Algebraic links in the Poincaré sphere and the Alexander polynomials *

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Abstract

The Alexander polynomial in several variables is defined for links in three-dimensional homology spheres, in particular, in the Poincaré sphere: the intersection of the surface $S = \{(z_1, z_2, z_3) \in \mathbb{C}^3 : z_1^5 + z_2^3 + z_3^2 = 0\}$ with the 5-dimensional sphere $\mathbb{S}^5_\varepsilon = \{(z_1, z_2, z_3) \in \mathbb{C}^3 : |z_1|^2 + |z_2|^2 + |z_3|^2 = \varepsilon^2\}$. An algebraic link in the Poincaré sphere is the intersection of a germ $(C, 0) \subset (S, 0)$ of a complex analytic curve in $(S, 0)$ with the sphere $\mathbb{S}^3_\varepsilon$ of radius ε small enough. Here we discuss to which extend the Alexander polynomial in several variables of an algebraic link in the Poincaré sphere determines the topology of the link. We show that, if the strict transform of a curve on $(S, 0)$ does not intersect the component of the exceptional divisor corresponding to the end of the longest tail in the corresponding E_8-diagram, then its Alexander polynomial determines the combinatorial type of the minimal resolution of the curve and therefore the topology of the corresponding link. Alexander polynomial of an algebraic link in the Poincaré sphere coincides with the Poincaré series of the filtration defined by the corresponding curve valuations. We show that, under conditions similar for those for curves, the Poincaré series of a collection of divisorial valuations determines the combinatorial type of the minimal resolution of the collection.

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1 Introduction

The three-dimensional sphere is $\mathbb{S}^3_\varepsilon = \{(z_1, z_2) \in \mathbb{C}^2 : |z_1|^2 + |z_2|^2 = \varepsilon^2\}$. An algebraic link in the three-dimensional sphere is the intersection of a germ $(C, 0) \subset (\mathbb{C}^2, 0)$ of a complex analytic plane curve with the sphere $\mathbb{S}^3_\varepsilon$ with ε small enough. The number of components of the link $K = C \cap \mathbb{S}^3_\varepsilon$ equals the number of the irreducible components of the curve $(C, 0)$. A link with r components in the three-sphere has the well-known topological invariant: the Alexander polynomial in r variables: see, e. g., [8]. It is known that the Alexander polynomial in several variables determines the topological type of an algebraic link (or, equivalently, the (local) topological type of the triple $(\mathbb{C}^2, C, 0)$): [12], see [4] for another proof of this statement.

The Alexander polynomial is defined for links in three-dimensional manifolds which are homology spheres. The Poincaré sphere $\mathbb{L}$ is the intersection of the surface $S = \{(z_1, z_2, z_3) \in \mathbb{C}^3 : z_1^5 + z_2^3 + z_3^2 = 0\}$ with the 5-dimensional sphere $\mathbb{S}^5_\varepsilon = \{(z_1, z_2, z_3) \in \mathbb{C}^3 : |z_1|^2 + |z_2|^2 + |z_3|^2 = \varepsilon^2\}$. It is a three-dimensional homology sphere. This definition describes the Poincaré sphere $\mathbb{L}$ as the link of a rational surface singularity of type E_8. The links of other rational surface singularities are rational homology spheres, but not homology spheres.

An algebraic link in the Poincaré sphere is the intersection of a germ $(C, 0) \subset (S, 0)$ of a complex analytic curve in $(S, 0)$ with the sphere $\mathbb{S}^5_\varepsilon$ of radius ε small enough. The number of components of the link $K = C \cap \mathbb{S}^5_\varepsilon$ equals the number of the irreducible components of the curve $(C, 0)$. For a link with r components in the Poincaré sphere $\mathbb{L} = S \cap \mathbb{S}^5_\varepsilon$ one has its Alexander polynomial $\Delta^K(t_1, \ldots, t_r)$ defined in the same way as for a link in the usual three-sphere $\mathbb{S}^3_\varepsilon$.

An irreducible curve germ $(C, 0)$ in a germ of a complex analytic variety $(V, 0)$ defines a valuation v_C on the ring $\mathcal{O}_{V,0}$ of germ of functions on $(V, 0)$ (called a curve valuation). Let $\varphi : (\mathbb{C}, 0) \to (V, 0)$ be a parametrization (an uniformization) of the curve $(C, 0)$, that is $\operatorname{Im} \varphi = (C, 0)$ and φ is an isomorphism between punctured neighbourhoods of the origin in $\mathbb{C}$ and in C. For a function germ $f \in \mathcal{O}_{V,0}$, the value $v_C(f)$ is defined as the degree of the leading term in the Taylor series of the function $f \circ \varphi : (\mathbb{C}, 0) \to \mathbb{C}$:

$$f \circ \varphi(\tau) = a\tau^{v_C(f)} + \text{ terms of higher degree,}$$

where $a \neq 0$; if $f \circ \varphi \equiv 0$, one defines $v_C(f)$ to be equal to $+\infty$.

A collection $\{(C_i, 0)\}$ of irreducible curves in $(V, 0)$, $i = 1, \ldots, r$, defines the collection $\{v_{C_i}\}$ of valuations. For a collection $\{v_i\}$ of discrete valuations on $\mathcal{O}_{V,0}$, $i = 1, \ldots, r$, there is defined its Poincaré series $P_{\{v_i\}}(t_1, \ldots, t_r) \in$

$\mathbb{Z}[[t_1, \ldots, t_r]]$: [6]. In [1] it was shown that, for $(V, 0) = (\mathbb{C}^2, 0)$, the Poincaré series $P_{\{v_{C_i}\}}(t_1, \ldots, t_r)$ of a collection of (different) curve valuations coincides with the Alexander polynomial $\Delta^C(t_1, \ldots, t_r)$ in r variables of the algebraic link defined by the curve $C = \bigcup_{i=1}^{r} C_i$ for $r > 1$. (For $r = 1$ one has $P_{v_C}(t) = \frac{\Delta_C(t)}{1-t}$) In [3] it was shown that the same holds for an algebraic link in the Poincaré sphere.

Here we discuss to which extend the Alexander polynomial in several variables of an algebraic link in the Poincaré sphere (that is the Poincaré series of the corresponding curve) determines the topology of the link. Two curves on $(S, 0)$ with the same (from the combinatorial point of view) minimal resolutions define topologically equivalent links in the Poincaré sphere. We show that two curves (even irreducible ones) on $(S, 0)$ with combinatorially different minimal resolutions may have equal Alexander polynomials. The (infinite-dimensional) space of arcs on $(S, 0)$ consists of 8 irreducible components. These components are in one-to-one correspondence with the components of the exceptional divisor of the minimal (good) resolution of $(S, 0)$. A component of the space of arcs consists of all arcs whose strict transforms intersect the corresponding component of the exceptional divisor. We show that, if the strict transform of a (possibly reducible) curve on $(S, 0)$ does not intersect one particular component of the exceptional divisor, namely the one corresponding to the end of the longest tail in the corresponding E_8-diagram, then its Poincaré series (that is the Alexander polynomial of the corresponding link) determines the combinatorial type of the minimal resolution of the curve and therefore the topology of the corresponding link.

We discuss an analogous question for a collection of divisorial valuations on the E_8 surface singularity. We show that, if no divisor from the collection is born by a sequence of blow-ups starting from a smooth point of the same component as above, the Poincaré series of the collection determines the combinatorial type of the minimal resolution of the collection of valuations.

The E_8 surface singularity is the quotient of the plane $\mathbb{C}^2$ by the binary icosahedral group. Therefore the results of this paper may have an interpretation it terms of equivariant topology of curves and/or divisors on the plane with the binary icosahedral group action.

2 Poincaré series of curve and divisorial valuations on the E_8-singularity

Let $(S, 0)$ be a normal surface singularity of type E_8 and let $(C_i, 0)$, $i = 1, 2, \ldots, r$, be (different) irreducible curves (branches) on $(S, 0)$. Let $(C, 0) = \bigcup_{i=1}^r (C_i, 0)$. The curves $(C_i, 0)$ define curve valuations on the ring $\mathcal{O}_{S,0}$ of germs of functions on $(S, 0)$ in the usual way (see Section 1). Let $P_C(t_1, \ldots, t_r)$ be the Poincaré series of this set of valuations.

Let $\pi : (\mathcal{X}, \mathcal{D}) \to (S, 0)$ be an embedded resolution of the curve $C = \bigcup_{i=1}^r C_i$. This means that:

1) $\mathcal{X}$ is a smooth surface, $\mathcal{D} = \pi^{-1}(0)$;

2) π is a proper complex analytic map;

3) the total transform $\pi^{-1}(C)$ of the curve C is a normal crossing divisor on $\mathcal{X}$ (this implies that the exceptional divisor $\mathcal{D}$ is a normal crossing divisor on $\mathcal{X}$ as well).

