Nonlinear Supersymmetry as a Hidden Symmetry

Mikhail S. Plyushchay

Abstract Nonlinear supersymmetry is characterized by supercharges to be higher order in bosonic momenta of a system, and thus has a nature of a hidden symmetry. We review some aspects of nonlinear supersymmetry and related to it exotic supersymmetry and nonlinear superconformal symmetry. Examples of reflectionless, finite-gap and perfectly invisible $\mathcal{PT}$-symmetric zero-gap systems, as well as rational deformations of the quantum harmonic oscillator and conformal mechanics, are considered, in which such symmetries are realized.

Keywords Hidden symmetry · Exotic supersymmetry · Nonlinear superconformal symmetry · Reflectionless and finite-gap systems · Perfect invisibility

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1 Introduction

Hidden symmetries are associated with integrals of motion of higher-order in momenta. They mix the coordinate and momenta variables in the phase space of a system, and generate a nonlinear, W-type algebras [1]. The best known examples of hidden symmetries are provided by the Laplace–Runge–Lenz vector integral in the Kepler–Coulomb problem, and the Fradkin–Jauch–Hill tensor in isotropic harmonic oscillator systems. Hidden symmetries also appear in anisotropic oscillator with commensurable frequencies, where they underlie the closed nature of classical trajectories and specific degeneration of the quantum energy levels. Hidden symmetry is responsible for complete integrability of geodesic motion of a test particle in the background of the vacuum solution to the Einstein's equation represented by the Kerr metric of the rotating black hole and its generalizations in the form of the Kerr-NUT-(A)dS solutions of the Einstein–Maxwell equations

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[2]. Another class of hidden symmetries underlies a complete integrability of the field systems described by nonlinear wave equations such as the Korteweg–de Vries (KdV) equation. Those symmetries are responsible for peculiar properties of the soliton and finite-gap solutions of the KdV system, whose equation of motion can be regarded as a geodesic flow on the Virasoro-Bott group [3, 4].

Nonlinear supersymmetry [5–45] is characterized by supercharges to be higher order in even (bosonic) momenta of a system, and thus has a nature of hidden symmetry. Here, we review some aspects of nonlinear supersymmetry, and related to it exotic supersymmetry and nonlinear superconformal symmetry.

Nonlinear supersymmetry appears, particularly, in purely parabosonic harmonic oscillator systems generated by the deformed Heisenberg algebra with reflection [12] as well as in a generalized Landau problem [15]. The peculiarity of supersymmetric parabosonic systems shows up in the nonlocal nature of supercharges to be of infinite order in the momentum operator as well as in the ladder operators but anti-commuting for a polynomial in Hamiltonian being quadratic in creation-annihilation operators. Similar peculiarities characterize hidden supersymmetry and hidden superconformal symmetry appearing in some usual quantum bosonic systems with a local Hamiltonian operator [20, 21, 24–26, 30–32, 35, 46–51]. Exotic supersymmetry emerges in superextensions of the quantum systems described by soliton and finite-gap potentials, in which the key role is played by the Lax–Novikov integrals of motion [30–33, 42]. A structure similar to that of the exotic supersymmetry of reflectionless and finite-gap quantum systems can also be identified in the "SUSY in the sky" type supersymmetry [52–55] based on the presence of the Killing–Yano tensors in the abovementioned class of the black hole solutions to the Einstein–Maxwell equations. Nonlinear superconformal symmetry appears in rational deformations of the quantum harmonic oscillator and conformal mechanics systems [49, 51]. Both exotic supersymmetry and nonlinear superconformal symmetry characterize the interesting class of the perfectly invisible zero-gap $\mathcal{PT}$-symmetric systems, which includes the $\mathcal{PT}$-regularized two-particle Calogero systems and their rational extensions with potentials satisfying the equations of the KdV hierarchy and exhibiting a behavior of extreme (rogue) waves [56, 57].

2 Nonlinear Supersymmetry and Quantum Anomaly

Classical analog of the Witten's supersymmetric quantum mechanics [58–61] is described by the Hamiltonian

$$\mathcal{H} = p^2 + W^2 + W'N\,, \tag{1}$$

where $N = \theta^+\theta^- - \theta^-\theta^+$, $W = W(x)$ is a superpotential, x and p are even canonical variables, $\{x, p\} = 1$, and $\theta^+, \theta^- = (\theta^+)^*$ are Grassmann variables with the only nonzero Poisson bracket $\{\theta^+, \theta^-\} = -i$. System (1) is characterized by the even, N, and odd, $Q_+ = (W + ip)\theta^+$ and $Q_- = (Q_+)^*$, integrals of motion

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satisfying the algebra of $\mathcal{N}=2$ Poincaré supersymmetry

$$\{Q_+, Q_-\} = -i\mathcal{H}\,, \quad \{\mathcal{H}, Q_\pm\} = 0\,, \quad \{N, \mathcal{H}\} = 0\,, \quad \{N, Q_\pm\} = \pm 2i Q_\pm\,. \tag{2}$$

For any choice of the superpotential, canonical quantization of this classical system gives rise to the supersymmetric quantum system in which quantum supercharges and Hamiltonian satisfy the $\mathcal{N}=2$ superalgebra given by a direct quantum analog of the corresponding Poisson bracket relations, with the quantum analog of the integral N playing simultaneously the role of the $\mathbb{Z}_2$-grading operator $\Gamma=\sigma_3$ of the Lie superalgebra.

A simple change of the last term in (1) for $n\mathcal{W}'N$ with n taking any integer value yields a system characterized by a nonlinear supersymmetry of order n generated by the supercharges $S_+ = (\mathcal{W}+ip)^n\theta^+$ and $S_- = (S_+)^*$ being the integrals of order n in the momentum p. Their Poisson bracket $\{S_+, S_-\} = -i(\mathcal{H})^n$ has order n in the Hamiltonian [12, 14, 62]

$$\mathcal{H} = p^2 + \mathcal{W}^2 + n\mathcal{W}'N\,. \tag{3}$$

System (3) can be regarded as a kind of the classical supersymmetric analog of the planar anisotropic oscillator with commensurable frequencies [63, 64]. Unlike a linear case (1) with $n=1$, canonical quantization of the system (3) with $n = 2, 3, \ldots$ faces, however, the problem of quantum anomaly: for arbitrary form of the superpotential, quantum analogs of the classical odd integrals $S_\pm$ cease to commute with the quantum analog of the Hamiltonian (3). In [14], it was found a certain class of superpotentials $\mathcal{W}(x)$ for which the supercharge S_+ has a polynomial structure in $z=\mathcal{W}+ip$ instead of monomial one so that the corresponding systems admit an anomaly-free quantization giving rise to quasi-exactly solvable systems [65–67].

If instead of the "holomorphic" dependence of the supercharge S_+ on the complex variable z we consider the supercharges with polynomial dependence on the momentum variable p, the case of quadratic supersymmetry turns out to be a special one. The Hamiltonian and supercharges then can be presented in the most general form

$$\mathcal{H} = zz^* - \frac{C}{\mathcal{W}^2} + 4\mathcal{W}'N + a\,, \tag{4}$$

$$S_+ = \left(z^2 + \frac{C}{\mathcal{W}^2}\right)\theta^+\,, \qquad S_- = (S_+)^*\,. \tag{5}$$

Here a and C are real constants, and we have

$$\{S_+, S_-\} = -i\left((\mathcal{H}-a)^2 + C\right)\,. \tag{6}$$

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Supersymmetry of the system (4), (5), (6) with an arbitrary superpotential can be preserved at the quantum level if to correct the direct quantum analog of the Hamiltonian and supercharges by adding to them the term quadratic in Plank constant [14, 62]:

$$\hat{\mathcal{H}} - a = -\hbar^2 \frac{d^2}{dx^2} + \mathcal{W}^2 - 2\hbar\sigma_3 \mathcal{W}' - \frac{C}{\mathcal{W}^2} + \Delta(\mathcal{W}) \,, \tag{7}$$

$$\hat{S}_+ = \hat{s}_+ \sigma_+ \,, \qquad \hat{s}_+ = \left(\hbar \frac{d}{dx} + \mathcal{W}\right)^2 + \frac{C}{\mathcal{W}^2} - \Delta(\mathcal{W}) \,, \tag{8}$$

$$\Delta(\mathcal{W}) = \frac{1}{2}\hbar^2 \left(\frac{\mathcal{W}''}{\mathcal{W}} - \frac{1}{2}\left(\frac{\mathcal{W}'}{\mathcal{W}}\right)^2 \right) = \hbar^2 \frac{1}{\sqrt{\mathcal{W}}} \left(\sqrt{\mathcal{W}}\right)'' , \tag{9}$$

