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Aggregating opinions in non-uniform ordered qualitative scales

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Abstract

This paper introduces a new voting system in the setting of ordered qualitative scales. The process is conducted in a purely ordinal way by considering an ordinal proximity measure that assigns an ordinal degree of proximity to each pair of linguistic terms of the qualitative scale. Once the agents assess the alternatives through the qualitative scale, the alternatives are ranked according to the medians of the ordinal degrees of proximity between the obtained individual assessments and the highest linguistic term of the scale. Since some alternatives may share the same median, an appropriate tie-breaking procedure is introduced. Some properties of the proposed voting system have been provided.

Keywords: group decision making; qualitative scales; ordinal proximity measures.

1. Introduction

Ordered qualitative scales are common in social sciences, engineering, computer sciences and other fields, because they are more appropriate than numerical scales for dealing with the vagueness and imprecision of human beings when evaluating different alternatives. Some ordered qualitative scales are uniform: the psychological proximity between each pair of consecutive terms of the scale

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is the same, e.g. the scale {'very bad', 'bad', 'regular', 'good', 'very good'}. Usually, this is the case of Likert-type scales [20]. However, not all ordered qualitative scales are uniform. For instance, the scale {'reject', 'major revision', 'minor revision', 'accept'}, that some scientific journals use for evaluating papers, may be considered as non-uniform (see García-Lapresta and Pérez-Román [14] for empirical evidence).

Although ordered qualitative scales consist of vague linguistic terms, sometimes these terms are represented by exact numerical values. For instance, the International Association of Oenologists considers that each attribute of a wine is evaluated in an ordered qualitative scale of seven linguistic terms: {'bad', 'mediocre', 'inadequate', 'passable', 'good', 'very good', 'excellent'} and each term is associated with an integer number (see Balinski and Laraki [4]). In spite of the fact of this practice has been widely used in the literature (see, for instance, Franceschini and Romano [10] and Averkin et al. [1]), it is meaningless because different codifications of the same ordered qualitative scale could generate different outcomes when aggregating individual assessments (see Roberts [23] and Franceschini et al. [9], among others).

In order to capture the vagueness of ordered qualitative scales, some authors assign other cardinal objects, such as intervals of real numbers or fuzzy numbers, to the linguistic terms of the scale (see, for instance, Zadeh [24], Bass and Kwakernaak [6] and Chen and Hwang [7]). Again, these cardinal representations may be considered as meaningless.

Herrera and Martínez [17, 18] introduce the 2-tuple linguistic model for aggregating linguistic information in the setting of uniform ordered qualitative scales. The authors identify each linguistic term of the scale with its position in the scale; after an aggregation process, the outcome is represented by a pair (2-tuple) consisting of a linguistic term and a numerical value that measures the deviation with respect to the linguistic term. Thus, in practice, 2-tuples and real numbers are identical. The procedure is completed with a linear order on the set of 2-tuples that permits rank order the outcomes generated by the aggregation process. Although the procedure manages linguistic information, it is mathematically equivalent to work with numerical values (see García-Lapresta [11]).

Herrera et al. [15, 16] extend the 2-tuple linguistic model to the case of unbalanced qualitative scales by considering additional linguistic terms and under a high computational cost (see also Martínez and Herrera [22]). Bartczuk et. al. [5] modify the previous model by introducing numerical correction factors in the extended linguistic terms. This new model is computationally less expensive than the previous one and provides a simpler semantics.

In the mentioned approaches, the linguistic information and its aggregation are managed through cardinal objects and techniques. In this paper, we do not represent linguistic terms of ordered qualitative scales by any mathematical object. Instead, we consider psychological proximities among linguistic terms in a purely ordinal way, without using numerical distances, but ordinal degrees.

In real life, it is usual to make comparisons between proximities of different pairs of objects in a vague and ordinal fashion. For instance, we say "Rome is closer to Naples than to Milan", "Budapest is closer to Vienna than Paris is to Athens", etc. An excellent example of the use of ordinal proximities can be found in the following sentence of the Amos Oz' novel *Suddenly in the Depth of the Forest*: "... fairly close to Maya's back but not as close as she was to the stranger, and slightly closer than she was to the opening of the cave".

These kinds of ordinal comparisons will be taken into account in the setting of ordered qualitative scales through the notion of ordinal proximity measure, introduced by García-Lapresta and Pérez-Román [14] to deal with the psychological proximities among linguistic terms of ordered qualitative scales.

