

Outliers and misleading leverage effect in asymmetric GARCH-type models

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Abstract

This paper illustrates how outliers can affect both the estimation and testing of leverage effect by focusing on the TGARCH model. Three estimation methods are compared through Monte Carlo experiments: Gaussian Quasi-Maximum Likelihood, Quasi-Maximum Likelihood based on the Student-t likelihood and Least Absolute Deviation method. The empirical behavior of the t-ratio and the Likelihood Ratio tests for the significance of the leverage parameter is also analyzed. Our results put forward the unreliability of Gaussian Quasi-Maximum Likelihood methods in the presence of outliers. In particular, we show that one isolated outlier could hide true leverage effect whereas two consecutive outliers bias the estimated leverage coefficient in a direction that crucially depends on the sign of the first outlier and could lead to wrongly reject the null of no leverage effect or to estimate asymmetries of the wrong sign. By contrast, we highlight the good performance of the robust estimators in the presence of one isolated outlier. However, when there are patches of outliers, our findings suggest that the sizes and powers of the tests as well as the estimated parameters based on robust methods may still be distorted in some cases. We illustrate these results by analyzing two series of daily returns, namely the exchange rate US Dollar/British Pound and futures contracts of Natural gas, as examples of series which include one isolated and two consecutive outliers, respectively.

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1 Introduction

One of the empirical stylized facts about the series of financial returns is the leverage effect, which refers to the asymmetric response of volatility to positive and negative past returns. In particular, the volatility tends to be higher following negative return shocks (‘bad’ news) than following positive shocks (‘good’ news) of the same magnitude. This feature conveys a generally negative cross-correlation between lagged asset returns and volatility; see Black (1976) who originally put it forward using the debt-to-equity ratio and Engle (2011) who provides the economic underpinning of volatility asymmetries following simple asset pricing theory. See also Hibbert *et al.* (2008) for a behavioral explanation of the negative asymmetric return–volatility relationship.

In the econometric literature there have been several methods proposed to represent and estimate conditional heteroscedasticity and leverage effect, like the asymmetric GARCH-type models (see the review in Rodríguez and Ruiz (2012)), the non-parametric high-frequency methods in Andersen *et al.* (2001) and the observation-driven models proposed by Creal *et al.* (2013) and Harvey (2013), to name but a few. In practice, when these methods are applied to estimate and test for the leverage effect, mixed results come up. For example, Andersen *et al.* (2001) found statistically significant leverage effect for most of the stocks returns of the DJIA stock market index by using realized volatilities, although they point out that this effect has a marginal economic importance. In turn, Zivot (2009) and Rodríguez and Ruiz (2012) also report significant leverage effect by applying GARCH-type methods to different series of returns, including a particular asset stock, a stock market index and exchange rates. However, Ait-Sahalia *et al.* (2013) discuss what they call *the leverage effect puzzle*, which relies on the fact that the empirical correlation between returns and changes in volatility estimated from high frequency data becomes nearly zero for most assets tested, despite the many economic reasons for expecting such correlation to be negative. In Energy markets, it is also common to find different results concerning the leverage effect. For example, Kristoufek (2014) finds the inverse leverage effect (positive correlation between returns and volatility) in future prices of the Natural gas by using correlation-based methods for non-stationary series, whereas Chkili *et al.* (2014) find the standard leverage effect when asymmetric and long-memory models are estimated to spot and future returns of the same commodity.

The mixed results on the sign and the significance of the leverage effect could be partly explained, among other reasons, by the harmful effect of the extreme observations usually encountered in the returns. For instance, it is already well known that the presence of outliers can have misleading effects on the identification of both conditional heteroscedasticity and leverage effect; see Carnero *et al.* (2007) and Carnero *et al.* (2016), respectively. Moreover, it is also proved that the presence of outliers renders the Gaussian Quasi Maximum Likelihood (QML) estimators unreliable in symmetric GARCH models; see Sakata and White (1998), Mendes (2000), Carnero *et al.* (2007) and Muler and Yohai (2008), among others. In this context, robust estimators based on maximizing non-Gaussian heavy-tailed likelihoods have been proposed and their main properties have been established; see, for instance, Newey and Steigerwald (1997), Sakata and White (1998), Berkes and Horvath (2004) and Fan *et al.* (2014). Another robust alternative for GARCH-type models is the log-transform-based Least Absolute Deviations (LAD) estimator proposed by Peng and Yao (2003) and further discussed in Huang *et al.* (2008). Related works also include Pan *et al.* (2008), Francq and Zakoian (2013) and Hill (2015), among others. Alternatively, some authors deal with this problem by applying methodologies based on detecting and correcting outliers; see, for example, Laurent *et al.* (2016) and the references therein.

In this paper, we face the problem of how outliers can affect the inference on the leverage parameter in asymmetric GARCH-type models: Does the sign of the outliers matter? How much the size of the outliers worsen the results? Is the effect of one isolated outlier comparable to that of consecutive outliers? Which estimation method is more resistant to outliers? To answer these questions, we conduct an extensive Monte Carlo study that includes models with high, low and none leverage and different types of outliers (isolated and in patches, positive and negative, big and small). For each setting, we compare the performance of three estimation methods: QML, Quasi-Maximum Likelihood estimator based on maximizing the Student- t likelihood (QML- t) and LAD. We also analyze the size and power of the t -ratio and the Likelihood Ratio tests for the significance of the leverage parameter. Among the several asymmetric GARCH-type models proposed in the literature, we focus on the Threshold GARCH (TGARCH) model of Zakoian (1994) since, according to Rodríguez and Ruiz (2012), this is more flexible than its competitors to properly represent the dynamics of financial returns.

Previous works related to our paper are Pan *et al.* (2008), who establish the asymptotic properties of both QML and LAD estimators for a general model that nests the TGARCH and compare, for a particular parametrization, the finite sample performance of both estimators, and Francq and Zakoian (2013), who compare the asymptotic relative efficiencies of QML and LAD estimators for a model that also nests the TGARCH. Our paper provides new insights on the topic in two ways. First, we also evaluate the QML-t estimator, that turns out to be more robust than LAD in most cases, and second, our Monte Carlo study is exhaustive enough to cover the multiple problems often found in real data where different type of outliers may come up.

Our results show that QML-based methods become unreliable in the presence of outliers and could lead to either hide true leverage effects (when there is one isolated outlier) or detect spurious leverage or leverage of the wrong sign (in the presence of two consecutive outliers). As expected, the robust estimators considered (QML-t and LAD) always outperform QML although they are still slightly biased in the presence of big consecutive outliers. In this case, the bias direction crucially depends on the sign of the first outlier and could lead to wrongly reject the null of no leverage effect. These results are further enhanced in our empirical application, where two particular series of financial returns are analyzed, namely a daily series of the exchange rate US Dollar/British Pound including one isolated negative outlier and a daily series of futures contracts of Natural gas including two consecutive outliers of opposite sign being the first one positive.

The rest of the paper is organized as follows. Section 2 reviews the TGARCH model and describes the three estimation methods to be analyzed (QML, QML-t and LAD) and the two significance tests considered (t-ratio and Likelihood Ratio tests). Section 3 is devoted to the finite sample performance of these estimators and tests in the presence of outliers for different parameter sets and different types of outliers. Section 4 illustrates our main results with an empirical application based on the two series of daily returns mentioned above. Finally, Section 5 concludes the paper with a summary of the main conclusions.

2 Statistical inference in the TGARCH model

2.1 TGARCH model: definition and main properties

As Rodríguez and Ruiz (2012) point out, the TGARCH model is an appropriate and flexible GARCH-type model to represent the features and dynamic properties of financial returns, namely excess kurtosis, conditional heteroscedasticity and leverage effect. In this model, the series of demeaned returns, y_t , is specified by the following equation:

$$y_t = \sigma_t \varepsilon_t, \quad (1)$$

where σ_t is the volatility and ε_t is a sequence of independent and identically distributed random variables with zero mean and unit variance. To accommodate the asymmetric relationship between past returns and volatility, σ_t is parametrized as a function of both the magnitude and the sign of past returns. In particular, the equation for the volatility in the TGARCH(1,1) model, as given in Rodríguez and Ruiz (2012), is the following¹

$$\sigma_t = \alpha_0 + \alpha |y_{t-1}| + \beta\sigma_{t-1} + \delta y_{t-1}. \quad (2)$$

When y_{t-1} is positive, the volatility response is linear in y_{t-1} with slope $(\delta + \alpha)$ but if y_{t-1} is negative, the slope of the response is $(\delta - \alpha)$. Thus, the volatility can respond asymmetrically to rises and falls in stock prices and the value of δ is expected to be negative. Under the constraints $\alpha_0 > 0$, $\beta \geq 0$ and $\alpha \geq |\delta|$, σ_t is always positive and represents the conditional standard deviation of y_t . Moreover, the model is covariance stationary if $\delta^2 < 1 - \alpha^2 - \beta^2 - 2\alpha\beta\nu_1$, where $\nu_1 = E|\varepsilon_t|$.

An advantage of the TGARCH model is that it parametrizes the conditional standard deviation rather than the conditional variance. This makes it easier to work out analytical expressions for its unconditional higher-order moments and cross-moments; see the results in He and Teräsvirta (1999), Hwang and Basawa (2004) and He *et al.* (2008). Moreover, as the TGARCH model involves absolute values rather than squares,

¹There are other (equivalent) parametrizations of the TGARCH(1,1) model proposed in the econometric literature and in the econometric software packages; see, for instance, Zakoian (1994), Hentschel (1995), He and Teräsvirta (1999) and He *et al.* (2008). Hence, in order to compare estimates and standard errors from different papers and/or different software packages, it is important to be very careful about which parametrization is being used in each case. It is also important to be aware that the GJR model, proposed by Glosten *et al.* (1993), is sometimes erroneously referred to as TGARCH.

it is expected to be less sensitive to extreme observations than similar models involving conditional variances and squared returns, like the GJR and the QARCH models in Glosten *et al.* (1993) and Sentana (1995), respectively.

