Portfolio optimization in a defined benefit pension plan where the risky assets are processes with constant elasticity of variance*

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Abstract

The paper studies the optimal asset allocation problem of a defined benefit pension plan that operates in a financial market composed of risky assets whose prices are constant elasticity variance processes. The benefits paid to the participants are deterministic. The contributions to the fund are designed by a spread amortization method, which takes into account the size of the unfunded actuarial liability, defined as the difference between the actuarial liability and the fund assets. We address the case where the fund manager wishes to minimize the solvency risk at the final date of the plan when the fund is underfunded, as well as the case where the fund manager wishes to maximize an increasing, constant elasticity utility function of the fund surplus, when the fund is overfunded. The optimal portfolio and contributions are obtained in both scenarios, with the help of the Hamilton-Jacobi-Bellman equation. A numerical illustration shows the evolution of the plan for several values of the elasticity parameter of the CEV price processes and the risk aversion of the manager, yielding some tips on the main properties of the optimal portfolio.

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1 Introduction

Pension funds are becoming fundamental tools in financial markets. Nowadays, pension fund investments represent a considerable percentage of financial market operations. According to the 2017 edition of the *Pension Markets in Focus* (OECD report), pension funds under management in OECD countries in 2016 are of USD 38 trillion assets, reaching the highest ever level up to date, and they have been constantly increasing since the financial crisis in 2008. Some of the reasons for the popularity of pension plans as vehicles for financial activities may be due to the fact that they have provided positive investment returns in a consistent way over the past decade. Moreover, population aging is becoming the most important problem to maintain public pensions in developed countries. To deal with this problem, social security reforms announced by different governments tend to devalue benefits, which may encourage private investment in pension funds.

Most pension funds invest in traditional assets such as bills, bonds and shares, being exposed, therefore, to (moderate) financial risks; an additional feature of *defined benefit* pension funds is that they are based on the sponsor's commitments with the participants, in the form of benefits at retirement that have to be honored. Thus, the management of such funds requires a careful control of the solvency risk, limiting the risk exposure by a continuous rebalancing of assets as well as a suitable design of the amortization scheme in the form of contributions.

In a defined benefit pension plan, benefits are defined in advance by the manager; whereas contributions are initially set, and subsequently adjusted if needed, to maintain a balanced fund[1]. In this paper, we assume a spread method of amortizing benefits, which takes into account the size of the unfunded actuarial liability, defined as the difference between the actuarial liability and the fund assets. At the disposal of the manager is also the construction of a portfolio composed of a riskless bond and of several risky assets. Usually, the literature on continuous time pension funding considers risky assets modeled by geometric Brownian motions, along the

[1]In contrast, in a defined contribution pension plan, the benefit is not fixed in advance, depending completely on the performance of the pension plan in the financial market. The risk is borne to the individual, not to the sponsor of the pension plan as in the defined benefit case, which is the objective of this paper.

lines of the seminal investment and consumption model of Merton (1971).

There are many papers in the literature about portfolio selection and pension funding, such as those of Chang (1999), Cairns (2000), Boulier *et al.* (2001), Josa-Fombellida and Rincón-Zapatero (2001, 2004, 2006, 2008, 2012, 2017), Chang *et al.* (2003), Deelstra *et al.* (2003), Battocchio and Menoncin (2004), Cairns *et al.* (2006), Xu *et al* (2007), Delong *et al.* (2008) and Le Cortois and Menoncin (2015). All these papers assume that the asset prices are geometric Brownian motions, with constant volatility therefore, as in the Black-Scholes model. This assumption does not reflect the sometimes observed real financial phenomena of skewness of the implied volatilities of the risky asset prices. To capture this feature, a possibility is to assume a stochastic local volatility that depends on the underlying asset price. The most well-known model of this kind is the Constant Elasticity of Variance model (CEV henceforth), introduced in Cox and Ross (1976), a natural extension of the geometric Brownian motion. The CEV model has been applied to analyzing option pricing problems, as in Beckers (1980), Davydov and Linetsky (2001), Detemple and Tian (2002) and Linetsky and Mendoza (2010).

Recently, the CEV process has also been used in portfolio selection problems, such as in Gu *et al.* (2012), Zhao and Rong (2012, 2017) and Shen *et al.* (2014). Gu *et al.* (2012) consider a reinsurance–investment problem, as well as an investment–only problem for an insurer, where the risky asset prices are CEV processes and the aim is to maximize the expected exponential utility of the terminal wealth. In Zhao and Rong (2012), a general exponential maximization portfolio selection problem with multiple risky assets following CEV processes and a risk–free asset is considered. Zhao and Rong (2017) extend the analysis to isoelastic utility functions and correlation between the risky assets. Shen *et al.* (2014) study a mean–variance portfolio selection problem where the risky asset is a CEV process, obtaining the efficient frontier explicitly.

The literature of defined contribution pension plans contains some models where the risky assets are CEV processes, such as those of Xiao *et al.* (2007), Gao (2009ab, 2010), Yang *et al.* (2015) and Li *et al.* (2017). All these papers take as primary model the one studied in Devolder *et al.* (2003) for geometric Brownian motion. The first paper using the CEV process in a portfolio optimization problem of a pension plan was Xiao *et al.* (2007). It analyzed the problem by

means of the Legendre transform and duality theory, but only includes the logarithmic utility case. Gao (2009ab) extends this paper to power or exponential utility functions, by applying the maximum principle and a change of variables technique. Further extensions to hybrid models mixing constant elasticity of variance and stochastic volatility are in Gao (2010) and Yang *et al.* (2015). Li *et al.* (2017) studies a mean-variance CEV model with default risk.

To the best of our knowledge, the literature has not paid attention to the influence of CEV asset prices in the optimal management of defined benefit pension plans. The objective of this paper is to fill this gap. With this aim, a contribution-investment defined benefit pension plan model is set, whose fund assets operates in a financial market where risky assets may be CEV processes. Regarding the preferences of the manager, when the fund is underfunded, the objective is to reduce the quadratic deviation of the assets from the actuarial liability, defined as the aggregation of all future benefits of retirees, at the end of the planning horizon. In more concise words, to minimize the solvency risk at the terminal date. This is the approach followed by Haberman and Sung (1994) or Josa-Fombellida and Rincón-Zapatero (2001, 2004), among other contributors. However, a manager operating with an overfunded pension plan will have a different goal. An acceptable assumption in this case is that the manager wishes to maximize an increasing and strictly concave utility function of the surplus, defined as the (positive) difference between the fund assets and the actuarial liability. This formulation has already been used in a defined benefit pension plan with heterogeneous workers in Josa-Fombellida and Rincón-Zapatero (2008), as well as in Josa-Fombellida and Rincón-Zapatero (2017), where a differential game between the sponsoring firm and workers' representatives (the union) is studied[2]. With respect to the paper of Zhao and Rong (2017), the environment is a defined benefit pension plan, and an important aim is the analysis of the effect of the elasticity parameter (in both pension plans), and the risk aversion parameter (in the overfunded plan), on the optimal contribution and investment, as well as on the fund wealth evolution.