In the *Majority Judgment* (MJ) voting system, introduced by Balinski and Laraki [2, 3]), agents evaluate the alternatives through the linguistic terms of an ordered qualitative scale. In MJ, the alternatives are ranked according to the medians of the obtained assessments. The authors also propose two different tie-breaking processes for obtaining the final ranking. Despite the fact that the qualitative scales considered by the authors are not necessarily uniform, the authors did not take this aspect into account.

In this paper, we use the new approach of ordinal proximity measures for designing a voting system that ranks the alternatives evaluated by the agents by means of an ordered qualitative scale. The proposed voting system is related to MJ, but we pay special attention to the ordinal proximities among the terms of the corresponding ordered qualitative scale. Concretely, alternatives are ranked according to the medians of the ordinal proximities between the individual assessments and the highest term of the scale. A tie-breaking procedure that takes into account the ordinal proximities among linguistic terms is also proposed. We also briefly show some properties of the devised voting system.

It should be noted that our approach shares with some soft computing methodologies the tolerance for imprecision, uncertainty and subjectivity, under a mathematical foundation (see Zadeh [25], Karray and De Silva [19] and

Magdalena [21], among others).

The rest of the paper is organized as follows. Section 2 is devoted to ordinal proximity measures. In Section 3 we introduce and analyze the proposed voting system. Section 4 includes an example that illustrates how the voting system works. In Section 5 we extend the voting system to the case of multiple criteria. Finally, Section 6 concludes the paper with some remarks.

2. Ordinal proximity measures

We consider that each individual of a group of agents assigns a linguistic term to every feasible alternative. These linguistic terms belong to an ordered qualitative scale $\mathcal{L} = \{l_1, \dots, l_g\}$, arranged from worst to best, $l_1 < \cdots < l_g$, where the granularity of $\mathcal{L}$ is at least 3, i.e., $g \geq 3$.

We now recall the notion of ordinal proximity measure, introduced by García-Lapresta and Pérez-Román [14]. It is a mapping that assigns an ordinal degree of proximity to each pair of linguistic terms of an ordered qualitative scale $\mathcal{L}$. These ordinal degrees of proximity belong to a linear order $\Delta = \{\delta_1, \dots, \delta_h\}$, with $\delta_1 \succ \cdots \succ \delta_h$, being δ_1 and δ_h the maximum and minimum degrees of proximity, respectively. It is important noticing that the elements of Δ are not numbers. In fact, they are only abstract objects, without meaning, representing different degrees of proximity.

As usual in the setting of linear orders, $\delta_r \succeq \delta_s$ means $\delta_r \succ \delta_s$ or $\delta_r = \delta_s$; and $\delta_r \prec \delta_s$ means $\delta_s \succ \delta_r$.

Definition 1. *([14]) An* ordinal proximity measure on $\mathcal{L}$ with values in Δ *is a mapping $\pi : \mathcal{L}^2 \longrightarrow \Delta$, where $\pi(l_r, l_s) = \pi_{rs}$ means the degree of proximity between l_r and l_s, satisfying the following conditions:*

1. Exhaustiveness*: For every $\delta \in \Delta$, there exist $l_r, l_s \in \mathcal{L}$ such that $\delta = \pi_{rs}$.*
2. Symmetry*: $\pi_{sr} = \pi_{rs}$, for all $r, s \in \{1, \dots, g\}$.*
3. Maximum proximity*: $\pi_{rs} = \delta_1 \Leftrightarrow r = s$, for all $r, s \in \{1, \dots, g\}$.*
4. Monotonicity*: $\pi_{rs} \succ \pi_{rt}$ and $\pi_{st} \succ \pi_{rt}$, for all $r, s, t \in \{1, \dots, g\}$ such that $r < s < t$.*

We note that the previous conditions are independent (see García-Lapresta and Pérez-Román [14, Prop. 1]).

We say that an ordinal proximity measure $\pi : \mathcal{L}^2 \longrightarrow \Delta$ is *uniform* if $\pi_{r(r+1)} = \pi_{s(s+1)}$ for all $r, s \in \{1, \dots, g-1\}$, and *totally uniform* if $\pi_{r(r+t)} = \pi_{s(s+t)}$ for all $r, s, t \in \{1, \dots, g-1\}$ such that $r+t, s+t \leq g$.