Another advantage of the TGARCH model is that it contains, as a special case, a model without leverage, namely the absolute-value GARCH (AVGARCH) model of Taylor (1986) and Schwert (1989), that comes up by taking $\delta = 0$ in (2). Thus, model selection can be easily performed by testing the significance of the leverage parameter δ with the usual t -ratio test and/or the Likelihood Ratio test. On the other hand, the TGARCH model is a particular case of the Asymmetric Power ARCH (A-PARCH) model proposed by Ding *et al.* (1993). Hence, the asymptotic theory in Pan *et al.* (2008) and Francq and Zakoian (2013) regarding estimation and hypothesis testing in the A-PARCH model can be applied. By contrast, the asymptotic theory for other popular asymmetric GARCH-type models is scarce; see, for instance, the results for the estimation of EGARCH models in Straumann and Mikosch (2006), Wintenberger (2013) and Hafner and Linton (2017).

2.2 Gaussian QML estimation

The TGARCH(1,1) model defined in equations (1)-(2) can be estimated by maximizing the conditional log-likelihood function, which, given initial values y_0 and σ_0 , is as follows

$$L(\theta) = \sum_{t=1}^T l_t(\theta) = \sum_{t=1}^T \left\{ -\frac{1}{2} \log \sigma_t^2 + \log f\left(\frac{y_t}{\sigma_t}\right) \right\}, \quad (3)$$

where $\theta = (\alpha_0, \alpha, \beta, \delta)'$ denotes the parameter vector to be estimated and $f(\cdot)$ is the probability density of ε_t . In particular, if ε_t is assumed to be $N(0, 1)$, the corresponding Gaussian log-likelihood function, denoted as $L_G(\theta)$, will come up, namely

$$L_G(\theta) = -\frac{T}{2} \log 2\pi - \frac{1}{2} \sum_{t=1}^T \left(\log \sigma_t^2 + \frac{y_t^2}{\sigma_t^2} \right).$$

The resultant estimator obtained from maximizing $L_G(\theta)$ is the well-known QML estimator. This estimator, that will be denoted as $\hat{\theta}_{QML}$, is the most commonly used one for GARCH-type models and, in particular, for the TGARCH model introduced above; see, for instance, Zivot (2009) and Francq and Zakoian (2010, chapter 10). Moreover,

this estimator is provided in most software packages, such as E-VIEWS, G@RCH4.0, MFE MATLAB Toolbox, SAS, Stata, Splus and R. Obviously, if the true distribution of ε_t is $N(0, 1)$, the resultant estimator will be the Maximum Likelihood estimator.

Pan *et al.* (2008) show that QML is consistent and asymptotically normal for an asymmetric power-transformed GARCH model that includes as a particular case the TGARCH model, provided that ε_t is symmetrically distributed with $E\varepsilon_t^2 = 1$ and $E\varepsilon_t^4 < \infty$ and some regularity assumptions hold. In such a framework, we can approximate the asymptotic variance of $\widehat{\theta}_{QML}$ by the so-called “sandwich” estimator

$$Var(\widehat{\theta}_{QML}) \approx H(\widehat{\theta}_{QML})^{-1} B(\widehat{\theta}_{QML}) H(\widehat{\theta}_{QML})^{-1}, \quad (4)$$

where $H(\theta)$ denotes the Hessian matrix of the log-likelihood and $B(\theta)$ is the inner product of the gradient (or score) of the log-likelihood.

On the other hand, Muler and Yohai (2008) show that the QML estimator could be regarded as an M-estimator defined as

$$\widehat{\theta}_{QML} = \arg \min_{\theta} M_{0,T}(\theta),$$

where $M_{0,T}$ is given by

$$M_{0,T}(\theta) = \frac{1}{T-2} \sum_{t=2}^T \rho_0(x_t - \log \sigma_t^2(\theta)),$$

where $x_t = \log(y_t^2)$ and ρ_0 is the auxiliary function defined as

$$\rho_0(x) = \log(\sqrt{2\pi}) + \frac{1}{2}(e^x - x). \quad (5)$$

This function as well as its 1st derivative, $\rho'_0 = \frac{1}{2}(e^x - 1)$, are both unbounded, rendering QML not robust, i.e., a few outliers can have a large influence on this estimator.

The lack of robustness of the QML estimator in symmetric GARCH models is already well documented; see Sakata and White (1998), Mendes (2000), Carnero *et al.* (2007) and Muler and Yohai (2008), among others. In Section 3 we analyze the effect of the outliers on QML in TGARCH models through Monte Carlo experiments. In particular, we investigate if the sign of the outliers, that is irrelevant in symmetric models, makes any difference when estimating the parameters of asymmetric models.

2.3 QML-t estimation

In order to gain resistance against outliers, some authors propose to use estimators based on maximizing non-Gaussian heavy-tailed log-likelihoods; see, for instance, Sakata and White (1998). Actually, in the seminal paper of Nelson (1991), the EGARCH model is estimated by maximizing the log-likelihood function in (3) assuming that ε_t follows a GED distribution normalized to have zero mean and unit variance. Another common practise is to maximize the conditional log-likelihood in (3) computed as if ε_t followed a Student-t distribution with ν degrees of freedom normalized to have zero mean and unit variance. In such a case, the log-likelihood function, denoted as L_{Stud} , becomes:

$$L_{Stud}(\theta) = T \log \left(\frac{\Gamma((\nu+1)/2)}{\sqrt{\pi(\nu-2)}\Gamma(\nu/2)} \right) - \frac{1}{2} \sum_{t=1}^T \left[\log \sigma_t^2 + (\nu+1) \log \left(1 + \frac{1}{\nu-2} \frac{y_t^2}{\sigma_t^2} \right) \right],$$

where the vector parameter is $\theta = (\alpha_0, \alpha, \beta, \delta, \nu)'$. The resultant estimator obtained from maximizing $L_{Stud}(\theta)$ is the so-called QML-t estimator denoted as $\hat{\theta}_{QML-t}$.

As far as we know, no asymptotic theory exists for QML-t estimation in the context of asymmetric GARCH models. Hence, we will assume the usual practice of researchers using GARCH-type models and we will approximate its asymptotic variance by the “sandwich” estimator in (4) replacing $\hat{\theta}_{QML}$ by $\hat{\theta}_{QML-t}$. Empirical applications performing QML-t estimation of the TGARCH model can be found in Zivot (2009), Francq and Zakoian (2010, chapter 10) and Rodríguez and Ruiz (2012).

Muler and Yohai (2008) point out that the QML-t estimator also corresponds to an M-estimator defined as

$$\hat{\theta}_{QML-t} = \arg \min_{\theta} M_{1,T}(\theta),$$

where $M_{1,T}$ is given by

$$M_{1,T}(\theta) = \frac{1}{T-2} \sum_{t=2}^T \rho_{1,\nu}(x_t - \log \sigma_t^2(\theta)),$$

where $x_t = \log(y_t^2)$ and $\rho_{1,\nu}$ is the following auxiliary function

$$\rho_{1,\nu}(x) = -\frac{x}{2} + \frac{\nu+1}{2} \log \left(1 + \frac{e^x}{\nu-2} \right) - \log \Gamma \left(\frac{\nu+1}{2} \right) + \log \left(\sqrt{\pi(\nu-2)} \Gamma \left(\frac{\nu}{2} \right) \right). \quad (6)$$

This function is unbounded but its 1st derivative, $\rho'_{1,\nu}(x) = \frac{\nu e^x - \nu + 2}{2(\nu - 2 + e^x)}$, is bounded; thus, QML-t is expected to be more robust than QML, although it can still be affected by some type of outliers.

Carnero *et al.* (2007) analyze the robustness of QML-t for symmetric ARCH and GARCH models. In Section 3 we analyze the finite sample behavior of the QML-t estimator, as compared to QML, in TGARCH models in the presence of outliers.

2.4 LAD estimation

As an alternative to QML, Peng and Yao (2003) propose the LAD estimator and they show that, in the context of GARCH models, this estimator is robust to heavy tails of the innovation distribution and it is asymptotically normal and unbiased under mild conditions for the error distribution. Huang *et al.* (2008) perform a comparison between both QML and LAD, where the latter is viewed as a quasi-maximum likelihood estimator based on the hypothesis that the log-squared innovations follow a Laplace distribution. Pan *et al.* (2008) establish the asymptotic properties of both QML and LAD estimators for an asymmetric power-transformed GARCH model that nests the TGARCH. They also compare, for a particular parametrization, the finite sample performance of both estimators and show that the LAD is more accurate than QML for heavy-tailed errors.

For the LAD estimator to be applied, the model should be reparametrized in such a way that the median (instead of the mean) of the squared innovations is equal to 1, while the mean of the innovations remains unchanged and equal to 0. In particular, for the TGARCH model we are interested in, the reparametrization is as follows. Let $M = \text{median}(\varepsilon_t^2) > 0$ and let $\sigma_t^* = M^{1/2}\sigma_t$ and $\varepsilon_t^* = M^{-1/2}\varepsilon_t$, so that $\text{median}(\varepsilon_t^{*2}) = 1$. Then, the TGARCH model defined in equations (1)-(2) may now be expressed as

$$\begin{aligned} y_t &= \sigma_t^* \varepsilon_t^*, \\ \sigma_t^* &= \alpha_0^* + \alpha^* |y_{t-1}| + \beta \sigma_{t-1}^* + \delta^* y_{t-1}, \end{aligned}$$

where $\alpha_0^* = M^{1/2}\alpha_0$, $\alpha^* = M^{1/2}\alpha$, $\delta^* = M^{1/2}\delta$ and the parameter vector to be estimated is $\theta^* = (\alpha_0^*, \alpha^*, \beta, \delta^*)'$. Notice that, under this new parametrization, the parameters α_0^* , α^* and δ^* differ from those in the old setting by a common positive constant factor while the parameter β remains unchanged. The LAD estimator is based on the regression equation for the log-squared transformation, namely

$$\log y_t^2 = \log \sigma_t^{*2} + \log \varepsilon_t^{*2},$$

where the error terms $\log \varepsilon_t^{*2}$ are independent and identically distributed with median equals to 0. Now, the LAD estimator of θ^* is defined as:

$$\widehat{\theta}_{LAD}^* = \arg \min_{\theta} \sum_{t=2}^T |\log y_t^2 - \log \sigma_t^{*2}(\theta)| .$$

Notice that, since some of the parameters estimated by LAD differ from the original parameters by a common positive factor, to compare the LAD estimates to those obtained by other methods, the former should be corrected by the corresponding scale factor $M^{-1/2}$. Moreover, for the standard errors of the LAD estimators to be computed, it is required to estimate the density of the log-squared innovations by the kernel method. Hence, the additional problem of choosing the bandwidth and the kernel function to be used should be addressed; see Pan *et al.* (2008). Alternatively, Zhu and Ling (2015) use a random weighting method to approximate the limiting distribution of the LAD estimator.