[2]In the game, the objective of the union is to maximize the expected discounted utility of the extra benefits claimed on the fund surplus, whereas the firm's objective is to maximize the expected discounted utility of the fund surplus.

The main result obtained in this paper is that the optimal investment decisions are proportional to the unfunded actuarial liability or to the fund surplus, respectively, with a variable coefficient that depends on time and on the vector of asset prices. This coefficient is the sum of two terms. The first summand reduces to the so called optimal growth portfolio strategy when the elasticity of variance is null. The optimal growth portfolio is modulated by the asset prices when this elasticity is not null, but retains a similar structure. The second summand is a correction term that will depend, in general, on the financial market parameters, including the elasticity of variance, as well as the amortization rate chosen by the manager. Besides, in the overfunded case, the solution also depends on the risk attitude of the manager.

The paper is organized as follows. In Section 2, we define the elements of the pension plan, describe the financial market composed of general CEV processes and establish the problem of the optimal management of the pension plan, in both the underfunded and overfunded case. The problem is formulated as a stochastic optimal control problem, where the fund is invested in a portfolio formed by a riskless asset and several risky assets. In Section 3, the optimal investment strategies are obtained, both in the underfunded and overfunded case. The problem is solved with the Hamilton-Jacobi-Bellman equation. Section 4 is devoted to a numerical illustration of the optimal evolution of the main elements of the pension plan for several values of the elasticity parameter of the CEV price processes. Finally, Section 5 establishes some conclusions. All proofs are shown in Appendix A.

2 The pension model

Consider a defined benefit pension plan of aggregated type where, at every instant of time, active participants coexist with retired participants. The promised liabilities (benefits) to the participants at the age of retirement are established in advance by the manager and are deterministic. The pension fund has a finite planning horizon $[0, T]$. To fulfill the obligations, the manager invests the fund assets in the financial market and makes contributions to the fund. Contributions are calculated based on actuarial principles, taking into account the characteris-

tics of the pension plan. The main elements intervening in the funding process and the essential hypotheses allowing its temporary evolution to be determined are as follows.

$F(t)$: Value of the fund assets at time t.

$P(t)$: Benefits promised to the participants at time t.

$C(t)$: Contribution made by the sponsor to the funding process at time t.

$AL\,(t)$: Actuarial liability at time t, or total liabilities of the sponsor.

$NC(t)$: Normal cost at time t.

$X(t)$: Fund surplus at time t, equal to $F(t) - AL\,(t)$; $-X(t)$ is the unfunded actuarial liability.

$SC(t)$: Supplementary cost at time t, equal to $C(t) - NC(t)$.

$M(u)$: Distribution functions of workers aged $u \in [a, d]$.

δ: Constant rate of valuation of the liabilities.

If the fund assets match the actuarial liability, AL, and if there are no uncertain elements in the plan, the normal cost, NC, is the value of the contributions, allowing equality between asset funds and obligations. It is a deterministic function. All workers enter the plan at the same age a and leave the plan at the same age d, with $a < d$. The valuation rate δ could be established by the regulatory authorities. We suppose that the functions P and M are both differentiable.

A spread method of fund amortization will be used, as in Haberman and Sung (1994) or in Josa-Fombellida and Rincón-Zapatero (2006). This means that the supplementary contribution rate is proportional to the unfunded actuarial liability, that is

$$C(t) = NC(t) + k(AL(t) - F(t)), \tag{1}$$

where k is a constant selected by the employer, representing the rate at which the surplus or deficit is amortized. A positive value of k means, when the fund is underfunded (*resp.* overfunded), that the contribution is above (*resp.* below) the normal cost.

The actuarial functions AL and NC are defined by

$$AL(t) = \int_a^d e^{-\delta(d-u)} P(t+d-u) M(u)\, du,$$

$$NC(t) = \int_a^d e^{-\delta(d-u)} P(t+d-u) M'(u)\, du,$$

respectively, and are linked by the ordinary differential equation

$$AL'(t) = \delta AL(t) + NC(t) - P(t), \quad t \geq 0, \tag{2}$$

as proven in Bowers *et al.* (1979).

The probability distribution function M satisfies, of course, $0 \leq M(u) \leq 1$ for all u, $M(u) = 0$ for $u \leq a$ and $M(u) = 1$ for $u \geq d$. The most simple case is the uniform distribution, $M(u) = \frac{u-a}{d-a}$, $a \leq u \leq d$. However, we do not restrict ourselves to this case.

Regarding benefits, we assume throughout the paper that they accumulate at an exponential rate. This model for the pension plan benefits have been studied, for instance, in Bowers *et al.* (1986).

Assumption 1 *There is a function $\mu : [0,T] \to \mathbb{R}$ such that benefits are given by*

$$P(t) = P_0 e^{\int_0^t \mu(s)\, ds}, \quad t \geq 0,$$

where P_0 represents the initial liabilities.

An easy consequence of the above hypothesis is the following proposition.

Proposition 2.1 *Under Assumption 1, the actuarial functions AL and NC are given by*

$$AL\,(t) = h(t) P(t), \tag{3}$$

$$NC\,(t) = (1 + h'(t)) P(t) + (\mu(t) - \delta) AL\,(t), \tag{4}$$

where

$$h(t) = \int_a^d e^{\int_t^{t+d-u} (\mu(v)-\delta) dv} M(u) du,$$

for all $t \in [0,T]$.

It follows from this result that $NC(t) - P(t) = \left(\frac{h'(t)}{h(t)} + \mu(t) - \delta\right) AL(t)$ holds for every $t \geq 0$. We will denote by AL_0 and NC_0 the initial values of the actuarial liability and the normal cost, respectively, that is, $AL_0 = h(0)P_0$ and $NC_0 = (1 + h'(0) + (\mu(0) - \delta)h(0))P_0$.