where $\sigma_+ = \frac{1}{2}(\sigma_1 + i\sigma_2)$. The quantum term $\Delta(\mathcal{W})$ can be presented as a Schwarzian, $\Delta = -\frac{1}{2}\hbar^2 S(\omega(x))$, $S(\omega(x)) = (\omega''/\omega')' - \frac{1}{2}(\omega''/\omega')^2$, of the function $\omega(x) = \int^x dy/\mathcal{W}(y)$. The quadratic in $\hbar$ terms in the quantum Hamiltonian (7) can be unified and presented in a form similar to that of the kinetic term of the quantum particle in a curved space: $-\hbar^2 \frac{d^2}{dx^2} + \Delta(\mathcal{W}) = \hat{\mathcal{P}}^\dagger \hat{\mathcal{P}}$, where $\hat{\mathcal{P}} = \hbar \zeta^{-1} \frac{d}{dx} \zeta$, $\zeta = 1/\sqrt{\mathcal{W}}$. Analogously, the first and third terms in $\hat{s}_+$ in (8) can be collected and presented in the form $\hat{z}^2 - \Delta(W) = (\zeta \hat{z} \zeta^{-1})(\zeta^{-1} \hat{z} \zeta)$, where $\hat{z} = \hbar \frac{d}{dx} + \mathcal{W}$ [62].

3 Exotic Nonlinear $\mathcal{N} = 4$ Supersymmetry

The anomaly-free prescription for quantization of the classical systems (3) with supersymmetry of order higher than two in general case is unknown, but there exist infinite families of the quantum systems described by supersymmetries of arbitrary order. They can be generated easily by applying the higher order Darboux–Crum (DC) transformations [68–70] to a given, for instance, exactly solvable quantum system instead of starting from a classical supersymmetric system of the form (3) followed by a search for the anomaly-free quantization scheme.

In general case the DC transformation of a given system described by the Hamiltonian operator $\hat{H}_- = -\frac{d^2}{dx^2} + V_-(x)$ is generated by selection of the set of physical or non-physical eigenstates $(\psi_1, \psi_2, \ldots, \psi_n)$ of $\hat{H}_-$ as the seed states. Here and below we put $\hbar = 1$. If they are chosen in such a way that their Wronskian $\mathbb{W}(\psi_1, \ldots, \psi_n)$ takes nonzero values in the region where $V_-(x)$ is defined, then the new potential

$$V_+ = V_- - 2(\ln \mathbb{W}(\psi_1, \ldots, \psi_n))'' \tag{10}$$

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will be regular in the same region as V_-. Physical and non-physical eigenstates of the new Hamiltonian operator $\hat{H}_+ = -\frac{d^2}{dx^2} + V_+$ are obtained from those of the original system $\hat{H}_-$ by the transformation

$$\psi_{+,\lambda} = \frac{\mathbb{W}(\psi_1, \ldots, \psi_n, \psi_\lambda)}{\mathbb{W}(\psi_1, \ldots, \psi_n)} = \mathbb{A}_n \psi_\lambda \,, \tag{11}$$

where ψ_λ is an eigenstate of $\hat{H}_-$ different from eigenstates in the set of the seed states with eigenvalue $E_\lambda \neq E_j$, $j = 1, \ldots, n$. The state $\psi_{+,\lambda}$ is of the same eigenvalue of $\hat{H}_+$ as ψ_λ of $\hat{H}_-$, $\hat{H}_-\psi_\lambda = \lambda\psi_\lambda \Rightarrow \hat{H}_+\psi_{+,\lambda} = \lambda\psi_{+,\lambda}$, and vice versa, from $\hat{H}_+\psi_{+,\lambda} = \lambda\psi_{+,\lambda}$ it follows that $\hat{H}_-\psi_\lambda = \lambda\psi_\lambda$. Operator $\mathbb{A}_n$ in (11) is a differential operator of order n,

$$\mathbb{A}_n = A_n \ldots A_1 \,, \quad A_j = (\mathbb{A}_{j-1}\psi_j)\frac{d}{dx}(\mathbb{A}_{j-1}\psi_j)^{-1} \,, \quad j = 1, \ldots, n \,, \quad \mathbb{A}_0 = 1 \,, \tag{12}$$

which is constructed recursively from the selected seed states. Operators $\mathbb{A}_n$ and $\mathbb{A}_n^\dagger$ intertwine Hamiltonian operators $\hat{H}_-$ and $\hat{H}_+$,

$$\mathbb{A}_n \hat{H}_- = \hat{H}_+ \mathbb{A}_n \,, \qquad \mathbb{A}_n^\dagger \hat{H}_+ = \hat{H}_- \mathbb{A}_n^\dagger \,, \tag{13}$$

and satisfy relations

$$\mathbb{A}_n^\dagger \mathbb{A}_n = \prod_{j=1}^{n} (\hat{H}_- - E_j) \,, \qquad \mathbb{A}_n \mathbb{A}_n^\dagger = \prod_{j=1}^{n} (\hat{H}_+ - E_j) \,, \tag{14}$$

where E_j is eigenvalue of the seed eigenstate ψ_j. Relations (13) and (14) underlie nonlinear supersymmetry of the extended system $\mathcal{H} = \mathrm{diag}\,(\hat{H}_+, \hat{H}_-)$, the supercharges of which are constructed from the operators $\mathbb{A}_n$ and $\mathbb{A}_n^\dagger$.

Using Eq. (11), one can prove the relation [71]

$$\mathbb{W}(\psi_*, \widetilde{\psi_*}, \psi_1, \ldots, \psi_n) = \mathbb{W}(\psi_1, \ldots, \psi_n) \,. \tag{15}$$

Here and in what follows equality between Wronskians is implied up to inessential multiplicative constant; ψ_* is some eigenstate of $\hat{H}_-$ with eigenvalue E_* different from E_j, $j = 1, \ldots, n$, and $\widetilde{\psi_*} = \psi_* \int^x dy/(\psi_*(y))^2$ is a linear independent eigenstate with the same eigenvalue E_* so that $\mathbb{W}(\psi_*, \widetilde{\psi_*}) = 1$.

Among supersymmetric quantum systems generated by DC transformations, there exists special class of infinite subfamilies in which the corresponding superextended systems are characterized simultaneously by supersymmetries of two different orders, one of which is of even order $n = 2l$, while another has some odd order $n = 2k + 1$ [30–32, 36, 37, 42, 56]. This corresponds to supersymmetrically extended finite-gap or reflectionless systems, which can be regarded as “instant

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photos" of solutions to the KdV equation [72] and are characterized by the presence of a nontrivial Lax–Novikov integrals to be operators of the odd differential order $n = 2\ell + 1 \geq 3$ with $\ell = l + k$. Factorization of Lax–Novikov integrals into two differential operators of orders $2l$ and $2k+1$ is reflected in the presence of the exotic nonlinear $\mathcal{N} = 4$ Poincaré supersymmetry generated by supercharges of orders $2l$ and $2k + 1$ instead of linear or nonlinear $\mathcal{N} = 2$ Poincaré supersymmetry obtained usually via the Darboux or Darboux–Crum transformation construction.

A simple example of a system with exotic nonlinear $\mathcal{N} = 4$ supersymmetry is generated via the construction of Witten's supersymmetric quantum mechanics with superpotential $W(x) = \kappa \tanh \kappa x$, where κ is a parameter of dimension of inverse length. The corresponding superextended system is described by the Hamiltonian $\hat{\mathcal{H}} = \mathrm{diag}\,(\hat{H}_+, \hat{H}_-)$ with $\hat{H}_- = -\frac{d^2}{dx^2} + \kappa^2$, $\hat{H}_+ = \hat{H}_- - 2\kappa^2/\cosh^2 \kappa x$, and first order supercharges $\hat{Q}_+ = (\frac{d}{dx} - W(x))\sigma_+$, $\hat{Q}_- = (\hat{Q}_+)^\dagger$. They generate the $\mathcal{N} = 2$ Poincaré superalgebra via the (anti)commutation relations

$$\{\hat{Q}_+, \hat{Q}_-\} = \hat{\mathcal{H}}\,, \qquad [\hat{\mathcal{H}}, \hat{Q}_\pm] = 0\,. \tag{16}$$

This system can also be obtained via the construction of the $n = 2$ supersymmetry by choosing $\mathcal{W}(x) = -\frac{1}{2}\kappa \tanh \kappa x$ and $C = -\frac{1}{16}\kappa^4$ [62]. In this case $\Delta = -\frac{\kappa^2}{\cosh^2 \kappa x}(1 + \frac{1}{4\sinh^2 \kappa x})$, and the operator in the second order supercharge (8) is factorized in the form

$$\hat{s}_+ = \left(\frac{d}{dx} - \kappa \tanh \kappa x\right)\frac{d}{dx}\,. \tag{17}$$