Interestingly, Muler and Yohai (2008) show that the LAD estimator can also be regarded as an M-estimator defined as

$$\widehat{\theta}_{LAD}^* = \arg \min_{\theta} M_{2,T}(\theta),$$

with

$$M_{2,T}(\theta) = \frac{1}{T-2} \sum_{t=2}^T \rho_2(x_t - \log \sigma_t^2(\theta)),$$

where $x_t = \log(y_t^2)$ and ρ_2 is the following auxiliary function

$$\rho_2(x) = |x - u_0|, \tag{7}$$

where $u_0 = \log M$. The function ρ_2 in (7) is unbounded but $\rho_2'(x) = \text{sign}(x - u_0)$ is bounded. Thus, as QML-t, this estimator is more robust than QML, although large outliers can still have a strong effect on it, as we will see in Section 3.

To summarize, Figure 1 plots the functions ρ_0 , $\rho_{1,v}$ and ρ_2 in (5), (6) and (7), respectively, as well as their 1st derivatives. For the QML-t, we consider $v = 3$ and for the LAD we take $u_0 = 0$. This figure clearly shows up the differences and similarities between the three functions: all of them are unbounded but, whereas ρ_0' is still unbounded, $\rho_{1,v}'$ and ρ_2' are both bounded, making the corresponding estimators, QML-t and LAD, more robust to outliers than QML.

INSERT FIGURE 1 HERE

2.5 Testing the significance of the leverage coefficient

The goal of estimating asymmetric GARCH models, like the TGARCH, is capturing the leverage effect through the leverage parameter δ . Hence, once the model has been estimated, the following significance testing problem is usually considered

$$H_0 : \delta = 0 \text{ against } H_1 : \delta \neq 0.$$

The most widely used test for this problem is the t -test based on the QML estimation of the model. This test employs the statistic $\mathbf{t} = \widehat{\delta}/(s.e.(\widehat{\delta}))$, where the standard error is computed as the square root of the corresponding diagonal element of the approximate asymptotic variance (4); see, for instance, Zivot (2009). At the asymptotic significance level α , the standard rejection region is thus

$$\{|\mathbf{t}| > \Phi^{-1}(1 - \alpha/2)\}. \quad (8)$$

where $\Phi(\cdot)$ is the cumulative distribution function of the standard normal. Alternatively, the Likelihood Ratio (LR) test can be performed. This test employs the statistic $LR = -2(L_c - L_u)$, where L_c is the constrained log-likelihood (estimating the model under the null) and L_u is the unconstrained log-likelihood. At the asymptotic significance level α , the standard rejection region of the LR test is

$$\{LR > \chi_1^2(1 - \alpha)\}, \quad (9)$$

where $\chi_1^2(1 - \alpha)$ is the $(1 - \alpha)$ -quantile of the χ_1^2 distribution. Other approaches are possible; see, for instance, the Wald test based on LAD estimation in Pan *et al.* (2008).

Despite the popularity of the t and LR tests, Francq and Zakoian (2010) warn about using their standard forms in (8) and (9), respectively, to test for the significance of GARCH coefficients. In particular, they point out that the standard version of the t -test in (8) is not of asymptotic level α but only $\alpha/2$. In turn, they propose modified versions of both tests which are appropriate for statistical inference in symmetric GARCH models. Whether those results apply to asymmetric GARCH models, like the TGARCH, is still an open question, but we conjecture that this would be the case. Furthermore, the problem will be enhanced in the presence of outliers, as we will see in the next section, where we analyze the empirical behavior of the standard versions of the t and LR tests in (8) and (9), respectively, in the context of TGARCH models contaminated with outliers.

3 Monte Carlo simulation

In this section we report the results on several Monte Carlo experiments comparing the performance of the three estimators described in Section 2, for two TGARCH models and a symmetric AVGARCH model contaminated with one isolated outlier and with two consecutive outliers of different sizes and sign. As far as we know, this is the first time that these three estimators are compared in such a framework. Previous work on standard GARCH models include Peng and Yao (2003), who compare QML and LAD in ARCH(2) and GARCH(1,1) models, and Muler and Yohai (2008), who compare, in the GARCH(1,1) model, the behavior of QML, QML-t (with fixed degrees of freedom $\nu = 3$) and LAD. Also, Pan *et al.* (2008) compare numerically QML and LAD for a general Power-Transformed and Threshold GARCH (PTTGARCH(1,1)) model that allows for leverage effect. However, their comparison is based on the average absolute error of all the model parameters, a criteria that misses the sign of the leverage parameter and does not allow to disentangle the effect of outliers on each parameter.

3.1 Data generation and estimators

In our Monte Carlo experiments, the data are generated by the following scheme:

$$y_t = \sigma_t \varepsilon_t,$$

$$z_t = \begin{cases} y_t + \omega & \text{if } t = \tau, \dots, \tau + k - 1 \\ y_t & \text{otherwise} \end{cases}$$

with $\varepsilon_t \sim N(0, 1)$, so that $E\varepsilon_t^2 = 1$ and $M = \text{median}(\varepsilon_t^2) = 0.454936$. The volatility process σ_t is generated by equation (2) with three true parameter sets:

Parameter set 1	$\alpha_0 = 0.0475, \alpha = 0.15, \beta = 0.83, \delta = -0.05$
Parameter set 2	$\alpha_0 = 0.0746, \alpha = 0.12, \beta = 0.825, \delta = -0.11$
Parameter set 3	$\alpha_0 = 0.0746, \alpha = 0.12, \beta = 0.825, \delta = 0$

The first two parameter sets define TGARCH models capturing leverage effects of different magnitude, while the latter defines a symmetric AVGARCH (without leverage) that is nested in the TGARCH model with parameter set 2. This allows us to evaluate the properties of some statistical tests for model selection, such as the LR test. Moreover,

the parameters values of the two TGARCH models have been chosen to resemble the values usually encountered in real empirical applications and to make them somehow comparable, since their marginal variance and kurtosis are very similar. In particular, with these parameter sets, the marginal variance, the kurtosis and the 1st-order cross-correlation between squared and past returns are the following:

	Marginal variance	Kurtosis	1st-order cross-correlation
Parameter set 1	1	5.389	-0.0647
Parameter set 2	1	5.727	-0.135
Parameter set 3	0.9175	3.492	0

Since we are interested in the effect of both the size and the sign of the outliers, we consider positive and negative outliers and we take the case of no outliers as a benchmark. In particular, the sizes of the outliers considered are the following: $\omega = \{0, \pm 10, \pm 20, \pm 30, \pm 40, \pm 50\}$. For each model, we generate 1000 independent samples of size $T = 1000$.² Then we contaminate each sample, first, with $k = 1$ isolated outlier of size ω at time $t = T/2$ and second, with $k = 2$ consecutive outliers of the same size but opposite signs placed at $t = \{T/2, T/2 + 1\}$. For instance, we contaminate with a negative outlier ($\omega = -10$) at time $t = 500$ and a second one positive ($\omega = 10$) at time $t = 501$ and then we repeat the experiments by contaminating first with a positive outlier ($\omega = 10$) at time $t = 500$ and a second one negative ($\omega = -10$) at time $t = 501$. Then, we repeat this procedure with each size of the outlier considered. For each replicate, we compute the estimates of the parameters $(\alpha_0, \alpha, \beta, \delta)$ by the three methods explained above and we also compute their average over all replicates. For QML and QML-t estimates, we also compute, for each replicate, the asymptotic standard deviation of each estimator and the corresponding t-ratios to test for the significance of each parameter. For comparison purposes, the LAD estimates are corrected by the corresponding scale factor $M^{-1/2}$ (see Section 2.4). Moreover, for QML-t, the parameter v is also estimated.

3.2 Simulation results on estimation

As we are interested in the possible effect of the outliers on the leverage effect, we focus our discussion on the Monte Carlo results for the parameter δ . Actually, our results for

²The results for $T = \{500, 5000\}$, not displayed to save space, are available upon request.

the parameters α_0 , α and β (not displayed here to save space) resemble those obtained in Carnero *et al.* (2007, 2012) for symmetric GARCH models. In particular, QML always overestimates α_0 and underestimates β in the presence of outliers, either isolated or in patches, and it overestimates α if there are two consecutive outliers. On the other hand, QML-t is robust to one isolated outlier without losing the good properties of QML for uncontaminated series, but it fails to be robust to patches of big outliers, especially when estimating parameters α and β . Moreover, we have also checked that the sign of the outliers does not make a great difference on estimating the parameters α_0 , α and β ; it is only the magnitude of the outlier rather than its sign what makes the difference. However, as Figures 2 and 3 will show, this is not the case when estimating the leverage parameter δ , where the sign of the outliers is essential.