2.1 Financial market and fund assets evolution

The uncertainty in the financial market is given by an n–dimensional standard Brownian motion $w = (w_1, \ldots, w_n)^\top$ generating a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$, where $\mathcal{F}$ is the filtration $\{\mathcal{F}_t\}_{t\geq 0}$, with $\mathcal{F}_t = \sigma\{w_1(s), \ldots, w_n(s);\, 0 \leq s \leq t\}$. The plan sponsor chooses a portfolio formed by a riskless asset (bond), whose price S_0 is given by

$$dS_0(t) = rS_0(t)dt, \quad S_0(0) = 1, \tag{5}$$

and n risky assets, whose prices $\{S_i\}_{i=1}^n$ are generated by w and satisfy:

$$dS_i(t) = S_i(t)\Big(b_i dt + \sum_{j=1}^{n} \sigma_{ij} S_i(t)^\beta dw_j(t)\Big), \quad S_i(0) = s_{i0}, \quad i = 1, 2, ..., n. \tag{6}$$

It is assumed that $r, b_i, \sigma_{ij} > 0$, $\beta \leq 0$ and $b_i > r$ for all i, j, so the sponsor has incentives to invest in the risky assets. Note that the risky assets are correlated CEV processes with elasticity parameter β (hereinafter, we often identify the asset with its price). In the case of only one risky asset and one Brownian motion, the parameter β is the elasticity of the local volatility function, σS^β, and σ is the volatility scale parameter. Since $\beta \leq 0$, the local volatility is a decreasing function on the risky asset price, hence the volatility increases as the stock price decreases. This behavior has been observed in real data of some financial markets, see Gao (2009b), Mendoza and Linetsky (2010) and Shen *et al.* (2014). When $\beta = 0$, the stock prices are geometric Brownian motions (GBMs henceforth), which is the classical assumption introduced in Merton (1971) or in Black and Scholes (1973). The cases $\beta = -1/2$ (square root process) and $\beta = -1$ (absolute diffusions) have been considered in detail in Cox and Ross (1976).

Figure 1 shows a path of a scalar CEV process with $b = 0.02$, $\sigma = 0.1$ and $s_0 = 50$, for several values of the elasticity parameter β.

In what follows, we designate the vector of risky asset prices by $S = (S_1, \ldots, S_n)$.

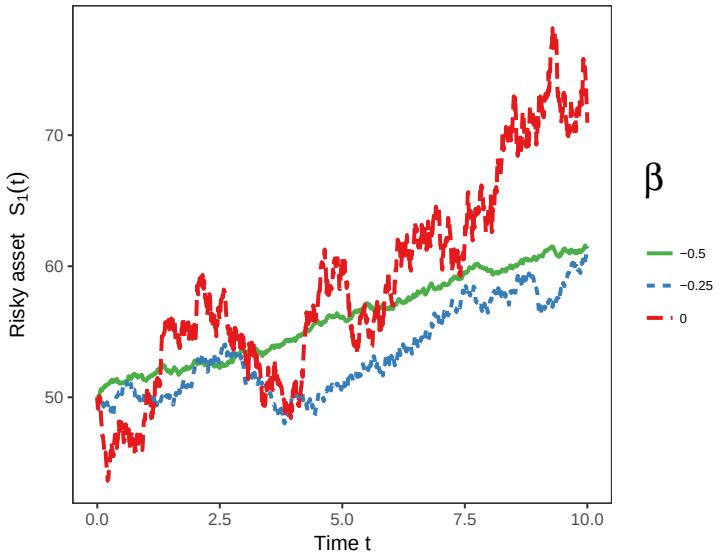

Figure 1: Evolution of the risky asset price over time. $\beta = -0.5, -0.25, 0$

The amount of fund invested at time t in the risky asset S_i is denoted by $\lambda_i(t)$, $i = 1, 2, \ldots, n$. The remainder, $F(t) - \sum_{i=1}^{n} \lambda_i(t)$, is invested in the bond. Borrowing and shortselling are allowed. A negative value of λ_i means that the sponsor sells a part of her/his risky asset S_i short, but if $\sum_{i=1}^{n} \lambda_i$ is larger than F, then he or she gets into debt to purchase the stocks, borrowing at the riskless interest rate r. We suppose that $\{\mathbf{\Lambda}(t) : t \geq 0\}$, with $\mathbf{\Lambda}(t) = (\lambda_1(t), \lambda_2(t), \ldots, \lambda_n(t))^\top$, is a control process adapted to the filtration $\{\mathcal{F}_t\}_{t \geq 0}$; it is $\mathcal{F}_t$–measurable, Markovian, satisfying

$$\mathbb{E} \int_0^t \mathbf{\Lambda}(s)^\top \mathbf{\Lambda}(s)\, ds < \infty, \tag{7}$$

for all $u \in [0, T]$. The symbol $^\top$ denotes matrix transposition.

Therefore, the fund evolution under the investment policy $\mathbf{\Lambda}$ is

$$dF(t) = \sum_{i=1}^{n} \lambda_i(t) \frac{dS_i(t)}{S_i(t)} + \left(F(t) - \sum_{i=1}^{n} \lambda_i(t) \right) \frac{dS_0(t)}{S_0(t)} + (C(t) - P(t))\, dt. \tag{8}$$

By substituting (5) and (6) into (8), we obtain the evolution of fund assets F

$$dF(t) = \left(rF(t) + \sum_{i=1}^{n} \lambda_i(t)(b_i - r) + C(t) - P(t) \right) dt + \sum_{i=1}^{n} \sum_{j=1}^{n} \lambda_i(t) \sigma_{ij} S_i(t)^\beta\, dw_j(t), \tag{9}$$

with initial condition $F(0) = F_0 > 0$.

In what follows we will use the notation: $\sigma = (\sigma_{ij})$, $\mathbf{b} = (b_1, b_2, \ldots, b_n)^\top$, $\mathbf{1} = (1, 1, \ldots, 1)^\top$ and $\Sigma = \sigma\sigma^\top$. We assume the invertibility of Σ. Finally, the vector of standardized risk premia of the portfolio, or Sharpe ratio, is denoted $\theta = \sigma^{-1}(\mathbf{b} - r\mathbf{1})$. With the notation just introduced, and substituting (1) into (9), this equation takes the form

$$dF(t) = \Big(rF(t) + \mathbf{\Lambda}^\top(t)(\mathbf{b} - r\mathbf{1}) + NC(t) + k(AL(t) - F(t)) - P(t)\Big)\, dt + \mathbf{\Lambda}^\top(t)\sigma\mathcal{S}(t)^\beta dw(t) \tag{10}$$

where, for $p \in \mathbb{R}$, $\mathcal{S}(t)^p$ is the diagonal matrix (see Zhao and Rong (2012))

$$\mathcal{S}(t)^p = \begin{pmatrix} S_1(t)^p & 0 & \ldots & 0 \\ 0 & S_2(t)^p & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \ldots & \ldots & S_n(t)^p \end{pmatrix}. \tag{11}$$

When $p = 1$, we will simply use the notation $\mathcal{S}$ instead of $\mathcal{S}^1$.

