})$) and $\mathcal{A}$ (for $(\Omega\times\mathbb{R}^2,\tau_{\mathbb{R}},\mathbb{R})$) described in Theorems 4.2 and 4.3.

Proposition 6.7. *Suppose that Hypotheses 6.1 hold. Then,*

(i) *the set $(\Omega\times\mathbb{P})_l^+$ is $\widetilde{\sigma}$-invariant and $(m_\Omega\times l_{\mathbb{P}})((\Omega\times\mathbb{P})_l^+)=1$.*
(ii) *Let Ω^* and Ω^+ be defined by (6.5) and (4.10). Then Ω^* is σ-invariant, $m_\Omega(\Omega^*)=1$, and $\Omega^*\subseteq\Omega^+$.*
(iii) *For every $\omega\in\Omega^*$ there exists $r_\omega>0$ such that $(\omega,\theta,r)\in\widetilde{\mathcal{B}}_l$ for all $\omega\in\Omega^*$, $\theta\in\mathbb{P}$ and $r\in[0,r_\omega]$. In particular, the section $(\mathcal{A}_l)_\omega\subset\mathbb{R}^2$ contains the closed disk centered at the origin and with radius $r_\omega>0$ for all $\omega\in\Omega^*$.*
(iv) *$\widetilde{\mathcal{B}}_l$ is $\widetilde{\tau}$-invariant, $\mathcal{A}_l$ is $\tau_{\mathbb{R}}$-invariant, and the restrictions of the flows to these sets are given by the family of linear systems (4.3).*

Proof. The set $(\Omega\times\mathbb{P})_l^+$ is composed by $\widetilde{\sigma}$-orbits, so that it is $\widetilde{\sigma}$-invariant. Let $\widetilde{\mu}$ be the $\widetilde{\sigma}$-invariant measure on $\Omega\times\mathbb{P}$ provided by Theorem 6.6, which we assume to be normalized. In the case that $\widetilde{\mu}$ is ergodic, we can repeat the arguments of Theorem 35 of [11] in order to check that $\widetilde{\mu}((\Omega\times\mathbb{P})_l^+)=1$ (see also Theorem 5.8 of [12]). The theorem of decomposition into ergodic components (see Theorem 6.4 or Corollary 6.5 of [32]) ensures that this property also holds in the general case. The last assertion (i) is an immediate consequence of the first one and the equivalence of the measures $\widetilde{\mu}$ and $m_\Omega\times l_{\mathbb{P}}$.

The σ-invariance of Ω^* follows from the fact that $\widetilde{\sigma}_(t,\omega)$ gives a C^1 transformation of $\mathbb{P}$ for each fixed (t,ω), and the property $m_\Omega(\Omega^*)=1$ follows from (i) and Fubini's theorem. Let us define $\mathcal{C}:=\{(\omega,\theta,r)\in\widetilde{\mathcal{B}}\,|\,(\omega,\theta)\in(\Omega\times\mathbb{P})_l^+\}$. We fix $\omega\in\Omega^*$ and choose θ_1 such that (ω,θ_1) and (ω,θ_2) are in $(\Omega\times\mathbb{P})_l^+$ for $\theta_2=\theta_1+\pi/2$. It follows from the definition of $(\Omega\times\mathbb{P})_l^+$ and Theorem 4.3(ii) that $\sup_{t\in\mathbb{R}}r(t,\omega,\theta_i,\widetilde{\beta}(\omega,\theta_i))\le\rho$ for $i=1,2$, so that $r(t,\omega,\theta_i,\widetilde{\beta}(\omega,\theta_i))=r_l(t,\omega,\theta_i,\widetilde{\beta}(\omega,\theta_i))$. Let us set $r_\omega:=(1/\sqrt{2})\inf(\widetilde{\beta}(\omega,\theta_1),\widetilde{\beta}(\omega,\theta_2))>0$ and take any $\theta\in\mathbb{S}$. Let us write $\left[\begin{smallmatrix}\sin\theta\\ \cos\theta\end{smallmatrix}\right]=c_1\left[\begin{smallmatrix}\sin\theta_1\\ \cos\theta_1\end{smallmatrix}\right]+c_2\left[\begin{smallmatrix}\sin\theta_2\\ \cos\theta_2\end{smallmatrix}\right]$, so that $c_1^2+c_2^2=1$. Since $r_\omega\le\rho/\sqrt{2}<\rho$, for t close to 0,