Each ordinal proximity measure $\pi : \mathcal{L}^2 \longrightarrow \Delta$ will be represented by a $g \times g$ symmetric matrix with coefficients in Δ, being the elements in the main diagonal $\pi_{rr} = \delta_1,\ r = 1, \dots, g$:

$$\begin{pmatrix} \pi_{11} & \cdots & \pi_{1s} & \cdots & \pi_{1g} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \pi_{r1} & \cdots & \pi_{rs} & \cdots & \pi_{rg} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \pi_{g1} & \cdots & \pi_{gs} & \cdots & \pi_{gg} \end{pmatrix}.$$

This matrix will be called *proximity matrix associated with* π.

If we consider the conditions appearing in Definition 1, we would only need to show the upper half proximity matrix

$$\begin{pmatrix} \delta_1 & \pi_{12} & \pi_{13} & \cdots & \pi_{1(g-1)} & \pi_{1g} \\ & \delta_1 & \pi_{23} & \cdots & \pi_{2(g-1)} & \pi_{2g} \\ & & & \cdots & \cdots & \cdots \\ & & & & \delta_1 & \pi_{(g-1)g} \\ & & & & & \delta_1 \end{pmatrix}.$$

It is important noticing that the minimum proximity between linguistic terms is only reached when comparing the extreme linguistic terms: $\pi_{rs} = \delta_h \Leftrightarrow (r, s) \in \{(1, g), (g, 1)\}$ (see García-Lapresta and Pérez-Román [14, Prop. 2]).

In the following example we illustrate an ordered qualitative scale of four linguistic terms with two extreme ordinal proximity measures.

Example 1. Consider $g = 4$, where five ordinal degrees, not necessarily different, have to be assigned (see Fig. 1) and $h \in \{4, 5, 6, 7\}$.

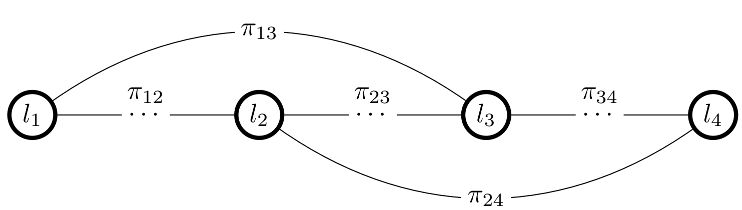

Figure 1: Ordinal degrees for $g = 4$.

It is worth mentioning that for $g = 4$ there are 51 different ordinal proximity measures (García-Lapresta et al. [13]).

1. The simplest case corresponds to $\Delta = \{\delta_1, \delta_2, \delta_3, \delta_4\}$, with $\pi_{rr} = \delta_1$, $\pi_{12} = \pi_{23} = \pi_{34} = \delta_2$, $\pi_{13} = \pi_{24} = \delta_3$ and $\pi_{14} = \delta_4$, i.e., the totally uniform ordinal proximity measure, with associated matrix[1]

$$A_{222} = \begin{pmatrix} \delta_1 & \delta_2 & \delta_3 & \delta_4 \\ & \delta_1 & \delta_2 & \delta_3 \\ & & \delta_1 & \delta_2 \\ & & & \delta_1 \end{pmatrix}$$

that can be visualized in Fig. 2.

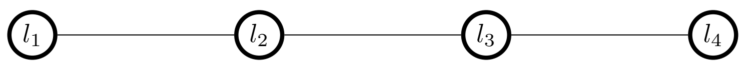

Figure 2: Ordinal proximity measure with associated matrix A_{222}.

2. We now consider $\Delta = \{\delta_1, \delta_2, \delta_3, \delta_4, \delta_5, \delta_6, \delta_7\}$, with $\pi_{rr} = \delta_1$, $\pi_{34} = \delta_2$, $\pi_{23} = \delta_3$, $\pi_{12} = \delta_4$, $\pi_{24} = \delta_5$, $\pi_{13} = \delta_6$ and $\pi_{14} = \delta_7$. In this case, the ordinal proximity measure has the following associated matrix

$$A_{432} = \begin{pmatrix} \delta_1 & \delta_4 & \delta_6 & \delta_7 \\ & \delta_1 & \delta_3 & \delta_5 \\ & & \delta_1 & \delta_2 \\ & & & \delta_1 \end{pmatrix}$$

that can be visualized in Fig. 3.

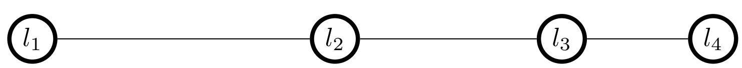

Figure 3: Ordinal proximity measure with associated matrix A_{432}.