We have here

$$\{\hat{S}_+, \hat{S}_-\} = \left(\hat{\mathcal{H}} - \frac{1}{2}\kappa^2\right)^2 - \frac{1}{16}\kappa^4\,, \qquad [\hat{\mathcal{H}}, \hat{S}_\pm] = 0\,. \tag{18}$$

The anti-commutators of the first and second order supercharges generate a nontrivial even integral of motion,

$$\{\hat{S}_+, \hat{Q}_-\} = -\{\hat{S}_-, \hat{Q}_+\} = i\hat{\mathcal{L}}\,, \tag{19}$$

$$\hat{\mathcal{L}} = \begin{pmatrix} \hat{q}_+ \hat{p}\, \hat{q}_+^\dagger & 0 \\ 0 & \hat{H}_- \hat{p} \end{pmatrix}, \tag{20}$$

where $\hat{q}_+ = \frac{d}{dx} - \kappa \tanh \kappa x$. Operator (20) satisfies the commutation relations

$$[\hat{\mathcal{L}}, \hat{Q}_\pm] = [\hat{\mathcal{L}}, \hat{S}_\pm] = [\hat{\mathcal{L}}, \hat{\mathcal{H}}] = 0\,, \tag{21}$$

which mean that the integral $\hat{\mathcal{L}}$ is the central element of the nonlinear superalgebra generated by $\hat{\mathcal{H}}$, $\hat{Q}_\pm$, $\hat{S}_\pm$, and $\hat{\mathcal{L}}$. The lower term in the diagonal operator $\hat{\mathcal{L}}$ is the

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momentum operator of a free quantum particle multiplied by $\hat{H}_-$, while the third order differential operator $\hat{q}_+\hat{p}\,\hat{q}_+^\dagger$ is the Lax–Novikov integral of reflectionless system described by the Hamiltonian operator $\hat{H}_+$.

Operator $\hat{\mathcal{L}}$ plays essential role in the description of the system $\hat{\mathcal{H}}$: it detects and annihilates a unique bound state in the spectrum of reflectionless subsystem $\hat{H}_+$, which is described by the wave function $\Psi_0 = \left(\sqrt{2\kappa^{-1}}\cosh\kappa x, 0\right)^t$ of zero energy. It also annihilates the doublet of states $\Psi_+ = (\tanh\kappa x,\ 0)^t$ and $\Psi_- = (0,\ 1)^t$ of the system $\hat{\mathcal{H}}$ of energy $E = \kappa^2$. Besides, operator $\hat{\mathcal{L}}$ distinguishes (with the aid of the integral σ_3) the states $\Psi_+^{\pm k} = (\pm ikx - \kappa\tanh\kappa x)e^{\pm ikx},\ 0)^t$ and $\Psi_-^{\pm k} = (0,\ e^{\pm ikx})^t$ in the four-fold degenerate scattering part of the spectrum of $\hat{\mathcal{H}}$: $\hat{\mathcal{L}}\Psi_+^{\pm k} = \pm k(\kappa^2 + k^2)\Psi_+^{\pm k}$, $\hat{\mathcal{L}}\Psi_-^{\pm k} = \pm k(\kappa^2 + k^2)\Psi_-^{\pm k}$. Zero energy state Ψ_0 is annihilated here by all the supercharges and by the Lax–Novikov integral $\hat{\mathcal{L}}$, and thus the system realizes exotic supersymmetry in the unbroken phase [36, 42].

Within the framework of the Darboux–Crum construction, the described reflectionless system $\hat{H}_+$ is obtained from the free particle system $\hat{H}_0 = -\frac{d^2}{dx^2}$ by taking its non-physical eigenstate $\psi_1(x) = \cosh\kappa x$ of eigenvalue $-\kappa^2$ as the seed state by constructing the operator

$$\hat{H}_+ = \hat{H}_- - 2(\ln \mathbb{W})'', \tag{22}$$

where $\hat{H}_- = \hat{H}_0 + \kappa^2$ and $\mathbb{W} = \psi_1(x)$. The supercharge $\hat{Q}_+$ is constructed then from the operator $\hat{q}_+ = \psi_1\frac{d}{dx}\frac{1}{\psi_1(x)} = \frac{d}{dx} - \kappa\tanh\kappa x$. The same superpartner system $\hat{H}_+$ can be generated via relation (22) by changing $\mathbb{W} = \psi_1(x)$ in it for Wronskian of the set of eigenstates $\psi_0 = 1$ and $\psi_1 = \sinh\kappa x$, which is equal, up to inessential multiplicative constant, to the same function $\mathbb{W} = \psi_1(x)$: $\mathbb{W}(1, \sinh\kappa x) = \kappa\cosh\kappa x$. This second DC scheme generates the intertwining operator (17) corresponding to the second order supercharge $\hat{S}_+$ via the chain of relations $\hat{s}_+ = A_2A_1$, where $A_1 = \psi_0\frac{d}{dx}\frac{1}{\psi_0} = \frac{d}{dx}$, $A_2 = (A_1\psi_1)\frac{d}{dx}\frac{1}{(A_1\psi_1)} = \hat{q}_+$. In this construction the third order Lax–Novikov integral $\hat{q}_+\hat{p}\,\hat{q}_+^\dagger$ of the subsystem $\hat{H}_+$ is the Darboux-dressed momentum operator of the free particle.

The described DC construction of superextended systems described by exotic $\mathcal{N} = 4$ supersymmetry is generalized for arbitrary case of the system of the form $\hat{\mathcal{H}} = \text{diag}\,(\hat{H}_+, \hat{H}_-)$, with reflectionless subsystems $\hat{H}_+$ and $\hat{H}_-$ having an arbitrary number and energies of bound states, but with identical continuous parts of their spectra [42]. The key point underlying the appearance of the two supersymmetries of different orders by means of which the partner systems $\hat{H}_+$ and $\hat{H}_-$ are related is that the same reflectionless system can be generated by two different Darboux–Crum transformations. One transformation is generated by the choice of the set of non-physical eigenstates

$$\psi_1 = \cosh\kappa_1(x + \tau_1),\ \psi_2 = \sinh\kappa_2(x + \tau_2),\ \ldots,\ \psi_n \tag{23}$$

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of the free particle system taken as the seed states. Here $\psi_{2l+1} = \cosh\kappa_{2l+1}(x + \tau_{2l+1})$, $\psi_{2l} = \sinh\kappa_{2l}(x + \tau_{2l})$, $1 \le 2l < 2l + 1 \le n$, and κ_j and τ_j, $j = 1, \ldots, n$, are arbitrary real parameters with restriction $0 < \kappa_j < \kappa_{j+1}$. The indicated choice of eigenstates guarantees that the Wronskian of these states takes nonzero values, and the potential produced via the Wronskian construction, $V(x) = -2(\ln \mathbb{W}(\psi_1, \ldots, \psi_n))''$, will be nonsingular reflectionless potential maintaining n bound states. The choice of the translation parameters τ_j in the form $\tau_j = x_{0j} - 4\kappa_j^2 t$ promotes the potential into the n-soliton solution to the KdV equation [43, 73]

$$u_t = 6uu_x - u_{xxx}\,. \tag{24}$$

Exactly the same reflectionless potential $V(x)$ is generated by taking the following set of eigenstates of the free particle Hamiltonian operator:

$$\phi_0 = 1,\ \phi_1 = \sinh\kappa_1(x + \tau_1),\ \phi_2 = \cosh\kappa_2(x + \tau_2), \ldots, \phi_n\,, \tag{25}$$

as the seed states for the Darboux–Crum transformation. Here

$$\phi_{2l+1} = \sinh\kappa_{2l+1}(x + \tau_{2l+1}), \qquad \phi_{2l} = \cosh\kappa_{2l}(x + \tau_{2l}),$$

and modulo the unimportant multiplicative constant, we have

$$\mathbb{W}(\psi_1, \ldots, \psi_n) = \mathbb{W}(1, \psi_1' \ldots, \psi_n')\,. \tag{26}$$