Let $\mathcal{D} = \bigcup\limits_{\sigma \in \Gamma} D_\sigma$ be the decomposition of the exceptional divisor into irreducible components. All the components D_σ are isomorphic to the complex projective line $\mathbb{CP}^1$. Let $(D_\sigma \cdot D_\delta)$ be the intersection matrix of the components D_σ. All the self-intersection numbers $D_\sigma \cdot D_\sigma$ are negative, for $\sigma \neq \delta$ the intersection number $D_\sigma \cdot D_\delta$ equals 1 if the components D_σ and D_δ intersect each other and equals 0 otherwise. The entries of the minus inverse matrix $(m_{\sigma\delta}) = -(D_\sigma \cdot D_\delta)^{-1}$ are positive integers. (Let us recall that we consider the case of an E_8-singularity.) Let $\widetilde{C}_i$ be the strict transform of the branch C_i, i. e., the closure of the preimage $\pi^{-1}(C_i \setminus \{0\})$, $i = 1, 2, \ldots, r$, $\widetilde{C} = \bigcup_{i=1}^r \widetilde{C}_i$. Let $D_{\sigma(i)}$ be the component of the exceptional divisor $\mathcal{D}$ intersecting the strict transform $\widetilde{C}_i$ and let $\underline{m}_\sigma := (m_{\sigma\sigma(1)}, \ldots, m_{\sigma\sigma(r)}) \in \mathbb{Z}^r_{>0}$. Let $\overset{\circ}{D}_\sigma$ be the "smooth part" of the component D_σ in the total transform $\pi^{-1}(C)$, i. e., the component D_σ minus the intersection points with all other components of the exceptional divisor and with the strict transforms $\widetilde{C}_i$. In [3] it was shown that

$$P_C(\underline{t}) = \prod_{\sigma \in \Gamma} (1 - \underline{t}^{\underline{m}_\sigma})^{-\chi(\overset{\circ}{D}_\sigma)}, \tag{1}$$

where $\underline{t} := (t_1, \ldots, t_r)$, $\underline{t}^{\underline{m}} := t_1^{m_1} \cdots t_r^{m_r}$, for $\underline{m} = (m_1, \cdots, m_r) \in \mathbb{Z}^r$, $\chi(\cdot)$ is the Euler characteristic.

Remarks. **1.** One has an essential difference between Equation 1 and the corresponding equation for all other rational surface singularities from [3]. For any other rational surface singularity the numbers $m_{\sigma\delta}$ are, generally speaking, not integers and the Poincaré series is obtained from a certain rational power series (somewhat similar to (1)) in variables T_σ corresponding to all the components of the exceptional divisor $\mathcal{D}$ by eliminating all the monomials with non-integer exponents and subsequent substitution of each variable T_σ by a product of variables $t_1, \ldots, t_r$ corresponding to the branches.

2. Equation 1 gives the Poincaré series $P_C(\underline{t})$ in the form

$$\prod_{\underline{m}\in(\mathbb{Z}_{\geq 0})^r\setminus\{\underline{0}\}} (1-\underline{t}^{\underline{m}})^{s_{\underline{m}}}, \tag{2}$$

where $s_{\underline{m}}$ are integers. For the E_8-singularity this product has finitely many factors. This does not hold, in general, for a curve on an arbitrary rational surface singularity. Any power series in the variables $t_1, \ldots, t_r$ with the free term equal to 1 has a unique representation of the form 2 (generally speaking, with infinitely many factors).

The dual graph of the minimal resolution of the E_8-singularity has the standard E_8 form (see Figure 1), all the self-intersection numbers of the components are equal to -2. An embedded resolution $\pi : (\mathcal{X}, \mathcal{D}) \to (S, 0)$ of the curve C is obtained from the minimal resolution of the singularity $(S, 0)$ by a sequence of blow-ups made (at each step) at intersection points of the strict transform of the curve C and the exceptional divisor. Some intersection points of the strict transform of the curve C and the exceptional divisor may be at the same time intersection points of components of the exceptional divisor. Let $\pi' : (\mathcal{X}', \mathcal{D}') \to (S, 0)$ be the resolution of $(S, 0)$ obtained only by all the blow-ups at the points of this sort. The dual graph of the resolution π' is of the "three-tails" form and is obtained from the one of the minimal resolution by inserting some (maybe zero) new vertices between the vertices of the minimal one. The strict transform $\overline{(\pi')^{-1}(C_i \setminus \{0\})}$ of the branch C_i, $i = 1, \ldots, r$, intersects the exceptional divisor $\mathcal{D}'$ at a smooth point of it, i. e., not at an intersection points of its components. Let $\Gamma_0 \subset \Gamma$ be the set of indices σ numbering the components of $\mathcal{D}'$.

A divisorial valuation v on $\mathcal{O}_{S,0}$ is defined by a component of the exceptional divisor $\mathcal{D} = \pi^{-1}(0)$ of a resolution $\pi : (\mathcal{X}, \mathcal{D}) \to (S, 0)$ of the singularity. For a function germ $f \in \mathcal{O}_{S,0}$, the value $v(f)$ is the multiplicity of the lifting $f \circ \pi$ of the function f to the space $\mathcal{X}$ of the resolution along the corresponding component. Let $v_1, \ldots, v_r$ be divisorial valuations on $\mathcal{O}_{S,0}$. A resolution of the collection $\{v_i\}$ of divisorial valuations is a resolution $\pi : (\mathcal{X}, \mathcal{D}) \to (S, 0)$ whose exceptional divisor contains all the components defining the valuations. For a resolution $\pi : (\mathcal{X}, \mathcal{D}) \to (S, 0)$ of the set $\{v_i\}$, let $\mathcal{D} = \bigcup\limits_{\sigma\in\Gamma} D_\sigma$ be the representation of the exceptional divisor as the union of its irreducible components, let the integers $m_{\sigma\delta}$ be defined as above, and let $\overset{\bullet}{D}_\sigma$ be the "smooth part" of the component D_σ in the exceptional divisor $\mathcal{D}$, i. e., the component D_σ minus the intersection points with all other components of the exceptional divisor. For $i \in \{1, \ldots, r\}$, let $D_{\sigma(i)}$ be the component of the exceptional divisor $\mathcal{D}$ defining the divisorial valuation v_i. Just as in [3] (see also [7] and [2]), one can show that the Poincaré series of the set $\{v_i\}$ of divisorial valuations is given by

$$P_{\{v_i\}}(t_1, \ldots, t_r) = \prod_{\sigma \in \Gamma} (1 - \underline{t}^{\underline{m}_\sigma})^{-\chi(\overset{\bullet}{D}_\sigma)} . \qquad (3)$$

For a component D_σ of the exceptional divisor $\mathcal{D}$ of the resolution π, let ℓ_σ be a germ of a smooth curve on $\mathcal{X}$ transversal to D_σ at a smooth point of $\mathcal{D}$ (i. e., at a point of $\overset{\bullet}{D}_\sigma$), let $(L_\sigma, 0) \subset (S, 0)$ be the blow-down $\pi(\ell_\sigma)$ of the curve ℓ_σ and let L_σ be given by an equation $h_\sigma = 0$ with $h_\sigma \in \mathcal{O}_{S,0}$. (Let us recall that an arbitrary curve germ on the E_8 surface singularity is Cartier, i. e., is defined by an equation.) The curve germ L_σ and/or the function germ h_σ are called *a curvette* at the component D_σ.

3 The Poincaré polynomial of an irreducible curve and the topological type

For a plane valuation centred at the origin (say, for a curve or for a divisorial one) the Poincaré series determines the combinatorial type of the minimal resolution: [4]. This does not hold, in general, for a valuation on a surface singularity. The problem is partially related with the following one. A resolution of a valuation (a curve or a divisorial one) on a surface singularity $(S, 0)$ is at the same time a resolution of the singularity itself. The minimal resolution of the valuation starts from a certain point on the exceptional divisor of the minimal resolution of the surface. Therefore a possibility to determine the combinatorial type of the minimal resolution of a valuation from its Poincaré series assumes that it is possible to determine the component (or the intersection of two components) of the exceptional divisor of the (minimal) resolution of the surface from which the resolution of the valuation starts (up to possible symmetries of the dual graph of the minimal resolution of the surface). However, in general this is not possible.

Example. The exceptional divisor of the minimal resolution of the A_k surface singularity consists of k irreducible components. One can show that a curvette at each of these components is smooth. This follows, e. g., from the results of

[9]. Also this can be deduced from the computation of the Poincaré series of the curvettes using [3, Theorem 2] (which gives $P(t) = \frac{1}{1-t}$).

It appears that the same problem can be met for valuations on the surface singularity of type E_8.

Examples. **1.** The dual graph of the minimal resolution of the E_8-singularity is shown on Figure 1. The exceptional divisor consists of 8 irreducible compo-

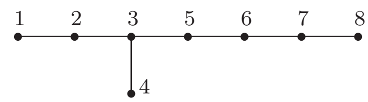

Figure 1: The dual graph of the E_8-singularity.

nents $D_1, \ldots, D_8$ numbered as in the order shown on Figure 1. Let C' be a curvette at the component D_6 and let C'' be the blow down of the plane curve singularity of type A_4 (that is with a local equation $u^5 + v^2 = 0$) transversal to the component D_8 at a smooth point of the exceptional divisor (that is, not at the intersection point of D_8 with D_7). The minimal resolution of the curve C' coincides with the minimal resolution of the surface. The dual graph of the minimal resolution of the curve C'' is shown on Figure 2. Using (1) one can show that

$$P_{C'}(t) = P_{C''}(t) = \frac{(1-t^{12})(1-t^{18})}{(1-t^4)(1-t^6)(1-t^9)}\,.$$

Thus the Poincaré series of a curve valuation on the E_8-singularity does not determine the combinatorial type of its minimal resolution. Moreover, it does not determine the component of the exceptional divisor of the minimal resolution of the surface singularity intersecting the strict transform of the curve.

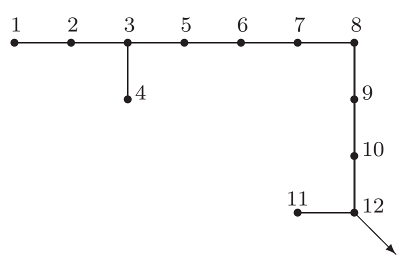

Figure 2: The dual graph of the minimal resolution of the curve L_2.

