ASYMPTOTIC BEHAVIOR OF SOLUTIONS OF NONAUTONOMOUS NEUTRAL DYNAMICAL SYSTEMS

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Abstract. This paper studies the dynamics of families of monotone nonautonomous neutral functional differential equations with nonautonomous operator, of great importance for their applications to the study of the long-term behavior of the trajectories of problems described by this kind of equations, such us compartmental systems and neural networks among many others. Precisely, more general admissible initial conditions are included in the study to show that the solutions are asymptotically of the same type as the coefficients of the neutral and non-neutral part.

1. Introduction

This paper studies the long-term behavior of the trajectories of a monotone skew-product semiflow, $\tau : \mathbb{R}^+ \times \Omega \times X \to \Omega \times X,\ (t,\omega,x) \mapsto (\omega{\cdot}t, u(t,\omega,x))$, generated by a family of nonautonomous differential equations. The base of the phase space, Ω, is a compact metric space endowed with a global recurrent flow $(\Omega,\sigma,\mathbb{R})$, and the fiber, X, is a Banach space with a positive cone X_+ that induces an order relation. The monotone character means that, if $\omega \in \Omega$ and $x_1, x_2 \in X$ with $x_1 \le x_2$, then $u(t,\omega,x_1) \le u(t,\omega,x_2)$ for each t in the common interval of definition of the trajectories. We denote $\omega{\cdot}t = \sigma(t,\omega)$ and u satisfies the cocycle identity $u(t+s,\omega,x) = u(t,\omega{\cdot}s, u(s,\omega,x))$ for every $t, s \ge 0$.

It is well known that the skew-product formalism is a powerful tool in the study of linear and nonlinear evolution systems. Frequently, this formalism is obtained from a single nonautonomous differential equation using a standard hull construction.

An important result for monotone uniformly stable recurrent skew-product semiflows is the convergence of relatively compact trajectories to their omega-limit sets, which define 1-coverings of the base space. This result was firstly proved by Jiang and Zhao [15] in an abstract setting applicable to cooperative systems of ordinary, finite delay and parabolic nonautonomous differential equations. In particular, when the skew-product semiflow comes from a single differential equation with a recurrence property in the coefficients as constancy, periodicity, almost-periodicity among others, it provides a unified generalization of the asymptotic constancy, periodicity or almost-periodicity of the solutions, studied in many previous papers.

This theory was extended to nonautonomous functional differential equations (FDEs for short) with infinite delay in Novo *et al.* [18] (see also Wang and Zhao [29]

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for other implications of this theory). In that paper, $X = BU$, i.e. the subset of functions of $C((-\infty, 0], \mathbb{R}^m)$ that are bounded and uniformly continuous, and the semiflow is generated by the solutions of a family $y' = F(\omega{\cdot}t, y_t)$ of FDEs defined by a continuous function $F : \Omega \times BU \to BU$ which is locally Lipschitz in its second variable. The space BU satisfies standard conditions of regularity that imply existence, uniqueness and continuous dependence of the solutions with respect to the initial data (see Hino *et al.* [12]).

Later, motivated by the applicability to the study of the long-term dynamics of compartmental systems, the paper by Muñoz-Villarragut *et al.* [17] is the starting point of an important effort to extend the previous results to nonautonomous neutral functional differential equations (NFDEs for short) with infinite delay.

Compartmental systems have been used as mathematical models for the study of the dynamical behavior of many processes in the biological and physical sciences which depend on local mass balance conditions (see Jacquez [13], Jacquez and Simon [14] and the references therein). Some initial results for models described by FDEs with finite and infinite delay can be found in Győri [6], and Győri and Eller [8]. The papers by Arino and Bourad [1], and Arino and Haourigui [2] prove the existence of almost periodic solutions for compartmental systems described by almost periodic FDEs and NFDEs with finite delay. Győri and Wu [9] modeled the dynamical properties of compartmental systems with active compartments by means of NFDEs with infinite delay, whose neutral term represents the net amount of material produced or swallowed by the compartments. This type of NFDEs equations have been investigated by Wu and Freedman [31], and Wu [30].

An important difficulty that appears in the monotone theory of NFDEs is that, in many applications, the order structure must be defined by means of an exponential ordering which provides a positive cone with empty interior. Krisztin and Wu [16] show the asymptotic periodicity of the solutions with Lipschitz continuous initial data, under appropriate conditions on the coefficients of scalar periodic NFDEs with finite delay and linear neutral term, which imply the monotonicity of the solutions for an exponential ordering. By means of monotone skew-product semiflow techniques, Novo *et al.* [19], and Obaya and Villarragut [21] generalize the previous results obtaining that, under appropriate assumptions, a family of nonautonomous NFDEs with infinite delay and linear neutral term induces a monotone skew-product semiflow on $\Omega\times BU$ for the exponential ordering, and the omega-limit sets of bounded trajectories with Lipschitz continuous initial data are copies of the base. The case of stable nonautonomous operator D for the neutral part is also considered in Obaya and Villarragut [20], where similar results are obtained for a new transformed exponential ordering.

The present paper provides new contributions to the core of the dynamical theory of monotone recurrent skew-product semiflows generated by FDEs and NFDEs with infinite delay, and improves the conditions of applicability of the theory to compartmental systems and other models of interest, which will be explained in detail in forthcoming publications. More precisely, this work provides a dynamical framework to study compartmental systems described by neutral functional differential equations analogous to those considered in [9, 16, 18, 19, 20, 21], under physical conditions that have not been previously considered in the literature. In addition, Wu and and Zhao [32] introduce the exponencial ordering and the associated monotone methods for abstract delayed reaction diffusion equations, and show a natural way to extend the conclusions of the above references to nonautonomous compartmental systems with spatial diffusion, to which our study may also be applied.

The structure and main goals of the paper are now described. Some basic notions and properties of the theory of nonautonomous dynamical systems are included in Section 2. Section 3 is devoted to the study of families $y' = F(\omega{\cdot}t, y_t)$ of FDEs with infinite delay defined by continuous functions $F : \Omega \times BC \to BC$, where $BC = \{y \in C((-\infty, 0], \mathbb{R}^m) \mid y \text{ is bounded}\}$, which are locally Lipschitz continuous in their second variable. Although initial data in $\Omega \times BC$ are physically admissible, the choice of this set as a phase space is problematic, because the existence of a solution of the Cauchy problem requires the measurability of the map $(-\infty, T] \to \mathbb{R}^m, t \to F(\omega \cdot t, y_t)$ for each $T \in \mathbb{R}$, $\omega \in \Omega$ and $y \in C((-\infty, T], \mathbb{R}^m]$, (see Driver [4] and Seifert [23]). In our setting, this is a consequence of assuming the continuity of $F : \Omega \times B_r \to \mathbb{R}^m$ when the closed ball $B_r \subset BC$ is endowed with the compact open topology, which, of course, is satisfied in all the physical models that we want to work with. Instead of considering Lipschitz continuous initial data in BC, we introduce the bigger set $\mathcal{R}$ of the elements in BC with uniformly bounded variation on the intervals $[-k, -k+1]$ for $k \geq 1$. By considering an appropriate exponential ordering $\leq_A$, assuming a quasimonotone condition on F, a componentwise separation property and the uniform stability of B_r for the order $\leq_A$, the main conclusion of this section is that omega-limit sets of bounded trajectories with initial data in $\mathcal{R}$ are 1-coverings of the base Ω, that is, the recurrent character is inherited.

Section 4 extends, for the transformed exponential ordering introduced in [20], the previous results to families $\frac{d}{dt}D(\omega{\cdot}t, z_t) = G(\omega{\cdot}t, z_t)$ of NFDEs defined by a stable neutral term $D : \Omega \times BC \to \mathbb{R}^m$, linear in the state component, and a function $G : \Omega \times BC \to \mathbb{R}^m$ that satisfies properties of regularity analogous to those considered in the previous section. The main idea is to deduce, from the stability of D, the invertibility of the operator $\widehat{D} \colon \Omega \times BC \to \Omega \times BC$, $(\omega, x) \mapsto (\omega, \widehat{D}_2(\omega, x))$, where $\widehat{D}_2(\omega, x) \colon (-\infty, 0] \to \mathbb{R}^m$, $s \mapsto D(\omega{\cdot}s, x_s)$, and to transform the NFDE into a FDE to which the conclusions of Section 3 can be applied. As a consequence, the omega-limit sets of bounded trajectories with initial datum x satisfying $\widehat{D}_2(\omega, x) \in \mathcal{R}$ are 1-coverings of the base Ω, or what is equivalent, the trajectories reproduce asymptotically the recurrent behavior of the coefficients of the neutral and non-neutral part, that is, the dynamics exhibited by the time variation of the equation.

2. Some preliminaries

Let (Ω, d) be a compact metric space. A real *continuous flow* $(\Omega, \sigma, \mathbb{R})$ is defined by a continuous mapping $\sigma : \mathbb{R} \times \Omega \to \Omega$, $(t, \omega) \mapsto \sigma(t, \omega)$ satisfying

(i) $\sigma_0 = \text{Id}$,
(ii) $\sigma_{t+s} = \sigma_t \circ \sigma_s$ for each $s, t \in \mathbb{R}$,

where $\sigma_t(\omega) = \sigma(t, \omega)$ for all $\omega \in \Omega$ and $t \in \mathbb{R}$. The set $\{\sigma_t(\omega) \mid t \in \mathbb{R}\}$ is called the *orbit* or the *trajectory* of the point ω. We say that a subset $\Omega_1 \subset \Omega$ is σ-*invariant* if $\sigma_t(\Omega_1) = \Omega_1$ for every $t \in \mathbb{R}$. A subset $\Omega_1 \subset \Omega$ is called *minimal* if it is compact, σ-invariant and its only nonempty compact σ-invariant subset is itself. Every compact and σ-invariant set contains a minimal subset; in particular, it is easy to prove that a compact σ-invariant subset is minimal if and only if every trajectory is dense. We say that the continuous flow $(\Omega, \sigma, \mathbb{R})$ is *recurrent* or *minimal* if Ω is minimal. Almost periodic and almost automorphic flows are relevant examples of recurrent flows. We refer to Ellis [5] and see Shen and Yi [24, part II] for the study of topological and ergodic properties of these flows.

Let E be a complete metric space and $\mathbb{R}^+ = \{t \in \mathbb{R} \,|\, t \geq 0\}$. A *semiflow* $(E, \Phi, \mathbb{R}^+)$ is determined by a continuous map $\Phi : \mathbb{R}^+ \times E \to E,\ (t,x) \mapsto \Phi(t,x)$ which satisfies

(i) $\Phi_0 = \mathrm{Id}$,
(ii) $\Phi_{t+s} = \Phi_t \circ \Phi_s$ for all $t,\, s \in \mathbb{R}^+$,

where $\Phi_t(x) = \Phi(t,x)$ for each $x \in E$ and $t \in \mathbb{R}^+$. The set $\{\Phi_t(x) \mid t \geq 0\}$ is the *semiorbit* of the point x. A subset E_1 of E is *positively invariant* (or just Φ-*invariant*) if $\Phi_t(E_1) \subset E_1$ for all $t \geq 0$. A semiflow $(E, \Phi, \mathbb{R}^+)$ admits a *flow extension* if there exists a continuous flow $(E, \widetilde{\Phi}, \mathbb{R})$ such that $\widetilde{\Phi}(t,x) = \Phi(t,x)$ for all $x \in E$ and $t \in \mathbb{R}^+$. A compact and positively invariant subset admits a flow extension if the semiflow restricted to it admits one.