When the number of bound states n in each partner reflectionless system $\hat{H}_+$ and $\hat{H}_-$ is the same but all the discrete energies of one subsystem are different from those of another subsystem, one pair of supercharges will have differential order $2n$ while another pair will have differential order $2n + 1$ independently on the values of translation parameters τ_j of subsystems. This corresponds to the nature of the described Darboux–Crum transformations. In this case one pair of the supercharges is constructed from intertwining operators which relate the partner system $\hat{H}_+$ via the "virtual" free particle system $\hat{H}_0$, and then $\hat{H}_0$ to $\hat{H}_-$. The corresponding intertwining operators are composed from intertwining operators obtained from the sets of the seed states of the form (23) used for the construction of each partner system. Another pair of supercharges of differential order $2n + 1$ is constructed from the intertwining operators of a similar form but with inserted in the middle free particle integral $\frac{d}{dx}$. This corresponds to the use of the set of the seed states of the form (25) for one of the partner subsystems. The Lax–Novikov integral being even generator of the exotic supersymmetry and having differential order $2n + 1$ is produced via anti-commutation of the supercharges of different differential orders. It, however, is not a central charge of the nonlinear superalgebra: commuting with one pair of supercharges it transforms them into another pair of supercharges multiplied by certain polynomials in Hamiltonian $\hat{\mathcal{H}}$ of corresponding orders [42]. The structure of exotic supersymmetry undergoes a reduction each time when some r discrete energies of one subsystem coincide with any r discrete

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energies of another subsystem. In this case the sum of differential orders of two pairs of supercharges reduces from $4n + 1$ to $4n - 2r + 1$, and nonlinear superlagebraic structure acquires a dependence on r relative translation parameters $\tau_j^+ - \tau_{j'}^-$ whose indexes j and j' correspond to coinciding discrete energy levels. When all the discrete energy levels of one subsystem coincide with those of the partner system, the Lax integral transforms into the bosonic central charge of the corresponding nonlinear superalgebra [42].

Different supersymmetric systems of the described nature can also be related by sending some of the translation parameters τ_j to infinity. In such a procedure exotic supersymmetry undergoes certain transmutations, particularly, between the unbroken and broken phases, and admits an interpretation in terms of the picture of soliton scattering [74].

In the interesting case of a superextended system unifying two finite-gap periodic partners described by the associated Lamé potentials shifted mutually for the half of the period of their potentials, the two corresponding Darboux–Crum transformations are constructed on the two sets of the seed states which correspond to the edges of the valence and conduction bands, one of which is composed from periodic states while another consists from antiperiodic states. One of such sets corresponding to antiperiodic wave functions has even dimension, while another that includes wave functions with the same period as the potentials has odd dimension. These sets generate the pairs of supercharges of the corresponding even and odd differential orders. On these sets of the states, certain finite-dimensional non-unitary representations of the $sl(2, \mathbb{R})$ algebra are realized of the same even and odd dimensions [30]. Lax–Novikov integral in such finite-gap systems with exotic nonlinear $\mathcal{N} = 4$ supersymmetry has a nature of the bosonic central charge and differential order equal to $2g + 1$, where g is the number of gaps in the spectrum of completely isospectral partners. The indicated class of the supersymmetric finite-gap systems admits an interpretation as a planar model of a non-relativistic electron in periodic magnetic and electric fields that produce a one-dimensional crystal for two spin components separated by a half-period spacing [30]. Exotic supersymmetry in such systems is in the unbroken phase with two ground states having the same zero energy, particularly, in the case when one pair of the supercharges has differential order one and corresponds to the construction of the Witten's supersymmetric quantum mechanics. The simplest case of such a system is given by the pair of the mutually shifted for the half-period one-gap Lamé systems,

$$\hat{H}_\pm = -\frac{d^2}{dx^2} + V_\pm(x), \quad V_-(x) = 2\mathrm{sn}^2(x|k) - k^2, \quad V_+(x) = V_-(x + \mathbf{K}), \tag{27}$$

where k is the modular parameter and $4\mathbf{K}$ is the period of the Jacobi elliptic function $\mathrm{sn}\,(x|k)$. The extended matrix system $\hat{\mathcal{H}}$ is described by the first order supercharges constructed on the base of the superpotential $W(x) = -(\ln \mathrm{dn}\, x)'$ generated by the ground state $\mathrm{dn}\, x$ of the subsystem $\hat{H}_-$ which has the same period $2\mathbf{K}$ as the potential $V_-(x)$. The second order supercharges are generated via the Darboux–Crum construction on the base of the seed states $\mathrm{cn}\, x$ and $\mathrm{sn}\, x$ which change sign

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under the shift for $2\mathbf{K}$, and describe the states of energies $1-k^2$ and 1 at the edges of valence and conduction bands of $\hat{H}_-$, respectively.

The superextended system composed from the same one-gap systems but shifted mutually for the distance less than half-period of their potentials is described by exotic nonlinear $\mathcal{N}=4$ supersymmetry with supercharges to be differential operators of the same first and second orders, and Lax–Novikov integral having differential order three. But in this case supersymmetry is broken, the positive energy of the doublet of the ground states depends on the value of the mutual shift, and though the Lax–Novikov integral is the bosonic central charge, the structure coefficients of the nonlinear superalgebra depend on the value of the shift parameter [37].

As was shown in [45], reflectionless and finite-gap periodic systems described by exotic nonlinear supersymmetry can also be generated in quantum systems with a position-dependent mass [75–78].

Very interesting physical properties are exhibited in the systems with the exotic nonlinear $\mathcal{N}=4$ supersymmetry realized on finite-gap systems with soliton defects [73, 79]. By applying Darboux–Crum transformations to a Lax pair formulation of the KdV equation, one can construct multi-soliton solutions to this equation as well as to the modified Korteweg–de Vries equation which represent different types of defects in crystalline background of the pulse and compression modulation types. These periodicity defects reveal a chiral asymmetry in their propagation. Exotic nonlinear supersymmetric structure in such systems unifies solutions to the KdV and modified KdV equations, it detects the presence of soliton defects in them, distinguishes their types, and identifies the types of crystalline backgrounds [73].

4 Perfectly Invisible $\mathcal{PT}$-Symmetric Zero-Gap Systems

Darboux–Crum transformations can be realized not only on the base of the physical or non-physical eigenstates of a system, but also by including into the set of the seed states of Jordan and generalized Jordan states [56, 57, 80–82], which, in turn, can be obtained by certain limit procedures from eigenstates of a system. For instance, one can start from the free quantum particle, and choose the set of the states $(x, x^2, x^3, \ldots x^n)$, $x^n = \lim_{k\to 0}(\sin kx/k)^n$. The first state x is a non-physical eigenstate of $\hat{H}_0 = -\frac{d^2}{dx^2}$ of zero eigenvalue. The states x^{2l}, x^{2l+1}, $l \geq 1$, are the Jordan states of order l of $\hat{H}_0$: $(\hat{H}_0)^l$ acting on both states transforms them into zero energy eigenstates $\psi_0 = 1$ and $\psi_1 = x = \widetilde{\psi}_0$, respectively. The Wronskian of these states is $\mathbb{W}(x, x^2, x^3, \ldots, x^n) = const \cdot x^n$, and the system generated via the corresponding Darboux–Crum transformation is $\hat{H}_n = -\frac{d^2}{dx^2} + \frac{n(n+1)}{x^2}$. Operator $\hat{H}_n$, however, is singular on the whole real line, and can be identified with the Hamiltonian of the two-particle Calogero [83, 84] model with the omitted center of mass coordinate, which requires for definition of its domain with $x \in (0, +\infty)$ the introduction of the Dirichlet boundary condition $\psi(0^+) = 0$. Systems $\hat{H}_0$ and $\hat{H}_n$ are intertwined by differential operators $\mathbb{A}_n = A_n \ldots A_1$ and $\mathbb{A}_n^\dagger$, $\mathbb{A}_n\hat{H}_0 = \hat{H}_n\mathbb{A}_n$,