Figure 2 displays, in its left-hand side panels, the Box-plots of the QML estimates of δ for the three models considered (one for each row) in series contaminated with one single outlier of size ω for all the sizes considered, from the most negative value ($\omega = -50$) to the highest positive one ($\omega = 50$). The central panels and the right-hand side panels of Figure 2 display the same plots for the QML-t and LAD estimates, respectively. In each plot, the horizontal line represents the true value of δ . Figure 3 displays similar plots for series contaminated with two consecutive outliers of the same magnitude but opposite sign for all the outlier sizes considered. In particular, the labels in the x -axis of these plots represent the sign and size of the first outlier, being the 2nd one of the same magnitude but opposite sign. For instance, the Box-plot labelled as “-50” represents the distribution of the 1000 estimated values of δ from 1000 series contaminated with two consecutive outliers, namely $\{-50, 50\}$, placed at $t = \{500, 501\}$, while the Box-plot labelled as “50” corresponds to the estimated values of δ from 1000 series contaminated with two consecutive outliers, namely $\{50, -50\}$, placed at $t = \{500, 501\}$.

INSERT FIGURE 2 HERE

INSERT FIGURE 3 HERE

Several conclusions emerge from these figures. First, QML is not robust to the presence of moderate-large outliers, as expected: both the bias and dispersion of QML estimates of δ increase with the size of the outliers. Actually, in the presence of moderate-large outliers, the estimates can take any value within the admissible parameter space,

rendering QML unreliable. Moreover, in models with leverage effect ($\delta \neq 0$), the presence of one isolated outlier of either sign pushes QML estimate of δ towards zero and hence could hide true leverage effect. This result agrees with the conclusions in Carnero *et al* (2016) regarding the identification of leverage based on the cross-correlogram between past returns and current squared returns. Another remarkable feature is that, in general, two consecutive outliers are more harmful in QML than one isolated outlier of the same magnitude. Comparing Figures 2 and 3, we can see that, for a given model, both the bias and the interquartile range of QML estimates are always larger in the presence of two consecutive outliers than in the presence of one isolated outlier of the same size. However, the effect of two consecutive outliers on QML estimates of δ clearly depends on the sign of the first outlier. If this is negative, QML underestimates δ and so it could yield spurious asymmetries (in models with no leverage) or it could overestimate the magnitude of the true leverage. By contrast, δ is overestimated in series contaminated with two consecutive outliers where the 1st one is positive. Hence, in this case, QML could estimate spurious asymmetries (in models with no leverage) or it could either estimate asymmetries of the wrong sign or even no asymmetries in models with true leverage effect. Again, this agrees with the results in Carnero *et al.* (2016) regarding the impact of consecutive outliers in the identification of the leverage effect.

Figures 2 and 3 also show that both QML-t and LAD always outperform QML in the presence of outliers, as expected. Moreover, in the presence of one isolated outlier, both estimators perform quite well, even if the outlier is very large. However, they become slightly downward biased in the presence of two big consecutive outliers with the first one being negative. By contrast, when there are no outliers, QML is the best one, as expected, whereas LAD is the worst. Noticeably, QML-t does not lose much efficiency with respect to QML in this case; a similar finding is reported in Muler and Yohai (2008) for standard GARCH models.

In order to better appreciate the differences between QML-t and LAD, in Figure 4 we compare in more detail the results from these two estimators for some selected outlier sizes, namely $\omega = \{0, \pm 10, \pm 20, \pm 30\}$. As we can see, QML-t outperforms LAD in all cases, but in the presence of two consecutive outliers, it still suffers from some biases, especially if the 1st outlier is negative. Moreover, in models with no leverage or even with low leverage, the presence of two consecutive outliers, being the first one

positive, pushed QML-t estimated δ upwards leading to a possible erroneous detection of inverse leverage. Hence, there is still place to improve robustness in the estimation of asymmetric GARCH-type models, a topic that is left for further research.

INSERT FIGURE 4 HERE

3.3 Simulation results on significance tests

In this section we analyze the empirical properties of both the t -test and LR test using the standard rejection regions (8) and (9), respectively, to test $H_0 : \delta = 0$ against $H_1 : \delta \neq 0$. We only perform this analysis with the two likelihood-based estimators, namely QML and QML-t. In all cases, the nominal size considered is 5%. In order to assess the empirical size of both tests, we simulated 1000 independent samples of size $T=1000$ of an AVGARCH (Parameter set 3) and, for each sample, we fitted a TGARCH model, by both QML and QML-t, and computed the corresponding t -ratio and log-likelihood. To calculate the LR statistic, we also fitted an AVGARCH and compare its log-likelihood with that of the fitted TGARCH. On the other hand, to analyze the empirical power of the tests, we simulated 1000 independent samples of size $T=1000$ of a TGARCH with Parameter set 2 ($\delta = -0.11$) and, for each sample, we proceed as before. We have also performed the same experiments with a TGARCH model with Parameter set 1 ($\delta = -0.05$) obtaining similar conclusions.

Figure 5 compares the empirical size of the t -test (left-hand side panels) and the LR test (right-hand side panels) based on both QML and QML-t estimates of TGARCH models contaminated with 1 isolated outlier (top panels) and with two consecutive outliers (bottom panels). That is, it represents the proportion of rejections under the null ($H_0 : \delta = 0$), based on 5% critical value of t and LR tests using the standard rejection regions (8) and (9), respectively. The main conclusions we can draw from this figure are as follows. As expected, both the LR and t -test based on QML-t are always more robust to outliers than those based on QML. Moreover, as the outlier size increases, the tests based on QML become more oversized, i.e., they erroneously reject the null more often that they should and so they tend to identify spurious leverage. This problem is especially remarkable in the LR test, which could reach an over-rejection as huge as 80% in the presence of outliers of size larger than 15. By contrast, the tests

based on QML-t keep the nominal size quite well and even better in the presence of 2 outliers than in the presence on 1 isolated outlier. It is also worth mentioning that, even with no outliers, the t -test does not reach the nominal size 5%. This could be related to the warning of Francq and Zakoian (2010), mentioned in Section 2.5, regarding the inappropriateness of the standard rejection region (8) in GARCH settings. This topic is out of the scope of this paper but deserves further research.

INSERT FIGURE 5 HERE

Figure 6 compares the empirical powers of the t -test (left-hand side panels) and the LR test (right-hand side panels) based on both QML and QML-t estimates of TGARCH models contaminated with 1 isolated outlier (top panels) and with two consecutive outliers (bottom panels). That is, this figure displays, for each test, the relative frequency of rejection of the hypothesis $H_0 : \delta = 0$ (no leverage) on 1000 independent realizations of length $T = 1000$ of the TGARCH model with parameter $\delta = -0.11$ (Parameter set 2). This figure shows that, in terms of power, both the LR and t -test based on QML-t are again more robust to outliers than those based on QML, as expected. The power of the tests based on QML decreases rapidly with the size of the outlier, although the loss of power is not that big in the LR test, which keeps the power around 80% when it is based on QML, even in the presence of huge outliers. However, the loss of power due to outliers of the t -test based on QML is dramatic. By contrast, the power of the tests based on QML-t is around 1 in all cases, regardless of the size, the sign and the amount of the outliers in the sample.

INSERT FIGURE 6 HERE

4 Empirical application

In this section we illustrate the previous results by fitting both the AVGARCH and the TGARCH models to two series of daily returns from different markets by using the estimation methods described above. For each series and estimated model, we check whether the leverage parameter is significant or not by applying both the t -test and LR test. The two series analyzed have been chosen to represent the possible effects than one isolated outlier and two consecutive outliers have on the inferential statistics.

4.1 Data description and dynamic properties

The series analyzed are daily returns of the exchange rate US Dollar/British Pound observed from January 14, 1999 to January 14, 2019, comprising 5218 observations, and daily returns of futures contracts of Natural gas from January 4, 2000 to June 28, 2013, comprising 3368 observations³. In both cases, the returns are computed as $y_t = 100 \times (\log(P_t) - \log(P_{t-1}))$, where P_t is the price at day t . The two series are plotted in Figure 7. As we can see, they both display volatility clustering and some occasional extreme values that could be regarded as outliers.

For instance, the exchange rate US Dollar/British Pound returns exhibit several extreme observations, the largest one corresponding to June 24, 2016, when the series drops by -8.3% after the referendum held on the previous day in which 51.9% of those voting supported leaving the EU. The Natural gas returns exhibit two extreme consecutive observations, the first one positive and the second one negative, of magnitudes about 12 times the standard deviation of the series. These outliers are due to large changes in the open price of the Natural gas on the 13th, 14th and 15th of April 2006, with values of 6.78, 11.26 and 7.15 dollars, respectively.

INSERT FIGURE 7 HERE

Table 1 contains descriptive statistics of the two series considered, as well as the Jarque-Bera and the Ljung-Box $Q(20)$ test statistics, the heteroscedastic-corrected $Q_c(20)$ test statistic, proposed by Diebold (1988), and the test statistic $CH(20)$, proposed by Cumby and Huizinga (1992), which is also robust to conditional heteroscedasticity. We also include the values of the statistics Q and CH applied to the squared returns, denoted by $Q_2(20)$ and $CH_2(20)$, respectively, to test for conditional heteroscedasticity. As expected, both series exhibit excess kurtosis and the Jarque-Bera test for Normality always rejects the null. The values of $Q_c(20)$ and $CH(20)$ never reject the null at 5% significance level, suggesting that both returns are uncorrelated, as expected. Also, as expected, the values of $Q_2(20)$ and $CH_2(20)$ are significant at 1% for both series, indicating strong evidence of conditional heteroscedasticity.

³The data for the exchange rate US Dollar/British Pound was downloaded from Datastream and the corresponding one for Natural gas was obtained from the additional files in Kristoufek (2014).