Write $\mathbb{R}^- = \{t \in \mathbb{R} \,|\, t \leq 0\}$. A *backward orbit* of a point $x \in E$ in the semiflow $(E, \Phi, \mathbb{R}^+)$ is a continuous map $\psi : \mathbb{R}^- \to E$ such that $\psi(0) = x$ and, for each $s \leq 0$, it holds that $\Phi(t, \psi(s)) = \psi(s+t)$ whenever $0 \leq t \leq -s$. If for $x \in E$ the semiorbit $\{\Phi(t,x) \mid t \geq 0\}$ is relatively compact, we can consider the *omega-limit set* of x,

$$\mathcal{O}(x) = \bigcap_{s\geq 0} \mathrm{closure}\{\Phi(t+s,x) \mid t \geq 0\}\,,$$

which is a nonempty compact connected and Φ-invariant set. Namely, it consists of the points $y \in E$ such that $y = \lim_{n\to\infty} \Phi(t_n, x)$ for some sequence $t_n \uparrow \infty$. It is well-known that every $y \in \mathcal{O}(x)$ admits a backward orbit inside this set. Actually, a compact positively invariant set M admits a flow extension if every point in M admits a unique backward orbit which remains inside the set M (see [24, part II]).

A compact positively invariant set M for the semiflow $(E, \Phi, \mathbb{R}^+)$ is *minimal* if it does not contain any other nonempty compact positively invariant set than itself. If E is minimal, we say that the semiflow is minimal.

A semiflow is *of skew-product type* when it is defined on a vector bundle and has a triangular structure; more precisely, a semiflow $(\Omega \times X, \tau, \mathbb{R}^+)$ is a *skew-product* semiflow over the product space $\Omega \times X$, for a compact metric space (Ω, d) and a complete metric space (X, d), if the continuous map τ is as follows:

$$\begin{array}{rccc} \tau: & \mathbb{R}^+ \times \Omega \times X & \longrightarrow & \Omega \times X \\ & (t, \omega, x) & \mapsto & (\omega{\cdot}t, u(t,\omega,x))\,, \end{array} \tag{2.1}$$

where $(\Omega, \sigma, \mathbb{R})$ is a real continuous flow $\sigma : \mathbb{R} \times \Omega \to \Omega,\ (t,\omega) \mapsto \omega{\cdot}t$, called the *base flow*. The skew-product semiflow (2.1) is *linear* if $u(t,\omega,x)$ is linear in x for each $(t,\omega) \in \mathbb{R}^+ \times \Omega$.

3. Functional differential equations with infinite delay

We consider the Fréchet space $X = C((-\infty, 0], \mathbb{R}^m)$ endowed with the compact-open topology, i.e. the topology of uniform convergence over compact subsets, which is a metric space for the distance

$$\mathsf{d}(x,y) = \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{\|x-y\|_n}{1 + \|x-y\|_n}\,, \quad x, y \in X\,,$$

where $\|x\|_n = \sup_{s\in[-n,0]} \|x(s)\|$, and $\|\cdot\|$ denotes the maximum norm in $\mathbb{R}^m$.

Let $(\Omega, \sigma, \mathbb{R})$ be a minimal flow over a compact metric space (Ω, d) and denote $\sigma(t, \omega) = \omega{\cdot}t$ for each $\omega \in \Omega$ and $t \in \mathbb{R}$. As usual, given $I = (-\infty, a] \subset \mathbb{R}$, $t \in I$ and a continuous function $x : I \to \mathbb{R}^m$, x_t will denote the element of X defined by $x_t(s) = x(t+s)$ for $s \in (-\infty, 0]$. We consider the family of nonautonomous infinite delay functional differential equations

$$z'(t) = F(\omega{\cdot}t, z_t)\,, \quad t \geq 0\,,\ \omega \in \Omega\,. \tag{3.1}$$

The first objective of this section is to provide an appropriate framework to study the dynamical behavior of the solutions of (3.1). One of the admissible phase spaces for the study of these equations is BU (see [12]), the Banach space of bounded and uniformly continuous functions in X, i.e.

$$BU = \{x \in X \mid x \text{ is bounded and uniformly continuous}\}$$

with the supremum norm $\|x\|_\infty = \sup_{s\in(-\infty,0]} \|x(s)\|$.

This is not the case for the Banach space

$$BC = \{x \in X \mid x \text{ is bounded}\}$$

where, in general, the family (3.1) does not induce a local skew-product semiflow on $\mathbb{R}^+ \times \Omega \times BC$. However, in many applications, initial data in BC are physically admissible. We show how to overcome this drawback to introduce a dynamical structure on $\Omega \times BC$.

Given $r > 0$, we will denote

$$B_r = \{x \in BC \mid \|x\|_\infty \leq r\}$$

and we consider the family of nonautonomous FDEs (3.1) defined by a function $F\colon \Omega \times BC \to \mathbb{R}^m$, $(\omega, x) \mapsto F(\omega, x)$ satisfying:

(**F1**) F is continuous on $\Omega \times BC$ when the the norm $\|\cdot\|_\infty$ is considered on BC, and Lipschitz continuous on $\Omega \times B_r$ in its second variable for each $r > 0$,

which in particular implies that

$$F(\Omega \times B_r) \text{ is a bounded subset of } \mathbb{R}^m \text{ for each } r > 0. \tag{3.2}$$

From this condition, the standard theory of infinite delay differential equations (see [12]) assures that, for each $x \in BU$ and each $\omega \in \Omega$, the system $(3.1)_\omega$ locally admits a unique solution $z(\cdot, \omega, x)$ with initial value x, i.e. $z(s, \omega, x) = x(s)$ for each $s \in (-\infty, 0]$. Therefore, the family (3.1) induces a local skew-product semiflow

$$\begin{array}{rlcl} \tau & : \mathcal{U} \subset \mathbb{R}^+ \times \Omega \times BU & \longrightarrow & \Omega \times BU \\ & (t, \omega, x) & \mapsto & (\omega{\cdot}t, u(t, \omega, x))\,, \end{array} \tag{3.3}$$

where $u(t, \omega, x) \in BU$ and $u(t, \omega, x)(s) = z_t(\omega, x)(s) = z(t+s, \omega, x)$ for $s \in (-\infty, 0]$.

When the initial data x belongs to BC the existence and uniqueness is not guaranteed from (**F1**). In addition, we impose the following condition satisfied in important applications, such as compartmental systems and neural networks.

(**F2**) for each $r > 0$, $F\colon \Omega \times B_r \to \mathbb{R}^m$ is continuous when we take the restriction of the compact-open topology to B_r, i.e. if $\omega_n \to \omega$ and $x_n \xrightarrow{\mathsf{d}} x$ as $n \uparrow \infty$ with $x \in B_r$, then $\lim_{n\to\infty} F(\omega_n, x_n) = F(\omega, x)$.

Next, we show that $\Omega \times BC$ is indeed a good phase space.

Proposition 3.1. *Under assumptions* (**F1**)–(**F2**), *for each $x \in BC$ and each $\omega \in \Omega$ the system* $(3.1)_\omega$ *locally admits a unique solution $z(\cdot,\omega,x)$ with initial value x, i.e. $z(s,\omega,x) = x(s)$ for each $s \in (-\infty, 0]$.*

Proof. As explained in [23] (see also Sawano [22]), the result can be deduced from [4] once we check the continuity of the map

$$(-\infty, T] \to \mathbb{R}^m, \quad t \mapsto F(\omega{\cdot}t, y_t)$$

for each $T \in \mathbb{R}$ and each bounded function $y \in C((-\infty, T], \mathbb{R}^m)$. This is an easy consequence of (**F2**), because if $t = \lim_{n\to\infty} t_n$ with $t_n \in (-\infty, T]$, then y_{t_n} belongs to some B_r for all $n \in \mathbb{N}$ and $y_{t_n} \xrightarrow{\mathsf{d}} y_t$ as $n \uparrow \infty$. □

As a consequence, the map (3.3) is extended to $\Omega \times BC$. Moreover, as shown next, this extension turns out to be continuous on bounded sets when the restriction of the compact-open topology to BC and the product metric topology on $\Omega \times BC$ are considered.

Proposition 3.2. *Under assumptions* (**F1**)–(**F2**), *the local map*

$$\begin{array}{rcl} \mathcal{U} \subset \mathbb{R}^+ \times \Omega \times B_r & \longrightarrow & \Omega \times BC \\ (t, \omega, x) & \mapsto & (\omega{\cdot}t, u(t,\omega,x)) \end{array}$$

is continuous when we take the restriction of the compact-open topology to B_r, i.e. if $t_n \to t$, $\omega_n \to \widetilde{\omega}$ and $x_n \xrightarrow{\mathsf{d}} \widetilde{x}$ as $n \uparrow \infty$ with $x_n, \widetilde{x} \in B_r$ for all $n \in \mathbb{N}$, then $\omega_n{\cdot}t_n \to \widetilde{\omega}{\cdot}t$ and $u(t_n, \omega_n, x_n) \xrightarrow{\mathsf{d}} u(t, \widetilde{\omega}, \widetilde{x})$ as $n \uparrow \infty$.

Proof. First we fix a $t \in \mathbb{R}^+$ such that $u(t,\widetilde{\omega},\widetilde{x})$ is defined and we check that $u(t,\omega_n,x_n) \xrightarrow{\mathsf{d}} u(t,\widetilde{\omega},\widetilde{x})$ as $n \uparrow \infty$. If F is a bounded function on $\Omega \times BC$, then $\sup_{\tau\in[0,t],n\geq 1} \|u(\tau,\omega_n, x_n)\|_\infty < \infty$ and the proof of Proposition 4.2 of [18] can be easily adapted to this case. Otherwise, take $\delta > 0$ such that $u(\tau,\widetilde{\omega},\widetilde{x})$ is defined for $\tau \in [0, t+\delta]$ and denote by $k = \sup_{\tau\in[0,t+\delta]}\{\|u(\tau,\widetilde{\omega},\widetilde{x})\|_\infty, r\}$. In addition, from (3.2), we know that $F(\Omega \times B_{k+1})$ is bounded and we can take $\rho = \sup_{(\omega,x)\in\Omega\times B_{k+1}} \|F(\omega,x)\|$. Now let $\varphi\colon \mathbb{R}^m \to \mathbb{R}^m$ be a C^∞ function such that

$$\varphi(y) = \begin{cases} y, & \text{if } \|y\| \leq \rho\,, \\ 0, & \text{if } \|y\| \geq \rho + 1 \end{cases}$$

and consider the family of equations

$$\widetilde{z}\,'(t) = \varphi(F(\omega{\cdot}t, \widetilde{z}_t))\,, \quad t \geq 0\,,\ \omega \in \Omega\,.$$

The boundedness of $\varphi \circ F$ provides $v(\tau,\omega_n,x_n) \xrightarrow{\mathsf{d}} v(\tau,\widetilde{\omega},\widetilde{x})$ as $n \uparrow \infty$ for each $\tau \in [0, t{+}\delta]$, where $v(\tau,\omega,x)(s) = \widetilde{z}(\tau{+}s,\omega,x)$ for $s \in (-\infty, 0]$, as usual. Therefore, the definitions of φ and ρ yield $v(\tau,\widetilde{\omega},\widetilde{x}) = u(\tau,\widetilde{\omega},\widetilde{x})$ for $\tau \in [0, t+\delta]$, that is,

$$\widetilde{z}(\tau,\omega_n,x_n) \to z(\tau,\widetilde{\omega},\widetilde{x}) \text{ as } n \uparrow \infty \text{ uniformly for } \tau \in [0, t+\delta]\,.$$

From this, together with $\|x_n\|_\infty \leq r$ for each $n \in \mathbb{N}$, we deduce that there is an $n_0 \in \mathbb{N}$ such that $\sup_{\tau\in[0,t+\delta]} \|v(\tau,\omega_n,x_n)\|_\infty \leq k+1$ for each $n \geq n_0$. Hence, $v(t,\omega_n,x_n) = u(t,\omega_n,x_n)$ for $n \geq n_0$ and $u(t,\omega_n,x_n) \xrightarrow{\mathsf{d}} u(t,\widetilde{\omega},\widetilde{x})$ as $n \uparrow \infty$, as claimed. Moreover, $v(\tau,\omega_n,x_n) = u(\tau,\omega_n,x_n)$ for $n \geq n_0$ and $\tau \in [0, t+\delta]$, and

$$\sup\{\|u(\tau,\omega_n,x_n)\|_\infty \mid \tau \in [0, t+\delta]\,,\ n \in \mathbb{N}\} < \infty\,. \tag{3.4}$$

Finally, if in addition $t_n \to t$ as $n \uparrow \infty$, then

$$\mathsf{d}(u(t,\widetilde{\omega},\widetilde{x}),u(t_n,\omega_n,x_n)) \le \mathsf{d}(u(t,\widetilde{\omega},\widetilde{x}),u(t,\omega_n,x_n)) + \mathsf{d}(u(t,\omega_n,x_n),u(t_n,\omega_n,x_n))$$

and we only have to check that the second term vanishes as $n \uparrow \infty$.