3. The voting system

Consider a set of agents $A = \{1, \ldots, m\}$, with $m \geq 2$, that have to evaluate a set of alternatives $X = \{x_1, \ldots, x_n\}$, with $n \geq 2$, through an ordered qualitative

[1] The subindices 222 of the matrix A_{222} correspond to the subindices of the δ's appearing in the coefficients just over the main diagonal. We follow the same pattern in subsequent matrices.

scale $\mathcal{L} = \{l_1, \ldots, l_g\}$, $l_1 < \cdots < l_g$, with $g \geq 3$, and an ordinal proximity measure $\pi : \mathcal{L}^2 \longrightarrow \Delta$.

The assessments provided by the agents to the alternatives are collected in a *profile*, that is a matrix

$$V = \begin{pmatrix} v_1^1 & \cdots & v_i^1 & \cdots & v_n^1 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ v_1^a & \cdots & v_i^a & \cdots & v_n^a \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ v_1^m & \cdots & v_i^m & \cdots & v_n^m \end{pmatrix}$$

that consists of m rows and n columns of linguistic terms, where the element $v_i^a \in \mathcal{L}$ is the linguistic assessment given by the agent $a \in A$ to the alternative $x_i \in X$.

3.1. Ranking the alternatives

To rank the alternatives, the procedure is divided in the following steps.

1. For each alternative $x_i \in X$, consider the assessments obtained by x_i for all the agents: $v_i^1, \ldots, v_i^m \in \mathcal{L}$ (column i of V).

2. For each alternative $x_i \in X$, calculate the ordinal proximities between the assessments obtained by x_i and the highest linguistic term l_g:

$$\pi\left(v_i^1, l_g\right), \ldots, \pi\left(v_i^m, l_g\right) \in \Delta.$$

In a different setting, Falcó et al. [8] rank order linguistic assessments taking into account their distances to the highest linguistic term of the ordered qualitative scale (the less, the better). However, in the present approach, when considering ordinal proximities between linguistic assessments and the highest linguistic term of the ordered qualitative scale, the pattern is just the opposite (the more, the better), because the notions of distance and proximity are antonyms.

3. For each alternative $x_i \in X$, arrange the previous ordinal degrees in a decreasing fashion and select the median(s)[2], M_i:

[2] When the number of elements is odd, the median is unique. However, if that number

(a) If the number of assessments is odd, then we duplicate the median. Thus, $M_i = (\delta_r, \delta_r)$ for some $r \in \{1, \ldots, h\}$.

(b) If the number of assessments is even, then we take into account the two medians. Thus, $M_i = (\delta_r, \delta_s)$ for some $r, s \in \{1, \ldots, h\}$ such that $r \leq s$.

Consequently, $M_i \in \Delta_2$, where Δ_2 is the *set of feasible medians*:

$$\Delta_2 = \{(\delta_r, \delta_s) \in \Delta^2 \mid r \leq s\}.$$

In the MJ voting system, Balinski and Laraki [2, 3] consider the lower median of the linguistic individual assessments as collective grade of each alternative when the number of assessments is odd (in MJ the individual assessments are arranged in an increasing manner). Choosing the lower median is not problematic when the number of agents is high, as happens in political elections. However, it can be considered as arbitrary when that number is low, as happens in small size committees.

In order to avoid loss of information, it is convenient to take into account the two medians. This requires to rank order feasible medians in a suitable way. For example, in our setting, (δ_2, δ_3) is clearly better than (δ_3, δ_3), and (δ_3, δ_3) can be considered better than (δ_2, δ_4) because $3+3 = 2+4$, but the dispersion (measured through the range of the subindices) is smaller in the first case than in the second one $(3-3 = 0 < 2 = 4-2)$.

In the next step we propose an appropriate linear order on the set of feasible medians.

4. To order the medians of ordinal proximities obtained by different alternatives in the previous step, consider the linear order $\succeq$ on Δ_2 defined as

$$(\delta_r, \delta_s) \succeq (\delta_t, \delta_u) \;\Leftrightarrow\; \begin{cases} r+s < t+u \\ \text{or} \\ r+s = t+u \;\text{ and }\; s-r \leq u-t, \end{cases} \tag{1}$$

for all $(\delta_r, \delta_s), (\delta_t, \delta_u) \in \Delta_2$.

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is even, then there exist two medians. When the elements of a list are real numbers, the median of that list is usually defined as the arithmetic mean of the two medians. That it is impossible when the elements of the list are abstract objects, as happens when considering ordinal proximities.