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$\mathbb{A}_n^\dagger \hat{H}_n = \hat{H}_0 \mathbb{A}_n^\dagger$ where $A_l = \frac{d}{dx} - \frac{l}{x}$, and construction of $\mathbb{A}_n$ corresponds to Eq. (12). The systems $\hat{H}_0$ and $\hat{H}_n$ can also be intertwined by the operators $\mathbb{B}_n = A_n \dots A_1 A_0$ and $\mathbb{B}_n^\dagger$, where $A_0 = \frac{d}{dx}$, which are obtained by realizing the Darboux–Crum transformation constructed on the base of the set of the states $(x^2, \dots, x^{n+1})$ extended with the state $\psi_0 = 1$. One could take then the extended system composed from $\hat{H}_+ = \hat{H}_n$ and $\hat{H}_- = \hat{H}_0$ with $\hat{H}_0$ restricted to the same domain as $\hat{H}_n$, and construct the supercharge operators of differential orders n and $n+1$ from the introduced intertwining operators. However, we find that the supercharge constructed on the base of the intertwining operators $\mathbb{B}_n$ and $\mathbb{B}_n^\dagger$ will be non-physical as the intertwining operator $\mathbb{B}_n$ acting on physical eigenstates $\sin kx$ of $\hat{H}_-$ of energy k^2 will transform them into non-physical eigenstates $\mathbb{B}_n \sin kx$ of the system $\hat{H}_+$ of the same energy but not satisfying the boundary condition $\psi(0^+) = 0$. In correspondence with this, differential operator of order $2n+1$, $\hat{\mathcal{L}} = \text{diag}\,(\hat{\mathcal{L}}_+, \hat{\mathcal{L}}_-)$, with $\hat{\mathcal{L}}_+ = \mathbb{B}_n \mathbb{A}_n^\dagger = \mathbb{A}_n \frac{d}{dx} \mathbb{A}_n^\dagger$ and $\hat{\mathcal{L}}_- = \mathbb{A}_n^\dagger \mathbb{B}_n = (\hat{H}_-)^n \frac{d}{dx}$ formally commutes with $\hat{\mathcal{H}}$, but it is not a physical operator for the system $\hat{\mathcal{H}}$ as acting on its physical eigenstates satisfying boundary condition at $x = 0^+$, it transforms them into non-physical eigenstates not satisfying the boundary condition. The situation can be "$\mathcal{PT}$-regularized" by shifting the variable x: $x \to \xi = x + i\alpha$, where α is a nonzero real parameter [56]. The obtained in such a way superextended system can be considered on the whole real line $x \in \mathbb{R}$, and boundary condition at $x = 0$ can be omitted. The system $\hat{H}_+(\xi)$ is $\mathcal{PT}$-symmetric [85–91]: $[PT, \hat{H}_+(\xi)] = 0$, where P is a space reflection operator, $Px = -Px$, and T is the operator defined by $T(x + i\alpha) = (x - i\alpha)T$. Subsystem $\hat{H}_+(\xi)$ has one bound eigenstate of zero eigenvalue described by quadratically integrable on the whole real line function $\psi_0^+ = \xi^{-n}$, which lies at the very edge of the continuous spectrum with $E > 0$. System $\hat{H}_+(\xi)$ therefore can be identified as $\mathcal{PT}$-symmetric zero-gap system. Moreover, it turns out that the transmission amplitude for this system is equal to one as for the free particle system, and $\hat{H}_+(\xi)$ can be regarded as a perfectly invisible $\mathcal{PT}$-symmetric zero-gap system. Exotic nonlinear supersymmetry of the system $\hat{\mathcal{H}}(\xi)$ will be described by two supercharges of differential order n constructed from the intertwining operators $\mathbb{A}_n(\xi)$ and $\mathbb{A}_n^\#(\xi) = A_1^\# \dots A_n^\#$, $A_j^\# = -\frac{d}{dx} - \frac{j}{\xi}$, by supercharges of the order $n+1$ constructed from the intertwining operators $\mathbb{B}_n(\xi)$ and $\mathbb{B}_n^\#(\xi)$, and by the Lax–Novikov integral $\hat{\mathcal{L}}(\xi)$ to be differential operator of order $2n+1$. Operator $\hat{\mathcal{L}}(\xi)$ annihilates the unique bound state of the system $\hat{\mathcal{H}}(\xi)$ and the state $\psi_0 = 1$ of zero energy in the spectrum of the free particle subsystem, and distinguishes plane waves e^{ikx} in the spectrum of the free particle subsystem and deformed plane waves $\mathbb{A}_n(\xi)e^{ik\xi}$ in the spectrum of the superpartner system $\hat{H}_+(\xi)$.

In the simplest case $n = 1$, the supercharges have the form

$$\hat{Q}_1 = \begin{pmatrix} 0 & A_1(\xi) \\ A_1^\#(\xi) & 0 \end{pmatrix}, \qquad \hat{Q}_2 = i\sigma_3 \hat{Q}_1\,, \tag{28}$$

$$\hat{S}_1 = \begin{pmatrix} 0 & -A_1(\xi)\frac{d}{dx} \\ \frac{d}{dx}A_1^\#(\xi) & 0 \end{pmatrix}, \qquad \hat{S}_2 = i\sigma_3 \hat{S}_1\,, \tag{29}$$

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where $\hat{Q}_1 = \hat{Q}_+ + \hat{Q}_-$, $\hat{S}_1 = \hat{S}_+ + \hat{S}_-$. The Lax–Novikov integral is

$$\hat{\mathcal{L}} = \begin{pmatrix} -iA_1(\xi)\frac{d}{dx}A_1^{\#}(\xi) & 0 \\ 0 & -i\frac{d}{dx}\hat{H}_0 \end{pmatrix}. \tag{30}$$

Together with Hamiltonian $\hat{\mathcal{H}} = \mathrm{diag}\,(\hat{H}_1(\xi), \hat{H}_0)$ they satisfy the following nonlinear superalgebra [56]:

$$[\hat{\mathcal{H}}, \hat{Q}_a] = 0\,, \qquad [\hat{\mathcal{H}}, \hat{S}_a] = 0\,, \tag{31}$$

$$\{\hat{Q}_a, \hat{Q}_b\} = 2\delta_{ab}\hat{\mathcal{H}}\,, \qquad \{\hat{S}_a, \hat{S}_b\} = 2\delta_{ab}\hat{\mathcal{H}}^2\,, \tag{32}$$

$$\{\hat{Q}_a, \hat{S}_b\} = 2\epsilon_{ab}\hat{\mathcal{L}}\,. \tag{33}$$

$$[\hat{\mathcal{L}}, \hat{\mathcal{H}}] = 0\,, \qquad [\hat{\mathcal{L}}, \hat{Q}_a] = 0\,, \qquad [\hat{\mathcal{L}}, S_a] = 0\,. \tag{34}$$

In the case of the superextended system $\hat{\mathcal{H}} = \mathrm{diag}\,(\hat{H}_1(\xi_2), \hat{H}_1(\xi_1))$, where $\xi_j = x + i\alpha_j$, $j = 1, 2$, and $\alpha_1 \neq \alpha_2$, exotic nonlinear supersymmetry is partially broken: the doublet of zero energy bound states is annihilated by the second order supercharges $\hat{S}_a$ and by the Lax–Novikov integral $\hat{\mathcal{L}}$, but they are not annihilated by the first order supercharges $\hat{Q}_a$ [56]. The first order supercharges $\hat{Q}_a$ are constructed in this case from the intertwining operators $A = \frac{d}{dx} + \mathcal{W}$, $\mathcal{W} = \xi_1^{-1} - \xi_2^{-1} - (\xi_1 - \xi_2)^{-1}$, and $A^{\#} = -\frac{d}{dx} + \mathcal{W}$. The second order supercharges $\hat{S}_a$ are composed from the intertwining operators $A_1(\xi_2)A_1^{\#}(\xi_1)$ and $A_1(\xi_1)A_1^{\#}(\xi_2)$. In the limit $\alpha_1 \to \infty$, the system $\hat{\mathcal{H}} = \mathrm{diag}\,(\hat{H}_1(\xi_2), \hat{H}_1(\xi_1)$ transforms into the system given by the $\mathcal{PT}$-symmetric Hamiltonian $\hat{\mathcal{H}} = \mathrm{diag}\,(\hat{H}_1(\xi_2), \hat{H}_0)$, and exotic nonlinear supersymmetry in the partially broken phase transmutes into the supersymmetric structure corresponding to the unbroken phase [56].

It is interesting to note that if to use the appropriate linear combinations of the Jordan states of the quantum free particle as the seed states for the Darboux–Crum transformations, one can construct $\mathcal{PT}$-symmetric time-dependent potentials which will satisfy equations of the KdV hierarchy and will exhibit a behavior typical for extreme (rogue) waves [56].