(We do not know whether or not the (algebraic) links in the Poincaré sphere corresponding to the curves L_1 and L_2 are topologically equivalent.)

2. Let D' be the divisor born under the blow-up of the component D_{12} of the resolution shown on Figure 2 (at a smooth point of the exceptional divisor) and let D'' be the divisor born after 7 blow-ups starting at a smooth point of the component D_6 and produced at each step at a smooth point of the previously born divisor. Let ν' and ν'' be the divisorial valuations defined by the divisors D' and D'' respectively. Using (3) one can show that

$$P_{\nu'}(t) = P_{\nu''}(t) = \frac{(1-t^{12})(1-t^{18})}{(1-t^4)(1-t^6)(1-t^9)(1-t^{19})}\,.$$

Thus the Poincaré series of a divisorial valuation on the E_8-singularity does not determine the combinatorial type of its minimal resolution. Other examples of this sort can be obtained by applying the same additional modifications at smooth points of the divisors D' and D''.

The examples show that one cannot restore, in general, the combinatorial type of the (minimal) resolution of a valuation (say, of an irreducible curve) on the E_8 surface singularity from its Poincaré series. However, often this is the case. According to [11] the space of arcs on the E_8-singularity consists of 8 irreducible components. Each of them is the closure of the subspace of arcs whose strict transforms intersect the exceptional divisor of the minimal resolution of the surface at one of the components $D_1, \ldots, D_8$. Let us denote these spaces of arcs by $\mathcal{E}_1, \ldots, \mathcal{E}_8$ respectively. One can show that only arcs from $\mathcal{E}_6$ and $\mathcal{E}_8$ may have the same Poincaré series. Moreover, if one restricts the consideration only to the arcs not intersecting the component D_8 at a smooth point of the exceptional divisor, one can determine the combinatorial type of the (minimal) resolution of the arc from its Poincaré series. The same holds for a reducible curve: if $(C,0) = \bigcup_{i=1}^{r}(C_i,0) \subset (S,0)$ and the strict transforms of the branches C_i do not intersect the component D_8 of the exceptional divisor at smooth points (that is, they belong to the union $\bigcup_{i=1}^{7}\mathcal{E}_i$), the Poincaré series of the curve C (in r variables) determines the combinatorial type of its minimal resolution. We shall show that analogues of these statements hold for divisorial valuations as well.

Let $(C,0) \subset (S,0)$ be an irreducible curve on the E_8 surface singularity $(S,0)$ such that its minimal embedded resolution does not start from a smooth point of the component D_8 of the minimal resolution of $(S,0)$. Let $P_C(t)$ be the Poincaré series of the corresponding (curve) valuation on $\mathcal{O}_{S,0}$. Let us recall that $P_C(t) = \frac{\Delta^C(t)}{1-t}$, where $\Delta^C(t)$ is the Alexander polynomial of the knot $C \cap \mathbb{L}$ in the Poincaré sphere $\mathbb{L}$.

Theorem 1 *The Poincaré series $P_C(t)$ determines the combinatorial type of the minimal embedded resolution of the irreducible curve $C \subset S$ (and therefore the topological type of the knot $(\mathbb{L}, C \cap \mathbb{L})$, where $\mathbb{L}$ is the link of the E_8 surface singularity $(S, 0)$, i. e., the Poincaré sphere).*

Proof. Let $\pi : (\mathcal{X}, \mathcal{D}) \to (S, 0)$ be the minimal embedded resolution of the curve $(C, 0)$ and let $\pi' : (\mathcal{X}', \mathcal{D}') \to (S, 0)$ be the resolution of the surface singularity $(S, 0)$ described in Section 2, $\mathcal{D}' = \bigcup_{\sigma \in \Gamma_0} D_\sigma$. The resolution π' either is the minimal resolution of the singularity $(S, 0)$, or is obtained from the minimal one by blow-ups points inbetween two particular components of it. In the latter case the dual graph of the resolution π' is obtained from the one of the minimal resolution by inserting several vertices inbetween two neighbouring vertices D_i and D_j of the E_8-graph. Each component D_{σ_0} ($\sigma_0 \in \Gamma_0$) inbetween D_i and D_j is characterized by a pair of coprime positive integers s_1 and s_2 so that one has $m_{k\sigma_0} = s_1 m_{ki} + s_2 m_{kj}$ for $1 \le k \le 8$.

Remark. If one knows the numbers i and j of the components and the ratio s_1/s_j for the component D_{σ_0}, one knows the resolution π' itself.

Either the resolution π coincides with the resolution π' (then C is a curvette at a component D_{σ_0} of $\mathcal{D}'$) or it is obtained from π' by a sequence of blow-ups starting at a smooth point of a component D_{σ_0}.

The matrix $(m_{ij}) = -(D_i \cdot D_j)^{-1}$ (minus the inverse of the intersection matrix of the components of the minimal resolution of the E_8 surface singularity) is equal to

$$\begin{pmatrix} 4 & 7 & 10 & 5 & 8 & 6 & 4 & 2 \\ 7 & 14 & 20 & 10 & 16 & 12 & 8 & 4 \\ 10 & 20 & 30 & 15 & 24 & 18 & 12 & 6 \\ 5 & 10 & 15 & 8 & 12 & 9 & 6 & 3 \\ 8 & 16 & 24 & 12 & 20 & 15 & 10 & 5 \\ 6 & 12 & 18 & 9 & 15 & 12 & 8 & 4 \\ 4 & 8 & 12 & 6 & 10 & 8 & 6 & 3 \\ 2 & 4 & 6 & 3 & 5 & 4 & 3 & 2 \end{pmatrix} \tag{4}$$

For $1 \le i \le 8$, let $Q_i = (m_{i8}, m_{i1}, m_{i4}) \in \mathbb{R}^3$. For i, j such that the components D_i and D_j intersect, let I_{ij} be the segment between the points Q_i and Q_j (that is, the set of points of the form $\lambda Q_i + (1 - \lambda) Q_j$ with $0 \le \lambda \le 1$). Let us consider the one-dimensional simplicial complex G in $\mathbb{R}^3$ with the vertices Q_i and the edges I_{ij}. (As an abstract graph G is isomorphic to the E_8-graph.)

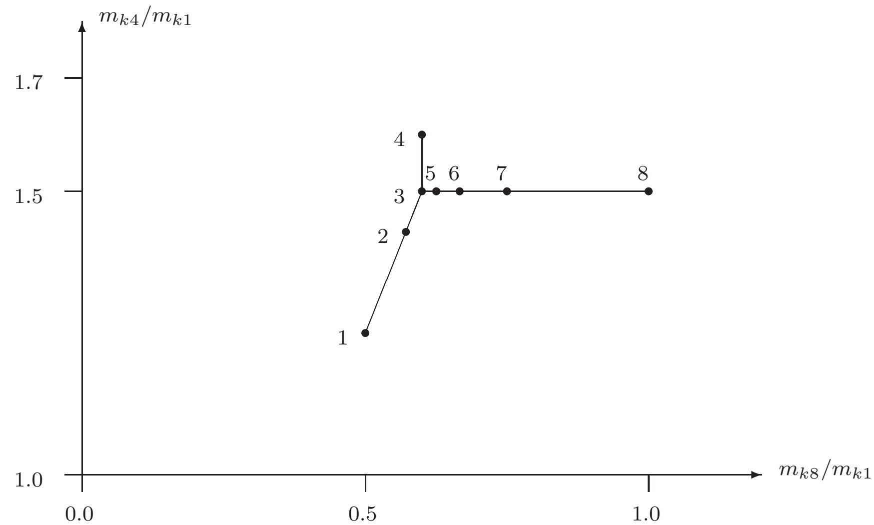

Figure 3: The graph G.

Lemma 1 *The image of the graph G in $\mathbb{RP}^2$ (under the natural quotient map $\mathbb{R}^3 \setminus \{0\} \to \mathbb{RP}^2$) is a planar graph, i. e. images in $\mathbb{RP}^2$ of different points of G are different.*

A proof is obtained by drawing the image of G in an affine chart of $\mathbb{RP}^2$: Figure 3.

Remark. One can see that the graph embedded into the projective plane consists of straight lines in between the rupture points and the deadends. This is a general property for the image in the projective space of the dual resolution graph of a surface singularity under the map which sends a vertex σ to the ratio of the "multiplicities" $m_{\sigma_i \sigma}$ (that is of the elements of the minus inverse of the intersection matrix) for deadends σ_i of the graph.

Lemma 1 says that the ratios of the three coordinates of different points of G never coincide.

Lemma 2 *The Poincaré series $P_C(t)$ of the curve C determines the resolution π' and the component D_{σ_0} in $\mathcal{D}'$ intersecting the strict transform of the curve C.*

Proof. Let us write the Poincaré series $P_C(t)$ in the form

$$\prod_{i=1}^{q}(1-t^{m_i})^{-1}\prod_{m>0}(1-t^m)^{s_m}, \tag{5}$$

where $m_1 \leq m_2 \leq \ldots \leq m_q$ (thus the first product may have repeated factors) and in the second product the (integer) exponents s_m are non-negative and are equal to zero for $m = m_i$, $i = 1, \ldots, q$. Let us recall that the representation of the Poincaré series in this form is unique. This follows from the fact that any series from $1 + t\mathbb{Z}[[t]]$ has a unique representation in the form of a (finite or infinite) product $\prod_{m=1}^{\infty} (1 - t^m)^{s_m}$ with integer exponents s_m.