INSERT TABLE 1 HERE

Figure 8 displays, in its first row, the correlograms of the returns with the corrected 95% confidence bands proposed by Diebold (1988) for conditionally heteroscedastic series. These bands are wider than the usual Barlett bands and show no evidence of autocorrelated returns. The correlograms of squared returns, displayed in the 2nd row of Figure 8, suggest correlation in the squared returns for both series but the pattern of both correlograms is different. For the Natural gas, we can see the typical pattern of the correlogram of the squared observations in the presence of two consecutive outliers, that is, a very high positive and significant 1st order correlation while the others are pushed downwards towards zero; see Carnero *et al.* (2007). However, when the robust autocorrelations of squares proposed by Teräsvirta and Zhao (2011) are computed (see the 3rd row of Figure 8), another picture comes up: the correlograms of both series have the same pattern, with all correlations being significantly different from zero, suggesting that the conditional variances of both returns are not constant over time, as expected. Notice that, for the US Dollar/British Pound, there is not such a difference between both the sample and the robust autocorrelations of squares, although the former are closer to zero. This is the expected pattern due to one isolated outlier; see Carnero *et al.* (2016).

INSERT FIGURE 8 HERE

Finally, the last two rows of Figure 8 display the sample cross-correlations between past and squared returns and their robust counterparts, respectively. The latter are computed using the proposal in Carnero *et al.* (2016) based on applying the Ramsay weighting scheme to the sample variances and cross-covariances. As expected, for the Natural gas, the picture changes depending on whether we look at the sample or the robust cross-correlations. Whereas all the robust cross-correlations are around zero, indicating no leverage effect, the 1st sample cross-correlation is pushed upwards to a significant positive value, suggesting inverse leverage effect (positive relationship between volatility and past returns). Again, this is the predicted pattern in the presence of two consecutive outliers, being the first one positive; see Carnero *et al.* (2016). Therefore, the inverse leverage effect found in the Natural gas by some authors (see Kristoufek

(2014) and the references therein) could be an artifact due to the misleading effect of outliers. By contrast, in the US Dollar/British Pound, the sample and robust cross-correlograms are similar, in agreement with the results in Carnero *et al.* (2016), who show that a single outlier needs to be larger to bias the sample cross-correlations.

4.2 Estimating and testing for leverage effect

We describe below the results from estimating the TGARCH and AVGARCH models to the two return series described above. Since the results in Section 3 suggest a better performance of QML-t over LAD, we focus our main comparison on QML and QML-t, although the estimation results from LAD will be also commented. Table 2 displays the estimation results obtained by QML, QML-t and LAD as well as some diagnostics based on the residuals, $\hat{\varepsilon}_t = y_t/\hat{\sigma}_t$, where $\hat{\sigma}_t$ is the estimated volatility for each model. The estimation has been carried out with Matlab. The QML and QML-t estimated parameter values and their corresponding standard errors were computed using the Oxford MFE Toolbox, taking into account the reparametrization used in this package as compared to the parametrization in (2) used in this paper.⁴ The LAD estimated parameter values were computed as explained in Section 2.4 and the method proposed by Zhu and Ling (2015) was used to calculate the standard errors.

INSERT TABLE 2 HERE

As we can see, there are remarkable differences between the results obtained for the two returns series and, for each series, there are also some differences between the estimation methods that are worth mentioning. Since we are interested in the effects of outliers on the leverage effect, our comments will be focused on the parameter δ .

For the exchange rate series, we can see that, regardless of the estimation method, the parameter δ in the TGARCH model is always estimated negative, as it is expected if there is leverage effect. Moreover, the leverage coefficient is statistically significant at 5% when both the QML and QML-t based *t*-tests are applied. In particular, the values (and *p*-values) of the *t*-statistics are -2.08 (0.0375) and -3.16 (0.002), respectively.

⁴To check for the robustness of our results, we have repeated the estimation by QML and QML-t in Stata, obtaining similar results. We have also estimated EGARCH models leading to similar conclusions.

Meanwhile, the values (and p-values) of the LR test statistic for $H_0 : \delta = 0$ (AVGARCH) against $H_1 : \delta \neq 0$ (TGARCH) are 11.182 (0.001) and 12.396 (0.001) when the models are estimated by QML and QML-t, respectively, confirming that δ is significant when both QML and QML-t are used. By contrast, when the LAD estimator is considered, the leverage coefficient is no longer statistically significant due to the large standard error. Notice also that δ is estimated closer to zero when QML is used compared to QML-t. This result agrees with our discussion in Section 3 where we show that a single isolated outlier (like the one existing in the exchange rate series) biases the QML estimated δ towards zero and could hide true leverage. Then, it seems that, in this case, QML-t is more reliable than QML suggesting that there is leverage effect in the exchange rate returns, although the negative return associated to the Brexit seems to be biasing towards zero the leverage coefficient, when the model is estimated by QML.

For the Natural gas series, the parameter δ is estimated positive by the three methods (QML, QML-t and LAD), suggesting inverse leverage effect. However, its statistical significance depends on the estimator used as well as the test statistic chosen. Using the LAD estimator, we can see that the standard error is huge rendering the leverage coefficient not statistically significant. On the other hand, when QML is used, the values (and p-values) of both the t -ratio and LR test statistics are 0.636 (0.5247) and 2.377 (0.123), respectively, indicating no evidence of leverage effect at any reasonable significance level. By contrast, when the model is estimated using QML-t, the values (and p-values) of these two test statistics are 2.550 (0.0108) and 8.458 (0.0036), respectively, showing evidence of inverse leverage effect at 5% significance level (or even at 1% if the LR test is selected). This somehow surprising result could be related to our findings in Section 3, where we show that QML-t could be still biased in the presence of two big consecutive outliers, with the estimated δ being pushed upwards if the first of these outliers is positive, as it is the case in the Natural gas series. Therefore, we should be very careful in concluding that there is actually inverse leverage effect in this series, as this could be a misleading effect caused by the two consecutive extreme observations present in this series. Finally, it is also worth mentioning that the thickness parameter ν of the Student error distribution in QML-t, indicates fat tails in both returns.

When looking at residuals diagnostics, as expected, the values of the statistics $Q_2(20)$ and $CH_2(20)$ for remaining autocorrelation in the squared residuals, have been reduced

remarkably in all estimated models, as compared to their values for the returns in Table 1. Only the values of $Q_2(20)$ remain significant for some models. However, the robust statistic $CH_2(20)$ is not longer significant for the TGARCH model, suggesting that this model has properly captured the dynamics in the conditional variance of the returns.

To further illustrate how the potential outliers can bias the estimation and testing of the leverage effect, we consider a rolling window scheme of size $T^* = 1000$ through the whole sample, starting at point $t = 1$ and moving forwards up to covering the last 1000 observations of the full sample. For the exchange rate and the Natural gas series, this amounts to analyzing 4217 and 2368 subsamples, respectively, covering periods of different volatilities and types and sizes of outliers, according to the following steps:

1. Select the 1st subsample of size $T^* = 1000$ for the exchange rate (14/01/1999 - 14/09/2002) and for Natural gas (04/01/2000 - 9/01/2004), estimate the parameter δ by the three methods considered (QML, QML-t and LAD) and compute the corresponding t -ratio and LR test statistics for these subsamples.⁵
2. Delete the 1st observation, add a new observation at the end of the subsample and re-estimate the model and test again for leverage.
3. Repeat the process until we reach the end of the full samples, where we cover the last subsamples of size $T^* = 1000$ for the exchange rate (13/03/2015 - 14/01/2019) and Natural gas (23/06/2009 - 28/06/2013).

Figure 9 plots, in its first row, the estimated values of δ across the rolling window for the two series considered and for the three estimation methods applied, i.e., it plots the values of $\widehat{\delta}_{QML}$, $\widehat{\delta}_{QML-t}$ and $\widehat{\delta}_{LAD}$ for each of the 4217 subsamples of the exchange rate series (left-hand side panel) and for each of the 2368 subsamples of the Natural gas series (right-hand side panel). As a benchmark, the zero line is also displayed to account for no leverage. Similarly, Figure 9 plots the corresponding p-values of both the t -ratio (2nd row) and the LR (3rd row) tests to test $H_0 : \delta = 0$ against $H_0 : \delta \neq 0$. The horizontal line represents the 5% significance level and consequently, those subsamples

⁵Notice that the t -ratio test can be computed using the three estimators considered but the LR test statistic is only calculated for QML and QML-t.

with p-values below this line are those where the leverage effect is significant at 5% significance level.

Several conclusions emerge from this figure. First, we observe remarkable differences among the three estimators considered: $\widehat{\delta}_{QML}$, $\widehat{\delta}_{QML-t}$ and $\widehat{\delta}_{LAD}$. Notice that, in both series, the leverage coefficient is never statistically significant when using the LAD estimator (see the p-values displayed in the 2nd row of Figure 9). This fact seems to be driven by the large standard errors of $\widehat{\delta}_{LAD}$, as we mentioned previously. Second, we observe how extreme observations can bias the estimated leverage parameter and the corresponding test statistics and could lead to a wrong conclusion about the sign and magnitude of the leverage effect. As expected, the QML estimated values of δ present several sharp drops and rises in both series. These sharp changes are usually due to the entrance and/or an exit of outlying observations in the corresponding subsample.

For instance, in the exchange rate series (left-hand side panels), the entrance of the observation in June 24, 2016, where the series sustained its largest drop ($y_{4551} = -8.3$), conveys a sudden jump in $\widehat{\delta}_{QML}$ from a negative value to a positive value around 0 (see the graph in the top left-hand side panel). Notice that, for most of the subseries before that date, where there are no outliers, the estimates $\widehat{\delta}_{QML}$ and $\widehat{\delta}_{QML-t}$ are very similar, in agreement with our simulation results in Section 3.2. However, in the last subseries around the end of the full sample, i.e., those including the isolated outlier due to the Brexit, there are big differences between $\widehat{\delta}_{QML}$ and the robust estimate, $\widehat{\delta}_{QML-t}$, which is mainly negative and similar to the other robust estimator, $\widehat{\delta}_{LAD}$. Actually, when QML-t is used, the p-values of both significance tests show that δ is statistically significant for these subsamples, indicating the presence of leverage effect. However, for these subsamples, both QML based tests are unable to reject the null, $H_0 : \delta = 0$. Recall that, according to our results in Section 3, these differences are expected in the presence of one isolated outlier, which seems to be the cause here of the upwards bias observed in $\widehat{\delta}_{QML}$, as compared to the robust estimator $\widehat{\delta}_{QML-t}$.