5 Nonlinear Superconformal Symmetry of the $\mathcal{PT}$-Symmetric Zero-Gap Calogero Systems

Free particle system is characterized by the Schrödinger symmetry generated by the first order integrals $\hat{P}_0 = \hat{p} = -i\frac{d}{dx}$ and $\hat{G}_0 = x + 2it\frac{d}{dx}$, and the second order integrals $\hat{H}_0 = -\frac{d^2}{dx^2}$, $\hat{D}_0 = \frac{1}{4}\{\hat{P}_0, \hat{G}_0\}$ and $\hat{K}_0 = \hat{G}_0^2$. Operators $\hat{G}_0$ as well as $\hat{D}_0$ and $\hat{K}_0$ are dynamical integrals of motion satisfying the equation of motion of the

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form $\frac{d}{dt}\hat{I} = \frac{\partial}{\partial t}\hat{I} - [\hat{H}_0, \hat{I}] = 0$. These time-independent and dynamical integrals generate the Schrödinger algebra

$$[\hat{D}_0, H_0] = i\hat{H}_0\,, \qquad [\hat{D}_0, \hat{K}_0] = -i\hat{K}_0\,, \qquad [\hat{K}_0, \hat{H}_0] = 8i\hat{D}_0\,, \tag{35}$$

$$[\hat{D}_0, \hat{P}_0] = \tfrac{i}{2}\hat{P}_0\,, \qquad \hat{[}D_0, \hat{G}_0] = -\tfrac{i}{2}\hat{G}_0\,, \tag{36}$$

$$[\hat{H}_0, \hat{G}_0] = -2i\hat{P}_0\,, \qquad [\hat{H}_0, \hat{P}_0] = 0\,, \tag{37}$$

$$[\hat{K}_0, \hat{P}_0] = 2i\hat{G}_0\,, \qquad [\hat{K}_0, \hat{G}_0] = 0\,, \tag{38}$$

$$[\hat{G}_0, \hat{P}_0] = i\,\mathbb{I}\,. \tag{39}$$

Equations (35) and (39) correspond to the $sl(2,\mathbb{R})$ and Heisenberg subalgebras, respectively. If we make a shift $x \to \xi = x + i\alpha$, and make Darboux-dressing of operators $\hat{P}_0$, $\hat{G}_0$, $\hat{D}_0$, and $\hat{K}_0$, we find the integrals of motion for the perfectly invisible zero-gap $\mathcal{PT}$-symmetric system $\hat{H}_1(\xi)$. These are $\hat{P}_1(\xi) = A_1(\xi)\hat{P}_0 A_1^{\#}(\xi)$, $\hat{G}_1(\xi) = A_1(\xi)\hat{G}_0 A_1^{\#}(\xi)$, and

$$\hat{D}_1(\xi) = -\frac{i}{2}\left(\xi\frac{d}{dx} + \frac{1}{2}\right) - t\hat{H}_1(\xi)\,, \tag{40}$$

$$\hat{K}_1(\xi) = \xi^2 - 8t\hat{D}_1(\xi) - 4t^2\hat{H}_1(\xi)\,, \tag{41}$$

where the dynamical integrals $\hat{D}_1(\xi)$ and $\hat{K}_1(\xi)$ have been extracted from the corresponding Darboux-dressed operators by omitting in them the operator factor $\hat{H}_1(\xi)$ [57]. Operators $\hat{H}_1(\xi)$, $\hat{D}_1(\xi)$, and $\hat{K}_1(\xi)$ generate the same $sl(2,\mathbb{R})$ algebra as in the case of the free particle. But now we have relations

$$[D_1, P_1] = \frac{3}{2}iP_1\,, \qquad [D_1, G_1] = \frac{i}{2}G_1\,, \qquad [K_1, P_1] = 6iG_1\,, \tag{42}$$

$$[G_1, P_1] = 3i(H_1)^2 \tag{43}$$

instead of the corresponding relations of the free particle system. In addition, two new dynamical integrals of motion,

$$V_1(\xi) = i\xi^2 A_1^{\#}(\xi) - 4tG_1(\xi) - 4t^2 P_1(\xi) \tag{44}$$

and

$$R_1(\xi) = \xi^3 - 6tV_1(\xi) - 12t^2 G_1(\xi) - 8t^3\xi_1\,, \tag{45}$$

are generated via the commutation relations

$$[\hat{K}_1, \hat{G}_1] = -4i\hat{V}_1\,, \qquad [\hat{K}_1, \hat{V}_1] = -2i\hat{R}_1\,, \tag{46}$$

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and we obtain additionally the commutation relations

$$[\hat{V}_1, \hat{H}_1] = 4i\hat{G}_1\,, \qquad [\hat{V}_1, \hat{D}_1] = \tfrac{i}{2}\hat{V}_1\,,$$
$$[\hat{V}_1, \hat{P}_1] = 12i\hat{H}_1\hat{D}_1 - 6\hat{H}_1\,, \qquad [\hat{V}_1, \hat{G}_1] = 12i(\hat{D}_1)^2 + \tfrac{3}{4}i\,\mathbb{I}\,,$$
$$[\hat{R}_1, \hat{H}_1] = 6i\hat{V}_1\,, \qquad [\hat{R}_1, \hat{D}_1] = \tfrac{3}{2}i\hat{R}_1\,, \quad [\hat{R}_1, \hat{K}_1] = 0\,,$$
$$[\hat{R}_1, \hat{P}_1] = 36i\,\hat{D}_1^2 + \tfrac{21}{4}i\,\mathbb{I}, \quad [\hat{R}_1, \hat{G}_1] = 12i\,\hat{D}_1\hat{K}_1 - 6\hat{K}_1\,, \quad [\hat{R}_1, \hat{V}_1] = 3i\,\hat{K}_1^2\,.$$

The Schrödinger algebra of the free particle is extended for its nonlinear generalization in the case of the $\mathcal{PT}$-symmetric system $\hat{H}_1(\xi)$, which is generated by the operators $\hat{H}_1(\xi)$, $\hat{P}_1(\xi)$, $\hat{G}_1(\xi)$, $\hat{D}_1(\xi)$, $\hat{K}_1(\xi)$, $\hat{V}_1(\xi)$, $\hat{R}_1(\xi)$, and central charge $\mathbb{I}$ (equals to 1 in the chosen system of units). All these integrals are eigenstates of the dilatation operator $\hat{D}_1(\xi)$ with respect to its adjoint action.

Now we can consider the generalized and extended superconformal symmetry of the system described by the matrix Hamiltonian operator $\hat{\mathcal{H}} = \mathrm{diag}\,(\hat{H}_1(\xi), \hat{H}_0)$. Supplying the Hamiltonian $\hat{\mathcal{H}}$ and Lax–Novikov integral (30) with the bosonic integrals $\hat{\mathcal{D}} = \mathrm{diag}\,(\hat{D}_1(\xi), \hat{D}_0(\xi))$, $\hat{\mathcal{K}} = \mathrm{diag}\,(\hat{K}_1(\xi), \hat{K}_0(\xi))$, and commuting them with supercharges (28) and (29), we obtain a nonlinear superalgebra that describes the symmetry of the system $\hat{\mathcal{H}}$, which corresponds to some nonlinear extension of the super-Schrödinger algebra. It is generated by the set of the even (bosonic) integrals $\hat{\mathcal{H}}$, $\hat{\mathcal{D}}$, $\hat{\mathcal{K}}$, $\hat{\mathcal{L}}$, $\hat{\mathcal{G}}$, $\hat{\mathcal{V}}$, $\hat{\mathcal{R}}$, $\hat{\mathcal{P}}_-$, $\hat{\mathcal{G}}_-$, $\Sigma = \sigma_3$, $\hat{\mathcal{I}} = \mathrm{diag}\,(1, 1)$, and by the odd (fermionic) integrals $\hat{\mathcal{Q}}_a$, $\hat{\mathcal{S}}_a$, and $\hat{\lambda}_a$, $\hat{\mu}_a$ and $\hat{\kappa}_a$, $a = 1, 2$, where

$$\hat{\mathcal{G}} = \mathrm{diag}\,\left(\hat{G}_1(\xi),\, \tfrac{1}{2}\{\hat{G}_0(\xi), \hat{H}_0\}\right), \qquad \mathcal{V} = i\xi^2 A_1^{\alpha\#}\mathcal{I} - 4t\mathcal{G} - 4t^2\mathcal{L}\,, \quad (47)$$

$$\hat{\mathcal{R}} = \xi^3\mathcal{I} - 6t\hat{\mathcal{V}} - 12t^2\hat{\mathcal{G}} - 8t^3\hat{\mathcal{L}}\,, \quad (48)$$

$$\hat{\mathcal{P}}_- = \tfrac{1}{2}(1-\sigma_3)\hat{P}_0\,, \qquad \hat{\mathcal{G}}_- = \tfrac{1}{2}(1-\sigma_3)\hat{G}_0(\xi), \quad (49)$$

$$\hat{\lambda}_1 = \begin{pmatrix} 0 & i\xi \\ -i\xi & 0 \end{pmatrix} - 2t\hat{\mathcal{Q}}_1\,, \qquad \hat{\lambda}_2 = i\sigma_3\hat{\lambda}_1\,, \quad (50)$$

$$\hat{\mu}_1 = \begin{pmatrix} 0 & \xi\hat{P}_0 \\ \hat{P}_0\xi & 0 \end{pmatrix} - 2t\hat{\mathcal{S}}_1\,, \qquad \hat{\mu}_2 = i\sigma_3\hat{\mu}_1\,, \quad (51)$$

$$\hat{\kappa}_1 = \begin{pmatrix} 0 & \xi^2 \\ \xi^2 & 0 \end{pmatrix} - 4t\hat{\mu}_1 - 4t^2\hat{\mathcal{S}}_1\,, \qquad \hat{\kappa}_2 = i\sigma_3\hat{\kappa}_1\,, \quad (52)$$

and we use the notation $\hat{G}_0(\xi) = \hat{G}_0(x + i\alpha)$. Explicit form of the nonlinear superalgebra generated by these integrals of motion of the system $\hat{\mathcal{H}}$ is presented in [57]. All the even and odd integrals here are eigenstates of the matrix dilatation operator $\hat{\mathcal{D}}$.