$$
\begin{aligned}
r(t,\omega,\theta,r_\omega) &= r_l(t,\omega,\theta,r_\omega)=r_\omega\big|\mathbf{y}_l\big(t,\omega,\left[\begin{smallmatrix}\sin\theta\\ \cos\theta\end{smallmatrix}\right]\big)\big|\\
&= r_\omega\left|\mathbf{y}_l\Big(t,\omega,c_1\left[\begin{smallmatrix}\sin\theta_1\\ \cos\theta_1\end{smallmatrix}\right]+c_2\left[\begin{smallmatrix}\sin\theta_2\\ \cos\theta_2\end{smallmatrix}\right]\Big)\right|\\
&\le r_\omega\left(\frac{|c_1|}{\widetilde{\beta}(\omega,\theta_1)}\,r(t,\omega,\theta_1,\widetilde{\beta}(\omega,\theta_1))+\frac{|c_2|}{\widetilde{\beta}(\omega,\theta_2)}\,r(t,\omega,\theta_2,\widetilde{\beta}(\omega,\theta_2))\right)\\
&\le \frac{|c_1|+|c_2|}{\sqrt{2}}\,\rho\le\rho\,.
\end{aligned}
$$

An easy contradiction argument shows that $\sup_{t\in\mathbb{R}} r(t,\omega,\theta,r_\omega) \le \rho$. This ensures that $\omega \in \Omega^+$, so that (ii) holds. It also ensures that $(\omega,\theta,r_\omega) \in \widetilde{\mathcal{B}}$ for all $\theta \in \mathbb{P}$. In particular, $(\omega,\theta,r) \in \mathcal{C}$ for all $\theta \in \mathcal{P}$ such that $(\omega,\theta) \in (\Omega\times\mathbb{P})_l^+$ and all $r \in [0,r_\omega]$. Since $\omega \in \Omega^*$, $l_{\mathbb{P}}(((\Omega\times\mathbb{P})_l^+)_\omega) = 1$, so that $((\Omega\times\mathbb{P})_l^+)_\omega$ is dense in $\mathcal{P}$. Therefore $(\omega,\theta,r) \in \widetilde{\mathcal{B}}_l$ for all $\theta \in \mathcal{P}$ and all $r \in [0,r_\omega]$, which is the first assertion in (iii). The second one follows immediately from the first one and Theorem 4.3.

The $\widetilde{\sigma}$-invariance of $(\Omega\times\mathbb{P})_l^+$ and the $\widetilde{\tau}$-invariance of $\widetilde{\mathcal{B}}$ ensure that the set $\mathcal{C}$ is $\widetilde{\tau}$-invariant, which in turn guarantees the invariance of the sets $\widetilde{\mathcal{B}}_l$ and $\mathcal{A}_l$. And the last assertion in (iv) follows from $r(t,\omega,\theta,t) \le \rho$ for all $(\omega,\theta,r) \in \widetilde{\mathcal{B}}_l$. □

In order to formulate our main result, we define

$$\Omega^0 := \{\omega \in \Omega \,|\, \widetilde{\beta}(\omega,\theta) = 0 \text{ for all } \theta \in \mathbb{P}\} \subseteq \Omega - \Omega^+. \tag{6.6}$$

It is obvious that if Ω^0 is nonempty then the condition $\Omega^+ \ne \Omega$, included in Hypotheses 6.1, holds. But the converse property cannot be guaranteed: see Remark 4.5.1. So, the conditions we have assumed so far do not guarantee that Ω^0 is nonempty. However this stronger requirement will be imposed in the formulation of our main result. At the end of the section we will show that these hypotheses are fulfilled in some situations which are easy to describe. For now, we recall once again that the condition $(m_\Omega\times l_{\mathbb{R}_2})(\mathcal{A}) > 0$ is equivalent to $m_\Omega(\Omega^+) = 1$: see Proposition 4.7 and Remark 4.8. All this means that Hypotheses 6.1 are included in those of the next results.