Given that there is a riskless rate of interest r in the financial market and benefits are deterministic, a rather natural selection of the technical rate of actualization is $\delta = r$. See Josa–Fombellida and Rincón–Zapatero (2001, 2004, 2006) for a more ample discussion of this topic, even when benefits are geometric Brownian motions. Hence we impose the following hypothesis.

Assumption 2 *The technical rate of actualization coincides with the riskless rate of interest,* $\delta = r$.

As a consequence, by (2), in terms of the surplus $X = F - AL$, equation (10) reads:

$$dX(t) = \Big((r - k)X(t) + \mathbf{\Lambda}^\top(t)(\mathbf{b} - r\mathbf{1})\Big)dt + \mathbf{\Lambda}^\top(t)\sigma\mathcal{S}(t)^\beta dw(t), \tag{12}$$

with the initial condition $X(0) = x_0$. Note that when $X < 0$ (underfunded plan), $k > 0$ in (1) has the effect of diminishing (augmenting when $k < 0$) the rate of interest that is being charged on the unfunded liability. The situation is reversed if $X > 0$ (overfunded plan).

In the following sections we use dynamic programming techniques to solve the problem, hence we will consider the optimization problem for every initial condition $X(0)$, that we denote

by x. Hence $X(t) = F(t) - AL(t)$ as a process, but $x = F - AL$ as a fixed initial value. Also, we consider arbitrary initial conditions $S_i(0)$, denoted by $s_i > 0$, for $i = 1, \ldots, n$, and let $s = (s_1, \ldots, s_n)$.

2.2 The optimization problem

In this section, we formulate the mathematical problem. The goal of the manager depends on whether the fund is underfunded, $F < AL$, or overfunded, $F > AL$. We consider these two scenarios separately. It is important to note that at the optimal solution, these two scenarios do not mix. That is, under an optimal management, a fund that starts in the underfunded (*resp.* overfunded) region, never becomes overfunded (*resp.* underfunded) with positive probability. Preferences of the manager in the underfunded region are directed at minimizing the solvency risk, that is, minimizing the quadratic deviation of the fund assets from the actuarial liability at the terminal time. On the contrary, in the overfunded case, the manager wishes to maximize a strictly increasing and strictly concave utility function of the fund assets at the terminal time. To get analytical solutions to the problem, we work with utilities of constant elasticity[3].

2.2.1 Underfunded plan

When the fund assets do not cover the liability, the manager's goal is to select the optimal investment strategy that minimizes the solvency risk at the terminal date of the pension plan. Thus, given initial values of time, t, surplus, $x < 0$, and asset prices $s = (s_1, \ldots, s_n)$, with $s_i > 0$, for $i = 1, \ldots, n$, the objective functional to be minimized over the class of admissible controls, $\mathcal{A}_{t,x,s}$, is given by the payoff functional

$$J((t, x, s); \mathbf{\Lambda}) = \mathbb{E}_{t,x,s}\left\{\alpha X(T)^2\right\}, \tag{13}$$

where α is a positive constant. Here, $\mathcal{A}_{t,x,s}$ is the set of measurable processes $\mathbf{\Lambda}$ satisfying (7) and where X and S satisfy (12) and (6), respectively. In the above, $\mathbb{E}_{t,x,s}$ denotes conditional

[3]This specification has also been used in Josa-Fombellida and Rincón-Zapatero (2017), where a differential pension plan game between the firm sponsor and (homogeneous) workers is studied in the overfunded region.

expectation with respect to the initial conditions $X(t) = x$ and $S(t) = s$. It is important to observe that the imposed admissibility condition (7) guarantees that the system of SDEs defining the fund, (6), (12), has a unique solution for each initial value of surplus and assets.

The dynamic programming approach is used to solve the problem. The value function is defined as

$$\widehat{V}(t,x,s) = \min_{\mathbf{\Lambda}\in\mathcal{A}_{t,x,s}} \big\{ J((t,x,s);\mathbf{\Lambda}) \,|\, \text{ s.t. } (6),(12) \big\}.$$

It is clear that the value function so defined is non–negative and strictly convex. The connection between value functions and optimal feedback controls in stochastic control theory is accomplished by the Hamilton–Jacobi–Bellman (HJB) equation; see Fleming and Soner (1993). For our problem, the value function $\widehat{V}$ satisfies the HJB equation

$$V_t + \min_{\mathbf{\Lambda}} \Big\{ \mathbf{b}^\top \mathcal{S} V_s + ((r-k)x + \mathbf{\Lambda}^\top(\mathbf{b} - r\mathbf{1}))V_x + \frac{1}{2}\operatorname{tr}\{\mathcal{S}^{\beta+1}\sigma\sigma^\top \mathcal{S}^{\beta+1} V_{ss}\}$$
$$+ \mathbf{\Lambda}^\top \mathcal{S}^\beta \sigma\sigma^\top \mathcal{S}^{\beta+1} V_{xs} + \frac{1}{2}\mathbf{\Lambda}^\top \mathcal{S}^\beta \sigma\sigma^\top \mathcal{S}^\beta \mathbf{\Lambda} V_{xx} \Big\} = 0, \tag{14}$$

with the final condition $V(T,x,s) = \alpha x^2$, for all $x \le 0$, $s > 0$, where the matrices $\mathcal{S}^p$, defined in (11), are given by

$$\mathcal{S}^p = \begin{pmatrix} s_1^p & 0 & \dots & 0 \\ 0 & s_2^p & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & \dots & s_n^p \end{pmatrix},$$

for $p = 1, \beta, \beta+1$, where $\mathcal{S} = \mathcal{S}^1$. We indicate partial derivatives of a function with respect to a variable by writing this variable as a subindex of the function. In the HJB equation above, $V_s = (V_{s_1}, \dots, V_{s_n})^\top$, $V_{xs} = (V_{xs_1}, \dots, V_{xs_n})^\top$ and $V_{ss} = (V_{s_i s_j})$ is the Hessian matrix of V with respect to s. Finally, tr{} is the trace operator.