Let $[a,b] \subset (-\infty,0]$ and $s \in [a,b]$. If $t+s>0$, we take an $n_1 \in \mathbb{N}$ such that the real interval I_n with extrema $t+s$ and t_n+s is contained in $(0,t+\delta)$ for $n \ge n_1$. Thus, from (3.4) and (3.2), we deduce that there is a constant M such that

$$\|u(t,\omega_n,x_n)(s)-u(t_n,\omega_n,x_n)(s)\| \le \int_{I_n} \|F(\omega_n{\cdot}\tau, z_\tau(\omega_n,x_n))\|\,d\tau \le M\,|t-t_n| \quad (3.5)$$

for each $n \ge n_1$. If $t+s<0$, there is also an $n_2 \ge n_1$ such that $t_n+s<0$ for each $n \ge n_2$ and, thus,

$$\|u(t,\omega_n,x_n)(s)-u(t_n,\omega_n,x_n)(s)\| = \|x_n(t+s)-x_n(t_n+s)\|\,. \quad (3.6)$$

We omit the case $t+s=0$ because it is a combination of the previous cases. Therefore, from (3.5), (3.6) and the convergence of $x_n \xrightarrow{\mathsf{d}} \widetilde{x}$ as $n \uparrow \infty$, it is easy to check that $\|u(t,\omega_n,x_n)(s)-u(t_n,\omega_n,x_n)(s)\|$ converges to 0 as $n \uparrow \infty$ uniformly for s in the compact set $[a,b]$, which finishes the proof. □

The next result proves that, under assumptions **(F1)** and **(F2)**, each bounded solution $z(\cdot,\omega_0,x_0)$ provides a relatively compact trajectory. Note that solutions that remain bounded are globally defined on the whole real line (see e.g. [22]).

Proposition 3.3. *Assume* **(F1)**–**(F2)**. *If $x_0 \in BC$ and $z(\cdot,\omega_0,x_0)$ is a solution of equation $(3.1)_{\omega_0}$ bounded for the norm $\|\cdot\|_\infty$, then $\mathcal{F} = \{u(t,\omega_0,x_0) \mid t \ge 0\}$ is a relatively compact subset of BC for the compact-open topology.*

Proof. Let $r = \sup_{t\ge 0}\|u(t,\omega_0,x_0)\|_\infty$. According to Theorem 8.1.4 of [12], $\mathcal{F}$ is relatively compact in X if, and only if, for every $s \in (-\infty,0]$ $\mathcal{F}$ is equicontinuous at s and $\mathcal{F}(s) = \{u(t,\omega_0,x_0)(s) \mid t \ge 0\}$ is relatively compact in $\mathbb{R}^m$.

The second condition holds because $\mathcal{F} \subset B_r$. As for the equicontinuity, let $\rho>0$, $\varepsilon>0$ and $M = \sup_{(\omega,x)\in\Omega\times B_r}\|F(\omega,x)\|$, which is finite thanks to (3.2). Then, for each $t \ge \rho$ and $s_1,s_2 \in [-\rho,0]$ with $|s_1-s_2| < \varepsilon/M$ and $s_1 \le s_2$ (the case $s_2 \le s_1$ is analogous), we have

$$\|u(t,\omega_0,x_0)(s_1)-u(t,\omega_0,x_0)(s_2)\| \le \int_{t+s_1}^{t+s_2} \|F(\omega_0{\cdot}\tau, z_\tau(\omega_{,0}\,,x_0))\|\,d\tau \le \varepsilon\,. \quad (3.7)$$

On the other hand, if $t \in [0,\rho]$, then, for each $s \in [-\rho,0]$,

$$t+s-\rho \in [-2\rho,0] \quad \text{and} \quad u(t,\omega_0,x_0)(s) = u(\rho,\omega_0,x_0)(t+s-\rho)\,.$$

Therefore, the equicontinuity of $\mathcal{F}$ follows from (3.7) and the uniform continuity of $u(\rho,\omega_0,x_0)$ on $[-2\rho,0]$, which finishes the proof. □

In the situation of the foregoing proposition, we can define the *omega-limit set* of the trajectory of the point (ω_0,x_0) as

$$\mathcal{O}(\omega_0,x_0) = \left\{(\omega,x) \mid \exists\, t_n \uparrow \infty \;\text{ with }\; \omega_0{\cdot}t_n \to \omega\,,\; u(t_n,\omega_0,x_0) \xrightarrow{\mathsf{d}} x\right\},$$

and the following proposition provides its main properties.

Proposition 3.4. *Assume* **(F1)**–**(F2)**. *If $(\omega_0,x_0) \in \Omega\times BC$ and $z(\cdot,\omega_0,x_0)$ is a solution of $(3.1)_{\omega_0}$ bounded for the norm $\|\cdot\|_\infty$, then $\mathcal{O}(\omega_0,x_0)$ is a nonempty, compact and invariant subset of $\Omega\times BU$ admitting a flow extension.*

Proof. Thanks to Proposition 3.3, $\mathcal{O}(\omega_0, x_0)$ is nonempty and relatively compact; in order to prove that $\mathcal{O}(\omega_0, x_0)$ is compact, it suffices to check that it is closed, which is omitted.

Next we show that $\mathcal{O}(\omega_0, x_0) \subset \Omega \times BU$. Let $r = \sup_{t\geq 0} \|u(t, \omega_0, x_0)\|_\infty$ and $M = \sup_{\tau\geq 0} \|F(\omega_0{\cdot}\tau, z_\tau(\omega_0, x_0)\|$, which is finite from (3.2) because $z_\tau(\omega_0, x_0) = u(\tau, \omega_0, x_0) \in B_r$. Take $(\omega, x) \in \mathcal{O}(\omega_0, x_0)$, i.e.

$$\exists\, t_n \uparrow \infty \;\text{ with } \omega = \lim_{n\to\infty} \omega_0{\cdot}t_n \text{ and } x \overset{\mathsf{d}}{=} \lim_{n\to\infty} u(t_n, \omega_0, x_0)\,. \tag{3.8}$$

Then, given $t, s \in (-\infty, 0]$ (assume without loss of generality that $t \leq s$), there is an $n_0 \in \mathbb{N}$ depending on them such that $t_n + t \geq 0$ and $t_n + s \geq 0$ for each $n \geq n_0$. Then, we have

$$\|z(t + t_n, \omega_0, x_0) - z(t_n + s, \omega_0, x_0)\| \leq \int_{t_n+t}^{t_n+s} \|F(\omega_0{\cdot}\tau, z_\tau(\omega_0, x_0))\|\, d\tau \leq M\, |t - s|\,,$$

which in turn implies that

$$\|x(t) - x(s)\| \leq \lim_{n\to\infty} \|z(t + t_n, \omega_0, x_0) - z(t_n + s, \omega_0, x_0)\| \leq M\, |t - s|$$

and proves that $x \in BU$, as claimed. The positive invariance, i.e. $\tau_t(\mathcal{O}(\omega_0, x_0)) \subset \mathcal{O}(\omega_0, x_0)$ for each $t > 0$, is deduced from Proposition 3.2 as follows:

$$\begin{matrix} \omega = \lim_{n\to\infty} \omega_0{\cdot}t_n \\ x \overset{\mathsf{d}}{=} \lim_{n\to\infty} u(t_n, \omega_0, x_0) \end{matrix} \quad\Longrightarrow\quad \begin{matrix} \omega{\cdot}t = \lim_{n\to\infty} \omega_0{\cdot}(t + t_n) \\ u(t, \omega, x) \overset{\mathsf{d}}{=} \lim_{n\to\infty} u(t + t_n, \omega_0, x_0) \end{matrix}$$

because $u(t_n, \omega_0, x_0) \in B_r$ and $u(t, \omega_0{\cdot}t_n, u(t_n, \omega_0, x_0)) = u(t + t_n, \omega_0, x_0)$, $n \in \mathbb{N}$.

Let us check that, in fact, $\tau_t(\mathcal{O}(\omega_0, x_0)) = \mathcal{O}(\omega_0, x_0)$ for each $t > 0$, i.e. $\mathcal{O}(\omega_0, x_0)$ is invariant. Fix $t > 0$ and $(\omega, x) \in \mathcal{O}(\omega_0, x_0)$, i.e. satisfying (3.8). Since there is an n_0 such that $t_n - t \geq 0$ for each $n \geq n_0$, from Proposition 3.3 we deduce that there exists a subsequence, which will be also denoted by $\{t_n\}_n$, and $(\omega_1, x_1) \in \mathcal{O}(\omega_0, x_0)$ such that

$$\omega_1 = \lim_{n\to\infty} \omega_0{\cdot}(t_n - t) \quad \text{ and } \quad x_1 \overset{\mathsf{d}}{=} \lim_{n\to\infty} u(t_n - t, \omega_0, x_0)\,.$$

Finally, as above from Proposition 3.2 we get

$$\omega_1{\cdot}t = \lim_{n\to\infty} \omega_0{\cdot}t_n = \omega \quad \text{ and } \quad u(t, \omega_1, x_1) \overset{\mathsf{d}}{=} \lim_{n\to\infty} u(t_n, \omega_0, x_0) = x\,,$$

and $(\omega, x) \in \tau_t(\mathcal{O}(\omega_0, x_0))$, as desired.

Once we have proved that $\mathcal{O}(\omega_0, x_0) \subset \Omega \times B_r$ is invariant, again from Proposition 3.2 we deduce that the semiflow τ is continuous on $\mathbb{R}^+ \times \mathcal{O}(\omega_0, x_0)$ when the product metric topology on $\mathcal{O}(\omega_0, x_0)$ is taken. To see that the semiflow over $\mathcal{O}(\omega_0, x_0)$ admits a flow extension, from Theorem 2.3 (part II) of [24] it suffices to show that every point in $\mathcal{O}(\omega_0, x_0)$ admits a unique backward orbit which remains inside the set $\mathcal{O}(\omega_0, x_0)$. See Proposition 4.4 of [18] for the details. □

As explained before, this paper provides a contribution to the dynamical theory of monotone recurrent skew-product semiflows. We consider a monotone structure on $\Omega \times BC$ determined by an exponential ordering and we enhance the theory started in [15], [18] and [17], where the 1-covering property of omega-limit sets of relatively compact trajectories was proved.

Let A be a diagonal matrix with negative diagonal entries $a_1, \ldots, a_m$. Notice that such A is a *quasipositive* matrix, i.e. there exists $\lambda > 0$ such that $A + \lambda I$ is a matrix whose entries are all nonnegative. As in [19], considering the componentwise partial ordering on $\mathbb{R}^m$, we introduce the positive cone with empty interior in BC

$$\begin{aligned} BC_A^+ &= \{x \in BC \mid x \geq 0 \ \text{ and } \ x(t) \geq e^{A(t-s)}x(s) \quad \text{for } -\infty < s \leq t \leq 0\} \\ &= \{x \in BC \mid x \geq 0 \ \text{ and } \ t \mapsto e^{-A\,t}x(t) \text{ is a nondecreasing function}\}\,, \end{aligned}$$

which induces the following partial order relation on BC:

$$\begin{aligned} &x \leq_A y \iff x \leq y \ \text{ and } \ y(t) - x(t) \geq e^{A(t-s)}(y(s) - x(s))\,, -\infty < s \leq t \leq 0\,, \\ &x <_A y \iff x \leq_A y \ \text{ and } \ x \neq y\,. \end{aligned} \tag{3.9}$$

Let us assume one additional quasimonotone condition on F:

(F3) If $x, y \in BC$ with $x \leq_A y$, then $F(\omega, y) - F(\omega, x) \geq A\,(y(0) - x(0))$ for each $\omega \in \Omega$ and the above quasipositive matrix A.