Theorem 6.8. *Suppose that $\Sigma_A = \{0\}$, that $(m_\Omega\times l_{\mathbb{R}_2})(\mathcal{A}) > 0$, and that the set Ω^0 given by (6.6) is nonempty. Then there exists a σ-invariant set $\Omega_0 \subseteq \Omega$ with $m_\Omega(\Omega_0) = 1$ such that $l_{\mathbb{R}^2}((\mathcal{A}_l)_\omega) > 0$ for all $\omega \in \Omega_0$, and such that any two different points $\big(\omega, \left[\begin{smallmatrix} r_1\sin\theta_1 \\ r_1\cos\theta_1\end{smallmatrix}\right]\big), \big(\omega, \left[\begin{smallmatrix} r_2\sin\theta_2 \\ r_2\cos\theta_2\end{smallmatrix}\right]\big)$ of $\mathcal{A}_l$ form a Li-Yorke pair for the flow $\tau_{\mathbb{R}}$ whenever $\omega \in \Omega_0$.*

Proof. Let $\left[\begin{smallmatrix} r_i\sin\theta_i \\ r_i\cos\theta_i\end{smallmatrix}\right]$ for $i = 1,2$ be two different points of $(\mathcal{A}_l)_\omega$, and let us call $\bar{\theta}_i = \mathfrak{p}(\theta_i) \in \mathbb{P}$. Note that $r_i \le \widetilde{\beta}(\omega,\bar{\theta}_i)$. We take $\widetilde{\omega}_0 \in \Omega^0$ and look for $(t_n) \uparrow \infty$ with $\widetilde{\omega}_0 = \lim_{n\to\infty} \omega{\cdot}t_n$ and such that there exist $\bar{\theta}_i^0 := \lim_{n\to\infty} \widetilde{\theta}(t_n,\omega,\bar{\theta}_i)$ for $i = 1,2$. Then

$$\begin{aligned} 0 &\le \limsup_{n\to\infty} r_l(t_n,\omega,\theta_i,r_i) = \limsup_{n\to\infty} r(t_n,\omega,\theta_i,r_i) = \limsup_{n\to\infty} r(t_n,\omega,\bar{\theta}_i,r_i) \\ &\le \limsup_{n\to\infty} r(t_n,\omega,\bar{\theta}_i,\widetilde{\beta}(\omega,\bar{\theta}_i)) = \limsup_{n\to\infty} \widetilde{\beta}(\widetilde{\sigma}(t_n,\omega,\bar{\theta}_i)) \le \widetilde{\beta}(\widetilde{\omega}_0,\bar{\theta}_i^0) = 0 \end{aligned}$$

for $i = 1,2$. Here we have used Proposition 6.7(iv), (4.7), the monotonicity of the flow $\widetilde{\tau}$, Theorem 4.3(ii), the semicontinuity of $\widetilde{\beta}$ also ensured by Theorem 4.3, and the definition of Ω^0. Consequently,

$$\liminf_{t\to\infty} \operatorname{dist}_{\mathbb{R}^2}\big(\mathbf{y}\big(t,\omega,\left[\begin{smallmatrix} r_1\sin\theta_1 \\ r_1\cos\theta_1\end{smallmatrix}\right]\big),\mathbf{y}\big(t,\omega,\left[\begin{smallmatrix} r_2\sin\theta_2 \\ r_2\cos\theta_2\end{smallmatrix}\right]\big)\big) = 0\,.$$

The goal is hence to find a σ-invariant set Ω_0 with $m_\Omega(\Omega_0) = 1$ such that two conditions hold for all $\omega \in \Omega_0$: $l_{\mathbb{R}^2}((\mathcal{A}_l)_\omega) > 0$, and

$$\limsup_{t\to\infty} \operatorname{dist}_{\mathbb{R}^2}\big(\mathbf{y}\big(t,\omega,\left[\begin{smallmatrix} r_1\sin\theta_1 \\ r_1\cos\theta_1\end{smallmatrix}\right]\big),\mathbf{y}\big(t,\omega,\left[\begin{smallmatrix} r_2\sin\theta_2 \\ r_2\cos\theta_2\end{smallmatrix}\right]\big)\big) > 0$$

whenever $\left[\begin{smallmatrix} r_1\sin\theta_1 \\ r_1\cos\theta_1\end{smallmatrix}\right]$ and $\left[\begin{smallmatrix} r_2\sin\theta_2 \\ r_2\cos\theta_2\end{smallmatrix}\right]$ belong to $(\mathcal{A}_l)_\omega$. Note that the first condition will be guaranteed by Proposition 6.7(iii) if we take (as we will do) Ω_0 contained in the set Ω^* given by (6.5), which satisfies $\Omega^* \subseteq \Omega^+$ and $m_\Omega(\Omega^*) = 1$; and that the second condition is equivalent to

$$\limsup_{t\to\infty} \mathrm{dist}_{\mathbb{R}^2}\big(\mathbf{y}_l\big(t,\omega,\left[\begin{smallmatrix} r_1\sin\theta_1\\ r_1\cos\theta_1\end{smallmatrix}\right]\big), \mathbf{y}_l\big(t,\omega,\left[\begin{smallmatrix} r_2\sin\theta_2\\ r_2\cos\theta_2\end{smallmatrix}\right]\big)\big) > 0 \tag{6.7}$$

whenever $\left[\begin{smallmatrix} r_1\sin\theta_1\\ r_1\cos\theta_1\end{smallmatrix}\right]$ and $\left[\begin{smallmatrix} r_2\sin\theta_2\\ r_2\cos\theta_2\end{smallmatrix}\right]$ belong to $(\mathcal{A}_l)_\omega$, as Proposition 6.7(iv) ensures.