When looking at the results for the Natural gas (right-hand side panels), the most remarkable differences between QML and QML-t arise in the subseries located around the middle of the sample, i.e., in those including the two huge consecutive outliers present in this series ($y_{1566} = 50.73$ and $y_{1567} = -45.41$). In these cases, our results in Section 3 show that even the robust estimators could still be slightly biased. In particular, when

these two observations, the first one being positive and the next one negative, are in a subsample, we expect $\widehat{\delta}_{QML}$ but also $\widehat{\delta}_{QML-t}$ to be upwards biased. Another remarkable feature from the Natural gas results in Figure 9, is that the values of $\widehat{\delta}_{QML}$ and $\widehat{\delta}_{QML-t}$ for the subseries at the end of the period, where there are no outliers (see the graph in the top right-hand side panel), are very similar to each other, as expected, but quite different from the values of $\widehat{\delta}_{LAD}$: the former take negative values (standard leverage) whereas the latter estimate $\delta > 0$ (inverse leverage). This feature could be related to the fact that LAD is underperforming if no outliers are present in the series, as discussed in Section 3. Accordingly, for most of the subseries at the end of the period, neither the t-ratio nor the LR test statistics based on QML and QML-t reject the null, $H_0 : \delta = 0$, in these subseries. Therefore, we wonder whether the inverse leverage effect found in the Natural gas returns by some authors (see, for example, Kristoufek (2014) and the references therein) could be due to the harmful effect of these consecutive outliers.

Alternatively, the patterns of the estimated parameters in Figure 9 could be indicating time-varying leverage effect, as suggested by Bandi and Reno (2012), Yu (2012) and Jensen and Maheu (2014), but this is a problem that is out of the scope of this paper.

5 Conclusions

This paper analyzes the effect of outliers on the estimation and testing for the leverage effect in TGARCH models. It is shown that QML-t and LAD always outperform QML in the presence of outliers, as expected. Actually, one isolated outlier could lead QML to hide true leverage effect whereas two consecutive outliers bias the QML estimated leverage in a direction that crucially depends on the sign of the first outlier. If this is negative (positive), QML underestimates (overestimates) the leverage parameter. Therefore, in these cases, QML could hide true leverage or estimate spurious asymmetries or asymmetries of the wrong sign. However, both robust estimators perform very well when there is one isolated outlier, but they are still slightly biased in the presence of patches of big outliers, leading, in some cases, to inaccurate estimates of the leverage coefficient. In general, QML-t seems to outperform LAD because it is robust to moderate-big outliers without losing much efficiency, as compared to QML, when there are no outliers. That is not the case for LAD, which performs much worse than QML with no outliers. These

results are further illustrated with the empirical analysis of two return series, including one isolated negative outlier and two consecutive outliers, respectively.

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Table 1: Descriptive statistics of the returns

Statistic	US Dollar/British Pound	Natural gas
Mean	-0.0048	0.0149
Std.Dev.	0.5826	3.7653
Skewness	-0.5696	0.7462
Kurtosis	14.402	21.765
Jarque-Bera	28502***	49631***
Sample size	5217	3368
$Q(20)$	55.407***	48.547***
$Q_c(20)$	28.479*	18.593
$CH(20)$	23.695	18.975
$Q_2(20)$	634.86***	631.56***
$CH_2(20)$	67.99***	75.94***

*, **, ***: statistically significant at 10%, 5% and 1% respectively

Table 2: Estimation of the TGARCH and AVGARCH models with QML, QML-t and LAD

Estimator	Parameter	US Dollar/British Pound		Natural gas	
		TGARCH	AVGARCH	TGARCH	AVGARCH
QML	ω	0.0065** (0.003)	0.0068** (0.003)	0.1019* (0.059)	0.1063 (0.075)
	α	0.0479*** (0.012)	0.0501*** (0.012)	0.0749*** (0.028)	0.0794** (0.036)
	β	0.9519*** (0.013)	0.9497*** (0.013)	0.9173*** (0.033)	0.9128*** (0.044)
	δ	-0.0097** (0.005)		0.0077 (0.018)	
	Log-Likelihood	-4109.0	-4114.6	-9028.7	-9029.9
Residuals	$Q(20)$	19.434	19.434	22.239	22.167
	$CH(20)$	19.921	19.844	15.991	15.875
	$Q_2(20)$	26.596	28.579*	51.180***	57.706***
	$CH_2(20)$	16.342	15.054	11.688	12.459
QML-t	ω	0.0053*** (0.002)	0.0056*** (0.002)	0.0833*** (0.021)	0.0788*** (0.018)
	α	0.0390*** (0.006)	0.0418*** (0.006)	0.0806*** (0.010)	0.0813*** (0.010)
	β	0.9608*** (0.007)	0.9580*** (0.0107)	0.9144*** (0.011)	0.9152*** (0.011)
	δ	-0.0111*** (0.004)		0.0185** (0.007)	
	v	7.7403	7.5673	6.7493	6.7402
Log-Likelihood	-4005.1	-4011.3	-8779.8	-8784.0	
Residuals	$Q(20)$	19.627	19.642	20.864	21.393
	$CH(20)$	19.956	19.886	16.273	15.904
	$Q_2(20)$	31.772**	35.232**	25.893	47.318***
	$CH_2(20)$	16.652	15.509	12.371	14.131
LAD	ω	0.0028 (0.136)	0.0024 (0.244)	0.0957 (0.143)	0.0667 (0.517)
	α	0.0445** (0.020)	0.0428 (0.085)	0.0581 (0.073)	0.0523 (0.168)
	β	0.9559*** (0.028)	0.9580*** (0.112)	0.9201*** (0.073)	0.9343*** (0.157)
	δ	-0.0056 (0.065)		0.0217 (0.108)	
Residuals	$Q(20)$	20.072	20.277	20.864	24.863
	$CH(20)$	20.482	20.589	16.273	16.631
	$Q_2(20)$	24.212	26.590	25.894	105.362***
	$CH_2(20)$	17.155	15.578	12.371	12.436

*, **, ***: statistically significant at 10%, 5%, and 1%, respectively

Figure 1: Function ρ (left panels) and its first derivative (right panels) in M-estimators

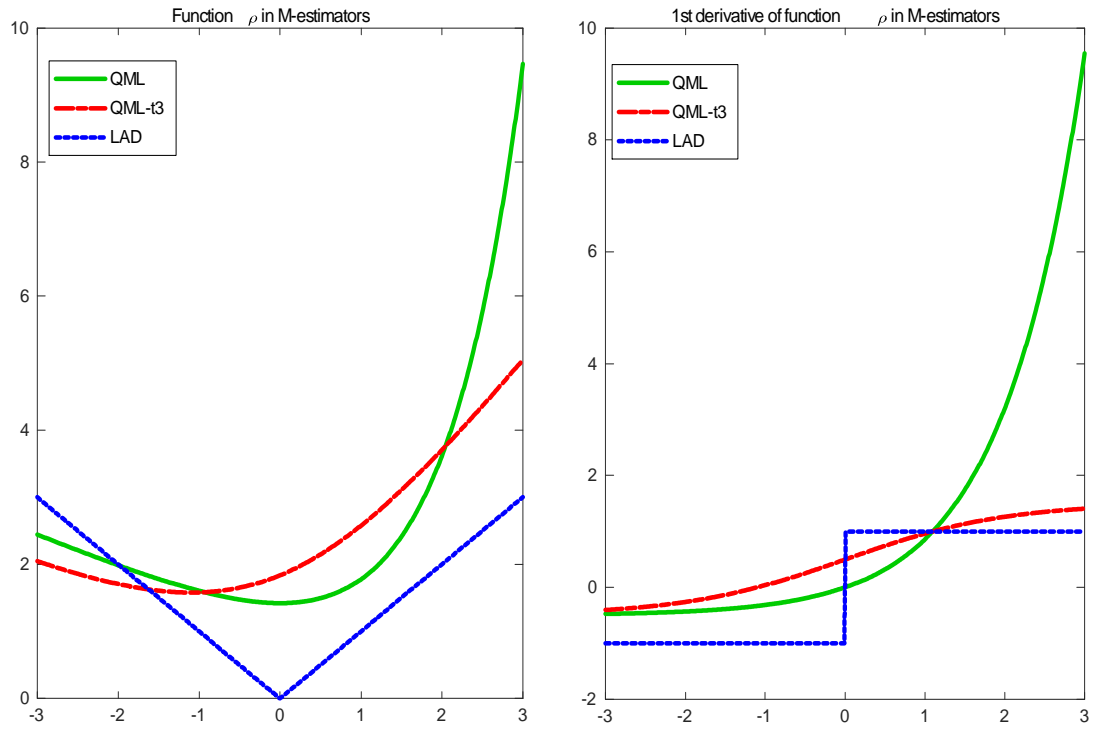


Figure 2: Boxplots of estimated δ with QML, QML-t and LAD in the presence of one isolated outlier