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Essentially different generalized nonlinear superconformal structure appears in the system described by the matrix Hamiltonian

$$\hat{\mathcal{H}} = \text{diag}\,(\hat{H}_1(\xi_2), \hat{H}_1(\xi_1))$$

and characterized by the partially broken exotic nonlinear $\mathcal{N} = 4$ supersymmetry. In that case the number of the even and odd integrals of motion is the same as in the system $\hat{\mathcal{H}} = \text{diag}\,(\hat{H}_1(\xi), \hat{H}_0)$ in the phase with unbroken supersymmetry. However, no odd (fermionic) integral of motion is eigenstate of the matrix dilatation operator $\hat{\mathcal{D}} = \text{diag}\,(\hat{D}_1(\xi_2), \hat{D}_1(\xi_1))$, and, as a result, the structure of the nonlinear superalgebra has more complicated form. When one of the shift parameters, α_1, is sent to infinity, the system $\hat{\mathcal{H}} = \text{diag}\,(\hat{H}_1(\xi_2), \hat{H}_1(\xi_1))$ transforms into the system $\hat{\mathcal{H}} = \text{diag}\,(\hat{H}_1(\xi), \hat{H}_0)$ in the unbroken phase of the exotic nonlinear $\mathcal{N} = 4$ super-Poincaré symmetry, and all the integrals of the latter system can be reproduced from the integrals of the former system. The relation between the integrals turns out to be rather nontrivial and requires some sort of a "renormalization" [57].

6 Rationally Extended Harmonic Oscillator and Conformal Mechanics Systems

Quantum harmonic oscillator (QHO) and conformal mechanics systems [92–122] described by de Alfaro-Fubini-Furlan (AFF) model [92] are characterized by conformal symmetry. In the case of harmonic oscillator, like in the free particle case, it extends to the Schrödinger symmetry [93–95, 123]. Heisenberg subalgebra in the free particle system is generated by the momentum operator being time-independent integral of motion, and by generator of the Galilean boosts $\hat{G}_0$, which is a dynamical integral of motion. In the case of the QHO, Heisenberg subalgebra is generated by two dynamical integrals of motion to be linear in the ladder operators. In correspondence with this, ladder operators are the spectrum-generating operators of the QHO having discrete equidistant spectrum instead of the continuous spectrum of the free particle. As a consequence of these similarities and differences between the free particle and QHO, exotic supersymmetry can also be generated by Darboux–Crum transformations applied to the latter system. Instead of the two pairs of time-independent supercharge generators in superextended reflectionless systems, in superextended systems constructed from the pairs of the rational extensions of the QHO, only two supercharges are time-independent integrals, while other two odd generators are dynamical integrals of motion. As a result, instead of the exotic nonlinear $\mathcal{N} = 4$ supersymmetry of the paired reflectionless (and finite-gap) systems, in the case of the deformed oscillator systems there appear some nonlinearly deformed and generalized super-Schrödinger symmetry. The superextended systems composed from the AFF model (with special values $g = n(n + 1)$ of the coupling constant in its additional potential term g/x^2) and

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its rational extensions are described by the nonlinearly deformed and generalized superconformal symmetry [51].

Let us consider first in more detail the case of rational deformations of the QHO system [5, 6, 49, 51, 71, 124–128]. To generate a rational deformation of the QHO, it is necessary to choose the set of its physical or non-physical eigenstates as seed states for the Darboux–Crum transformation so that their Wronskian will take nonzero values. In this way we generate an almost isospectral quantum system with difference only in finite number of added or eliminated energy levels. The QHO Hamiltonian $\hat{H}_{osc} = -\frac{d^2}{dx^2} + x^2$ possesses the same symmetry under the Wick rotation as the quantum free particle system: if $\psi(x)$ is a solution of the time-independent Schrödinger equation $\hat{H}_{osc}(x)\psi(x) = E\psi(x)$, then $\psi(ix)$ is a solution of equation $\hat{H}_{osc}(x)\psi(ix) = -E\psi(ix)$. To construct a rational deformation of the QHO described by a nonsingular on the whole real line potential, one can take the following set of the non-physical eigenstates of $\hat{H}_{osc}$ as the seed states for the Darboux–Crum transformation:

$$(\psi^-_{j_1}, \ldots, \psi^-_{j_1+l_1}),\ (\psi^-_{j_2}, \ldots, \psi^-_{j_2+l_2}),\ \ldots,\ (\psi^-_{j_r}, \ldots, \psi^-_{j_r+l_r}), \tag{53}$$

where $j_1 = 2g_1$, $j_{k+1} = j_k + l_k + 2g_{k+1}$, $g_k = 1, \ldots$, $l_k = 0, 1, \ldots$, $k = 1, \ldots, r-1$. Here $\psi^-_n(x) = \psi_n(ix)$, $n = 0, \ldots$, is a non-physical eigenstate of $\hat{H}_{osc}$ of eigenvalue $E^-_n = -(2n+1)$, obtained by Wick rotation from a (non-normalized) physical eigenstate $\psi_n(x) = H_n(x)e^{-x^2/2}$ of energy $E_n = 2n+1$, where $H_n(x)$ is Hermite polynomial of order n. The indicated set of non-physical eigenstates of $\hat{H}_{osc}$ guarantees that the Wronskian of the chosen seed states, $\mathbb{W} = \mathbb{W}(-n_m, \ldots, -n_1)$, takes nonzero values for all $x \in \mathbb{R}$ [129]. Here we assume that $n_m > \ldots > n_1 > 0$, and in what follows we use the notation for physical and non-physical eigenstates $n = \psi_n$ and $-n = \psi^-_n$, respectively. The DC scheme based on the set of the non-physical states having negative eigenvalues was called "negative" in [71]. Wronskian $\mathbb{W} = \mathbb{W}(-n_m, \ldots, -n_1)$ is equal to some polynomial multiplied by $\exp(n_- x^2/2)$, where $n_- = (l_1+1) + \cdots + (l_r+1)$ is the number of the chosen seed states, and according to Eq. (10), the DC transformation generates the system described by the harmonic term x^2 extended by some rational in x term. Transformation based on the negative scheme $(-n_m, \ldots, -n_1)$ introduces effectively into the spectrum of the QHO the n_- bound states of energy levels $-2n_m-1, \ldots, -2n_1-1$. These additional energy levels are grouped into r "valence" bands with $l_k + 1$ levels in the band with index k, which are separated by gaps of the size $4g_k$, with the first valence band separated from the infinite equidistant part of the spectrum by the gap of the size $4g_1$. The same structure of the spectrum can be achieved alternatively by eliminating $n_+ = 2(g_1 + \cdots + g_r)$ energy levels from the spectrum of the QHO by taking n_+ physical states

$$(\psi_{l_r+1}, \ldots, \psi_{l_r+2g_r}),\ \ldots,\ (\psi_{n_m-2g_1+1}, \ldots, \psi_{n_m}), \tag{54}$$

organized into n_- groups.

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The duality of the negative and positive schemes based on the sets of the seed states (53) and (54) can be established as follows. Applying Eq. (15) with $\psi_* = -0$, and equalities $\psi_0^- \frac{d}{dx}\frac{1}{\psi_0^-} = -a^+$, $a^+\widetilde{\psi_0^-} = \psi_0$, $a^+(-n) = -(n-1)$, where $a^+ = -\frac{d}{dx} + x$ is the raising ladder operator of the QHO, we obtain the relation [71]

$$\mathbb{W}(-n_m, \ldots, -n_1) = \mathbb{W}(-0, \widetilde{-0}, -n_m, \ldots, -n_1)$$
$$= e^{x^2/2}\mathbb{W}(0, -(n_m-1), -(n-1)). \qquad (55)$$

It means that the negative scheme generated by the set of the n_- non-physical seed states $(-n_m, \ldots, -n_1)$ and the "mixed" scheme based on the set of the seed states $(0, -(n_m-1), -(n-1))$ involving the ground eigenstate generate, according to Eq. (10), the same quantum system but given by the Hamiltonian operator shifted for the additive constant term: the potential obtained on the base of the indicated mixed scheme will be shifted for the constant $+4$ in comparison with the potential generated via the DC transformation based on the negative scheme. Eq. (55) is analogous to the Wronskian relation (26) for the free particle states, with the state $\psi_0 = 1$ and operator $\psi_0\frac{d}{dx}\frac{1}{\psi_0} = \frac{d}{dx}$ there to be analogous to the ground state and raising ladder operator of the QHO here. In (26), however, the Wronskian equality does not contain any nontrivial functional factor in comparison with the exponential multiplier appearing in (55). As a result, as we saw before, in the case of the free particle any reflectionless system can be generated from it by means of the two DC transformations, which produce exactly the same Hamiltonian operator. Consequently, we construct there two pairs of the supercharges for the corresponding superextended system which are the integrals of motion not depending explicitly on time. On the other hand, in the case of a superextended system produced from the QHO we shall have two fermionic integrals to be true, time-independent integrals of motion, but two other odd generators of the superalgebra will be time-dependent, dynamical integrals of motion.