The representation (5) may not have less than two binomial factors with the exponent (-1). As a rational function the Poincaré series $P_C(t)$ has the form of a polynomial (in fact the Alexander polynomial of the corresponding algebraic link) divided by $(1 - t)$. One cannot have a degree of the binomial $(1 - t)$ in (5) since all the entries of the matrix (4) are greater than 1. If there is one factor $(1 - t^{m_1})^{-1}$ with $m_1 > 1$, its poles at the degree m_1 roots of 1 different from 1 have to be zeroes of the second product $\prod_m (1 - t^m)^{s_m}$. This implies that this product contains a binomial $(1 - t^{km_1})$ with a non-zero exponent s_{km_1} and therefore the series itself is a polynomial.

If the strict transform $\widetilde{C}$ (in the space $\mathcal{X}'$ of the resolution π') intersects the component D_1, then the ratio m_2/m_1 is greater than 2. This follows from the fact that the exponent m_1 is equal to $\ell m_{18} = 2\ell$, where ℓ is the intersection number $\widetilde{C} \cdot D_1$, and the exponent m_2 is either equal to $\ell m_{14} = 5\ell$ or corresponds to a divisor born from D_1 under some blow-ups. In the last case it is greater that $\ell m_{11} = 4\ell$. The ratio $m_2/m_1 > 2$ cannot be met in other cases: see below.

If the strict transform $\widetilde{C}$ does not intersect the component D_1 (and intersects a component D_{σ_0} of the exceptional divisor $\mathcal{D}$), one has $m_1 = \ell m_{\sigma_0 8}$, $m_2 = \ell m_{\sigma_0 1}$, where $\ell = \widetilde{C} \cdot D_{\sigma_0}$. (This follows from the fact that in the matrix (4) one has $m_{i8} \leq m_{i1} \leq m_{ik}$ for $1 < k < 8$ and all i.) From Figure 3 one can see that $1 < m_2/m_1 < 2$. Moreover, if $m_2/m_1 \neq 5/3$, the ratio m_2/m_1 determines the component D_{σ_0} (that is the components D_i and D_j from the minimal resolution graph of the E_8 surface singularity and the corresponding ratio s_1/s_2). The ratio m_2/m_1 is equal to $5/3$ if and only if the strict transform $\widetilde{C}$ intersects either D_3, or D_4, or a component inbetween D_3 and D_4 in $\mathcal{D}'$. In this case if the strict transform $\widetilde{C}$ does not intersect D_4, there are at least three binomial factors with the exponent -1 and one has $m_1 = \ell m_{\sigma_0 8}$, $m_2 = \ell m_{\sigma_0 1}$, $m_3 = \ell m_{\sigma_0 4}$. Lemma 1 implies that the ratio $m_3 : m_2 : m_1$ determines the component D_{σ_0}. If there are less than three binomial factors with the exponent -1 or the ratio $m_3 : m_2 : m_1$ is different from $8 : 5 : 3$, the strict transform $\widetilde{C}$ intersects the component D_4. $\square$

Remark. One can avoid arguments from the first paragraph of the proof formulating the criterium for the strict transform $\widetilde{C}$ not to intersect the component D_1 in the following form: this holds if and only if the Poincaré series $P_C(t)$ contains at least two binomial factors with the exponent (-1) and $m_2/m_1 \leq 2$. The formal negation says that the strict transform $\widetilde{C}$ intersects the component D_1 if and only if either the Poincaré series $P_C(t)$ contains less than two binomial factors with the exponent (-1) or $m_2/m_1 > 2$. The first option does not take place, but this does not contradict the statement. This sort of formulation can be useful for the proof of a version of this Lemma for a divisorial valuation in Section 4.

Lemma 2 says that the Poincaré series $P_C(t)$ determines the (minimal) modification $\pi' : (\mathcal{X}', \mathcal{D}') \to (S, 0)$ and the component D_{σ_0} containing the point $P = \widetilde{C} \cap \mathcal{D}'$ (a smooth point of $\mathcal{D}'$). In particular, one knows the multiplicity $m_{\sigma_0\sigma_0}$. In terms of the decomposition (5) one has $m_1 = \ell m_{8\sigma_0}$ and therefore the intersection number $\ell = \widetilde{C} \cdot D_{\sigma_0}$ is determined by the Poincaré series $P_C(t)$. Let

$$D(t) := (1 - t^{\ell m_{1\sigma_0}})^{-1}(1 - t^{\ell m_{4\sigma_0}})^{-1}(1 - t^{\ell m_{8\sigma_0}})^{-1}(1 - t^{\ell m_{3\sigma_0}})\,.$$

Let $\pi'' : (\mathcal{X}, \mathcal{D}'') \to (\mathcal{X}', P)$ be the minimal embedded resolution of the germ $(\widetilde{C} \cup D_{\sigma_0}, P) \subset (\mathcal{X}', \mathcal{D}')$, $\mathcal{D}'' = \bigcup_{\delta \in \Gamma_1} F_\delta$. The dual graph Γ_1 (with an arrow representing the strict transform of the curve $\widetilde{C}$ by π'') is of the form shown on Figure 4. Here τ_0 marks the divisor corresponding to the first blow-

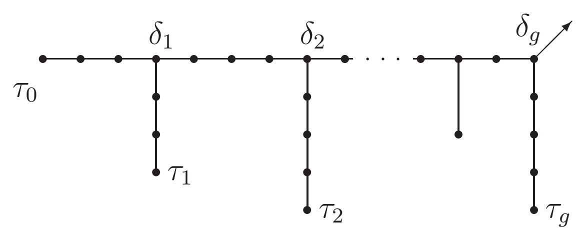

Figure 4: The graph Γ_1.

up at the point P, g is the number of Puiseux pairs of the curve $\widetilde{C}$ and F_{δ_g} is the last component of $\mathcal{D}''$ (i. e., the component with self-intersection -1): the strict transform of $\widetilde{C}$ intersects F_{δ_g}; τ_i, $i = 0, 1, \ldots, g$, are the deadends of the graph Γ_1. Let $(\ell_{\tau_i}, P) \subset (\mathcal{X}', P)$ be a curvette at the component D_{τ_i} and let $\overline{\beta}_i$ be the intersection number $\widetilde{C} \cdot \ell_{\tau_i}$. One has $\overline{\beta}_0 < \overline{\beta}_1 < \cdots < \overline{\beta}_g$ and $\{\overline{\beta}_i | i = 0, 1, \ldots, g\}$ is the minimal set of generators of the semigroup of values of the curve germ $(\widetilde{C}, P)$ (in particular $\overline{\beta}_0$ is the multuiplicity of $\widetilde{C}$). Moreover, the sequence $\overline{\beta}_0, \ldots, \overline{\beta}_g$ determines the graph Γ_1.

Let

$$Q(t) := \prod_{j=0}^{s}(1 - t^{m_{\delta_g\tau_j}})^{-1} \;\cdot\; \prod_{j=1}^{s}(1 - t^{m_{\delta_g\delta_j}})\,.$$

If $\widetilde{C}$ is a curvette at the component D_{σ_0} one defines $Q(t)$ to be equal to 1 and Γ_1 is only an arrow without any vertex. If $\widetilde{C}$ is smooth but $\widetilde{C} \cdot D_{\sigma_0} = \ell > 1$ (that is $\widetilde{C}$ is tangent to D_{σ_0}), we put $Q(t) = (1 - t^{m_{\delta_g \tau_0}})^{-1}$ (the dual graph is shown in Figure 5). Pay attention that

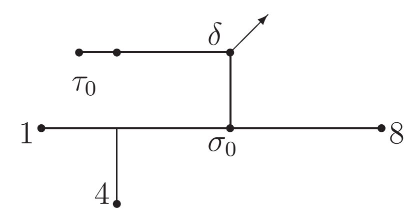

Figure 5: Case 2

$$m_{\delta_g \tau_0} < m_{\delta_g \tau_1} < \ldots < m_{\delta_g \tau_g} ,$$
$$m_{\delta_g \tau_j} < m_{\delta_g \delta_j} < m_{\delta_g \tau_{j+1}} .$$

The modification $\pi = \pi' \circ \pi'' : (\mathcal{X}', \mathcal{D}) \to (S, 0)$ is the minimal resolution of the curve C. The dual graph of this resolution (i. e., the one of the modification π with an arrow corresponding to C added) is obtained by joining the graphs Γ_0 and Γ_1 by an edge between σ_0 and a vertex $\delta \in \Gamma_1$. If $\widetilde{C}$ is a curvette at the component D_{σ_0}, the graph Γ is obtained from the graph Γ_0 by attaching an arrow to the vertex $\sigma_0 \in \Gamma_0$. This case (we refer to it as **Case 1** in the sequel) is characterized by the condition $\ell = 1$. One has

$$P_C(t) = D(t)(1 - t^{m_{\sigma_0 \sigma_0}}) .$$

We will consider several cases corresponding to essentially different possibilities for the position of the vertex δ in Γ_1. Taking into account the fact that $m_{\delta_g \sigma} = \ell m_{\sigma\sigma}$ for all $\sigma \in \Gamma_0$, one can show that in all these cases one has

$$P_C(t) = D(t)Q(t)(1 - t^{m_{\delta_g \delta}})(1 - t^{m_{\delta_g \sigma_0}})$$

(see the discussion of the cases below). The exponent $m_{\delta_g \sigma_0}$ is equal to $\ell m_{\sigma_0 \sigma_0}$ and the series $D(t)$ is known. Therefore the remaining part of the proof consists in the computation of δ, $m_{\delta_g \delta}$ and $\overline{\beta}_0, \ldots, \overline{\beta}_g$ from the (known) series $B(t) = Q(t)(1 - t^{m_{\delta_g \delta}})$.