From this hypothesis, the monotone character of the semiflow (3.3) and its extension to $\Omega \times BC$ are deduced. We omit the proof, analogous to that of Proposition 3.1 of Smith and Thieme [25].

Theorem 3.5. *Under assumptions* **(F1)**–**(F3)**, *for each $\omega \in \Omega$ and x, $y \in BC$ such that $x \leq_A y$, it holds that*

$$u(t, \omega, x) \leq_A u(t, \omega, y)$$

for all $t \geq 0$ where they are defined.

Next, let us recall the definition of uniform stability for the order $\leq_A$.

Definition 3.6. A subset K of BC is said to be *uniformly stable for the order* $\leq_A$ if, given $\varepsilon > 0$, there is a $\delta > 0$ such that, if x, $y \in K$ satisfy $\mathsf{d}(x, y) < \delta$ and $x \leq_A y$ or $y \leq_A x$, then $\mathsf{d}(u(t, \omega, x), u(t, \omega, y)) < \varepsilon$ for each $t \geq 0$.

In order to obtain the 1-covering property of some omega-limit sets, in addition to Hypotheses **(F1)**–**(F3)**, the componentwise separating property and the uniform stability are assumed.

(F4) If (ω, x), $(\omega, y) \in \Omega \times BC$ admit a backward orbit extension, $x \leq_A y$, and there is a subset $J \subset \{1, \ldots, m\}$ such that

$$\begin{aligned} x_i &= y_i &&\text{for each } i \notin J\,, \\ x_i(s) &< y_i(s) &&\text{for each } i \in J \ \text{ and } s \leq 0\,, \end{aligned}$$

then $F_i(\omega, y) - F_i(\omega, x) - (A\,(y(0) - x(0)))_i > 0$ for each $i \in J$.

(F5) For each $k \in \mathbb{N}$, B_k is uniformly stable for the order $\leq_A$.

The following result follows from Theorem 5.6 of [19].

Theorem 3.7. *Under assumptions* **(F1)**–**(F5)**, *we consider the monotone skew-product semiflow* (3.3) *induced by* (3.1). *Fix $(\omega_0, x_0) \in \Omega \times BC$ such that x_0 is Lipschitz continuous and $z(\cdot, \omega_0, x_0)$ is bounded for the norm $\|\cdot\|_\infty$. Then $\mathcal{O}(\omega_0, x_0) = \{(\omega, c(\omega)) \mid \omega \in \Omega\}$ is a copy of the base and*

$$\lim_{t \to \infty} \mathsf{d}(u(t, \omega_0, x_0), c(\omega_0 {\cdot} t)) = 0\,,$$

where $c : \Omega \to BU$ is a continuous equilibrium, i.e. $u(t, \omega, c(\omega)) = c(\omega {\cdot} t)$ for each $\omega \in \Omega$ and $t \geq 0$.

The aim of the rest of this section is to extend the previous characterization to a more general class of initial data, not necessarily Lipschitz continuous. More precisely, the functions x of BC satisfying the following property:

(R) x is of bounded variation componentwise on $[-k,-k+1]$ for all $k\in\mathbb{N}$ and

$$\sup\left\{V_{[-k,-k+1]}(x_i)\mid i\in\{1,\dots,m\},\,k\geq 1\right\}<\infty\,,$$

where $V_{[a,b]}(f)$ denotes the total variation of the scalar function $f\colon[a,b]\to\mathbb{R}$ on the interval $[a,b]$.

Note that the subset $\mathcal{R}$ of all the functions in BC satisfying property **(R)** is a vector subspace of BC. Moreover, $\mathcal{R}$ is a Banach space when endowed with the norm defined for $x\in\mathcal{R}$ by

$$\|x\|_R=\|x\|_\infty+\sup\left\{V_{[-k,-k+1]}(x_i)\mid i\in\{1,\dots,m\},\,k\geq 1\right\}\,.$$

Let us prove a useful characterization of this property in terms of the existence of a common upper bound of x and 0 for the exponential ordering $\leq_A$. It is noteworthy that property **(R)** does not depend on the choice of the quasipositive matrix A. We will denote by $e^{a\cdot}$ the function $(-\infty,0]\to\mathbb{R}$, $t\mapsto e^{at}$ for each $a\in\mathbb{R}$.

Proposition 3.8. *Let $x\in BC$. The following statements are equivalent:*

(i) *x satisfies property* **(R)***;*
(ii) *there exists $h\in BC$ such that $h\geq_A x$ and $h\geq_A 0$.*

Proof. Since A is a diagonal matrix, we may assume without loss of generality that we are dealing with a scalar problem, i.e. $m=1$ and $A=(-a)$ for some $a>0$.

(i) $\Rightarrow$ (ii) Let $c=\sup_{k\geq 1}V_{[-k,-k+1]}(x)$. We fix $t\in(-\infty,0]$ and let $\lceil t\rceil$ denote the integer part of the negative real number t, i.e. $\lceil t\rceil-1<t\leq\lceil t\rceil$. Then, taking into account the properties of the bounded variation of the product of two functions and the increasing character of $e^{a\cdot}$, we deduce that

$$\begin{aligned}V_{(-\infty,t]}(e^{a\cdot}\,x)\leq V_{(-\infty,\lceil t\rceil]}(e^{a\cdot}x)&\leq\sum_{j=-\lceil t\rceil+1}^{\infty}V_{[-j,-j+1]}(e^{a\cdot}x)\\&\leq\sum_{j=-\lceil t\rceil+1}^{\infty}\Big[e^{a\,(-j+1)}V_{[-j,-j+1]}(x)+(e^{a\,(-j+1)}-e^{-a\,j})\|x\|_\infty\Big]\\&\leq[c\,e^{a}+\|x\|_\infty(e^{a}-1)]\sum_{j=-\lceil t\rceil+1}^{\infty}e^{-aj}=C\,e^{a\,(\lceil t\rceil-1)}\leq C\,e^{a\,t}\,,\end{aligned}$$

where $C=(c\,e^{a}+\|x\|_\infty(e^{a}-1))/(1-e^{-a})$. This proves that $e^{a\cdot}\,x$ is a function of bounded variation on $(-\infty,0]$ and we can define h as follows:

$$\begin{array}{rccl}h\colon&(-\infty,0]&\longrightarrow&\mathbb{R}\\&t&\mapsto&e^{-a\,t}\,V_{(-\infty,t]}(e^{a\cdot}\,x),\end{array}$$

which is clearly bounded by C. The continuity of h follows from that of $e^{a\cdot}x$ (see Ex. 4 on p. 137 of Cohn [3]). Moreover, from Proposition 4.4.2 of [3] the functions $e^{a\cdot}\,h:t\mapsto V_{(-\infty,t]}(e^{a\cdot}\,x)$ and $e^{a\cdot}(h-x):t\mapsto V_{(-\infty,t]}(e^{a\cdot}\,x)-e^{at}x(t)$ are nonnegative and nondecreasing, which implies that $h\geq_A x$ and $h\geq_A 0$ and (ii) holds.

(ii) $\Rightarrow$ (i) Since $A=(-a)$, from $h\geq_A 0$ and $h\geq_A x$, we deduce that $e^{a\cdot}\,h$ and $e^{a\cdot}\,(h-x)$ are nonnegative and nondecreasing. Consequently, the function

$e^{a\cdot}\,x = e^{a\cdot}\,h - e^{a\cdot}\,(h-x)$ is of bounded variation on $(-\infty,0]$ and hence on $[-k,-k+1]$ for each $k\in\mathbb{N}$. Moreover,

$$\begin{aligned} V_{[-k,-k+1]}(e^{a\cdot}x) &\le V_{[-k,-k+1]}(e^{a\cdot}h) + V_{[-k,-k+1]}(e^{a\cdot}(h-x)) \\ &\le e^{a\,(-k+1)}h(-k+1) + e^{a\,(-k+1)}(h(-k+1)-x(-k+1)) \le D\,e^{-a\,k}\,, \end{aligned}$$

where $D = e^{a}(2\|h\|_\infty + \|x\|_\infty)$. In addition, $e^{-a\cdot}$ is also of bounded variation on $[-k,-k+1]$, whence we deduce the same for $x = e^{-a\cdot}e^{a\cdot}x$. Therefore,

$$V_{[-k,-k+1]}(x) \le e^{ak}\,D\,e^{-a\,k} + e^{a\,k}e^{a\,(-k+1)}\|x\|_\infty = D + e^{a}\|x\|_\infty\,,$$

which is a bound irrespective of k and (i) holds. □

Remark 3.9. Notice that all Lipschitz continuous initial data satisfy property **(R)**, but the converse does not hold. Actually, there exist functions in BC satisfying property **(R)** and which are not in BU.

A special choice of the function h of Proposition 3.8 will be important for later purposes.

Lemma 3.10. *Fix $x_0\in BC$ satisfying property* **(R)**. *Then there exists $h_0\in BC$ which, in addition to $h_0\ge_A x_0$ and $h_0\ge_A 0$, satisfies*

$$h_0 \ge_A x_0 - x_0(0)\,,$$

where $x_0(0)$ represents the constant function with that value.

Proof. If $\widetilde{h}_0\in BC$ is the function given in Proposition 3.8(ii), then the function $h_0=\widetilde{h}_0+\|x_0\|_\infty$, where $\|x_0\|_\infty$ represents the constant function from $(-\infty,0]$ into $\mathbb{R}^m$ with that value in all the components, satisfies the desired properties. □

The main theorem of the section provides the 1-covering property of omega-limit sets when the initial data x_0 are in BC and satisfies property **(R)**.

Theorem 3.11. *Let $(\omega_0,x_0)\in\Omega\times BC$. Under assumptions* **(F1)**–**(F5)**, *if x_0 satisfies property* **(R)** *and $z(\cdot,\omega_0,x_0)$ is a bounded solution of $(3.1)_{\omega_0}$, the omega-limit set $\mathcal{O}(\omega_0,x_0)=\{(\omega,c(\omega))\mid\omega\in\Omega\}$ is a copy of the base and*

$$\lim_{t\to\infty}\mathsf{d}(u(t,\omega_0,x_0),c(\omega_0{\cdot}t))=0\,,$$

where $c:\Omega\to BU$ is a continuous equilibrium, i.e. $u(t,\omega,c(\omega))=c(\omega{\cdot}t)$ for each $\omega\in\Omega$ and $t\ge 0$, and it is continuous for the compact-open topology on BU.