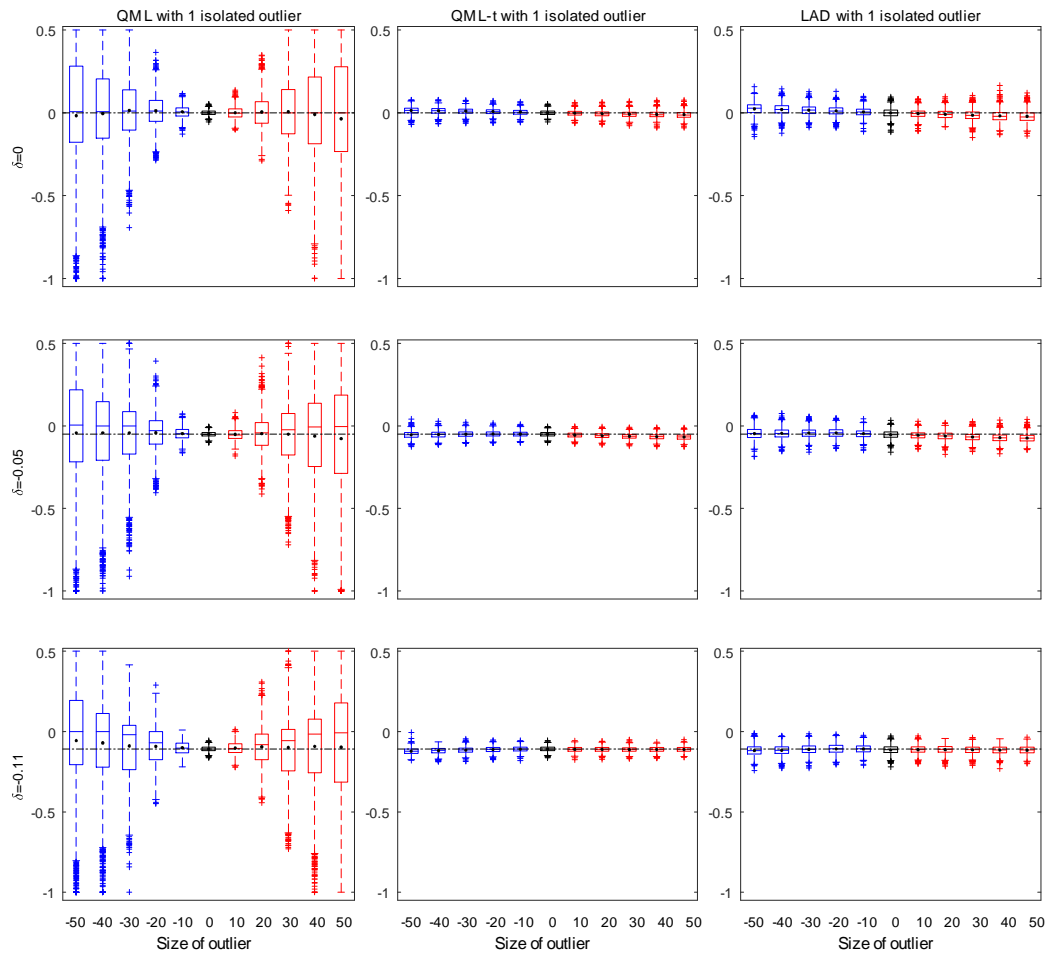


Figure 3: Boxplots of estimated δ with QML, QML-t and LAD in the presence of two consecutive outliers

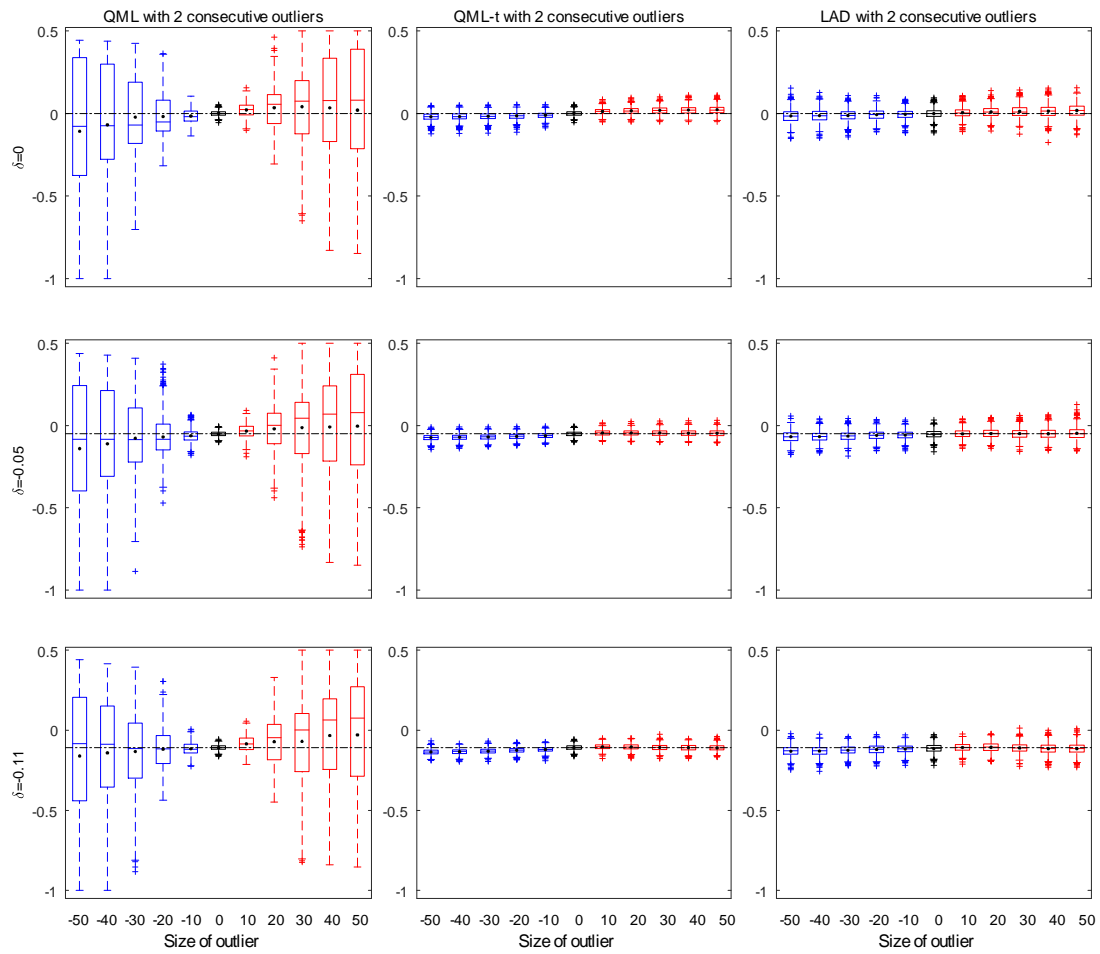


Figure 4: Boxplots of estimated δ with QML-t and LAD in the presence of one isolated and two consecutive outliers

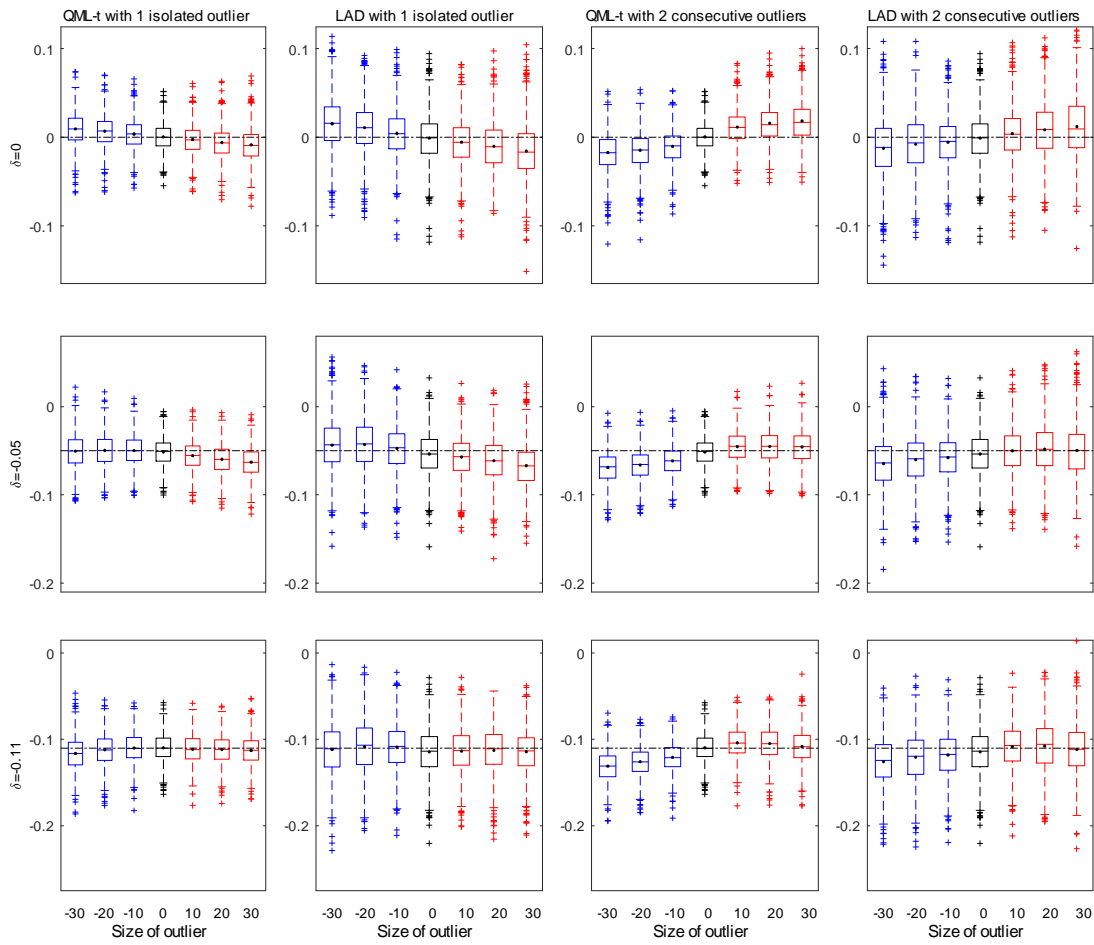


Figure 5: Monte Carlo size. Proportion of rejections based on 5% critical value of t test (left panels) and LR test (right panels) under $H_0 : \delta = 0$, with one isolated outlier (top panels) and two consecutive outliers (bottom panels)