Recall that our hypotheses guarantee that $m_\Omega(\Omega^+) = 1$ and $\Omega^+ \neq \Omega$. Hence the map $e = (1/2)\,\mathrm{tr}\,A$ satisfies the assertions of Propositions 6.5 and 6.4. In addition, the Saker and Sell spectrum of the family of systems (5.5) given by $\widetilde{A} = A - e\,I_2$ is $\{0\}$ (see Remark 5.5.3). Hence we can distinguish two cases: either all the solutions of all the systems (5.5) are bounded, or the family is in the weakly elliptic case.

Case 1. We begin by assuming that all the solutions of all the systems (5.5) are bounded, hypothesis which we will use twice later. Remark 5.5.2 ensures that

$$r_l(t,\omega,\theta_i,r_i) = r_i\,\widetilde{r}_l(t,\omega,\theta_i,1)\exp\left(\int_0^t e(\omega{\cdot}s)\,ds\right) = r_i\,\widetilde{r}_l(t,\omega,\theta_i,1)\,\frac{\exp(h_e(\omega{\cdot}t))}{\exp(h_e(\omega))}$$

for $i = 1,2$ if $\omega \in \Omega^* \subseteq \Omega^+$, where h_e is given by Propositions 6.5 and 6.4. Lusin's theorem provides a compact set $\mathcal{K} \subset \Omega$ with $m_\Omega(\mathcal{K}) > 0$ such that h_e is continuous on $\mathcal{K}$, and Birkhoff's ergodic theorem ensures that the σ-invariant set $\Omega_0 \subseteq \Omega^*$ of points ω for which there exists $(s_n) \uparrow \infty$ with $\omega{\cdot}s_n \in \mathcal{K}$ for all $n \geq 0$ satisfies $m_\Omega(\Omega_0) = 1$. We take $\omega \in \Omega_0$ and two different points $\left[\begin{smallmatrix} r_i\sin\theta_i\\ r_i\cos\theta_i\end{smallmatrix}\right]$ of $(\mathcal{A}_l)_\omega$. We also take $(s_n) \uparrow \infty$ with $\omega{\cdot}s_n \in \mathcal{K}$ such that there exist $(\omega_0, \theta_i^0) := \lim_{n\to\infty}(\omega{\cdot}s_n, \widehat{\theta}(s_n,\omega,\theta_i))$. All the solutions of all the systems (5.5) are also bounded away from zero: the determinant of the fundamental matrix solution of (5.5) is 1 (see (2.9)), so that its inverse is also globally bounded. Hence the two sequences $(\widetilde{r}_l(s_n,\omega,\theta_i,1))$ are contained in an interval $[\kappa_1,\kappa_2] \subset (0,\infty)$. So, there is no restriction in assuming that there exist $\bar{r}_i := \lim_{n\to\infty}\widetilde{r}_l(s_n,\omega,\theta_i,1) > 0$, and there also exist $r_i^* := \lim_{n\to\infty} r_l(s_n,\omega,\theta_i,r_i) = r_i\,\bar{r}_i\,\exp(h_e(\omega_0))/\exp(h_e(\omega))$ with $r_i^* \neq 0$ if $r_i \neq 0$.

The assumed boundedness also ensures that the flow $(\Omega\times\mathbb{S},\widehat{\sigma},\mathbb{R})$ (see Remark 5.5.1) is distal (see e.g. the proof of Theorem 3.1 of [39]). Consequently, $\lim_{n\to\infty}\mathrm{dist}_{\mathbb{S}}\big(\widehat{\theta}(s_n,\omega,\theta_1),\widehat{\theta}(s_2,\omega,\theta_2)\big) > 0$ if $\theta_1 \neq \theta_2$, so that (6.7) holds in this case. If, on the contrary, $\theta_1 = \theta_2$, we have $r_1 \neq r_2$ and $\bar{r}_1 = \bar{r}_2$, which ensures that $r_1^* \neq r_2^*$ and hence that (6.7) also holds.