Proof. Fix a point $(\widetilde{\omega}_0,\widetilde{x}_0)\in\mathcal{O}(\omega_0,x_0)$. As explained before, every point in the omega-limit set $\mathcal{O}(\omega_0,x_0)$ admits a unique backward orbit which remains inside the set $\mathcal{O}(\omega_0,x_0)$. From this fact, we deduce that $u(t,\widetilde{\omega}_0,\widetilde{x}_0)$ is defined for each $t\in\mathbb{R}$ and $\widetilde{x}_0$ is continuously differentiable because $\widetilde{x}_0(t)=z(t,\widetilde{\omega}_0,\widetilde{x}_0)$ for each $t\in(-\infty,0]$. Therefore, from $(3.1)_{\widetilde{\omega}_0}$ and (3.2), the Lipschitz character of $\widetilde{x}_0$ is deduced. An application of Theorem 3.7 yields $\mathcal{O}(\widetilde{\omega}_0,\widetilde{x}_0)=\{(\omega,c(\omega))\mid\omega\in\Omega\}$ for a continuous equilibrium $c:\Omega\to BU$. We claim that $\mathcal{O}(\widetilde{\omega}_0,\widetilde{x}_0)=\mathcal{O}(\omega_0,x_0)$, which will finish the proof. We know that $\mathcal{O}(\widetilde{\omega}_0,\widetilde{x}_0)\subset\mathcal{O}(\omega_0,x_0)$; to check the coincidence of both sets, it is enough to prove that, given $\varepsilon>0$, there is a $T>0$ such that

$$\mathsf{d}(u(t,\omega_0,x_0),c(\omega_0{\cdot}t))<\varepsilon\qquad\text{for each } t>T\,.$$

Since $(\omega_0,c(\omega_0))\in\mathcal{O}(\omega_0,x_0)$, there exists a sequence $t_n\uparrow\infty$ such that

$$\omega_0=\lim_{n\to\infty}\omega_0{\cdot}t_n\qquad\text{and}\qquad c(\omega_0)\overset{\mathsf{d}}{=}\lim_{n\to\infty}u(t_n,\omega_0,x_0)\,.$$

However, since $u(t_n,\omega_0,c(\omega_0))=c(\omega_0{\cdot}t_n)\xrightarrow{\mathsf{d}} c(\omega_0)$ as $n\uparrow\infty$, we deduce that

$$\lim_{n\to\infty}\mathsf{d}(u(t_n,\omega_0,x_0),c(\omega_0{\cdot}t_n))=0\,. \tag{3.10}$$

Next, we will approximate $u(t,\omega_0,x_0)$ by the Lipschitz continuous function of BU $v(t,\omega_0,x_0)$ defined by

$$v(t,\omega_0,x_0)(s)=\begin{cases}u(t,\omega_0,x_0)(s) & \text{if } s\in[-t,0]\,,\\ x_0(0) & \text{if } s\in(-\infty,-t]\,.\end{cases} \tag{3.11}$$

It is immediate to see that

$$\lim_{t\to\infty}\mathsf{d}\big(u(t,\omega_0,x_0),v(t,\omega_0,x_0)\big)=0\,. \tag{3.12}$$

We introduce the following auxiliary continuous functions defined from the function h_0 provided by Lemma 3.10:

$$\overline{h}\colon\mathbb{R}\to\mathbb{R}^m\,,\qquad s\mapsto\begin{cases}e^{As}\,h_0(0) & \text{if } s>0\,,\\ h_0(s) & \text{if } s\le 0\,,\end{cases}$$

$$\overline{h}_T\colon(-\infty,0]\to\mathbb{R}^m\,,\quad s\mapsto\overline{h}(s+T)\,,\quad T\in\mathbb{R}\,.$$

It is easy to check that

$$\overline{h}_T\ge_A 0\ \text{ for each } T\in\mathbb{R}\quad\text{and}\quad \overline{h}_T\xrightarrow{\mathsf{d}}0\ \text{ as }\ T\to\infty\,. \tag{3.13}$$

From (3.10), (3.12) and (3.13), it follows that

$$\lim_{n\to\infty}\mathsf{d}(u(t_n,\omega_0,x_0),c(\omega_0{\cdot}t_n))=0\,, \tag{3.14}$$

$$\lim_{n\to\infty}\mathsf{d}\big(u(t_n,\omega_0,x_0),v(t_n,\omega_0,x_0)\big)=0\,, \tag{3.15}$$

$$\lim_{n\to\infty}\mathsf{d}\big(\overline{h}_{t_n},0\big)=0\,, \tag{3.16}$$

Therefore, denoting for simplicity by c_n and v_n the functions of BU

$$c_n=c(\omega_0{\cdot}t_n)\quad\text{and}\quad v_n=v(t_n,\omega_0,x_0)\,,$$

from (3.14) and (3.15), it follows that

$$\lim_{n\to\infty}\mathsf{d}(c_n,v_n)=0\,. \tag{3.17}$$

Next, as in Proposition 4.4 of [19], we define the functions a_{v_n,c_n}, b_{v_n,c_n} of BU by

$$\begin{aligned} a_{v_n,c_n}\colon(-\infty,0] &\longrightarrow \mathbb{R}^m\\ s &\mapsto \int_{-\infty}^s e^{A(s-\tau)}\inf\{v_n'(\tau)-A\,v_n(\tau),c_n'(\tau)-A\,c_n(\tau)\}\,d\tau\,,\end{aligned}$$

$$\begin{aligned} b_{v_n,c_n}\colon(-\infty,0] &\longrightarrow \mathbb{R}^m\\ s &\mapsto \int_{-\infty}^s e^{A(s-\tau)}\sup\{v_n'(\tau)-A\,v_n(\tau),c_n'(\tau)-A\,c_n(\tau)\}\,d\tau\,,\end{aligned}$$

which satisfy $a_{v_n,c_n}\le_A v_n\le_A b_{v_n,c_n}$ and $a_{v_n,c_n}\le_A c_n\le_A b_{v_n,c_n}$. Moreover, for each $s\le 0$, we have $\|b_{v_n,c_n}(s)-c_n(s)\|\le\|a_{v_n,v_n}(s)-b_{v_n,c_n}(s)\|$, whence

$$\|b_{v_n,c_n}(s)-c_n(s)\|\le\int_{-\infty}^s\big\|e^{A(s-\tau)}\big\|\big(\|v_n'(\tau)-c_n'(\tau)\|+\|A\|\,\|v_n(\tau)-c_n(\tau)\|\big)\,d\tau\,.$$

Thanks to the definition of v_n and (3.11),

$$\begin{aligned} v_n'(\tau) &= \begin{cases} F(\omega_0{\cdot}(t_n+\tau), u(t_n+\tau,\omega_0,x_0)) & \text{if } \tau\in(-t_n,0]\,,\\ 0 & \text{if } \tau\in(-\infty,-t_n)\,, \end{cases} \\ c_n'(\tau) &= F(\omega_0{\cdot}(t_n+\tau), c(\omega_0{\cdot}(t_n+\tau)) \qquad \text{for each } \tau\le 0\,, \end{aligned} \tag{3.18}$$

whence (3.2) provides the uniform boundedness of v_n, v_n', c_n and c_n' for all $n\in\mathbb{N}$. Now, for each $s\le 0$, $\|e^{A\,(s-\tau)}\|\le e^{-a(s-\tau)}$ for some positive $a>0$ and $\int_{-\infty}^s e^{-a(s-\tau)}\,d\tau = 1/a$. As a result, we deduce the existence of $T_0>0$ such that

$$\int_{-\infty}^{-T_0} \big\|e^{A(s-\tau)}\big\|\,\big(\|v_n'(\tau)-c_n'(\tau)\|+\|A\|\,\|v_n(\tau)-c_n(\tau)\|\big)\,d\tau<\varepsilon$$

for each $n\in\mathbb{N}$ and, hence,

$$\|b_{v_n,c_n}(s)-c_n(s)\|\le\varepsilon+\int_{-T_0}^0\big(\|v_n'(\tau)-c_n'(\tau)\|+\|A\|\,\|v_n(\tau)-c_n(\tau)\|\big)\,d\tau\,.$$

Since $\lim_{n\to\infty}\mathsf{d}(v_n,c_n)=0$, the second part of the integral vanishes as $n\uparrow\infty$. In addition, if we fix $n_0\in\mathbb{N}$ with $t_{n_0}>T_0$, the inequality $-t_n\le -t_{n_0}\le -T_0\le 0$ holds for each $n\ge n_0$ and, thanks to (3.18), (3.17), **(F2)** and the relative compactness of the trajectory $\{\tau(t,\omega_0,x_0): t\ge 0\}$,

$$\begin{aligned}\int_{-T_0}^0 \|v_n'(\tau)-c_n'(\tau)\|\,d\tau = \int_{-T_0}^0 \big\|&F(\omega_0{\cdot}(t_n+\tau),u(t_n+\tau,\omega_0,x_0))\\ &- F(\omega_0{\cdot}(t_n+\tau),c(\omega_0{\cdot}(t_n+\tau))\big\|\,d\tau\end{aligned}$$

also tends to 0 as $n\uparrow\infty$. Thus, $\lim_{n\to\infty}\mathsf{d}(b_{v_n,c_n},c_n)=0$ and, consequently, $\lim_{n\to\infty}\mathsf{d}(b_{v_n,c_n},v_n)=0$. Next, we consider the function $g_n=b_{v_n,c_n}+\overline{h}_{t_n}$, which satisfies $\lim_{n\to\infty}\mathsf{d}(g_n,b_{v_n,c_n})=0$ thanks to (3.16). Therefore,

$$\lim_{n\to\infty}\mathsf{d}(g_n,c_n)=0 \quad\text{and}\quad \lim_{n\to\infty}\mathsf{d}(g_n,v_n)=0\,. \tag{3.19}$$

From Lemma 3.10, we know that $h_0\ge_A x_0-x_0(0)$ and $h_0\ge_A 0$, whence

$$u(t_n,\omega_0,x_0)\le_A v(t_n,\omega_0,x_0)+\overline{h}_{t_n}=v_n+\overline{h}_{t_n}\,,$$

i.e. the function $v: s\mapsto e^{-A\,s}\big(v_n(s)+\overline{h}_{t_n}(s)-u(t_n,\omega_0,x_0)(s)\big)$ is nondecreasing because we can write

$$v(s)=\begin{cases} e^{-A\,s}\,\overline{h}_{t_n}(s) & \text{if } s\in[-t_n,0]\,,\\ e^{A\,t_n}\,e^{-A(s+t_n)}\big[\,x_0(0)-x_0(s+t_n)+h_0(s+t_n)\big] & \text{if } s\in(-\infty,-t_n]\,. \end{cases}$$

Therefore, $u(t_n,\omega_0,x_0)\le_A g_n$; hence, (3.15) and (3.19) yield

$$\mathsf{d}(u(t_n,\omega_0,x_0),g_n)\le\mathsf{d}(u(t_n,\omega_0,x_0),v_n)+\mathsf{d}(v_n,g_n)\underset{n\to\infty}{\longrightarrow}0\,. \tag{3.20}$$

Thanks to the uniform boundedness of g_n and c_n, there exist a constant $c>0$ such that $u(t_n,\omega_0,x_0)$, $c(\omega_0{\cdot}t_n)$ and $g_n\in B_c$ for each $n\in\mathbb{N}$. Let $\delta>0$ be the modulus of uniform stability of B_c for the order $\le_A$ for $\varepsilon/2$. From (3.20) and (3.19), it follows that there exists $n_0\in\mathbb{N}$ such that

$$\mathsf{d}(u(t_n,\omega_0,x_0),g_n)<\delta \quad\text{and}\quad \mathsf{d}(c(\omega_0{\cdot}t_n),g_n)<\delta$$

for each $n \geq n_0$. Therefore, from $u(t_n, \omega_0, x_0) \leq_A g_n$, $c(\omega_0{\cdot}t_n) = c_n \leq_A g_n$ and **(F5)**, we deduce that

$$\mathsf{d}\big(u(t, \omega_0{\cdot}t_n, u(t_n, \omega_0, x_0)), u(t, \omega_0{\cdot}t_n, g_n)\big) < \frac{\varepsilon}{2}\,,$$
$$\mathsf{d}\big(u(t, \omega_0{\cdot}t_n, c(\omega_0{\cdot}t_n)), u(t, \omega_0{\cdot}t_n, g_n)\big) < \frac{\varepsilon}{2}\,,$$

that is,

$$\mathsf{d}\big(u(t+t_n, \omega_0, x_0), c(\omega_0{\cdot}(t+t_n))\big) < \varepsilon \quad \text{for each } t \geq 0 \text{ and } n \geq n_0\,.$$

Finally, taking $T = t_{n_0}$ yields the expected result. □

4. Transformed exponential order for NFDEs with infinite delay

In this section we will extend the previous results to NFDEs with infinite delay and nonautonomous stable operator. This extension requires the definition of a transformed exponential ordering.