It is easy to see that if $r+s=t+u$, then $s-r \leq u-t \Leftrightarrow r \geq t \Leftrightarrow s \leq u$. Notice that $(\delta_r, \delta_r) \succeq (\delta_t, \delta_t) \Leftrightarrow r \leq t$.

5. Finally, the alternatives are ranked according to the weak order $\succcurlyeq$ on X defined as $x_i \succcurlyeq x_j \Leftrightarrow M_i \succeq M_j$.

3.2. Breaking ties

Since some alternatives can share the same median(s), it is necessary to devise a tie-breaking process for ordering the alternatives. We propose to use a sequential procedure based on Balinski and Laraki [2] (see Balinski and Laraki [4] for practical examples). It consists of dropping the median(s) of the respective alternatives that are in a tie, and then select the new median(s) of the remaining ordinal degrees for the corresponding alternatives and applying the procedure given in (1).

Formally, when $M_i = M_j$:

- If m is odd, let $M_i^{(1)}, M_j^{(1)} \in \Delta_2$ be the medians obtained after dropping in $\pi\left(v_i^1, l_g\right), \ldots, \pi\left(v_i^m, l_g\right)$ and $\pi\left(v_j^1, l_g\right), \ldots, \pi\left(v_j^m, l_g\right)$ the ordinal degree appearing in $M_i = M_j$, respectively.
- If m even, let $M_i^{(1)}, M_j^{(1)} \in \Delta_2$ be the medians obtained after dropping in $\pi\left(v_i^1, l_g\right), \ldots, \pi\left(v_i^m, l_g\right)$ and $\pi\left(v_j^1, l_g\right), \ldots, \pi\left(v_j^m, l_g\right)$ the pair of ordinal degrees appearing in $M_i = M_j$, respectively.

Then, the procedure given in (1) is applied again. If $M_i^{(1)} = M_j^{(1)}$, then the process continues with the remaining ordinal degrees for the corresponding alternatives until the ties are broken[3]. It is important noticing that alternatives with different assessments never become in a final tie.

3.3. Properties

We now enumerate some properties that the proposed voting system satisfies.

1. *Anonymity*: All individuals are treated in the same way.
2. *Neutrality*: All alternatives are treated in the same way.
3. *Independence of irrelevant alternatives*: The ranking between two alternatives only depends on the individual assessments obtained by these alternatives, being irrelevant the assessments obtained by other alternatives.

[3]Notice that in the following steps, the number of ordinal degrees is always even.

4. *Unanimity*: If all agents assign the same or a better assessment to an alternative than to another one, then the second alternative cannot be ranked ahead the first one.
5. *Monotonicity*: Given two profiles with the only difference that an alternative receives a better assessment from an agent in the second profile, then that alternative cannot be ranked worse in the the second profile than in the first one.
6. *Replication invariance*: If all agents are replicated a number of times with the same assessments, then the outcome does not change.

4. An illustrative example

Consider five agents assessing the alternatives of $X = \{x_1, x_2, x_3\}$ through the ordered qualitative scale $\mathcal{L} = \{l_1, l_2, l_3, l_4\}$ and the profile

$$\begin{pmatrix} v_1^1 & v_2^1 & v_3^1 \\ v_1^2 & v_2^2 & v_3^2 \\ v_1^3 & v_2^3 & v_3^3 \\ v_1^4 & v_2^4 & v_3^4 \\ v_1^5 & v_2^5 & v_3^5 \end{pmatrix} = \begin{pmatrix} l_3 & l_2 & l_1 \\ l_1 & l_3 & l_4 \\ l_3 & l_3 & l_2 \\ l_4 & l_3 & l_3 \\ l_3 & l_3 & l_4 \end{pmatrix}.$$

In Table 1 the number of linguistic terms obtained for each alternative is summarized.

	x_1	x_2	x_3
l_1	1		1
l_2		1	1
l_3	3	4	1
l_4	1		2

Table 1: Number of linguistic terms obtained for each alternative.

In order to show that the ordinal proximity measure matters, we provide three different cases.

1. Consider the ordinal proximity measure with associated matrix A_{222} appearing in Example 1.

The ordinal proximities between the assessments obtained by the alternatives and l_4 are included in Table 2.

	$\pi\left(v_i^1, l_4\right)$	$\pi\left(v_i^2, l_4\right)$	$\pi\left(v_i^3, l_4\right)$	$\pi\left(v_i^4, l_4\right)$	$\pi\left(v_i^5, l_4\right)$
x_1	δ_2	δ_4	δ_2	δ_1	δ_2
x_2	δ_3	δ_2	δ_2	δ_2	δ_2
x_3	δ_4	δ_1	δ_3	δ_2	δ_1

Table 2: Ordinal proximities between assessments and l_4.