Let us write the series $B(t)$ in the form

$$B(t) = \prod_{k=1}^{r} (1 - t^{m_k})^{-1} \cdot \prod_{k=1}^{r} (1 - t^{n_k})$$

with $m_1 < \cdots < m_r$ and $n_1 < \cdots < n_r$. In the following μ will denote the integer $\mu = m_1 - m_{\delta_g\sigma_0}$.

Case 2. The curve $\widetilde{C}$ is smooth but $\ell > 1$. In this case one has $\delta = \delta_g$, $m_{\delta_g\tau_0} = \ell m_{\sigma_0\sigma_0} + 1$ and $m_{\delta_g\delta} = \ell(m_{\delta_g\sigma_0} + 1) > m_{\delta_g\tau_0}$. The graph Γ_1 is shown in Figure 5 and the series $B(t)$ is equal to

$$B(t) = (1 - t^{m_{\delta_g\tau_0}})^{-1} \cdot (1 - t^{\ell m_{\delta_g\tau_0}})\,.$$

Note that in this case $\mu = 1$.

Case 3. The curve $\widetilde{C}$ is non smooth but it is transversal to the component D_{σ_0}. In this case $\delta = \tau_0$ and $m_{\delta_g\delta} = m_{\delta_g\tau_0}$. The binomial factor $(1 - t^{m_{\delta_g\delta}}) = (1 - t^{m_{\delta_g\tau_0}})$ does not participate in the decomposition of $B(t)$. The integer $\ell = \overline{\beta}_0$ coincides with the multiplicity of $\widetilde{C}$ at P and one has

$$m_1 = m_{\delta_g\tau_1} = \ell m_{\sigma_0\sigma_0} + \overline{\beta}_1 = m_{\delta_g\sigma_0} + \overline{\beta}_1\ .$$

For $\mu = m_1 - m_{\delta_g,\sigma_0}$ one has $\mu > \ell > 1$. Comparing with all the other cases below, one can see that these conditions characterize the case 3. One has $\overline{\beta}_0 = \ell$, $\overline{\beta}_1 = \mu$ and the graph Γ is shown on Figure 6. Simple computations

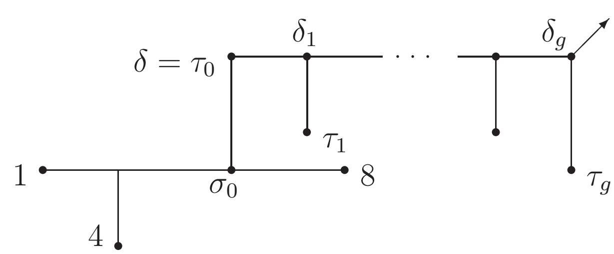

Figure 6: Case 3

using the well-known description of the resolution process of a plane curve singularity give us

$$\begin{aligned} m_{\delta_g\tau_j} &= N_{j-1}\cdots N_1 m_{\delta_g\sigma_0} + \overline{\beta}_j\,, \\ m_{\delta_g\delta_j} &= N_j\cdots N_1 m_{\delta_g\sigma_0} + N_j\overline{\beta}_j = N_j m_{\delta_g\tau_j} \end{aligned}$$

for $j = 1, \ldots, g$. Here $N_k = e_{k-1}/e_k$, where $e_k = \gcd(\overline{\beta}_0, \ldots, \overline{\beta}_k)$ for $k = 0, 1, \ldots, g$. Note that N_k depends only on the numbers $\overline{\beta}_0, \ldots, \overline{\beta}_k$. The equations above permit to compute the sequence $\overline{\beta}_0, \ldots, \overline{\beta}_g$ starting from the (known) sequence $\overline{\beta}_0 < m_{\delta_g\tau_1} < \cdots < m_{\delta_g\tau_g}$.

The expression for $B(t)$ implies that $g = r$, $m_j = m_{\delta_g\tau_j}$ for $j \geq 1$.

Case 4. The component D_{σ_0} has the maximal contact with $\widetilde{C}$, i. e., $\ell = \widetilde{C} \cdot D_{\sigma_0} = \overline{\beta}_1$. In this case $\delta = \tau_1$ and $m_{\delta_g \delta} = m_{\delta_g \tau_1}$. As a consequence the binomial factor $(1 - t^{m_{\delta_g \delta}}) = (1 - t^{m_{\delta_g \tau_1}})$ does not participate in the decomposition of the series $B(t)$. In this case one has

$$m_1 = m_{\delta_g \tau_0} = \ell m_{\sigma_0 \sigma_0} + \overline{\beta}_0 = m_{\delta_g \sigma_0} + \overline{\beta}_0 \ .$$

This case is characterized by the conditions $\mu = m_1 - m_{\delta_g \sigma_0} < \ell$ and μ does not divide ℓ. The graph Γ is shown in Figure 7. As in the previous case one

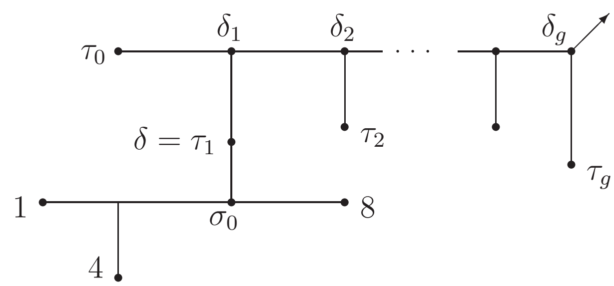

Figure 7: Case 4

has

$$m_{\delta_g \tau_j} = N_{j-1} \cdots N_2 \frac{\overline{\beta}_1}{e_1} m_{\delta_g \sigma_0} + \overline{\beta}_j \ ,$$

$$m_{\delta_g \delta_j} = N_j \cdots N_2 \frac{\overline{\beta}_1}{e_1} m_{\delta_g \sigma_0} + N_j \overline{\beta}_j = N_j m_{\delta_g \tau_j}$$

for $j = 1, \ldots, g$. These equations permit to compute the sequence $\overline{\beta}_0, \ldots, \overline{\beta}_g$ starting from the sequence $m_{\delta_g \tau_2} < \cdots < m_{\delta_g \tau_g}$ and the (known) integers μ and ℓ.

The expression for $B(t)$ implies that $g = r$, $m_j = m_{\delta_g \tau_j}$ for $j \geq 2$, $n_j = m_{\delta_g \delta_j}$ for $j \geq 1$. Thus the integers $m_1, \ldots, m_r$ permit to compute the integers $\overline{\beta}_0, \ldots, \overline{\beta}_g$.

Case 5. The component D_{σ_0} is tangent to $\widetilde{C}$ but does not have the maximal contact with it. This case is equivalent to the condition $\ell = k \cdot \overline{\beta}_0$ for some integer k with $1 < k < \overline{\beta}_1 / \overline{\beta}_0$. The vertex δ is the k-th vertex of the geodesic in Γ_1 from τ_0 to δ_1. In this case one has

$$m_1 = m_{\delta_g \tau_0} = \ell m_{\sigma_0 \sigma_0} + \overline{\beta}_0 = m_{\delta_g \sigma_0} + \overline{\beta}_0 \ .$$

The case is characterized by the conditions $\mu = m_1 - m_{\delta_g \sigma_0} < \ell$ and $\mu (= \overline{\beta}_0)$ divides $\ell (= k \overline{\beta}_0)$. The graph Γ is shown in Figure 8. As in the previous case

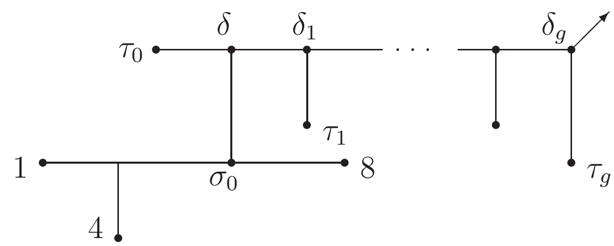

Figure 8: Case 5

one has

$$\begin{aligned} m_{\delta_g\delta} &= k m_{\delta_g\tau_0} = k(m_{\delta_g\sigma_0} + \overline{\beta}_0)\,, \\ m_{\delta_g\tau_j} &= N_{j-1}\cdots N_1 k m_{\delta_g\sigma_0} + \overline{\beta}_j\,, \\ m_{\delta_g\delta_j} &= N_j\cdots N_1 k m_{\delta_g\sigma_0} + N_j\overline{\beta}_j = N_j m_{\delta_g\tau_j} \end{aligned}$$

for $j = 1, \ldots, g$. The expression for $B(t)$ implies that $r = g+1$, $m_j = m_{\delta_g\tau_j}$ for $j \geq 0$; $n_1 = m_{\delta_g\delta}$ and $n_j = m_{\delta_g\delta_{j-1}}$ for $j \geq 2$. Thus the integers $m_1, \ldots, m_r$ permit to compute the intergers $\overline{\beta}_0, \ldots, \overline{\beta}_g$.