4.1. **Nonautonomous stable operators.** Let $D\colon \Omega \times BC \to \mathbb{R}^m$ be an operator satisfying the following hypotheses:

(D1) D is linear and continuous in its second variable for the norm $\|\cdot\|_\infty$ and the map $\Omega \to \mathcal{L}(BC, \mathbb{R}^m)$, $\omega \mapsto D(\omega, \cdot)$ is continuous;

(D2) for each $r > 0$, $D\colon \Omega \times B_r \to \mathbb{R}^m$ is continuous when we take the restriction of the compact-open topology to B_r, i.e. if $\omega_n \to \omega$ and $x_n \xrightarrow{\mathsf{d}} x$ as $n \uparrow \infty$ with $x \in B_r$, then $\lim_{n\to\infty} D(\omega_n, x_n) = D(\omega, x)$.

Under these assumptions, by adapting the proof of Lemma 3.1 of [20] for each $\omega \in \Omega$, we deduce that

$$D(\omega, x) = \int_{-\infty}^{0} [d\mu(\omega)(s)]\, x(s) = B(\omega)\, x(0) - \int_{-\infty}^{0} [d\nu(\omega)(s)]\, x(s)\,,$$

where $\mu(\omega) = [\mu_{ij}(\omega)]$, $\mu_{ij}(\omega)$ is a real regular Borel measure with finite total variation $|\mu_{ij}(\omega)|((-\infty, 0]) < \infty$, for $i,\, j \in \{1, \dots, m\}$ and $\omega \in \Omega$, $B(\omega) = \mu(\omega)(\{0\})$ and $\nu(\omega) = B(\omega)\, \delta_0 - \mu(\omega)$ where δ_0 is the Dirac measure at 0. As in Corollary 3.4 of [20], it follows that

$$\lim_{\rho\to 0^+} \|\nu(\omega)\|_\infty([-\rho, 0]) = 0 \text{ and } \lim_{\rho\to\infty} \|\nu(\omega)\|_\infty((-\infty, -\rho]) = 0 \tag{4.1}$$

uniformly for $\omega \in \Omega$, where $\|\nu(\omega)\|_\infty(E)$ denotes the matrix norm associated to the maximum norm of the $m \times m$ matrix $[|\nu_{ij}(\omega)|(E)]$ of total variations over the Borel subset $E \subset (-\infty, 0]$, i.e. $\|\nu(\omega)\|_\infty(E) = \max\limits_{1\leq i\leq m} \sum_{j=1}^{m} |\nu_{ij}(\omega)|(E)\,.$

In addition, we will assume a natural generalization of the atomic character at 0 of the operator D, as seen in Hale [10], Hale and Verduyn-Lunel [11], and [20].

(D3) $B(\omega)$ is the identity matrix for all $\omega \in \Omega$.

Thus, under assumptions **(D1)**–**(D3)**, D takes the form

$$D(\omega, x) = x(0) - \int_{-\infty}^{0} [d\nu(\omega)(s)]\, x(s)\,,$$

where ν has a continuous variation with respect to ω and satisfies (4.1).

We omit the proof of the following result, which can be easily adapted to this case from Theorem 2.5 of Muñoz-Villarragut [27]. As stated before, given a continuous function $x \in C(\mathbb{R}, \mathbb{R}^m)$, $x_t(\cdot)$ denotes the continuous function $x_t\colon (-\infty, 0] \to \mathbb{R}^m$ defined by $x_t(s) = x(t+s)$ for $s \in (-\infty, 0]$.

Proposition 4.1. *Under assumptions* **(D1)**–**(D3)**, *for each* $h \in C([0,\infty), \mathbb{R}^m)$ *and* $(\omega, \varphi) \in \Omega \times BC$ *with* $D(\omega, \varphi) = h(0)$, *the nonhomogeneous equation*

$$\begin{cases} D(\omega{\cdot}t, x_t) = h(t)\,, \quad t \geq 0\,, \\ x_0 = \varphi\,, \end{cases} \tag{4.2}$$

has a solution $x \in C(\mathbb{R}, \mathbb{R}^m)$.

Next the definition of stability for D is stated.

(D4) D is *stable*, that is, there is a continuous function $c \in C([0,\infty), \mathbb{R})$ with $\lim_{t\to\infty} c(t) = 0$ such that, for each $(\omega, \varphi) \in \Omega \times BC$ with $D(\omega, \varphi) = 0$, the solution of the homogeneous problem

$$\begin{cases} D(\omega{\cdot}t, x_t) = 0\,, \quad t \geq 0\,, \\ x_0 = \varphi\,, \end{cases}$$

satisfies $\|x(t)\| \leq c(t)\,\|\varphi\|_\infty$ for each $t \geq 0$.

The next result provides a condition to check the stability of D. Its proof is similar to the one for BU done in Theorem 3.9(iii) of [20]. We will denote $\widehat{D}_2(\omega, x)\colon (-\infty, 0] \to \mathbb{R}^m$, $s \mapsto D(\omega{\cdot}s, x_s)$.

Proposition 4.2. *Under assumptions* **(D1)**–**(D3)**, *if for each* $r > 0$ *and each sequence* $\{(\omega_n, x^n)\}_n \subset \Omega \times BC$ *such that* $\|\widehat{D}_2(\omega_n, x^n)\|_\infty \leq r$, $\omega_n \to \omega \in \Omega$ *and* $\widehat{D}_2(\omega_n, x^n) \stackrel{\mathsf{d}}{\to} 0$ *as* $n \uparrow \infty$, *it holds that* $x^n(0) \to 0$ *as* $n \to \infty$, *then* D *is stable.*

Although the definition of stability is given for the homogeneous equation, it is easy to deduce quantitative estimates for the solution of a non-homogeneous equation in terms of the initial data. The proof of the next proposition is analogous to the one for BU done in Theorem 2.11 and Proposition 3.2 of [27].

Proposition 4.3. *Under assumptions* **(D1)**–**(D4)**, *there are a positive constant* $k > 0$ *and a continuous function* $c \in C([0,\infty), \mathbb{R})$ *with* $\lim_{t\to\infty} c(t) = 0$ *such that*

(i) $\|x^h(s)\| \leq c(t)\,\|x^h\|_\infty + k \sup_{s-t\leq\tilde{s}\leq s} \|h(\tilde{s})\|$ *for all* $s \leq 0 \leq t$, *and hence*

(ii) $\|x^h\|_\infty \leq k\,\|h\|_\infty$,

for each $h \in BC$, $\omega \in \Omega$ *and* $x^h \in BC$ *satisfying* $D(\omega{\cdot}s, x^h_s) = h(s)$ *for* $s \leq 0$.

The following statement associates the stability of D to the invertibility of its convolution operator $\widehat{D}$ and will allow us to transform the family of NFDEs with infinite delay (4.5) and nonoautonomous stable operator D into a family of FDEs with infinite delay. We refer to Staffans [26] for the case of autonomous stable operators D in an appropriate fading memory space, to Haddock *et al.* [7] for an application of these ideas, and to [17] (resp. [20]) for the cases of autonomous (resp. nonautonomous) stable operators D in BU.

Theorem 4.4. *Under assumptions* **(D1)**–**(D4)**, *we define the map*

$$\begin{array}{rccc} \widehat{D}\colon & \Omega \times BC & \longrightarrow & \Omega \times BC \\ & (\omega, x) & \mapsto & (\omega, \widehat{D}_2(\omega, x)) \end{array}$$

where $\widehat{D}_2(\omega, x)\colon (-\infty, 0] \to \mathbb{R}^m$, $s \mapsto D(\omega{\cdot}s, x_s)$. *Then*

(i) $\widehat{D}$ *is well defined and invertible;*
(ii) $\widehat{D}_2$ *and* $(\widehat{D}^{-1})_2$ *are linear and continuous for the norm in their second variable for all* $\omega\in\Omega$*; and*
(iii) *for all* $r>0$, $\widehat{D}$ *and* $\widehat{D}^{-1}$ *are uniformly continuous on* $\Omega\times B_r$ *when we take the restriction of the compact-open topology to* B_r.

Proof. (i) We check that $\widehat{D}_2(\omega,x)\in BC$. The continuity follows from **(D2)**, and the boundedness from **(D1)** because

$$\|\widehat{D}_2(\omega,x)\|_\infty=\sup_{s\le 0}\|D(\omega{\cdot}s,x_s)\|\le\sup_{\widetilde{\omega}\in\Omega}\|D(\widetilde{\omega},\cdot)\|\,\|x\|_\infty<\infty\,.$$

$\widehat{D}$ is injective because, if we have (ω,x), $(\widehat{\omega},\widehat{x})\in\Omega\times BU$ with $\widehat{D}(\omega,x)=\widehat{D}(\widehat{\omega},\widehat{x})$, then $\omega=\widehat{\omega}$ and, from Proposition 4.3(ii) and the fact that $D(\omega{\cdot}s,x_s-\widehat{x}_s)=0$ for $s\le 0$, we get $x=\widehat{x}$.

In order to show that $\widehat{D}$ is surjective, let $(\omega,h)\in\Omega\times BC$. As in Theorem 3.9 of [20], we take a sequence of continuous functions with compact support $\{h_n\}_n\subset B_r$ for some $r>0$ such that $h_n\xrightarrow{\mathsf{d}}h$ as $n\uparrow\infty$, and a sequence $\{x^n\}_n$ of continuous functions with compact support such that $\widehat{D}_2(\omega,x^n)=h_n$, i.e. $D(\omega{\cdot}s,x^n_s)=h_n(s)$ for each $s\le 0$. The next step of the proof differs from the one in Theorem 3.9 of [20] because now h belongs to BC instead of BU. We will check that given $\rho>0$ and $\varepsilon>0$ there is a $\delta>0$ such that

$$\|x^n-x^n_\tau\|_{[-\rho,0]}<\varepsilon\quad\forall\, n\in\mathbb{N}\;\text{ and }\;\tau\in[-\delta,0]\,. \tag{4.3}$$

For each $\tau\le 0$ we define

$$g^\tau_n\colon(-\infty,0]\to\mathbb{R}^m\,,\;s\mapsto D(\omega{\cdot}s,(x^n-x^n_\tau)_s)\,,$$

that is, $D(\omega{\cdot}s,(x^n-x^n_\tau)_s)=g^\tau_n(s)$ for each $s\le 0$, and from Proposition 4.3(i) we deduce that

$$\|x^n(s)-x^n_\tau(s)\|\le c(t)\,\|x^n-x^n_\tau\|_\infty+k\sup_{s-t\le\widetilde{s}\le s}\|g^\tau_n(\tilde{s})\|\quad\text{for each } t\ge 0,\; s\le 0\,.$$

Since $c(t)\to 0$ as $t\uparrow\infty$ and from Proposition 4.3(ii) we have $\|x^n_\tau\|_\infty\le k\,r$ for each $n\in\mathbb{N}$ and $\tau\le 0$, we can find a $T>0$ such that $c(T)\,\|x^n-x^n_\tau\|_\infty<\varepsilon/2$ and if $s\in[-\rho,0]$ then

$$\|x^n-x^n_\tau\|_{[-\rho,0]}\le\frac{\varepsilon}{2}+k\,\|g^\tau_n\|_{[-\tilde{\rho},0]}\,. \tag{4.4}$$

for $\tilde{\rho}=\rho+T$. From the equicontinuity of $\{h_n\}_n$ and **(D2)** there is a $\delta>0$ such that for each $\tau\in[-\delta,0]$, $s\in[-\tilde{\rho},0]$ and $n\in\mathbb{N}$

$$\|h_n-(h_n)_\tau\|_{[-\tilde{\rho},0]}<\frac{\varepsilon}{4\,k}\quad\text{and}\quad\|D(\omega{\cdot}(s+\tau),\cdot)-D(\omega{\cdot}s,\cdot)\|<\frac{\varepsilon}{4\,k^2\,r}\,.$$