We now arrange the previous ordinal degrees in a decreasing fashion:

$$x_1:\ \delta_1, \delta_2, \delta_2, \delta_2, \delta_4\,; \quad x_2:\ \delta_2, \delta_2, \delta_2, \delta_2, \delta_3\,; \quad x_3:\ \delta_1, \delta_1, \delta_2, \delta_3, \delta_4\,.$$

The three alternatives have the same median, δ_2, i.e., $M_1 = M_2 = M_3 = (\delta_2, \delta_2)$. Then, it is necessary to use the tie-breaking process. After removing the median, we obtain

$$x_1:\ \delta_1, \delta_2, \delta_2, \delta_4\,; \quad x_2:\ \delta_2, \delta_2, \delta_2, \delta_3\,; \quad x_3:\ \delta_1, \delta_1, \delta_3, \delta_4\,.$$

The new medians are $M_1^{(1)} = M_2^{(1)} = (\delta_2, \delta_2)$ and $M_3^{(1)} = (\delta_1, \delta_3)$. By (1), $M_1^{(1)} = M_2^{(1)} \succ M_3^{(1)}$; then, $x_1 \succ x_3$ and $x_2 \succ x_3$. We use again the tie-breaking process with x_1 and x_2; thus, we remove the medians and obtain

$$x_1:\ \delta_1, \delta_4\,; \quad x_2:\ \delta_2, \delta_3\,.$$

The new medians are $M_1^{(2)} = (\delta_1, \delta_4)$ and $M_2^{(2)} = (\delta_2, \delta_3)$. By (1), $M_2^{(2)} \succ M_1^{(2)}$; then, $x_2 \succ x_1$ and, finally, we have $x_2 \succ x_1 \succ x_3$.

2. We now consider the ordinal proximity measure with associated matrix

$$A_{223} = \begin{pmatrix} \delta_1 & \delta_2 & \delta_4 & \delta_6 \\ & \delta_1 & \delta_2 & \delta_5 \\ & & \delta_1 & \delta_3 \\ & & & \delta_1 \end{pmatrix}$$

that can be visualized in Fig. 4.

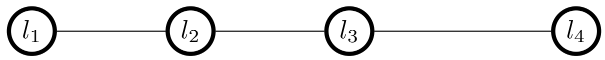

Figure 4: Ordinal proximity measure with associated matrix A_{223}.

The ordinal proximities between the assessments obtained by the alternatives and l_4 are included in Table 3.

	$\pi\left(v_i^1, l_4\right)$	$\pi\left(v_i^2, l_4\right)$	$\pi\left(v_i^3, l_4\right)$	$\pi\left(v_i^4, l_4\right)$	$\pi\left(v_i^5, l_4\right)$
x_1	δ_3	δ_6	δ_3	δ_1	δ_3
x_2	δ_5	δ_3	δ_3	δ_3	δ_3
x_3	δ_6	δ_1	δ_5	δ_3	δ_1

Table 3: Ordinal proximities between assessments and l_4.

We now arrange the previous ordinal degrees in a decreasing fashion:

$$x_1 : \ \delta_1, \delta_3, \delta_3, \delta_3, \delta_6 \, ; \quad x_2 : \ \delta_3, \delta_3, \delta_3, \delta_3, \delta_5 \, ; \quad x_3 : \ \delta_1, \delta_1, \delta_3, \delta_5, \delta_6 \, .$$

The three alternatives have the same median, δ_3, i.e., $M_1 = M_2 = M_3 = (\delta_3, \delta_3)$. After applying the tie-breaking process, the new medians are $M_1^{(1)} = M_2^{(1)} = (\delta_3, \delta_3)$ and $M_3^{(1)} = (\delta_1, \delta_5)$. By (1), $M_1^{(1)} = M_2^{(1)} \succ M_3^{(1)}$; then, $x_1 \succ x_3$ and $x_2 \succ x_3$. Applying the tie-breaking process on x_1 and x_2, we obtain $M_1^{(2)} = (\delta_1, \delta_6)$ and $M_2^{(2)} = (\delta_3, \delta_5)$; then, by (1), we have $x_1 \succ x_2 \succ x_3$.

3. We now consider the ordinal proximity measure with associated matrix

$$A_{224} = \begin{pmatrix} \delta_1 & \delta_2 & \delta_3 & \delta_6 \\ & \delta_1 & \delta_2 & \delta_5 \\ & & \delta_1 & \delta_4 \\ & & & \delta_1 \end{pmatrix}$$

that can be visualized in Fig. 5.