Remark. The characterizations of the different possibilities for the location of the vertex δ are already made in the course of the analysis of the different cases above. For a convenience of understanding let us summarize these characterizations. Let $\ell = \widetilde{C}\cdot D_{\sigma_0}$ be the intersection multiplicity between $\widetilde{C}$ and D_{σ_0} and let $\mu := m_1 - m_{\delta_g\sigma_0}$. Then one has

Case	ℓ	μ	$\overline{\beta}_0$	$\overline{\beta}_1$
1	1		1	
2	>1	1	1	
3	>1	$\mu > \ell$	ℓ	μ
4	>1	$\mu < \ell$ and $\mu \not| \ell$	μ	ℓ
5	>1	$\mu < \ell$ and $\mu | \ell$	μ	

□

Remark. The Poincaré series of a curve (reducible or irreducible) on the E_8 surface singularity being either the Alexander polynomial of the corresponding link or the Alexander polynomial divided by $1-t$, is a topological invariant of the curve. Therefore Theorem 1 means that for $i \neq j$, $i, j \leq 7$, all arcs from the component $\mathcal{E}_i$ are not topologically equivalent to arcs from the component $\mathcal{E}_j$ (except those from the intersection of the components).

4 The Poincaré polynomial of one divisorial valuation and the topological type

Let v be a divisorial valuation defined by a component D_{σ^*} of the exceptional divisor $\mathcal{D}$ of a resolution $\pi : (\mathcal{X}, \mathcal{D}) \to (S, 0)$ of the E_8 surface singularity $(S, 0)$. We assume that π is the minimal modification containing the component defining the valuation, i. e., the minimal resolution of the valuation v. Let $\pi' : (\mathcal{X}', \mathcal{D}') \to (S, 0)$ be the resolution of the surface described above and let D_{σ_0} be the component of the exceptional divisor $\mathcal{D}'$ which either coincides with D_{σ^*} or is such that D_{σ^*} is born by a sequence of blow-ups starting at a point of D_{σ_0} (smooth in $\mathcal{D}'$). Assume that $\sigma_0 \neq 8$. This means that the component D_{σ^*} does not originate from a point of D_8 (smooth in $\mathcal{D}'$).

Theorem 2 *In the described situation the Poincaré series $P_v(t)$ of the divisorial valuation v determines the combinatorial type of the minimal resolution of the valuation.*

Proof. One has the following analogue of Lemma 2.

Lemma 3 *The Poincaré series $P_v(t)$ of the divisorial valuation v determines the resolution π' and the component D_{σ_0} in $\mathcal{D}'$.*

Proof. The proof is essentially the same as of Lemma 2. The component D_{σ_0} coincides with D_1 if and only if either the Poincaré series $P_v(t)$ contains less than two binomial factors with the exponent (-1) or $m_2/m_1 > 2$. (Again, as in the curve case, the first option does not take place, but it is easier to avoid a proof of that.) If $D_{\sigma_0} \neq D_1$, the Poincaré series $P_v(t)$ contains at least two binomial factors with the exponent (-1) and one has $m_1 = \ell m_{\sigma_0 8}$, $m_2 = \ell m_{\sigma_0 1}$, where ℓ is the intersection number of the strict transform in $\mathcal{X}'$ of a curvette at the component defining the valuation with the component D_{σ_0}. One has $1 < m_2/m_1 < 2$. If $m_2/m_1 \neq 5/3$, the ratio m_2/m_1 determines the component D_{σ_0}. The ratio m_2/m_1 is equal to 5/3 if and only if the component D_{σ_0} either coincides with D_3, or coincides with D_4, or is produced by blow-ups inbetween D_3 and D_4. If $D_{\sigma_0} \neq D_4$, the Poincaré series contains at least three binomial factors with the exponent -1 and the ratio $m_3 : m_2 : m_1$ determines the component D_{σ_0} (Lemma 1). If the Poincaré series contains less than three binomial factors with the exponent -1 or the ratio $m_3 : m_2 : m_1$ is different from $8 : 5 : 3$, one has $D_{\sigma_0} = D_4$. □

The vertices σ^* and σ_0 coincide if and only if

$$P_v(t) = (1 - t^{m_{1\sigma_0}})^{-1}(1 - t^{m_{4\sigma_0}})^{-1}(1 - t^{m_{8\sigma_0}})^{-1}(1 - t^{m_{3\sigma_0}})\,.$$

Let $\sigma^* \neq \sigma_0$ and let $\pi'' : (\mathcal{X}, \mathcal{D}'') \to (\mathcal{X}', P)$ be the minimal modification of $\mathcal{X}'$ containing the component D_{σ^*} defining the valuation v ($P \in D_{\sigma_0}$). Let $(\ell_{\sigma^*}, P) \subset (\mathcal{X}', P)$ be a curvette at the component D_{σ^*}. The dual graph of the modification π'' differs from the dual graph of the minimal resolution of the curve $\widetilde{C} = \ell_{\sigma^*}$ by a tail of length $k \geq 0$ attached to the vertex δ_g corresponding to the curve $\widetilde{C}$: see Figure 9. The only difference with the case

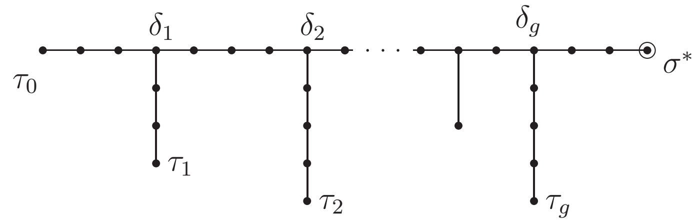

Figure 9: Resolution graph for a divisorial valuation.

of an irreducible curve treated in Theorem 1 above (and applied to the curve $C = \pi'(\widetilde{C})$) consists in the necessity to find the length k of the tail. In the case $k = 0$ the vertex σ^* just coincides with δ_g. In this case the Poincaré series $P_v(t)$ does not contain the factor $(1 - t^{m_{\delta_g \delta_g}})$ (since now there is no arrow at the vertex δ_g). If $k > 0$ then, in order to obtain the Poincaré series $P_v(t)$, one has to add the factor $(1 - t^{m_{\sigma^* \sigma^*}})^{-1}$ to the decomposition of the Poincaré series $P_C(t)$. In any case one has

$$P_v(t) = P_C(t)(1 - t^{m_{\sigma^* \sigma^*}})^{-1}.$$

The intersection number $\ell = \widetilde{C} \cdot D_{\sigma_0}$ can be determined from the Poincaré series $P_v(t)$ just in the same way as for a curve valuation.

As a consequence, the proof for a divisorial valuation almost repeats the one of Theorem 1 for an irreducible curve with the additional duty to determine the component (vertex) σ^* and the multiplicity $m_{\sigma^* \sigma^*}$. The analogue of the series $B(t)$ considered in the proof of Theorem 1 is the series

$$B_v(t) = B(t)(1 - t^{m_{\sigma^* \sigma^*}})^{-1} = Q(t)(1 - t^{m_{\delta_g \delta}})(1 - t^{m_{\sigma^* \sigma^*}})^{-1}.$$

(Notice that for all σ not in the tail one has $m_{\sigma^* \sigma} = m_{\delta_g \sigma}$.) Let us write the series $B_v(t)$ in the form

$$B_v(t) = \prod_{k=1}^{r} (1 - t^{m_k})^{-1} \cdot \prod_{k=1}^{r-1} (1 - t^{n_k})$$

with $m_1 < \cdots < m_r$ and $n_1 < \cdots < n_{r-1}$.

Case 2. The component of the exceptional divisor corresponding to the vertex σ^* is produced by k supplementary blow-ups at smooth points starting at a point of δ_g. If $k = 0$, i. e., if $\sigma^* = \delta_g = \delta$, the series $B_v(t)$ consists only of one term $(1 - t^{m_{\sigma^*\tau_0}})^{-1}$. Otherwise

$$B_v(t) = (1 - t^{m_1})^{-1}(1 - t^{m_2})^{-1}(1 - t^{n_1})$$

and $k = m_2 - n_1$.

Cases 3, 4, 5. From the proof of Theorem 1, it follows that the integers $m_1, \ldots, m_i$ alongside with $\overline{\beta}_0$ and $\overline{\beta}_1$ permit to determine the numbers $\overline{\beta}_0, \ldots, \overline{\beta}_i$. For $i = 1, \ldots, r$ let $\varepsilon_i := \gcd(\overline{\beta}_0, \overline{\beta}_1, m_1, \ldots, m_i)$. One can see that $k = 0$ if and only if $\varepsilon_r < \varepsilon_{r-1}$. In this case $\sigma^* = \delta_g$. Otherwise one has $k = m_r - n_{r-1}$, $m_{\sigma^*\sigma^*} = m_{\delta_g\delta_g} + k$. $\square$

5 The Poincaré polynomial of a collection of divisorial valuations and the topological type

Let v_i, $i = 1, \ldots, r$, be divisorial valuations defined by components of the exceptional divisor $\mathcal{D}$ of a resolution $\pi : (\mathcal{X}, \mathcal{D}) \to (S, 0)$ of the E_8 surface singularity $(S, 0)$. We assume that π is the minimal modification containing the components defining the valuations, i. e., the minimal resolution of the collection $\{v_i\}$ of valuations. The resolution π can be obtained from the minimal resolution of the E_8 surface singularity $(S, 0)$ by a sequence of blow-ups such that at first some of them are made at intersection points of the components of the exceptional divisor (and produce a resolution $\pi' : (\mathcal{X}', \mathcal{D}') \to (S, 0)$ with a "three tails" dual graph) and later additional blow-ups do not touch the intersection points of the components of $\mathcal{D}'$, but start from smooth points of $\mathcal{D}'$. Assume that the modification π of π' does not include blow-ups of smooth (in $\mathcal{D}'$) points of the component D_8.