Case 2. Now we assume that the family (5.5) is weakly elliptic. This fact and Theorem 6.6 ensure that all the hypotheses of Theorem 3.4 hold. As at the beginning of step 2 in its proof, we choose a $\widehat{\sigma}$-minimal set $\mathcal{M} \subseteq \Omega\times\mathbb{S}$ and consider the flow $(\mathcal{M}\times\mathbb{S},\widehat{\vartheta}_{\mathcal{M}},\mathbb{R})$. We also fix a $\widehat{\sigma}$-ergodic measure $m_{\mathcal{M}}$ concentrated on $\mathcal{M}$. Let $\mathcal{K} \subseteq \mathcal{M}$ be the compact set with $m_{\mathcal{M}}(\mathcal{K}) > 1/2$ described in Remark 3.5. It follows from Proposition 6.7(i) that $\mathcal{C}^0 := (\mathcal{K}\times\mathbb{S}^1)\cap\{(\omega,\theta^1,\theta)\in\mathcal{M}\times\mathbb{S}\,|\,(\omega,\mathfrak{p}(\theta))\in(\Omega\times\mathbb{P})_l^+\}$ satisfies $(m_{\mathcal{M}}\times l_{\mathbb{S}})(\mathcal{C}^0) = m_{\mathcal{M}}(\mathcal{K}) > 1/2$. Let us define $\widehat{\beta}_*\colon \mathcal{M}\times\mathbb{S}\to[0,\rho]$, $(\omega,\theta^1,\theta)\mapsto\widetilde{\beta}(\omega,\mathfrak{p}(\theta))$. Lusin's theorem provides a compact set $\mathcal{C}^1 \subseteq \mathcal{C}^0$ with $(m_{\mathcal{M}}\times l_{\mathbb{S}})(\mathcal{C}^1) > 1/2$ such that the restriction of $\widehat{\beta}_*$ to $\mathcal{C}^1$ is continuous. The definitions of $(\Omega\times\mathbb{P})_l^+$ and $\widetilde{\beta}$ ensure that $\widehat{\beta}_*(\omega,\theta^1,\theta) = \widetilde{\beta}(\omega,\mathfrak{p}(\theta)) \in (0,\rho]$ for all $(\omega,\theta^1,\theta)\in\mathcal{C}^1$, and hence that $\beta_* := \inf_{(\omega,\theta^1,\theta)\in\mathcal{C}^1}\widehat{\beta}_*(\omega,\theta^1,\theta)\in(0,\rho]$. In addition, the set $\mathcal{M}_1 := \{(\omega,\theta^1)\in\mathcal{M}\,|\,l_{\mathbb{S}}(\mathcal{C}^1_{(\omega,\theta^1)}) > 1/2\}$ satisfies $m_{\mathcal{M}}(\mathcal{M}_1) > 0$, since $(m_{\mathcal{M}}\times l_{\mathbb{S}})(\mathcal{C}^1) > 1/2$. It is clear that $\mathcal{M}_1 \subseteq \mathcal{K}$. It is also clear that, if $(\omega,\theta^1)\in\mathcal{M}_1$, we can find $\theta_1 \in \mathbb{S}$ such that the points $(\omega, \theta^1, \theta_1)$ and $(\omega, \theta^1, \theta_2)$ belong to $\mathcal{C}^1$ for $\theta_2 = \theta_1 + \pi/2$. As in the proof of Proposition 6.7(iii), we check that $\widehat{\beta}_*(\omega, \theta^1, \theta) \geq \beta^*/\sqrt{2}$ for all $\theta \in \mathbb{S}^1$. That is,

$$\widetilde{\beta}(\omega, \theta) \geq \beta^*/\sqrt{2} \quad \text{for all } \theta \in \mathbb{P} \text{ if there exists } \theta^1 \in \mathbb{S} \text{ with } (\omega, \theta^1) \in \mathcal{M}_1\,. \tag{6.8}$$

It follows form Birkhoff's ergodic theorem that the $\widehat{\sigma}$-invariant set $\mathcal{M}_2$ of points $(\omega, \theta^1) \in \mathcal{M}$ for which there exists a sequence $(t_n) \uparrow \infty$ with $\widehat{\sigma}(t_n, \omega, \theta^1) \in \mathcal{M}_1$ for all $n \geq 0$ satisfies $m_{\mathcal{M}}(\mathcal{M}_2) > 0$. Since $m_{\mathcal{M}}$ is $\widehat{\sigma}$-ergodic, $m_{\mathcal{M}}(\mathcal{M}_1) = 1$.