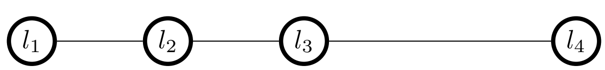

Figure 5: Ordinal proximity measure with associated matrix A_{224}.

The ordinal proximities between the assessments obtained by the alternatives and l_4 are included in Table 4.

	$\pi\left(v_i^1, l_4\right)$	$\pi\left(v_i^2, l_4\right)$	$\pi\left(v_i^3, l_4\right)$	$\pi\left(v_i^4, l_4\right)$	$\pi\left(v_i^5, l_4\right)$
x_1	δ_4	δ_6	δ_4	δ_1	δ_4
x_2	δ_5	δ_4	δ_4	δ_4	δ_4
x_3	δ_6	δ_1	δ_5	δ_4	δ_1

Table 4: Ordinal proximities between assessments and l_4.

We now arrange the previous ordinal degrees in a decreasing fashion:

$$x_1: \ \delta_1, \delta_4, \delta_4, \delta_4, \delta_6 \, ; \quad x_2: \ \delta_4, \delta_4, \delta_4, \delta_4, \delta_5 \, ; \quad x_3: \ \delta_1, \delta_1, \delta_4, \delta_5, \delta_6 \, .$$

The three alternatives have the same median, δ_4, i.e., $M_1 = M_2 = M_3 = (\delta_4, \delta_4)$. After applying the tie-breaking process, the new medians are $M_1^{(1)} = M_2^{(1)} = (\delta_4, \delta_4)$ and $M_3^{(1)} = (\delta_1, \delta_5)$. By (1), we have $M_3^{(1)} \succ M_1^{(1)} = M_2^{(1)}$; then, $x_3 \succ x_1$ and $x_3 \succ x_2$. By using again the tie-breaking process with x_1 and x_2, the new medians are $M_1^{(2)} = (\delta_1, \delta_6)$ and $M_2^{(2)} = (\delta_4, \delta_5)$. Since $M_1^{(2)} \succ M_2^{(2)}$, we finally obtain $x_3 \succ x_1 \succ x_2$.

It is important emphasizing that the three ordinal proximity measures generate different rankings on the set of alternatives. One of the reasons is that the ordinal proximities between l_3 and l_4 are δ_2, δ_3 and δ_4, respectively; and the ordinal proximities between l_2 and l_4 are δ_3, δ_5 and δ_5, respectively. This means that the assessments have different "value" in each case.

If we apply MJ, in the proposal of Balinski and Laraki [3], to the considered profile, the outcome is $x_1 \sim x_2 \succ x_3$, different to the ones obtained in the three cases we have analyzed.

5. Extensions

García-Lapresta and González del Pozo [12] devise a multi-criteria decision-making procedure in the context of uniform qualitative scales. In their proposal, agents evaluate the alternatives regarding several criteria by assigning one or two consecutive terms of the scale to each alternative in each criterion. Weights assigned to criteria are managed through replications of the corresponding ratings, and alternatives are ranked according to the medians of their ratings after the replications.

It is easy to check that the voting system proposed in this paper coincides with the one given in García-Lapresta and González del Pozo [12] when agents assign a single linguistic term to each alternative, only a criterion is considered and the qualitative scale is equipped with the corresponding totally uniform ordinal proximity measure.

It is also easy to extend the voting system proposed in this paper to the cases where there are multiple criteria and agents are allowed to assign one or two consecutive terms of a qualitative scale equipped with a non necessarily uniform ordinal proximity measure. To this purpose, it would be necessary to duplicate each proximity $\pi(l_r, l_g)$ when a single linguistic term l_r is assigned, and consider the two proximities $\pi(l_r, l_g)$ and $\pi(l_{r+1}, l_g)$ when two consecutive linguistic terms, l_r and l_{r+1} are assigned.

Since distinct criteria may have different importance in the global decision, we consider a weighting vector $\boldsymbol{w} = (w_1, \ldots, w_q) \in [0,1]^q$, with $w_1 + \cdots + w_q = 1$, where $C = \{c_1, \ldots, c_q\}$ is the set of criteria. For practical reasons, we assume that these weights have at most two decimals, i.e., the percentages $100 \cdot w_1, \ldots, 100 \cdot w_q$ are integer numbers.