Thus, from the definitions of g^τ_n and h_n we deduce that

$$\begin{aligned}\|g^\tau_n(s)\|&\le\|h_n(s)-(h_n)_\tau(s)\|+\|D(\omega{\cdot}(s+\tau),x^n_{s+\tau})-D(\omega{\cdot}s,x^n_{s+\tau})\|\\&\le\frac{\varepsilon}{4\,k}+\frac{\varepsilon}{4\,k^2\,r}\|x^n\|_\infty\le\frac{\varepsilon}{2\,k}\end{aligned}$$

for each $s\in[-\tilde{\rho},0]$, which together with (4.4) yields (4.3), as stated. Thus, $\{x^n\}_n$ is equicontinuous and, consequently, relatively compact for the compact-open topology. Hence, there is a convergent subsequence of $\{x^n\}_n$ (let us assume it is the whole sequence), i.e. there is a continuous function x such that $x^n\xrightarrow{\mathsf{d}}x$ as $n\to\infty$. Therefore, we have that $\|x\|_\infty\le k\,r$, which implies that $x\in BC$. From this, $x^n_s\xrightarrow{\mathsf{d}}x_s$

for each $s \le 0$ and the expression of D yields $D(\omega{\cdot}s, x^n_s) = h_n(s) \to D(\omega{\cdot}s, x_s)$, i.e. $D(\omega{\cdot}s, x_s) = h(s)$ for $s \le 0$ and $\widehat{D}_2(\omega, x) = h$. Then $\widehat{D}$ is surjective, as claimed.

(ii) The continuity of $\widehat{D}_2$ for the norm in the second variable is a consequence of **(D1)**, and the corresponding property for $(\widehat{D}^{-1})_2$ follows from Proposition 4.3(ii).

(iii) The proof of the uniform continuity of $\widehat{D}$ (resp. $\widehat{D}^{-1}$) on $\Omega \times B_r$ for the compact-open topology is omitted because it follows, adapted to this case, the same steps of Theorem 3.6 of [20] (resp. Theorem 3.9 of [20]). □

4.2. **Neutral functional differential equations.** Let us consider the family of NFDEs with infinite delay

$$\frac{d}{dt} D(\omega{\cdot}t, z_t) = G(\omega{\cdot}t, z_t), \quad t \ge 0,\, \omega \in \Omega, \tag{4.5}$$

defined by an operator $D \colon \Omega \times BC \to \mathbb{R}^m$ and a function $G \colon \Omega \times BC \to \mathbb{R}^m$.

With the notation of the previous subsection and a diagonal matrix A with negative diagonal entries as is Section 3, we define the *transformed exponential order* relation introduced in [20] on each fiber of the product $\Omega \times BC$: if (ω, x), $(\omega, y) \in \Omega \times BC$, then

$$(\omega, x) \le_{D,A} (\omega, y) \iff \widehat{D}_2(\omega, x) \le_A \widehat{D}_2(\omega, y), \tag{4.6}$$

based on the partial order relation $\le_A$ on BC given in (3.9).

Let us assume the following hypotheses:

(N1) $G : \Omega \times BC \to \mathbb{R}^m$ is continuous on $\Omega \times BC$ when the norm $\|\cdot\|_\infty$ is considered on BC and its restriction to $\Omega \times B_r$ is Lipschitz continuous in its second variable for each $r > 0$;

(N2) for each $r > 0$, the restriction of G to $\Omega \times B_r$ is continuous when the compact-open topology is considered on B_r;

(N3) if (ω, x), $(\omega, y) \in \Omega \times BC$ and $(\omega, x) \le_{D,A} (\omega, y)$, then $G(\omega, y) - G(\omega, x) \ge A(D(\omega, y) - D(\omega, x))$ for the usual componentwise partial order relation on $\mathbb{R}^m$.

Notice that assumption **(N1)** implies

$$G(\Omega \times B_r) \text{ is a bounded subset of } \mathbb{R}^m \text{ for each } r > 0.$$

As in the case of conditions **(F1)** and **(F2)**, condition **(N1)** does not imply **(N2)**. The reason for writing them separately is that **(N1)** provides existence and uniqueness in BU, while **(N2)** is used in the case of BC.

Under assumptions **(D1)**–**(D4)** and **(N1)**, as seen in Wang and Wu [28] and [30], for each $\omega \in \Omega$, the local existence and uniqueness of the solutions of equation $(4.5)_\omega$ is guaranteed if we assume some hypotheses on the phase space that, in particular, are satisfied by BU. Moreover, given $(\omega, x) \in \Omega \times BU$, if $z(\cdot, \omega, x)$ represents the solution of equation $(4.5)_\omega$ with initial datum x, then $u(t, \omega, x) : (-\infty, 0] \to \mathbb{R}^m$, $s \mapsto z(t + s, \omega, x)$ is an element of BU for all $t \ge 0$ where $z(\cdot, \omega, x)$ is defined.

As a result, a local skew-product semiflow can be defined on $\Omega \times BU$:

$$\begin{array}{rccc} \tau : & \mathcal{U} \subset \mathbb{R}^+ \times \Omega \times BU & \longrightarrow & \Omega \times BU \\ & (t, \omega, x) & \mapsto & (\omega{\cdot}t, u(t, \omega, x)). \end{array}$$

Next, let $(\omega, y) \in \Omega \times BU$. For each $t \geq 0$ such that $u(t, \widehat{D}^{-1}(\omega, y))$ is defined, we define $\widehat{u}(t, \omega, y) = \widehat{D}_2(\omega{\cdot}t, u(t, \widehat{D}^{-1}(\omega, y)))$. As seen in [20], it can be checked that

$$\widehat{z}(t, \omega, y) = \begin{cases} y(t) & \text{if } t \leq 0, \\ \widehat{u}(t, \omega, y)(0) & \text{if } t \geq 0, \end{cases}$$

is the solution of

$$\widehat{z}\,'(t) = F(\omega{\cdot}t, \widehat{z}_t), \quad t \geq 0,\, \omega \in \Omega \tag{4.7}$$

with initial datum y, where $F = G \circ \widehat{D}^{-1}$.

A similar proof to that of Proposition 4.1 of [20] provides the following result.

Proposition 4.5. *Under assumptions* **(D1)**–**(D4)** *and* **(N1)**–**(N3)***, the map* $F = G \circ \widehat{D}^{-1}$ *satisfies conditions* **(F1)**–**(F3)***.*

As a result, thanks to Theorem 4.4, we can deduce results concerning the local existence and uniqueness of solutions of equation (4.5) on $\Omega \times BC$ which are analogous to those obtained in Section 3 for equation (4.7). Therefore, we can deduce the following result concerning the existence of solutions of (4.5) with initial data in BC.

Proposition 4.6. *Under assumptions* **(D1)**–**(D4)**, **(N1)**–**(N2)***, for each* $\omega \in \Omega$ *and each* $x \in BC$*, the system* $(4.5)_\omega$ *locally admits a unique solution* $z(\cdot, \omega, x)$ *with initial value* x*, i.e.* $z(s, \omega, x) = x(s)$ *for each* $s \in (-\infty, 0]$*.*

Proof. Since $\widehat{D}(\omega, x) \in \Omega \times BC$, from Proposition 4.5 and Proposition 3.1, we deduce that system $(4.7)_\omega$ locally admits a unique solution $\widehat{z}(\cdot, \widehat{D}(\omega, x))$. Hence, taking $\widehat{u}(t, \widehat{D}(\omega, x)) = \widehat{z}_t(\widehat{D}(\omega, x))$ and $u(t, \omega, x) = (\widehat{D}^{-1})_2(\omega{\cdot}t, \widehat{u}(t, \widehat{D}(\omega, x)))$ for $t \geq 0$ as above, we conclude that

$$z(t, \omega, x) = \begin{cases} x(t) & \text{if } t \leq 0, \\ u(t, \omega, x)(0) & \text{if } t \geq 0, \end{cases}$$

satisfies the statement. □

Analogously, from Proposition 3.2, we deduce the continuous dependence for the product metric topology on sets of the form $\Omega \times B_r$ for each $r \geq 0$.

Proposition 4.7. *Under assumptions* **(D1)**–**(D4)** *and* **(N1)**–**(N2)***, the local map*

$$\begin{array}{rcl} \mathcal{U} \subset \mathbb{R}^+ \times \Omega \times B_r & \longrightarrow & \Omega \times BC \\ (t, \omega, x) & \mapsto & (\omega{\cdot}t, u(t, \omega, x)) \end{array}$$

is continuous when we take the restriction of the compact-open topology to B_r*, i.e. if* $t_n \to t$*,* $\omega_n \to \widetilde{\omega}$ *and* $x_n \xrightarrow{\mathsf{d}} \widetilde{x}$ *as* $n \uparrow \infty$ *with* $x_n, x \in B_r$ *for all* $n \in \mathbb{N}$*, then* $\omega_n{\cdot}t_n \to \widetilde{\omega}{\cdot}t$ *and* $u(t_n, \omega_n, x_n) \xrightarrow{\mathsf{d}} u(t, \widetilde{\omega}, \widetilde{x})$ *as* $n \uparrow \infty$*.*

Proof. It is an easy consequence of the relation

$$\widehat{D}(\omega{\cdot}t, u(t, \omega, x)) = (\omega{\cdot}t, \widehat{u}(t, \widehat{D}(\omega, x)))\,, \tag{4.8}$$

Theorem 4.4 and Proposition 3.2. □

As in Theorem 4.2 of [20], the following monotonicity theorem, whose proof is omitted, is an immediate consequence of Theorem 3.5 and Proposition 4.5.

Theorem 4.8. *Under assumptions* **(D1)**–**(D4)** *and* **(N1)**–**(N3)**, *for each* $\omega \in \Omega$ *and* $x,\, y \in BC$ *such that* $(\omega, x) \leq_{D,\,A} (\omega, y)$, *it holds that*

$$\tau(t, \omega, x) \leq_{D,\,A} \tau(t, \omega, y)$$

for all $t \geq 0$ *where they are defined.*

Lemma 4.9. *Under assumptions* **(D1)**–**(D4)**, *there exist positive constants* K_D *and* K'_D *such that*

$$K_D = \sup_{\omega\in\Omega} \|D(\omega, \cdot)\| = \sup_{\omega\in\Omega} \|\widehat{D}_2(\omega, \cdot)\| \quad \textit{and} \quad K'_D = \sup_{\omega\in\Omega} \|(\widehat{D}^{-1})_2(\omega, \cdot)\|\,.$$

Proof. The map $\Omega \to \mathbb{R}$, $\omega \to \|D(\omega, \cdot)\|$ is continuous thanks to **(D1)**. Since Ω is compact, there is a $K_D > 0$ such that $K_D = \sup_{\omega\in\Omega} \|D(\omega, \cdot)\|$. Fix $(\omega, x) \in \Omega \times B_1$; then

$$\begin{aligned} \|\widehat{D}_2(\omega, x)\|_\infty &= \sup_{s\leq 0} \|D(\omega{\cdot}s, x_s)\| \leq K_D\, \|x_s\|_\infty \leq K_D \quad \text{and} \\ \|D(\omega, x)\| &= \|\widehat{D}_2(\omega, x)(0)\| \leq \|\widehat{D}_2(\omega, x)\|_\infty\,, \end{aligned}$$

whence $K_D = \sup_{\omega\in\Omega} \|\widehat{D}_2(\omega, \cdot)\|$. As for the bound for $\widehat{D}^{-1}$, it follows immediately from Proposition 4.3(ii). □

The next result provides the main properties of the trajectory and the omega-limit set of a bounded solution.