Now we define Ω_0 as the projection of $\mathcal{M}_1$ on Ω intersected with the set Ω^* defined by (6.5), so that Ω_0 is σ-invariant, $m_\Omega(\Omega_0) = 1$, and $l_{\mathbb{R}^2}((\mathcal{A}_l)_\omega) > 0$ for all $\omega \in \Omega_0$. We fix $\omega \in \Omega_0$, look for $\theta^1 \in \mathbb{S}$ with $(\omega, \theta^1) \in \mathcal{M}_2$, and take $\theta_1, \theta_2 \in \mathbb{S}$ and $r_1, r_2 \geq 0$ such that $\big(\omega, \left[\begin{smallmatrix} r_1 \sin\theta_1 \\ r_1 \cos\theta_1 \end{smallmatrix}\right]\big)$ and $\big(\omega, \left[\begin{smallmatrix} r_2 \sin\theta_2 \\ r_2 \cos\theta_2 \end{smallmatrix}\right]\big)$ are two different points of $\mathcal{A}_l$. Our goal is to prove (6.7).

We fix a sequence of positive numbers $(t_n) \uparrow \infty$ with $\widehat{\sigma}(t_n, \omega, \theta^1) \in \mathcal{M}_1$ for all $n \geq 0$. Since $\mathcal{M}_1 \subseteq \mathcal{K}$, using the notation established in Remark 3.5 we have

$$\mathrm{dist}_{\mathbb{S}}(\widehat{\theta}_{\mathcal{M}}(t_n, \omega, \theta^1, \theta_1), \widehat{\theta}_{\mathcal{M}}(t_n, \omega, \theta^1, \theta_2)) \geq \mathrm{d}(\eta_1, \eta_2)\,,$$

where η_1 and η_2 are determined by $(\omega, \theta^1, \theta_i) \in \mathcal{S}_{\eta_i}$ for $i = 1, 2$, and $\mathrm{d}(\eta_1, \eta_2) := \inf_{(\omega,\theta^1) \in \mathcal{K}} \mathrm{dist}_{\mathbb{P}}(\varphi_{\eta_1}(\omega, \theta^1), \varphi_{\eta_2}(\omega, \theta^1))$. Here $\widehat{\theta}_{\mathcal{M}}$ is the fiber component of the skew-product flow $(\mathcal{M} \times \mathbb{S}, \widehat{\vartheta}_{\mathcal{M}}, \mathbb{R})$, which agrees with that of $(\Omega \times \mathbb{S}, \widehat{\sigma}, \mathbb{R})$. Hence,

$$\mathrm{dist}_{\mathbb{S}}(\widehat{\theta}(t_n, \omega, \theta_1), \widehat{\theta}(t_n, \omega, \theta_2)) \geq \mathrm{d}(\eta_1, \eta_2)\,. \tag{6.9}$$

On the other hand, $r_l(t_n, \omega, \mathfrak{p}(\theta_i), r_i) = (r_i/\widetilde{\beta}(\omega, \mathfrak{p}(\theta_i))\, r_l(t_n, \omega, \mathfrak{p}(\theta_i), \widetilde{\beta}(\omega, \mathfrak{p}(\theta_i))$ and $r_l(t_n, \omega, \mathfrak{p}(\theta_i), \widetilde{\beta}(\omega, \mathfrak{p}(\theta_i)) \geq r(t_n, \omega, \mathfrak{p}(\theta_i), \widetilde{\beta}(\omega, \mathfrak{p}(\theta_i))$ (since $g \geq g - k_\rho$ and $t_n > 0$). Therefore, Theorem 4.3(ii) and (6.8) yield

$$\limsup_{n\to\infty} r_l(t_n, \omega, \theta_i, r_i) \geq \frac{r_i}{\widetilde{\beta}(\omega, \mathfrak{p}(\theta_i))} \limsup_{n\to\infty} \widetilde{\beta}(\widetilde{\sigma}(t_n, \omega, \mathfrak{p}(\theta_i))) \geq \frac{r_i\, \beta^*}{\sqrt{2}\, \widetilde{\beta}(\omega, \mathfrak{p}(\theta_i))} > 0$$

if $r_i \neq 0$. Since r_1 and r_2 cannot be simultaneously 0, there is no restriction in assuming that