Theorem 3 *In the described situation the Poincaré series $P_{\{v_i\}}(\underline{t})$, $\underline{t} = (t_1, \ldots, t_r)$, of the collection $\{v_i\}$ of divisorial valuations determines the combinatorial type of the minimal resolution of collection.*

Proof. Equation 3 implies the following *projection formula*: if $\{i_1, \ldots, i_\ell\} \subset \{1, \ldots, r\}$, then the Poincaré series of the ℓ-index filtration corresponding to the divisorial valuations $v_{i_1}, \ldots, v_{i_\ell}$ is obtained from the Poincaré series $P_{\{v_i\}}(t_1, \ldots, t_r)$ by substituting the variables t_i with $i \notin \{i_1, \ldots, i_\ell\}$ by 1.

Remark. The last property does not hold for the Poincaré series of the filtration defined by a collection of curve valuations. This makes the proof of the corresponding statement for curves valuations (Theorem 4 below) somewhat more complicated.

The dual graph of the minimal resolution of a set of divisorial valuations is determined by the dual graph of the minimal resolution for each divisor plus the deviation points of the resolutions for each pair of divisors. The projection formula alongside with Theorem 2 imply that the Poincaré series $P_{\{v_i\}}(\underline{t})$ determines the minimal resolution graph of each valuation from the collection and, in particular, the component of the exceptional divisor $\mathcal{D}'$ of the modification π' which the resolution of the divisorial valuation start from. If, for two divisorial valuations from the collection, these starting components are different, one does not need to find the deviation point. Assume that the starting components coincide. In order to find the deviation point, without lost of generality (due to the projection formula) one may assume that $r = 2$, i. e., that the collection consists of these two valuations: v_1 and v_2. In this situation one has the following picture. In the dual resolution graph on the geodesic inbetween the vertices σ_1 and σ_2 (defining the divisorial valuation) the ratio $m_{\sigma\sigma_1}/m_{\sigma\sigma_2}$ (as a function on σ) is strictly monotonous being maximal at the vertex σ_1 and minimal at σ_2; on the components of the closure of the complement to this geodesic in the dual graph this ratio is constant: see [4, page 43], see also Proposition 1 below for the same statemant in a somewhat different setting. One of the components includes all the vertices corresponding to the components of the exceptional divisor $\mathcal{D}'$ and, in particular, the vertex 8. The factor $(1-t_1^{m_{8\sigma_1}}t_2^{m_{8\sigma_2}})^{-1}$ participates in the decomposition of the Poincaré series $P_{\{v_1,v_2\}}(t_1,t_2)$. Moreover, its exponent $(m_{8\sigma_1}, m_{8\sigma_2})$ is the minimal one in it. The splitting point of the resolutions of the valuations v_1 and v_2 also participates in the decomposition and is maximal in the described components, i. e., among those with $m_{\sigma\sigma_1}/m_{\sigma\sigma_2} = m_{8,\sigma_1}/m_{8\sigma_2}$. This determines the splitting point between the resolutions of the valuations v_1 and v_2. $\square$

6 The Poincaré polynomial of a reducible curve and the topological type

Let $C = \bigcup_{i=1}^r C_i$ be a (reducible: $r > 1$) curve germ on the surface $(S, 0)$ and let $\pi : (\mathcal{X}, \mathcal{D}) \to (S, 0)$ be the minimal embedded resolution of the curve $(C, 0) \subset (S, 0)$. Let us assume that the resolution process does not contain a blow-up of a smooth (in the exceptional divisor) point of the component D_8 of the minimal resolution of the surface $(S, 0)$.

Theorem 4 *In the described situation the Poincaré series $P_C(\underline{t})$, $\underline{t} = (t_1, \ldots, t_r)$, of the curve $C = \bigcup_{i=1}^{r} C_i$ determines the combinatorial type of the minimal resolution of the curve.*

Proof. We have to show that the Poincaré series $P_C(\underline{t})$ determines the minimal resolution graph Γ of C. In the case under consideration one has a projection formula different of the one for divisorial valuations.

In what follows, let us denote $m_{\sigma\sigma(i)}$ ($\sigma(i)$ is the vertex of Γ such that the component $D_{\sigma(i)}$ of the exceptional divisor $\mathcal{D}$ intersects the strict transform $\widetilde{C}_i$ of the curve C_i) by m^i_σ. Therefore one has $\underline{m}_\sigma = (m^1_\sigma, \ldots, m^r_\sigma)$. The reason (somewhat psychological) for that is the fact that, for a multi-exponent of a term of the Poincaré series $P_C(t_1, \ldots, t_r)$ or of a factor of its decomposition, one knows its components $m^1_\sigma, \ldots, m^r_\sigma$, but does not know the vertex σ. One can say that our aim is to find vertices $\sigma(i)$ corresponding to the curve.

Let $i_0 \in \{1, \ldots, r\}$. The A'Campo type formula (1) for $P_C(\underline{t})$ implies that

$$P_C(\underline{t})_{|t_{i_0}=1} = P_{C\setminus\{C_{i_0}\}}(t_1, \ldots, t_{i_0-1}, t_{i_0+1}, \ldots, t_r) \cdot (1 - \underline{t}^{\underline{m}_{\sigma(i_0)}})_{|t_{i_0}=1} \,. \quad (6)$$

Applying (6) several times one gets

$$P_C(\underline{t})_{|t_j=1 \text{ for } j\neq i_0} = P_{C_{i_0}}(t_{i_0}) \cdot \prod_{i\neq i_0} (1 - t_{i_0}^{m^{i_0}_{\sigma(i)}}) \,. \quad (7)$$

Pay attention to the fact that $m^{i_0}_{\sigma(i)} = m^i_{\sigma(i_0)}$ and therefore the series $P_{C_{i_0}}(t_{i_0})$ can be determined from the Poincaré series $P_C(\underline{t})$ if one knows the multiplicity $\underline{m}_{\sigma(i_0)}$. The strategy of the proof follows the steps from [4] (see also [5]):

1) To detect an index i_0 for which one can find the corresponding multiplicity $\underline{m}_{\sigma(i_0)}$ from the A'Campo type formula for $P_C(\underline{t})$. Then Theorem 2 and equation (7) permit to recover the minimal resolution graph Γ_{i_0} of the curve C_{i_0}. Equation (6) gives the possibility to compute the Poincaré series $P_{C\setminus\{C_{i_0}\}}(t_1, \ldots, t_{i_0-1}, t_{i_0+1}, \ldots, t_r)$ of the curve $C\setminus\{C_{i_0}\}$. By induction one can assume that the resolution graph Γ^{i_0} of the curve $C \setminus \{C_{i_0}\}$ is known.

2) To determine the separation vertex of the curves C_{i_0} and C_j for $j \neq i_0$ in order to join the graphs Γ_{i_0} and Γ^{i_0} to obtain the resolution graph Γ.

Once we finish the first step, the second one almost repeats the same steps in the proof of Theorem 3 (for divisorial valuations). Therefore we omit the analysis of 2).

Let $[\sigma(j), \sigma(i)] \subset \Gamma$ be the (oriented) geodesic from $\sigma(j)$ to $\sigma(i)$ and let $\{\Delta_p\}$, $p \in \Pi$, be the connected components of $\Gamma \setminus [\sigma(j), \sigma(i)]$. For each $p \in \Pi$ there exists an unique $\rho_p \in [\sigma(j), \sigma(i)]$ connecting Δ_p with $[\sigma(j), \sigma(i)]$, i. e., such that $\Delta_p^* = \Delta_p \cup \{\rho_p\}$ is connected.

Proposition 1 *Let $q : \Gamma \to \mathbb{Q}$ be the function defined by $q(\alpha) = m_\alpha^j / m_\alpha^i$ for $\alpha \in \Gamma$. Then one has:*

1. *The function q is strictly decreasing along the geodesic $[\sigma(j), \sigma(i)]$.*
2. *For each $p \in \Pi$, the function q is constant on Δ_p^*.*

Proof. Let $\overline{C}_k$ $(k = 1, \ldots, r)$ be the total transform of the curve C_k in $\mathcal{X}$. One has

$$\overline{C}_k = \widetilde{C}_k + \sum_{\sigma \in \Gamma} m_\sigma^k D_\sigma \ ,$$

where $\widetilde{C}_k$ is the strict transform of the curve C_k. For each component D_α, $\alpha \in \Gamma$, one has $\overline{C}_k \cdot D_\alpha = 0$ and therefore

$$\widetilde{C}_k \cdot D_\alpha + \sum_{\sigma \in \Gamma} m_\sigma^k \, D_\sigma \cdot D_\alpha = 0. \tag{8}$$

This equation is a consequence of the Mumford formula (see [10, Equation (1)]) applied to the function defining the curve C_k. (Let us recall that on the E_8 surface singularity $(S, 0)$ all divisors are Cartier ones.)