Proposition 4.10. *Assume* **(D1)**–**(D4)** *and* **(N1)**–**(N2)**. *If* $z(\cdot, \omega_0, x_0)$ *is a solution of* $(4.5)_{\omega_0}$ *bounded for the norm* $\|\cdot\|_\infty$, *then the set* $\{u(t, \omega_0, x_0) \mid t \geq 0\}$ *is relatively compact when the compact-open topology is considered on* BC *and the omega-limit set of the trajectory of the point* $(\omega_0, x_0) \in \Omega \times BC$, *defined as*

$$\mathcal{O}(\omega_0, x_0) = \{(\omega, x) \mid \exists\, t_n \uparrow \infty \;\text{ with }\; \omega_0{\cdot}t_n \to \omega\,,\; u(t_n, \omega_0, x_0) \xrightarrow{d} x\}$$

is a nonempty, compact and invariant subset of $\Omega \times BU$ *admitting a flow extension.*

Proof. From (4.8), we deduce that $\widehat{u}(t, \widehat{D}(\omega_0, x_0)) = \widehat{D}_2(\omega_0{\cdot}t, u(t, \omega_0, x_0))$, whence

$$\mathcal{O}(\omega_0, x_0) = \widehat{D}^{-1}\big(\widehat{\mathcal{O}}(\omega_0, \widehat{D}_2(\omega_0, x_0))\big)\,, \tag{4.9}$$

where $\widehat{\mathcal{O}}(\omega_0, \widehat{D}_2(\omega_0, x_0))$ denotes the omega-limit set corresponding to the transformed system $(4.7)_{\omega_0}$ with initial datum $\widehat{D}_2(\omega_0, x_0)$. From the boundedness of $u(t, \omega_0, x_0)$ and Lemma 4.9, we deduce the boundedness of $\widehat{u}(t, \widehat{D}(\omega_0, x_0))$ and, consequently, Propositions 4.5, 3.3 and 3.4 imply that $\{\widehat{u}(t, \widehat{D}(\omega_0, x_0)) \mid t \geq 0\}$ is a relatively compact subset of BC and $\widehat{\mathcal{O}}(\omega_0, \widehat{D}_2(\omega_0, x_0))$ is a compact and invariant subset of $\Omega \times BU$ admitting a flow extension. Finally, (4.9) and Theorem 4.4(iii) imply that $\{u(t, \omega_0, x_0) \mid t \geq 0\}$ is a relatively compact subset of BC and $\mathcal{O}(\omega_0, x_0)$ is a compact subset of $\Omega \times BU$. The invariance and flow extension follow the proof of Proposition 3.4, taking into account that, now, Proposition 4.7 holds. □

In order to obtain the 1-covering property of some omega-limit sets, in addition to Hypotheses **(N1)**–**(N3)**, the uniform stability and the componentwise separating property are assumed.

(N4) If (ω, x), $(\omega, y) \in \Omega \times BC$ admit a backward orbit extension, $(\omega, x) \leq_{D,\,A} (\omega, y)$, and there is a subset $J \subset \{1, \ldots, m\}$ such that

$$\begin{aligned} \widehat{D}_2(\omega, x)_i &= \widehat{D}_2(\omega, y)_i \quad \text{ for each } i \notin J\,, \\ \widehat{D}_2(\omega, x)_i(s) &< \widehat{D}_2(\omega, y)_i(s) \quad \text{ for each } i \in J \text{ and } s \leq 0\,, \end{aligned} \tag{4.10}$$

then $G_i(\omega, y) - G_i(\omega, x) - [A\,(D(\omega, y) - D(\omega, x))]_i > 0$ for each $i \in J$.

(N5) For each $k \in \mathbb{N}$, B_k is uniformly stable for the order $\leq_{D,\,A}$.

As in Proposition 4.5, these two assumptions provide **(F4)** and **(F5)** for $F = G \circ \widehat{D}^{-1}$.

Proposition 4.11. *Under assumptions* **(D1)**–**(D4)** *and* **(N1)**–**(N3)***, if G satisfies* **(N4)** *and* **(N5)***, then the map $F = G \circ \widehat{D}^{-1}$ satisfies* **(F4)** *and* **(F5)***.*

Proof. First, we check **(F5)**. Notice that $\widehat{u}(t, \omega, x) = \widehat{D}_2(\omega{\cdot}t, u(t, \widehat{D}^{-1}(\omega, x)))$. Fix a $k > 0$. From Lemma 4.9, there is a $\widetilde{k} > 0$ such that, if $x \in B_k$, then $(\widehat{D}^{-1})_2(\omega, x) \in B_{\widetilde{k}}$ for each $\omega \in \Omega$. In addition, it is not hard to check that

$$\begin{matrix} x, y \in B_k \\ x \leq_A y \text{ or } y \leq_A x \end{matrix} \implies \begin{matrix} \widetilde{x} = (\widehat{D}^{-1})_2(\omega, x),\ \widetilde{y} = (\widehat{D}^{-1})_2(\omega, y) \in B_{\widetilde{k}} \\ (\omega, \widetilde{x}) \leq_{D,\,A} (\omega, \widetilde{y}) \text{ or } (\omega, \widetilde{y}) \leq_{D,\,A} (\omega, \widetilde{x}) \end{matrix}\,. \tag{4.11}$$

Next, let $r = \max(2\,\widetilde{k}, 1)$ and $\varepsilon > 0$. From Theorem 4.4, there is a $\delta_1 > 0$ such that

$$\{x \in BC \mid \mathsf{d}(x, 0) < \delta_1\} \subset \{x \in BC \mid \|x(0)\| \leq 1\} \text{ and} \tag{4.12}$$

$$\mathsf{d}(\widehat{D}_2(\omega, x), \widehat{D}_2(\omega, y)) < \varepsilon \quad \forall\, \omega \in \Omega \text{ and } x, y \in B_r \text{ with } \mathsf{d}(x, y) < \delta_1\,. \tag{4.13}$$

Now, from assumption **(N5)**, given this $\delta_1 > 0$, there is a $\delta_2 > 0$ such that if $\widetilde{x}$, $\widetilde{y} \in B_{\widetilde{k}}$ satisfy $\mathsf{d}(\widetilde{x}, \widetilde{y}) < \delta_2$ and $(\omega, \widetilde{x}) \leq_{D,\,A} (\omega, \widetilde{y})$ or $(\omega, \widetilde{y}) \leq_{D,\,A} (\omega, \widetilde{x})$, then $\mathsf{d}(u(t, \omega, \widetilde{x}), u(t, \omega, \widetilde{y})) < \delta_1$ for each $t \geq 0$.

Moreover, with the notation of (4.11), again Theorem 4.4 provides a $\delta > 0$ such that, for each $\omega \in \Omega$ and x, $y \in B_k$ with $\mathsf{d}(x, y) < \delta$, it holds

$$\mathsf{d}(\widetilde{x}, \widetilde{y}) = \mathsf{d}\big((\widehat{D}^{-1})_2(\omega, x), (\widehat{D}^{-1})_2(\omega, y)\big) < \delta_2\,. \tag{4.14}$$

Altogether, if x, $y \in B_k$ satisfy $\mathsf{d}(x, y) < \delta$ and $x \leq_A y$ or $y \leq_A x$, from (4.11) and (4.14), we deduce that $(\omega, \widetilde{x}) \leq_{D,\,A} (\omega, \widetilde{y})$ or $(\omega, \widetilde{y}) \leq_{D,\,A} (\omega, \widetilde{x})$ and $\mathsf{d}(\widetilde{x}, \widetilde{y}) < \delta_2$. Then, as seen above, $\mathsf{d}(u(t, \omega, \widetilde{x}), u(t, \omega, \widetilde{y})) < \delta_1$ for each $t \geq 0$ and, from this and (4.12), we get $\|(u(t, \omega, \widetilde{x}) - u(t, \omega, \widetilde{y}))(0)\| \leq 1$ for each $t \geq 0$, which together with $\widetilde{x}$, $\widetilde{y} \in B_{\widetilde{k}}$ yields $u(t, \omega, \widetilde{x}) - u(t, \omega, \widetilde{y}) \in B_r$. Finally, from (4.13) and the linear character of $\widehat{D}_2$, we conclude that

$$\mathsf{d}(\widehat{D}_2(\omega{\cdot}t, u(t, \omega, \widetilde{x}), \widehat{D}_2(\omega{\cdot}t, u(t, \omega, \widetilde{y}))) = \mathsf{d}(\widehat{u}(t, \omega, x), \widehat{u}(t, \omega, y)) < \varepsilon$$

for each $t \geq 0$ and **(F5)** holds, as claimed. The verification that **(F4)** is fulfilled is easier and it is omitted. ☐

As a consequence, the asymptotic behavior of bounded trajectories with initial datum x_0 such that $\widehat{D}_2(\omega_0, x_0)$ satisfies **(R)**, reproduces exactly the dynamics exhibited by the time variation of the equation, as claimed.

Theorem 4.12. *Let $(\omega_0, x_0) \in \Omega \times BC$. Under assumptions* **(D1)**–**(D4)** *and* **(N1)**–**(N5)***, if $\widehat{D}_2(\omega_0, x_0)$ satisfies property* **(R)** *and $z(\cdot, \omega_0, x_0)$ is a solution of* $(4.5)_{\omega_0}$

bounded for the norm $\|\cdot\|_\infty$*, then the omega-limit set* $\mathcal{O}(\omega_0, x_0) = \{(\omega, c(\omega)) \mid \omega \in \Omega\}$ *is a copy of the base and*

$$\lim_{t\to\infty} \mathsf{d}(u(t,\omega_0,x_0), c(\omega_0{\cdot}t)) = 0\,,$$

where $c : \Omega \to BU$ *is a continuous equilibrium, i.e.* $u(t,\omega,c(\omega)) = c(\omega{\cdot}t)$ *for each* $\omega \in \Omega$ *and* $t \geq 0$*, and it is continuous for the compact-open topology on* BU*.*

Proof. From Propositions 4.5 and 4.11, $F = G \circ \widehat{D}^{-1}$ and the corresponding family of systems (4.7) satisfies assumptions (**F1**)–(**F5**). In addition, as above, $\widehat{u}(t,\omega_0,\widehat{D}_2(\omega_0,x_0)) = \widehat{D}_2(\omega_0{\cdot}t, u(t,\omega_0,x_0))$ and Lemma 4.9 yield the boundedness of $\widehat{z}(\cdot,\omega_0,\widehat{D}_2(\omega_0,x_0))$, the solution of $(4.7)_{\omega_0}$. As a consequence, from Theorem 3.11, we deduce that the omega-limit $\widehat{\mathcal{O}}\big(\omega_0,\widehat{D}_2(\omega_0,x_0)\big)$ is a copy of the base, that is,

$$\widehat{\mathcal{O}}\big(\omega_0,\widehat{D}_2(\omega_0,x_0)\big) = \{(\omega,\widehat{c}(\omega)) \mid \omega\in\Omega\}\,,$$

where $\widehat{c} : \Omega \to BU$ is a continuous equilibrium and

$$\lim_{t\to\infty} \mathsf{d}\big(\widehat{u}(t,\omega_0,\widehat{D}_2(\omega_0,x_0)), \widehat{c}(\omega_0{\cdot}t)\big) = 0\,.$$

Finally, from (4.9), we have $\mathcal{O}(\omega_0,x_0) = \widehat{D}^{-1}\widehat{\mathcal{O}}(\omega_0,\widehat{D}_2(\omega_0,x_0))$, and we conclude that $\mathcal{O}(\omega_0,x_0) = \{(\omega,c(\omega)) \mid \omega\in\Omega\}$ with $c(\omega) = (\widehat{D}^{-1})_2(\omega,\widehat{c}(\omega))$ or, equivalently, $\widehat{D}^{-1}(\omega,\widehat{c}(\omega)) = (\omega, c(\omega))$, from which the proof is easily finished. ☐

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