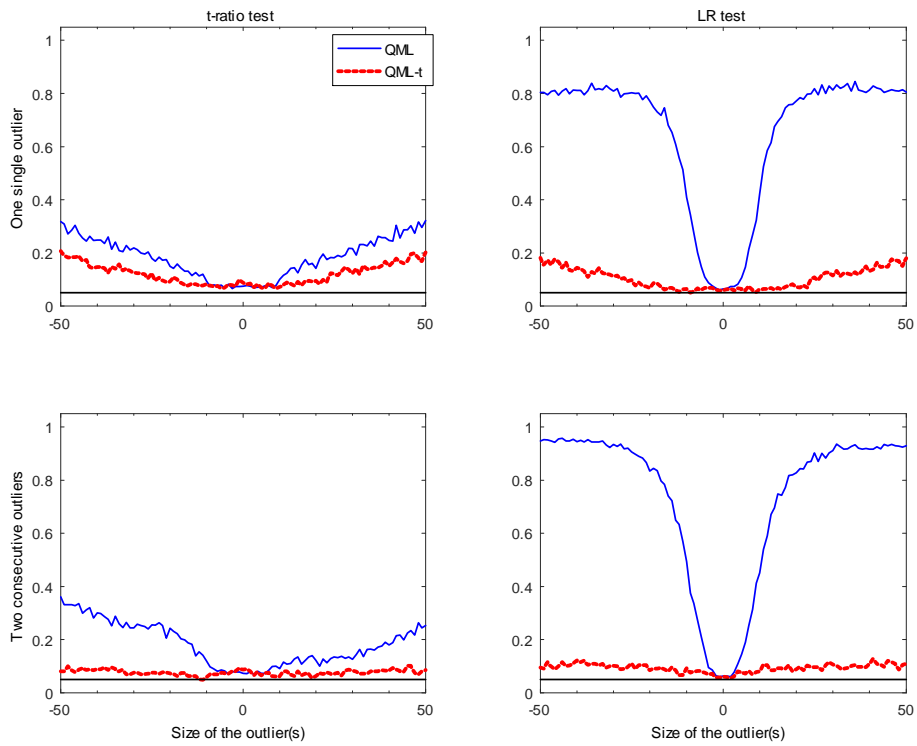


Figure 6: Monte Carlo power. Proportion of rejections based on 5% critical value of t test (left panels) and LR test (right panels) under $H_1 : \delta = -0.11$, with one isolated outlier (top panels) and two consecutive outliers (bottom panels)

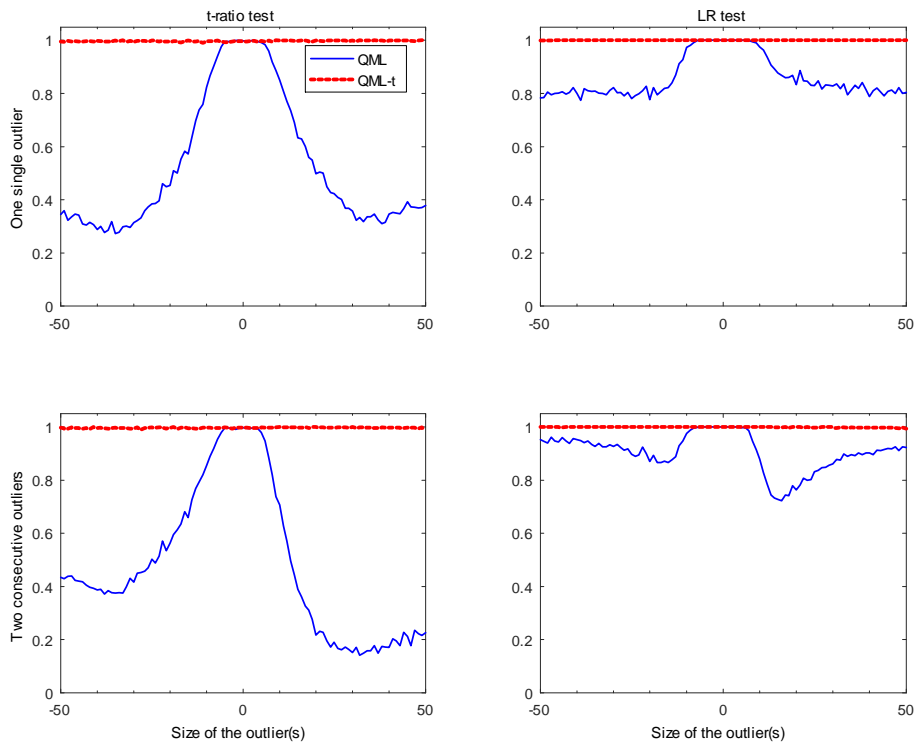


Figure 7: Daily returns for the exchange rate US Dollar/British Pound and future contracts of Natural gas

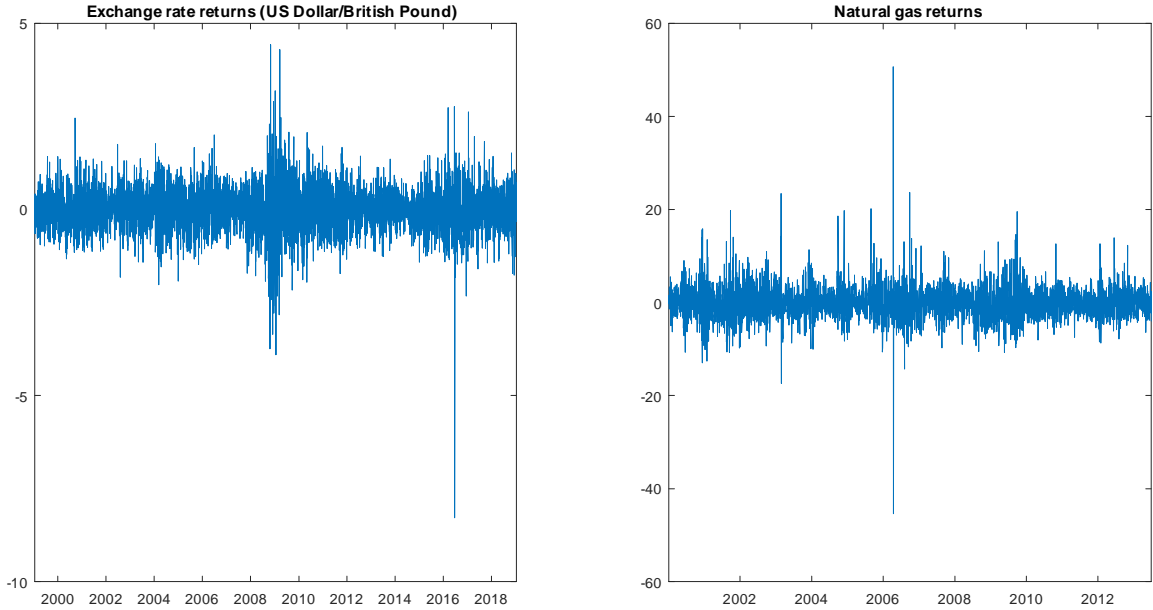


Figure 8: Correlograms of returns and squared returns and cross-correlograms between lagged returns and squared returns

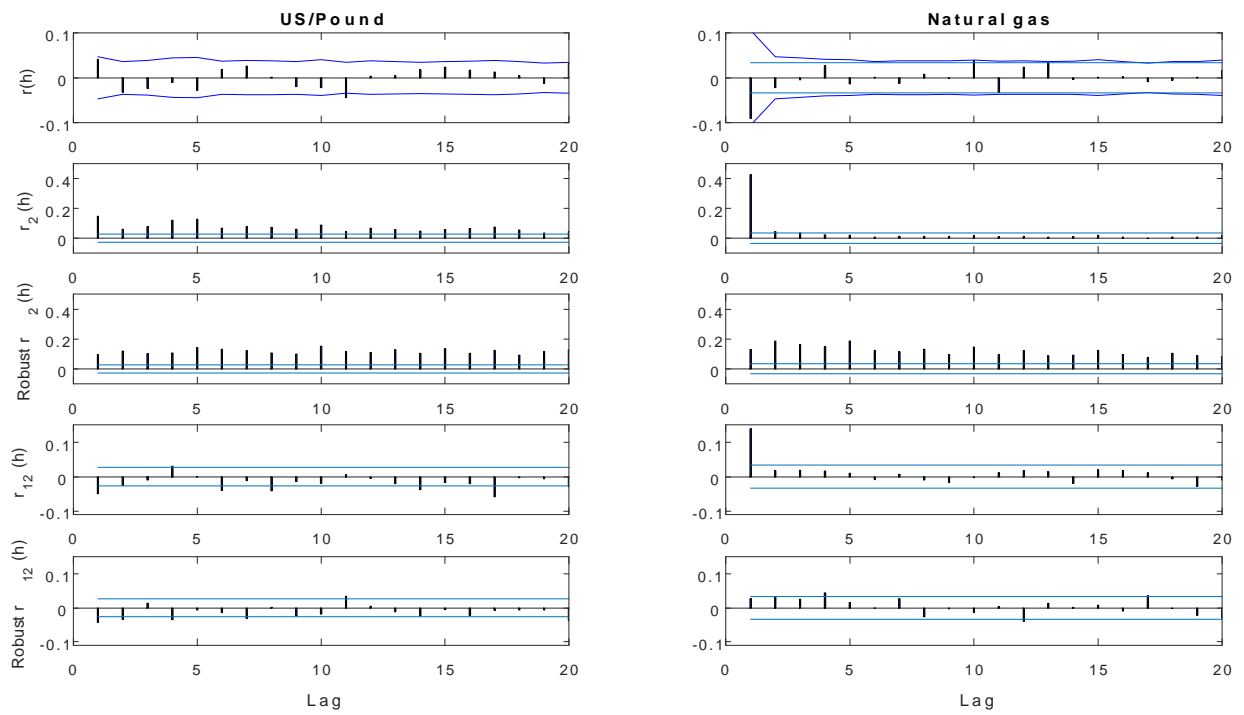


Figure 9: Leverage parameter δ in a TGARCH model estimated with subsamples of size $T = 1000$ using a rolling window of both the Exchange rate US Dollar/British Pound and Natural gas daily returns (top panels) and the corresponding p-values of the t and LR tests for the statistical significance of δ .