Applying repeatedly the procedure of Eq. (55), we obtain finally the relation [71]

$$\mathbb{W}(-n_m, \ldots, -n_1) = e^{(n_m+1)x^2/2}\mathbb{W}(n'_1, \ldots, n'_m = n_m), \qquad (56)$$

where $0 < n'_1 < \cdots < n'_m = n_m$. This relation means that the negative scheme $(-n_m, \ldots, -n_1)$ with n_- seed states is dual to the positive scheme $(n'_1, \ldots, n'_m = n_m)$ with $n_+ = n_m + 1 - n_- = 2(g_1 + \cdots + g_k)$ seed states representing physical eigenstates of the QHO. The two dual schemes can be unified in one "mirror" diagram, in which any of the two schemes can be obtained from another by a kind of a "charge conjugation," see ref. [71]. In this way we obtain, as an example, the pairs of dual schemes $(-2) \sim (1, 2)$ and $(-2, -3) \sim (2, 3)$. Eq. (56) means that the dual schemes generate the same rationally extended QHO system but the Hamiltonian corresponding to the positive scheme will be shifted in comparison to the Hamiltonian produced on the base of the negative scheme for additive constant equal to $2(n_+ + n_-) = 2(n_m + 1)$. One can also note that in comparison with the

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free particle case, the total number of the seed states in both dual schemes can be odd or even.

We denote by $\mathbb{A}^-_{(-)}$ the intertwining operator $\mathbb{A}_{n_-}$ constructed on the base of the negative scheme, and $\mathbb{A}^+_{(-)} \equiv (\mathbb{A}^-_{(-)})^\dagger$, see Eq. (12). These are differential operators of order n_-. Analogously, the intertwining operators constructed by employing the dual positive scheme we denote as $\mathbb{A}^-_{(+)}$, and $\mathbb{A}^+_{(+)} \equiv (\mathbb{A}^-_{(+)})^\dagger$; they are differential operators of order n_+. We denote by $\hat{L}_{(-)}$ and $\hat{L}_{(+)}$ the Hamiltonian operators generated from the QHO Hamiltonian $\hat{H}_- = \hat{H}_{osc}$ by means of the DC transformation realized on the base of the negative and positive dual schemes, respectively. Then $\hat{L}_{(+)} = \hat{L}_{(-)} + 2(n_+ + n_-)$, $\mathbb{A}^-_{(-)}\hat{H}_- = \hat{L}_{(-)}$, $\mathbb{A}^-_{(+)}\hat{H}_- = \hat{L}_{(+)}$. For the rationally deformed QHO system $\hat{L}_{(-)}$ one can construct three pairs of the ladder operators, two of which are obtained by Darboux-dressing of the ladder operators of the QHO system $\mathcal{A}^\pm = \mathbb{A}^-_{(-)} a^\pm \mathbb{A}^+_{(-)}$, and $\mathcal{B}^\pm = \mathbb{A}^-_{(+)} a^\pm \mathbb{A}^+_{(+)}$, while the third pair is obtained by gluing different intertwining operators, $\mathcal{C}^- = \mathbb{A}^-_{(+)}\mathbb{A}^+_{(-)}$, $\mathcal{C}^+ = \mathbb{A}^-_{(-)}\mathbb{A}^+_{(+)}$. These ladder operators detect all the separated states in the rationally deformed QHO system $\hat{L}_{(-)}$ (or $\hat{L}_{(+)}$) organized into the valence bands; they also distinguish the valence bands themselves, and any of the two sets $(\mathcal{C}^\pm, \mathcal{A}^\pm)$ or $(\mathcal{C}^\pm, \mathcal{B}^\pm)$ represents the complete spectrum-generating set of the ladder operators of the system $\hat{L}_{(-)}$. The operators $\mathcal{A}^\pm e^{\mp 2it}$, $\mathcal{B}^\pm e^{\mp 2it}$, $\mathcal{C}^\pm e^{\pm 2(n_+ + n_-)it}$ are the dynamical integrals of motion of the system $L_{(-)}$. Being higher derivative differential operators, they have a nature of generators of a hidden symmetry. If we construct now the extended system $\hat{\mathcal{H}} = \text{diag}\,(\hat{L}_{(-)}, \hat{H}_-)$, the pair of the supercharges constructed from the intertwining operators $\mathbb{A}^\pm_{(-)}$ will be its time-independent odd integrals of motion, while from the intertwining operators $\mathbb{A}^\pm_{(+)}$ we obtain a pair of the fermionic dynamical integrals of motion. Proceeding from these odd integrals of motion and the Hamiltonian $\hat{\mathcal{H}}$, one can generate a nonlinearly deformed generalized super-Schrödinger symmetry of the superextended system $\hat{\mathcal{H}}$. In the superextended system $\hat{\mathcal{H}} = \text{diag}\,(\hat{L}_{(+)}, \hat{H}_-)$, the pair of the time-independent supercharges is constructed from the pair of intertwining operators $\mathbb{A}^\pm_{(+)}$, while the dynamical fermionic integrals of motion are obtained from the intertwining operators $\mathbb{A}^\pm_{(-)}$. This picture with the nonlinearly deformed generalized super-Schrödinger symmetry can also be extended for the case of a superextended system $\hat{\mathcal{H}}$ composed from any pair of the rationally deformed quantum harmonic oscillator systems.

In [71], it was shown that the AFF model $\hat{H}_g = -\frac{d^2}{dx^2} + x^2 + \frac{g}{x^2}$ with special values $g = n(n+1)$ of the coupling constant can be obtained by applying the appropriate CD transformation to the half-harmonic oscillator obtained from the QHO by introducing the infinite potential barrier at $x = 0$. As a consequence, rational deformations of the AFF conformal mechanics model can be obtained by employing some modification of the described DC transformations based on the dual schemes applied to the QHO system. The corresponding superextended systems composed from rationally deformed versions of the conformal mechanics are described by the nonlinearly deformed generalized superconformal symme-

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try instead of the nonlinearly deformed generalized super-Schrödinger symmetry appearing in the case of the superextended rationally deformed QHO systems, see [51]. The construction of rational deformations for the AFF model can be generalized for the case of arbitrary values of the coupling constant $g = \nu(\nu + 1)$ [130].

7 Conclusion

We considered nonlinear supersymmetry of one-dimensional mechanical systems which has the nature of the hidden symmetry generated by supercharges of higher order in momentum. In the case of reflectionless, finite-gap, rationally deformed oscillator and conformal mechanics systems, as well as in a special class of the $\mathcal{PT}$-regularized Calogero systems, the nonlinear $\mathcal{N} = 2$ Poincaré supersymmetry expands up to exotic nonlinear $\mathcal{N} = 4$ supersymmetric and nonlinearly deformed generalized super-Schrödinger or superconformal structures.

Classical symmetries described by the linear Lie algebraic structures are promoted by geometric quantization to the quantum level [131, 132]. Though nonlinear symmetries described by W-type algebras can be produced from linear symmetries via some reduction procedure [64], the problem of generation of nonlinear quantum mechanical supersymmetries from the linear ones was not studied in a systematic way. It would be interesting to investigate this problem bearing particularly in mind the problem of the quantum anomaly associated with nonlinear supersymmetry [14]. Some first steps were realized in this direction in [62] in the light of the so-called coupling constant metamorphosis mechanism [133]. Note also that, as was shown in [12], nonlinear supersymmetry of purely parabosonic systems can be obtained by reduction of parasupersymmetric systems.

Hidden symmetries can be associated with the presence of the peculiar geometric structures in the corresponding systems [1, 2, 134]. It would be interesting to investigate nonlinear supersymmetry and related exotic nonlinear supersymmetric and superconformal structures from a similar perspective.

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