2.2.2 Overfunded plan

When the fund assets are above the liability, the manager's goal is to maximize the utility derived from the fund surplus. We let the utility function $U(x) = \frac{x^{1-\gamma}}{1-\gamma}$, where $\gamma > 0$ is constant[4], known

[4]We suppose $\gamma \neq 1$. The case $\gamma = 1$ is the logarithmic utility, $U(x) = \ln x$.

as the relative risk aversion index of the agent. Given initial values of time, t, surplus, $x > 0$, and assets, $s = (s_1, \ldots, s_n)$ with $s_i > 0$ for all $i = 1, \ldots, n$, the payoff functional to be maximized over the class of admissible controls, $\mathcal{A}_{t,x,s}$, is given by

$$J((t,x,s);\mathbf{\Lambda}) = \mathbb{E}_{t,x,s}\left\{\frac{X(T)^{1-\gamma}}{1-\gamma}\right\}.$$

The value function, defined by

$$\widehat{V}(t,x,s) = \max_{\mathbf{\Lambda}\in\mathcal{A}_{t,x,s}} \big\{J((t,x,s);\mathbf{\Lambda}) \,|\, \text{ s.t. } (6),(12)\big\},$$

is non–negative and strictly concave, and is characterized as the solution of the HJB equation

$$V_t + \max_{\mathbf{\Lambda}}\Big\{\mathbf{b}^\top \mathcal{S} V_s + ((r-k)x + \mathbf{\Lambda}^\top(\mathbf{b} - r\mathbf{1}))V_x + \frac{1}{2}\operatorname{tr}\{\mathcal{S}^{\beta+1}\sigma\sigma^\top \mathcal{S}^{\beta+1} V_{ss}\}$$
$$+\mathbf{\Lambda}^\top \mathcal{S}^\beta \sigma\sigma^\top \mathcal{S}^{\beta+1} V_{xs} + \frac{1}{2}\mathbf{\Lambda}^\top \mathcal{S}^\beta \sigma\sigma^\top \mathcal{S}^\beta \mathbf{\Lambda} V_{xx}\Big\} = 0,$$

with the final condition $V(T,x,s) = \frac{x^{1-\gamma}}{1-\gamma}$, for all $x > 0, s > 0$.

3 The optimal strategies

In this section, we solve the problem for each of the two scenarios considered. First we provide the solution to the general case, in which the asset prices are correlated. Then we focus on the independent case, which allows for more explicit expressions.

3.1 General case

3.1.1 Underfunded plan

The following result collects the optimal investment strategies and the evolution of the unfunded actuarial liability in the underfunded case.

Theorem 3.1 *Suppose that Assumptions 1 and 2 hold. The optimal investment vector is given by*

$$\mathbf{\Lambda}^*(t,x,s) = -\left((\mathcal{S}^\beta \sigma\sigma^\top \mathcal{S}^\beta)^{-1}(\mathbf{b} - r\mathbf{1}) + \frac{1}{g(t,s)}\mathcal{S} g_s(t,s)\right) x \tag{15}$$

where $g : [0,T] \times \mathbb{R}^n \longrightarrow \mathbb{R}$ *is a positive solution of the non–linear PDE (partial differential equation)*

$$g_t(t,s) + 2(r-k)g(t,s) + (2r\mathbf{1} - \mathbf{b})^\top \mathcal{S} g_s(t,s) + \frac{1}{2}\operatorname{tr}\{\mathcal{S}^{\beta+1}\sigma\sigma^\top \mathcal{S}^{\beta+1} g_{ss}(t,s)\} \tag{16}$$

$$-(\mathbf{b} - r\mathbf{1})^\top (\mathcal{S}^\beta \sigma\sigma^\top \mathcal{S}^\beta)^{-1}(\mathbf{b} - r\mathbf{1})g(t,s) - \frac{1}{g(t,s)} g_s(t,s)^\top \mathcal{S}^{\beta+1}\sigma\sigma^\top \mathcal{S}^{\beta+1} g_s(t,s) = 0, \tag{17}$$

with the final condition $g(T,s) = \alpha$*, for all* s*.*

Optimal investment decisions (15) are proportional to the unfunded actuarial liability, $-X$, with a variable coefficient that depends on time and asset prices. This coefficient is the sum of two terms. The first summand, $(\mathcal{S}^\beta \sigma\sigma^\top \mathcal{S}^\beta)^{-1}(\mathbf{b} - r\mathbf{1})$, reduces to the so called optimal growth portfolio strategy, when $\beta = 0$. The optimal growth portfolio is modulated by the asset prices when β is not null, but retains a similar structure. The second summand is a correction term that is also absent when the elasticity parameter β is null[5]. When β is not zero, this term depends on the financial market parameters, including the elasticity parameter β, as well as the amortization rate k chosen by the manager. The expression $\frac{1}{g(t,s)}\mathcal{S} g_s(t,s)$ is the vector of elasticities of the function $g(t,\cdot)$ with respect to the asset prices, that is, the ratio of the percentage variation in $g(t,\cdot)$ to the percentage variation in asset prices. This elasticity is obviously affected, by the existing correlation between risky assets.

Substituting Λ, from (15) into (12), we find that the surplus evolution is given by the stochastic differential equation (SDE)

$$\begin{aligned} dX(t) = &\Big(r - k - (\mathbf{b} - r\mathbf{1})^\top (\mathcal{S}(t)^\beta \sigma\sigma^\top \mathcal{S}(t)^\beta)^{-1}(\mathbf{b} - r\mathbf{1}) - \frac{1}{g(t,S(t))} g_s(t,S(t))^\top \mathcal{S}(t)(\mathbf{b} - r\mathbf{1})\Big) X(t)dt \\ &- \Big((\mathbf{b} - r\mathbf{1})^\top (\mathcal{S}(t)^\beta \sigma\sigma^\top \mathcal{S}(t)^\beta)^{-1}\sigma \mathcal{S}(t)^\beta + \frac{1}{g(t,S(t))} g_s(t,S(t))^\top \mathcal{S}(t)\sigma \mathcal{S}(t)^\beta\Big) X(t)dw(t), \end{aligned} \tag{18}$$

where $S = (S_1, \ldots, S_n)$, the vector of risky asset prices, satisfies (6). Thus, the SDE satisfied by the fund, $F = X + AL$, is coupled with the system (6) of SDEs for the risky assets S. It is

[5]Note that when $\beta = 0$, the theorem gives the solution of a classical Merton problem with quadratic utility index, since the unique solution of (16) is independent of s, and given by $g(t) = \alpha \exp\{2(r - k - \theta^\top\theta/2)(T-t)\}$; see Remark 3.1 below.

not possible to obtain explicitly the optimal fund since (17) is non linear. Also, from (1), the optimal contribution rate is $C(t) = NC(t) - kX(t)$, where X follows (18) and the normal cost is obtained in Proposition 2.1.