$$r_1^\infty := \limsup_{n\to\infty} r_l(t_n, \omega, \theta_i, r_1) > 0\,.$$

Suppose now that $\theta_1 \neq \theta_2$. Then $\eta_1 \neq \eta_2$, so that $\mathrm{d}(\eta_1, \eta_2) > 0$ and (6.9) yields $\mathrm{dist}_{\mathbb{S}}(\widehat{\theta}(t_n, \omega, \theta_1), \widehat{\theta}(t_n, \omega, \theta_2)) > 0$. Hence (6.7) holds unless $\lim_{n\to\infty} r(t_n, \omega, \theta_i, r_i) = 0$ for $i = 1, 2$, which is not the case. Hence the condition (6.7) holds in this case. If, on the contrary, $\theta_1 = \theta_2$, then $r_1 \neq r_2$ and

$$\begin{aligned}\limsup_{n\to\infty} &\,\mathrm{dist}_{\mathbb{R}^2}\big(\mathbf{y}_l\big(t_n, \omega, \left[\begin{smallmatrix} r_1 \sin\theta_1 \\ r_1 \cos\theta_1 \end{smallmatrix}\right]\big), \mathbf{y}_l\big(t_n, \omega, \left[\begin{smallmatrix} r_2 \sin\theta_1 \\ r_2 \cos\theta_1 \end{smallmatrix}\right]\big)\big) \\ &= \limsup_{n\to\infty} |r_l(t_n, \omega, \theta_1, r_1) - r_l(t_n, \omega, \theta_1, r_2)| = |r_1^\infty - (r_2/r_1) r_1^\infty| > 0\,,\end{aligned}$$

so that (6.7) also holds in this case. The proof is complete.

□

The information provided by Proposition 6.7, Theorem 6.8 and Proposition 4.6 leads us to:

Corollary 6.9. *Suppose that $\Sigma_A = \{0\}$, that $(m_\Omega \times l_{\mathbb{R}_2})(\mathcal{A}) > 0$, and that the set Ω^0 given by (6.6) is nonempty. Then the restricted flow $(\mathcal{A}, \tau_{\mathbb{R}}, \mathbb{R})$ is Li-Yorke fiber-chaotic in measure. And, in addition, there exists a σ-invariant residual set of points of Ω for which the section $\mathcal{A}_\omega$ is given by $\{\mathbf{0}\}$.*

Let us now analyse the hypotheses we have worked with in this section. All its results require Hypotheses 6.1. In particular, according to Proposition 6.5, the map $e = (1/2)\,\mathrm{tr}\,A$ must belong to the set $\mathcal{R}(\Omega)$ of Definition 6.2. How restrictive is this condition? The set $\mathcal{R}(\Omega)$ is a subset of the set of continuous functions with mean value zero which admit a measurable primitive, which is known to be (at least in the case of minimal almost periodic base flow) of the first Baire category: this is proved by Johnson in [20], and it is expected to be true in more general settings. Therefore $\mathcal{R}(\Omega)$ is not a "large" set, at least from a topological point of view. As a matter of fact, $\mathcal{R}(\Omega)$ is empty in the autonomous and periodic cases. However it is nonempty for more complex nonautonomous cases: there are well known examples of functions $e \in \mathcal{R}(\Omega)$. For instance, Johnson shows in [22] that this is the case for an example based in a previous one constructed by Anosov in [5]. The main ideas of the description of this example in [22] are used by Ortega and Tarallo in [40] in order to optimize the construction of functions with these characteristics. In both cases, $(\Omega, \sigma, \mathbb{R})$ is a minimal quasi periodic but not periodic flow. And recently Campos *et al.* [10] have shown the existence of elements of $\mathcal{R}(\Omega)$ whenever the base flow is nonperiodic, uniquely ergodic and minimal.

The condition $e \in \mathcal{R}(\Omega)$ does not suffice to guarantee Hypothesis 6.1, or the more restrictive conditions of Theorem 6.8 and Corollary 6.9. However,

Proposition 6.10. *Suppose that $e \in \mathcal{R}(\Omega)$, that $\widetilde{A}\colon \Omega \to \mathbb{M}_{2\times 2}(\mathbb{R})$ is continuous and with $\mathrm{tr}\,\widetilde{A} = 0$, and that all the solutions of all the systems $\mathbf{y}' = \widetilde{A}(\omega{\cdot}t)\,\mathbf{y}$ are bounded. Let $k_\rho\colon \mathbb{R}^+ \to \mathbb{R}^+$ be given by (4.1). Then the family of systems*

$$\mathbf{y}' = \big(\widetilde{A}(\omega{\cdot}t) + e(\omega{\cdot}t)\, I_2\big)\,\mathbf{y} - k_\rho(|\mathbf{y}|)\,\mathbf{y} \tag{6.10}$$

satisfies the hypotheses of Theorem 6.8 and Corollary 6.9.