Lemma 4 *Let D_α be a component of the exceptional divisor $\mathcal{D}$ such that $\widetilde{C}_i \cdot D_\alpha = 0$ and let $\{\rho_1, \ldots, \rho_s\} \subset \Gamma$ be the set of all vertices connected by an edge with α. Let us assume that either $\widetilde{C}_j$ intersects D_α or there exists ρ_{i_0} such that $q(\rho_{i_0}) > q(\alpha)$. Then there exists ρ_k such that $q(\alpha) > q(\rho_k)$.*

Proof. Assume that $q(\rho_k) \geq q(\alpha)$ for any $k = 1, \ldots, s$. Applying (8) to C_j and C_i one gets:

$$\begin{aligned} 0 &= \widetilde{C}_j \cdot D_\alpha + m_\alpha^i D_\alpha^2 + \sum_{k=1}^{s} m_{\rho_k}^j \geq \\ &\geq \widetilde{C}_j \cdot D_\alpha + m_\alpha^i D_\alpha^2 + \sum_{k=1}^{s} q(\alpha) m_{\rho_k}^i \geq \\ &= \widetilde{C}_j \cdot D_\alpha + q(\alpha)(m_\alpha^j D_\alpha^2 + \sum_{k=1}^{s} m_{\rho_k}^i) = \widetilde{C}_j \cdot D_\alpha \geq 0 \end{aligned}$$

The inequality is strict if $\widetilde{C}_j \cdot D_\alpha > 0$ or if there exists i_0 such that $q(\rho_{i_0}) > q(\alpha)$. This implies the statement. $\square$

Let α and β be two vertices of Γ connected by an edge and let $q(\alpha) > q(\beta)$. Lemma 4 permits to construct a maximal sequence $\alpha_0, \alpha_1, \ldots, \alpha_k$ of consecutive vertices starting with α and β (i. e., $\alpha_0 = \alpha$, $\alpha_1 = \beta$) such that $q(\alpha_i) > q(\alpha_{i+1})$. (We will call a sequence of this sort *a decreasing path*. If the inequality is in the other direction, the path will be called *increasing*.) The maximality means that either α_k is a deadend of Γ or $\widetilde{C}_i \cdot D_{\alpha_k} \neq 0$. If α_k is a deadend, α_{k-1} is the only vertex connected with α_k and Lemma 4 implies that $q(\alpha_k) = q(\alpha_{k-1})$. Therefore the constructed path finishes by the vertex $\alpha_k = \sigma(i)$. Note that, if $\alpha \in [\sigma(j), \sigma(i)]$ and $\beta \notin [\sigma(j), \sigma(i)]$, the end of a maximal decreasing (or increasing) path has to finish at a deadend and therefore $q(\alpha) = q(\beta)$. In particular, this implies that the function q is constant on each connected set Δ_p^*.

Assume that $\sigma(i) \neq \sigma(j)$. Lemma 4 implies that there exists a vertex α_1 connected with $\sigma(j)$ such that $q(\sigma(j)) > q(\alpha_1)$. Therefore the maximal decreasing path starting with $\sigma(j)$ and α_1 coincides with the geodesic $[\sigma(j), \sigma(i)]$. $\square$

Proposition 1 implies that, for any fixed i_0 and for any $j \neq i_0$ and $\sigma \in \Gamma$, one has $m_\sigma^j / m_\sigma^{i_0} \geq m_{\sigma(i_0)}^j / m_{\sigma(i_0)}^{i_0}$. Therefore one has

$$\frac{1}{m_\sigma^{i_0}} \underline{m}_\sigma \geq \frac{1}{m_{\sigma(i_0)}^{i_0}} \underline{m}_{\sigma(i_0)} \, .$$

Let $P_C(\underline{t}) = \prod_{k=1}^{p} (1 - \underline{t}^{\underline{n}_k})^{s_k}$ be the Poincaré series of the curve C, where $s_k \neq 0$ for all k. For $i \in \{1, \ldots, r\}$ let $k = k(i)$ be such that

$$\frac{1}{n_j^i} \underline{n}_j \geq \frac{1}{n_k^i} \underline{n}_k$$

for all j. Let $E \subset \{1, \ldots, p\}$ be the set of indices k such that $k = k(i)$ for some $i \in \{1, \ldots, r\}$ and for $k \in E$ let $A(k) \subset \{1, \ldots, r\}$ denote the set of indices i such that $k = k(i)$. Note that $A(k)$ contains all the indices $i \in \{1, \ldots, r\}$ such that $\underline{n}_k = \underline{m}_{\sigma(i)}$. Let $B(k)$ be the subset of such indices. Our aim is to show that one can find $k \in E$ such that $B(k) \neq \emptyset$.

Let $j \in A(k)$, $j \notin B(k)$. One has

$$\frac{1}{n_k^j} \underline{n}_k > \frac{1}{m_{\sigma(j)}^j} \underline{m}_{\sigma(j)}$$

and therefore $\chi(\overset{\circ}{D}_{\sigma(j)}) = 0$. This implies that $\sigma(j)$ is connected with only one vertex in Γ (plus the arrow corresponding to $\widetilde{C}_j$), i. e., $\sigma(j)$ is a deadend of the resolution graph of the curve $C \setminus \{C_j\}$. In particular, there are at most two indices $i, j \in A(k)$ such that $\underline{m}_{\sigma(i)}$ and $\underline{m}_{\sigma(j)}$ are different from $\underline{n}_k$. Moreover, if there are two indices of this sort, the vertex $\sigma \in \Gamma$ such that $\underline{n}_k = \underline{m}_\sigma$ is the vertex 3 corresponding to the divisor D_3 of the minimal resolution of $(S, 0)$. In fact in this case the strict transforms $\widetilde{C}_i$ and $\widetilde{C}_j$ are curvettes at the divisors D_1 and D_4. Therefore, if $\#A(k) \geq 3$, there exists $i_0 \in B(k)$.

Let $k \in E$ be such that $B(k) = \emptyset$ and let us assume that $\underline{n}_k = \underline{m}_3$ (i. e., that the multiplicity $\underline{n}_k$ is the multiplicity of the divisor D_3). Let $\underline{m}_8 = (m_8^1, \ldots, m_8^r)$ be the multiplicity of the divisor D_8. Notice that $\underline{m}_8$ is determined by the Poincaré series $P_C(\underline{t})$ because the decomposition of the Poincaré series contains the factor $(1 - \underline{t}^{\underline{m}_8})^{-1}$ and, moreover, the multiplicity $\underline{m}_8$ is the smallest one appearing in it.

If there exists $i \in A(k)$ such that $\widetilde{C}_i$ is a curvete at D_1, then $m_8^i = 2$ (note that $m_8^i = 2$ implies that $\widetilde{C}_i$ is a curvette at D_1). In this case, if $A(k) \neq \{1, \ldots, r\}$, one can choose any other $k' \in E$ instead of k. If $A(k) = \{1, \ldots, r\}$, then $r = 2$ and the branches C_1 and C_2 are curvettes at the divisors D_1 and D_4 (see Figure 10). This situation is equivalent to have the Poincaré series of the form $P_C(t_1, t_2) = (1 - \underline{t}^{(2,3)})^{-1}(1 - \underline{t}^{(10,15)})$, what gives the statement in this case. Note also that in this case $m_8^2 = 3$. If $A(k) = \{i\}$ and $\widetilde{C}_i$ is a curvette at the divisor D_4 then one has $m_8^i = 3$. However this condition does not characterize completely the situation described: for $\underline{n}_{k'} = \underline{m}_7$ and $A(k') = \{j\}$ with $\widetilde{C}_j$ a curvette at D_7 one has also that $m_8^j = 3$. If the both multiplicities appear simultaneously, one can distinguish the first one because $\underline{n}_{k'} = \underline{m}_7$ is always a multiple of $\underline{m}_8$ (see Proposition 1) but $\underline{n}_k$ is not (in the presence of k'). This permits to determine the index k in this case from the information given by the series $P_C(\underline{t})$.

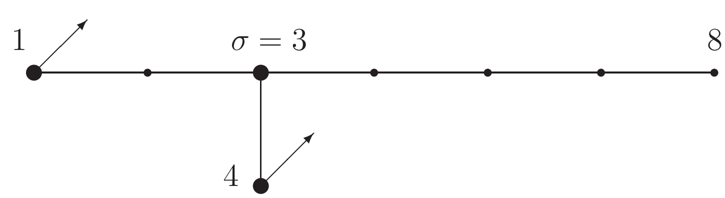

Figure 10: The case $\underline{n}_k = \underline{m}_3$, $r = 2$, $B(k) = \emptyset$

Let us now consider the case when one has $B(k) = \emptyset$ and $\underline{n}_k = \underline{m}_\sigma \neq \underline{m}_3$ for some $\sigma \in \Gamma$. In this case one has $A(k) = \{i\}$ and $\sigma(i)$ is a deadend of the dual resolution graph of the curve $C \setminus \{C_i\}$. In particular, the vertex σ appears after $\sigma(i)$ in the resolution process of a certain branch C_j, $j \neq i$, which is not a curvette at D_σ. It is clear that in this case $\underline{n}_k < \underline{m}_{\sigma(j)}$ and also $\underline{n}_k < \underline{n}_{k(j)}$.

Thus in this case we take $k' = k(j)$ and $\underline{n}_{k'}$ instead of k and $\underline{n}_k$. Iterating this procedure one gets k' such that $B(k') \neq \emptyset$. Note that this situation can be determined from $P_C(\underline{t})$ taking $k \in E$ such that $\underline{n}_k$ is maximal among the elements $\underline{n}_k$ for $k \in E$ not excluded on the previous stages.

Once we have an index $k \in E$ such that $B(k) \neq \emptyset$ we have to choose an index $i_0 \in B(k)$. Since $n_k^i > n_k^j$ for $i \in B(k)$ and $j \in A(k) \setminus B(k)$ and $n_k^i = n_k^j$ for $i, j \in B(k)$, for the role of i_0 one can take an index from $A(k)$ such that $n_k^{i_0}$ is the maximal one in $\{n_k^i : i \in A(k)\}$. This finishes the step 1) of the proof and thus the proof itself. $\square$

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