Remark 3.1 (GBM case) *When $\beta = 0$, $g(t,s) = g(t) = \alpha \exp\{2(r - k - \theta^\top\theta/2)(T-t)\}$, the value function is $V(t,x,s) = V(t,x) = g(t)x^2$, the optimal investment strategy is $\mathbf{\Lambda}^*(t,x,s) = -\Sigma^{-1}(\mathbf{b} - r\mathbf{1})x > 0$, and the optimal surplus is a GBM whose evolution is given by the SDE $dX(t) = (r-k-\theta^\top\theta)X(t)dt - \theta^\top X(t)dw(t)$, with $X(0) = x_0$. Its expected value, given $X(0) = 0$, is given by $\mathbb{E}X(t) = x_0 \exp\{(r - k - \theta^\top\theta)t\}$, which is increasing on t and converges to 0 when t goes to ∞, if $r < k + \theta^\top\theta$. Thus, under that condition, the expected fund $\mathbb{E}F(t)$ is nearest to AL when t goes to T. However, if $r > k + \theta^\top\theta$, the contrary effect holds.*

3.1.2 Overfunded plan

The following result collects the optimal investment strategies and the evolution of the fund surplus in the overfunded case. Notice that the optimal investment decisions differ from the previous case only in the inverse of the relative risk aversion parameter, $\frac{1}{\gamma}$. Values of γ within the interval $(0,1)$ characterize agents with small risk aversion; a value $\gamma > 1$ indicates a more risk averse player, and this attitude towards risk becomes sharper as γ increases. The logarithmic case $\gamma = 1$ can be considered as one of moderate risk aversion.

Theorem 3.2 *Suppose that Assumptions 1 and 2 hold. The optimal investment vector is given by*

$$\mathbf{\Lambda}^*(t,x,s) = \frac{1}{\gamma}\left((\mathcal{S}^\beta\sigma\sigma^\top\mathcal{S}^\beta)^{-1}(\mathbf{b} - r\mathbf{1}) + \frac{1}{g(t,s)}\mathcal{S}g_s(t,s)\right)x \tag{19}$$

where $g : [0,T] \times \mathbb{R}^n \longrightarrow \mathbb{R}$ is a positive solution of the non–linear PDE

$$g_t(t,s) + (1-\gamma)(r-k)g(t,s) + \frac{1}{\gamma}(-(1-\gamma)r\mathbf{1} + \mathbf{b})^\top\mathcal{S}g_s(t,s) + \frac{1}{2}\operatorname{tr}\{\mathcal{S}^{\beta+1}\sigma\sigma^\top\mathcal{S}^{\beta+1}g_{ss}(t,s)\}$$
$$+ \frac{1-\gamma}{2\gamma}(\mathbf{b} - r\mathbf{1})^\top(\mathcal{S}^\beta\sigma\sigma^\top\mathcal{S}^\beta)^{-1}(\mathbf{b} - r\mathbf{1})g(t,s) + \frac{1-\gamma}{2\gamma}\frac{1}{g(t,s)}g_s(t,s)^\top\mathcal{S}^{\beta+1}\sigma\sigma^\top\mathcal{S}^{\beta+1}g_s(t,s) = 0, \tag{20}$$

with the final condition $g(T,s) = \frac{1}{1-\gamma}$, for all s.

As said above, the vector of optimal investment decisions, (19), has an identical structure to that of the underfunded case, except for the fact that it is weighted by the inverse of the relative risk aversion parameter, $1/\gamma$. A word of caution is needed here: the PDE for g in the underfunded case is different from the overfunded case, since in the latter case, γ explicitly appears in the PDE. So the effect of γ on the optimal decision investments is not straightforward to determine at this level of generality.

Substituting Λ from (19) into (12), we find that the surplus evolution is given by

$$\begin{aligned} dX(t) = &\Big(r - k + \frac{1}{\gamma}(\mathbf{b} - r\mathbf{1})^\top (\mathcal{S}(t)^\beta \sigma\sigma^\top \mathcal{S}(t)^\beta)^{-1}(\mathbf{b} - r\mathbf{1}) + \frac{1}{\gamma g(t, S(t))} g_s(t, S(t))^\top \mathcal{S}(t)(\mathbf{b} - r\mathbf{1})\Big) X(t)dt \\ &+ \frac{1}{\gamma}\Big((\mathbf{b} - r\mathbf{1})^\top (\mathcal{S}(t)^\beta \sigma\sigma^\top \mathcal{S}(t)^\beta)^{-1}\sigma\mathcal{S}(t)^\beta + \frac{1}{g(t, S(t))} g_s(t, S(t))^\top \mathcal{S}(t)\sigma\mathcal{S}(t)^\beta\Big) X(t)dw(t), \end{aligned} \tag{21}$$

where $S = (S_1, \ldots, S_n)$ satisfies (6). As in the underfunded case, the contribution rate $C(t)$ depends on surplus and benefits, $C(t) = NC(t) - kX(t)$.

Remark 3.2 (GBM and logarithmic cases) *When $\beta = 0$, $g(t,s) = g(t) = \frac{1}{1-\gamma}\alpha \exp\{(1-\gamma)(r - k + \theta^\top\theta/(2\gamma))(T-t)\}$ and the optimal investment strategy is $\mathbf{\Lambda}^*(t,x,s) = \frac{1}{\gamma}\Sigma^{-1}(\mathbf{b} - r\mathbf{1})x > 0$. When $U(x) = \ln x$, the optimal investment is $\mathbf{\Lambda}^*(t,x,s) = \Big((\mathcal{S}^\beta\sigma\sigma^\top\mathcal{S}^\beta)^{-1}(\mathbf{b} - r\mathbf{1}) + \frac{1}{g(t,s)}\mathcal{S}g_s(t,s)\Big)\, x$, where g satisfies $g_t(t,s) + \mathbf{b}^\top \mathcal{S} g_s(t,s) + \frac{1}{2}\operatorname{tr}\{\mathcal{S}^{\beta+1}\sigma\sigma^\top\mathcal{S}^{\beta+1} g_{ss}(t,s)\} = 0$ with $g(T,s) = 1$, and the surplus evolution is given by (21), but taking $\gamma = 1$. If $\beta = 0$ and $U(x) = \ln x$, then $g(t,s) = g(t) = (r - k - \theta^\top\theta/2)(T-t)$, $\mathbf{\Lambda}^*(t,x,s) = \Sigma^{-1}(\mathbf{b} - r\mathbf{1})x > 0$ and the optimal surplus evolution is given by $dX(t) = (r - k + \theta^\top\theta)X(t)dt + \theta^\top X(t)dw(t)$, with $X(0) = x_0$.*

3.2 Case where the risky assets are uncorrelated

In order to explicitly solve the problem, we make the simplifying assumption that the risky assets are uncorrelated. A particular case is when there exists a unique risky asset, $n = 1$.

