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A note on the Moll–Arias de Reyna integral

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Abstract
The Moll–Arias de Reyna integral

$$\int_0^\infty \frac{\mathrm{d}x}{(x^2+1)^{3/2}} \frac{1}{\sqrt{\varphi(x)+\sqrt{\varphi(x)}}} \quad \text{where } \varphi(x) = 1 + \frac{4}{3}\left(\frac{x}{x^2+1}\right)^2$$

is generalized and several values are given.

Keywords Definite integral · Elliptic integral · Elliptic modulus · Algebraic integrand

Mathematics Subject Classification Primary 33E05, 33E20

1 Introduction

We define

$$f(a,b) = \int_0^\infty \frac{\mathrm{d}x}{(x^2+1)^a} \frac{1}{\sqrt{\varphi(x)+\sqrt{\varphi(x)}}} \tag{1.1}$$

where

$$\varphi(x) = 1 + 4b^{-2}u^2, \qquad u = \frac{x}{x^2+1}. \tag{1.2}$$

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The value $f(3/2,\sqrt{3})=\frac{\pi}{2\sqrt{6}}$ appeared as entry 3.248.5 in [5] and was shown to be incorrect by Moll et al. [1]. Unable to find the correct evaluation, the editors decided to take this entry out of later editions of the table [6]. The exact value

$$f(3/2,\sqrt{3})=\frac{\sqrt{3}-1}{2}\Pi\Big(\frac{\pi}{2},3-\sqrt{3},3^{-1/2}\Big)-6^{-1/2}F\Big(\sin^{-1}\sqrt{2-\sqrt{3}},3^{-1/2}\Big) \tag{1.3}$$

was recently provided in a mathematical *tour de force* by Arias de Reyna [2]. The aim of the present note is to provide further values of (1.1) and to suggest that the incorrect value in [5] is not merely a misprint within the scope of the parametrization (1.2).

2 Calculation

Factor $\sqrt{\varphi(x)}$ from the denominator of the integrand of (1.1), multiply the numerator and denominator by $\sqrt{\sqrt{\varphi(x)}-1}$, change the integration variable to u (note that the range of x must be divided into $[0,1]\cup[1,\infty]$) and set $s=2u$ to obtain

$$f(a,b) = 2^{-a}b\int_0^1\frac{\mathrm{d}s}{s\sqrt{1-s^2}}\left\{[1+\sqrt{1-s^2}]^{a-1}+[1-\sqrt{1-s^2}]^{a-1}\right\}\sqrt{1-\frac{b}{\sqrt{b^2+s^2}}} \tag{2.1}$$

Since both quadratic surds can be rationalized by the elliptic substitution $s=\mathrm{cn}(\kappa,x)$ for a suitable modulus, $f(a,b)$ should be expressible in terms of elliptic integrals for integer and half-integer values of a, even in the trigonometric case $\kappa=0$. For example, it is clear that

$$f(2,b)=\frac{1}{2}f(1,b) \tag{2.2}$$

and with the substitution $t=b/\sqrt{b^2+s^2}$

$$f(1,b)=k\int_\kappa^1\frac{\mathrm{d}t}{(t+1)\sqrt{(1-t)(t^2-k^2)}},\quad k=\frac{b}{\sqrt{b^2+1}}. \tag{2.3}$$

which is a complete elliptic integral of the third kind which can be manipulated into standard form [4]

$$f(1,b)=\frac{\kappa}{\sqrt{k+1}}\Pi\left(\frac{\pi}{2},\alpha^2,\kappa\right)$$

with

$$\alpha^2 = \frac{\sqrt{b^2+1}+b}{2\sqrt{b^2+1}}, \qquad \kappa = \sqrt{b^2+1}-b. \tag{2.4}$$

For $a = 3, 4, 5, \ldots,\ f(a, b)$, with $x = s^2$, can be seen to be a multiple of $f(1, b)$ plus an integral of the form

$$\int_0^1 \frac{P(x)}{\sqrt{1-x}}\sqrt{1-\frac{b}{\sqrt{b^2+x}}}\,\mathrm{d}x, \tag{2.5}$$

where P is a polynomial. Such an integral can be transformed into a sum of elliptic integrals by the substitutions $x \to 1-x^2$, $x \to \sqrt{b^2+1}\sin t$. For example,

$$f(3, b) = \frac{1}{2}f(1, b) - \frac{b^2}{4k}\int_k^1 \sqrt{\frac{x(x-k)}{1-x^2}}\mathrm{d}x. \tag{2.6}$$

For $a = 3/2$ (2.4) yields

$$f(3/2, b) = \frac{b}{4}\int_0^{\pi/2} [\csc(t/2)+\sec(t/2)]\sqrt{1-\frac{b}{\sqrt{b^2+\sin^2 t}}}\,\mathrm{d}t. \tag{2.7}$$

This can be further simplified by the substitutions $\sin t = b\sin u,\ \ \sin u = x$, $\sqrt{1+x^2} = 1/y$ to

$$f(3/2, \sqrt{3}) = \frac{3}{\sqrt{8}}\sum_{\pm}\int_{\sqrt{3}/2}^1 \frac{\mathrm{d}y}{\sqrt{y(1+y)(4y^2-3)(y\pm\sqrt{4y^2-3})}} \tag{2.8}$$

which may be reduced further to

$$f(2/3, \sqrt{3}) = \frac{3^{1/4}}{2}\int_0^{1/\sqrt{3}} \frac{\mathrm{d}x}{X^{3/2}\sqrt{4-3X^2}}\,\frac{\sqrt{X-2x}+\sqrt{X+2x}}{\sqrt{X+2/\sqrt{3}}} \tag{2.9}$$

with $X = \sqrt{x^2+1}$ and which offers an alternative approach to (1.3).

3 Discussion

Very recently a preprint by Blaschke [3] has appeared pointing out that if the nested square root in (1.1) is replaced by the three halves power the value $\pi/2\sqrt{6}$ is obtained. Thus the error in [5] is indeed merely a misprint. Nevertheless, we examine the possibility of reproducing this value by a specific choice of (a, b) in (1.1). To keep things simple, take $b = i$, so that $b\sqrt{s^2+b^2} = (1-s^2)^{-1/2}$ thus eliminating one of the surds in (1.1). Then one has

$$f(a,i) = 2^{-a}\int_0^1 \frac{\mathrm{d}s}{s\sqrt{1-s^2}}\left\{[1+\sqrt{1-s^2}]^{a-1}+[1-\sqrt{1-s^2}]^{a-1}\right\}$$
$$\times\sqrt{\frac{1}{\sqrt{1-s^2}}-1}$$
$$= 2^{1-a}\int_0^1\left[\frac{(1+x^2)^{a-2}}{\sqrt{1-x^2}}+\frac{(1-x^2)^{a-1/2}}{\sqrt{1+x^2}}\right]\mathrm{d}x$$
$$= 2^{-a}\pi\left[{}_2F_1(1/2,2-a;1;-1)+\frac{\Gamma(1-a)}{\sqrt{\pi}\Gamma(a-1/2)}\right].$$

On solving $f(a,i) = \pi/2\sqrt{6}$ numerically one finds the two values

$$f(0.701935\ldots,i) = f(11.8052\ldots,i) = \frac{\pi}{2\sqrt{6}}.$$

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