The criteria weighting scheme should follow the replication proposal given by García-Lapresta and González del Pozo [12]: the profiles associated with the criteria are replicated according to the corresponding percentages, $100 \cdot w_1$, $\ldots$, $100 \cdot w_q$. In practice, it should be convenient to calculate the greatest common divisor (gcd) of percentages associated with the weights, and divide each percentage by the gcd. Then, the minimum number of replications of each profile is obtained.

Example 2. Consider that five agents assess the alternatives of $X = \{x_1, x_2, x_3\}$ regarding the criteria of $C = \{c_1, c_2, c_3\}$, with associated weighting vector $\boldsymbol{w} = (0.3, 0.3, 0.4)$, through the ordered qualitative scale $\mathcal{L} = \{l_1, l_2, l_3, l_4\}$ equipped with the ordinal proximity measure with associated matrix A_{223}.

Consider the profiles V_1, V_2 and V_3 corresponding to the criteria c_1, c_2 and

c_3, respectively:

$$V_1 = \begin{pmatrix} l_3 & l_2 & l_1 \\ l_1 & l_3 & l_4 \\ l_3 & l_3 & l_2 \\ l_4 & l_3 & l_3 \\ l_3 & l_3 & l_4 \end{pmatrix}, \quad V_2 = \begin{pmatrix} l_4 & l_2 & l_1 \\ l_3 & l_4 & l_4 \\ l_4 & l_1 & l_3 \\ l_4 & l_3 & l_1 \\ l_1 & l_4 & l_1 \end{pmatrix}, \quad V_3 = \begin{pmatrix} l_1 & l_2 & l_1 \\ l_1 & l_2 & l_4 \\ l_3 & l_1 & l_3 \\ l_4 & l_1 & l_1 \\ l_1 & l_2 & l_4 \end{pmatrix}.$$

Taking into account the percentages $100 \cdot w_1 = 30$, $100 \cdot w_2 = 30$ and $100 \cdot w_3 = 40$, since $\gcd(30, 30, 40) = 10$, the profiles corresponding to each criterion should be replicated $30/10 = 3$, $30/10 = 3$ and $40/10 = 4$ times, respectively.

After some computations, in the first step we obtain the following medians: $M_1 = M_3 = (\delta_3, \delta_3)$ and $M_2 = (\delta_5, \delta_5)$. By (1), $M_1 = M_3 \succ M_2$; then, $x_1 \succ x_2$ and $x_3 \succ x_2$. Then, it is necessary to use the tie-breaking process. After removing the medians, the new medians are $M_1^{(1)} = M_3^{(1)} = (\delta_3, \delta_3)$ and by using again the tie-breaking process we obtain $M_1^{(2)} = (\delta_3, \delta_3)$ and $M_3^{(2)} = (\delta_3, \delta_5)$. Since $M_1^{(2)} \succ M_3^{(2)}$, we finally have $x_1 \succ x_3 \succ x_2$.

6. Concluding remarks

This paper proposes a voting system in the setting of ordered qualitative scales non-necessarily uniform. The novelty of the proposal lies on the purely ordinal approach, where the ordinal proximities among the linguistic terms of the qualitative scale are essential for obtaining the ranking on the set of alternatives generated by the individual assessments. This ranking is based on the median(s) of the ordinal proximities between the individual assessments and the highest linguistic term of the scale, through an appropriate linear order on the set of feasible medians.

Although this aggregation procedure only takes into account the proximities among the individual assessments and the highest linguistic term of the scale, all the ordinal proximities among the linguistic terms of the qualitative scale are relevant. This is due to the fact that the degree of proximity between each linguistic term and the highest linguistic term of the scale depends on the rest of ordinal comparisons, as shown in the illustrative example included in Section 4.

In the aggregation procedure and in the proposed tie-breaking process, when the number of corresponding assessments is even, we have considered the two medians. This avoids loss of information and it is also a novelty with respect to other ordinal approaches ([2, 3]).

As shown in Section 4, given an ordered qualitative scale, the outcome of the voting system could depend on the ordinal proximity measure fixed for describing the proximities among the linguistic terms of the scale. Then, a relevant problem is how to determine what is the most appropriate ordinal proximity measure in that scale. It is not a trivial problem and the solution may depend on the society where the voting system is applied. If several experts provide their opinions about the mentioned proximities, then an aggregation procedure is needed. This issue has been analyzed in García-Lapresta et al. [13].

The properties included in Subsection 3.3 ensure that the proposed voting system is suitable for group decision making applications in the setting of ordered qualitative scales.

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