Assumption 3 *The risky assets satisfy $\sigma_{ij} = 0$, for all $i \neq j$.*

The previous assumption simplifies the system (6) to

$$dS_i(t) = S_i(t)\Big(b_i dt + \sigma_i S_i(t)^\beta dw_i(t)\Big), \quad S_i(0) = s_{i0}, \quad i = 1, 2, ..., n, \tag{22}$$

where we have denoted $\sigma_i = \sigma_{ii}$, for all $i = 1, \ldots, n$. This assumption allows us to solve analytically the nonlinear system (17).

3.2.1 Underfunded plan

The following result explicitly provides the optimal investment policies.

Proposition 3.1 *Suppose that Assumptions 1, 2 and 3 hold. Then the optimal investment vector is given by*

$$\lambda_i^*(t,x,s) = -\left(\frac{\theta_i}{\sigma_i} + 2\beta B_i(t)\right) s_i^{-2\beta} x, \quad i = 1, \ldots, n, \tag{23}$$

where θ_i the Sharpe ratio of the asset i, that is $\theta_i = \frac{b_i - r}{\sigma_i}$, and where $B_i(t)$, $i = 1, \ldots, n$, is

$$B_i(t) = -\frac{b_i - 2r}{2\beta\sigma_i^2} + \frac{\sqrt{b_i^2 - 2r^2}}{2\beta\sigma_i^2} \tan\left(\beta\sqrt{b_i^2 - 2r^2}(T-t) + \arctan\left(\frac{b_i - 2r}{\sqrt{b_i^2 - 2r^2}}\right)\right), \tag{24}$$

when $b_i > \sqrt{2}r$,

$$B_i(t) = \frac{(2-\sqrt{2})^2 r^2 (T-t)}{2\sigma_i^2\left(\beta(2-\sqrt{2})r(T-t) + 1\right)}, \tag{25}$$

when $b_i = \sqrt{2}r$, and

$$B_i(t) = \frac{m^- m^+ \left(1 - e^{2\beta\sqrt{2r^2 - b_i^2}(T-t)}\right))}{m^- - m^+ e^{2\beta\sqrt{2r^2 - b_i^2}(T-t)}}, \tag{26}$$

with

$$m_-^+ = \frac{-(b_i - 2r)_-^+ \sqrt{2r^2 - b_i^2}}{2\beta\sigma_i^2}, \tag{27}$$

when $b_i < \sqrt{2}r$.

The observations made about the content of Theorem 3.1 in Section 3.1.1 take here a more specific form. The second summand in (15) is now $\frac{1}{g(t,s)}\mathcal{S}g_s(t,s) = 2\beta(B_1(t)s_1^{-2\beta}, \ldots, B_n(t)s_n^{-2\beta})$. Each λ_i^* is affected only by its own price process through the power term $s_i^{-2\beta}$, for $i = 1, \ldots, n$. Depending on the parameter values of the model, the optimal investment policy may require

shortselling or borrowing[6]. Note that the optimal portfolio is not affected by benefits[7], contrary to the amortization rate k.

The optimal solvency risk is obtained from the value function. Following the proofs of Theorem 3.1 and Proposition 3.1 in Appendix A, and with the same notation used there, we obtain

$$V(t,s,x) = g(t,s)x^2 = \frac{1}{f(t,y)}x^2 = \frac{1}{A(t)}e^{-\sum_{i=1}^n B_i(t)y_i}x^2,$$

hence the solvency risk is

$$\mathbb{E}_{t,x,s}(\mathit{UAL}^*(T)^2) = \frac{1}{\alpha A(t)}e^{-\sum_{i=1}^n B_i(t)s_i^{-2\beta}}x^2,$$

where $A(t) = K\exp\{2(r-k)t - \beta(2\beta+1)\sum_{i=1}^n \sigma_i^2\phi_i(t)\}$, with $\phi_i(t)$ a primitive function of $B_i(t)$ and K a constant determined by the condition $A(T) = \frac{1}{\alpha}$. Appendix A shows the explicit expression of $A(t)$.

Substituting $g = f^{-1}$ and $f(t,y) = A(t)e^{\sum_{i=1}^n B_i(t)y_i}$ in (18) (see again the proof of Proposition 3.1 in Appendix A), we arrive to the SDE for the optimal surplus

$$\begin{aligned} dX(t) = &\left(r - k - \sum_{i=1}^n \left(\frac{\theta_i}{\sigma_i} + 2\beta B_i(t)\right)S_i(t)^{-2\beta}(b_i - r)\right)X(t)dt \\ &- \sum_{i=1}^n \sigma_i\left(\frac{\theta_i}{\sigma_i} + 2\beta B_i(t)\right)S_i(t)^{-\beta}X(t)dw_i(t), \end{aligned}$$

where S_i satisfies (22), for $i = 1,\ldots,n$. From here, taking into account the relations $F = X + AL$, $C = NC - kX$ and $SC = C - NC$, and Proposition 2.1, we can study the evolution of the fund assets, the contributions and the supplementary cost, as well as their expected values, along

[6]In a general portfolio selection problem, Zhang and Rong (2017) analyze the dependence of the risky investments with respect to the elasticity parameter in the different cases and by means of a numerical illustration they show a sensitivity analysis, but for a fixed time. This study could be carried out on our pension model but, though we find it interesting, we consider it is more convenient to study the time evolution not only of the investment strategies, but also of the variables defining the pension plan, such as the fund surplus and supplementary cost, since they provide useful information about the stability and safety properties of the pension plan.

[7]This is no longer true in defined benefit pension plans with stochastic benefits, as in Josa-Fombellida and Rincón-Zapatero (2004), where the value of the actuarial liability enters into the optimal portfolio.