Proof. Let us call $A := \widetilde{A} + e\, I_2$. With the notation established in Remark 5.5.2, $r_l(t,\omega,\theta,1) = \widetilde{r}_l(t,\omega,\theta,1)\exp\big(\int_0^t e(\omega{\cdot}s)\,ds\big)$. This relation and Proposition 6.4 have two consequences. The first one is that the solutions of the systems $\mathbf{y}' = A(\omega{\cdot}t)\,\mathbf{y}$ are bounded for all ω in a σ-invariant set of full measure. In particular, $\Sigma_A = \{0\}$ (see Theorem 2.8) and $m_\Omega(\Omega^+) = 1$ (see Proposition 4.4), so that $(m_\Omega{\times}l_{\mathbb{R}_2})(\mathcal{A}) > 0$ (see Proposition 4.7 and Remark 4.8). The second consequence is that all the nonzero solutions of the systems $\mathbf{y}' = A(\omega{\cdot}t)\,\mathbf{y}$ are unbounded on $(-\infty, 0]$ for ω in a residual subset of Ω. (Here we are using that the functions $\widetilde{r}_l(t,\omega,\theta,1)$ are bounded away from 0: see e.g. CASE 1 in the proof of Theorem 6.8.) This property and Proposition 4.4 ensure that the set Ω^0 given by (6.6) contains a residual set. □

Remark 6.11. The hypotheses of Theorem 6.8 and Corollary 6.9 are also fulfilled by some families of systems of the form (6.10) with $e \in \mathcal{R}(\Omega)$ and $\mathbf{y}' = \widetilde{A}(\omega{\cdot}t)\,\mathbf{y}$ in the weakly elliptic case (see Definition 3.1). For instance, we take $e \in \mathcal{R}(\Omega)$ and $\widetilde{A} = \left[\begin{smallmatrix} e/2 & 0 \\ 0 & -e/2 \end{smallmatrix}\right]$, call $A := e\, I_2 + \widetilde{A}$, and note that

$$U(t,\omega) = \begin{bmatrix} \exp(\int_0^t (3e(\omega{\cdot}s)/2)\,ds) & 0 \\ 0 & \exp(\int_0^t (e(\omega{\cdot}s)/2)\,ds) \end{bmatrix}$$

is the fundamental matrix solution of $\mathbf{y}' = A(\omega{\cdot}t)\,\mathbf{y}$. It follows from Theorem 2.8 that $\Sigma_A = \{0\}$. In addition, the solutions of the systems corresponding to the set Ω^e given by Proposition 6.4 are bounded, so that $m_\Omega(\Omega^+) = 1$; and all the nonzero solutions of the systems corresponding to a residual subset of Ω, also given by Proposition 6.4, are unbounded, so that the set Ω^0 is nonempty.

We conclude by recalling that the carried-on analysis gives rise to examples fitting in what we called *nonautonomous Hopf bifurcation pattern with zero spectrum* in the Introduction. To construct one of these examples it is enough to take a family of systems of the form (6.10) satisfying the hypotheses of Theorem 6.8 and Corollary 6.9 (as those provided by Proposition 6.10 and Remark 6.11), and to consider the one-parametric family of families of ODEs

$$\mathbf{y}' = \big(\widetilde{A}(\omega{\cdot}t) + (e(\omega{\cdot}t) + \varepsilon)\, I_2\big)\, \mathbf{y} - k_\rho(|\mathbf{y}|)\, \mathbf{y}$$

for $\omega \in \Omega$ and $\varepsilon \in \mathbb{R}$. The Sacker and Sell spectrum of the associated linear family given by ε is $\{\varepsilon\}$. Hence, if $\varepsilon < 0$, the global attractor $\mathcal{A}$ is $\Omega \times \{\mathbf{0}\}$; and if $\varepsilon > 0$, then $\mathcal{A}$ contains a set $\Omega \times \{\mathbf{y} \in \mathbb{R}^2 \mid |\mathbf{y}| \le \beta\}$ for a $\beta > 0$ and, as a matter of fact, it is homeomorphic to this set. These assertions follow from the discussion carried-on in Section 5. Finally, the situation at the bifurcation point $\varepsilon = 0$ is that described in Theorem 6.8 and Corollary 6.9, with the occurrence of Li-Yorke chaos.

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