$[0,T]$. From Proposition 2.1, the optimal rate of contribution is given by

$$\begin{aligned} C(t) = NC(t) - kX(t) &= (1 + h'(t) + (\mu(t) - \delta)h(t))e^{\int_0^t \mu(s)ds} P_0 - kX(t) \\ &= (1 + h'(t) + (\mu(t) - \delta + k)h(t))e^{\int_0^t \mu(s)ds} P_0 - kF(t). \end{aligned}$$

Remark 3.3 (GBM case) *When $\beta = 0$, the optimal investment in the ith asset is $\lambda_i^*(t,x,s) = -\frac{\theta_i}{\sigma_i} s_i^{-2\beta} x$, $i = 1,\ldots,n$, thus shortselling is not necessary. The equations* (34) *and* (35) *in Appendix A are linear and decoupled, and in consequence the solutions are easily obtained,*

$$A(t) = \frac{1}{\alpha} e^{2(k-r)(T-t)}, \qquad B_i(t) = \theta_i^2 (T-t),$$

for all $i = 1,\ldots,n$. The optimal surplus evolution is given by

$$dX(t) = \left(r - k - \sum_{i=1}^n \theta_i^2\right) X(t)dt - \sum_{i=1}^n \theta_i X(t) dw_i(t),$$

with $X(0) = x_0$. Thus the expected fund surplus is increasing over time if $r < k + \sum_{i=1}^n \theta_i^2$ and decreasing otherwise.

3.2.2 Overfunded plan

The following result explicitly provides the optimal investment policies.

Proposition 3.2 *Suppose that Assumptions 1, 2 and 3 hold. Then the optimal investment vector is given by*

$$\lambda_i^*(t,x,s) = \frac{1}{\gamma}\left(\frac{\theta_i}{\sigma_i} + 2\beta B_i(t)\right) s_i^{-2\beta} x, \quad i = 1,\ldots,n, \tag{28}$$

where

$$\begin{aligned} B_i(t) &= -\frac{b_i - (1-\gamma)r}{2\beta\sigma_i^2} \\ &+ \frac{\sqrt{\gamma((1-\gamma)r^2 - b_i^2)}}{2\beta\sigma_i^2} \tan\left(\frac{-\beta}{\gamma}\sqrt{\gamma((1-\gamma)r^2 - b_i^2)}(T-t) + \arctan\left(\frac{b_i - (1-\gamma)r}{\sqrt{\gamma((1-\gamma)r^2 - b_i^2)}}\right)\right), \end{aligned}$$

when $b_i^2 < (1-\gamma)r^2$,

$$B_i(t) = \frac{-((1-\gamma) - \sqrt{1-\gamma})^2 r^2 (T-t)}{2\sigma_i^2\left(\beta(\sqrt{1-\gamma} - (1-\gamma))r(T-t) + \gamma\right)},$$

when $b_i^2 = (1-\gamma)r^2$, *and*

$$B_i(t) = \frac{m^- m^+ \left(1 - e^{-2\frac{\beta}{\gamma}\sqrt{\gamma(b_i^2-(1-\gamma)r^2)}(T-t)}\right)}{m^- - m^+ e^{-2\frac{\beta}{\gamma}\sqrt{\gamma(b_i^2-(1-\gamma)r^2)}(T-t)}},$$

with

$$m_-^+ = \frac{-(b_i-(1-\gamma)r)_-^+\sqrt{\gamma(b_i^2-(1-\gamma)r^2)}}{2\beta\sigma_i^2},$$

when $b_i^2 > (1-\gamma)r^2$, *for* $i = 1, \ldots, n$.

Now the value function is $V(t,s,x) = \frac{1}{A(t)}e^{-\sum_{i=1}^n B_i(t)y_i}x^{1-\gamma}$, where $A(t) = K\exp\{-(1-\gamma)(r-k)t - \beta(2\beta+1)\sum_{i=1}^n \sigma_i^2\phi_i(t)\}$, with $\phi_i(t)$ being a primitive function of $B_i(t)$ and K a constant determined by the condition $A(T) = 1-\gamma$. The SDE for the optimal surplus is

$$dX(t) = \left(r - k + \frac{1}{\gamma}\sum_{i=1}^n \left(\frac{\theta_i}{\sigma_i} + 2\beta B_i(t)\right)S_i(t)^{-2\beta}(b_i - r)\right)X(t)dt$$
$$+ \frac{1}{\gamma}\sum_{i=1}^n \sigma_i\left(\frac{\theta_i}{\sigma_i} + 2\beta B_i(t)\right)S_i(t)^{-\beta}X(t)dw_i(t).$$

From this SDE, we can study the evolution of the pension plan as in the underfunded case.

Remark 3.4 (Logarithmic case) *When* $U(x) = \ln x$, *then* $g = 1$, *which leads us to positive risky investments* $\lambda_i^*(t,x,s) = \frac{\theta_i}{\sigma_i}s_i(t)^{-2\beta}x$, $i = 1, \ldots, n$, *and fund evolution*

$$dX(t) = \left(r - k + \sum_{i=1}^n \theta_i^2 S_i(t)^{-2\beta}\right)X(t)dt + \sum_{i=1}^n \theta_i S_i(t)^{-\beta}X(t)dw_i(t),$$

with $X(0) = x_0$.

Remark 3.5 (GBM case) *When* $\beta = 0$, *the equations* (41) *and* (42) *in Appendix A are linear and decoupled, and in consequence the solutions are easily obtained:*

$$A(t) = \frac{1}{1-\gamma}e^{-(1-\gamma)(k-r)(T-t)}, \qquad B_i(t) = -\frac{1-\gamma}{2\gamma}\theta_i^2(T-t),$$

for all $i = 1, \ldots, n$. *The optimal investments are* $\lambda_i^*(t,x,s) = \frac{\theta_i}{\gamma\sigma_i}s_i(t)^{-2\beta}x$, $i = 1, \ldots, n$, *and shortselling is avoided. The optimal surplus is the GBM given by*

$$dX(t) = \left(r - k + \frac{1}{\gamma}\sum_{i=1}^n \theta_i^2\right)X(t)dt + \frac{1}{\gamma}\sum_{i=1}^n \theta_i X(t)dw_i(t),$$

with $X(0) = x_0$. Depending of the drift sign, the optimal expected surplus is increasing or decreasing over time, that is to say, the optimal expected fund $\mathbb{E}F$ is moving away from or approaching the actuarial liability AL. For instance, it decreases when the amortization rate is greater than the riskless rate of interest, $k > r$, and the risk aversion is greater than the quotient of the sum of the Sharpe ratios of the assets and $k - r$, $\gamma > \sum_{i=1}^{n} \theta_i^2/(k-r)$. The following section will show a sensitivity analysis of the optimal expected fund surplus, $\mathbb{E}X(t)$, with respect to the risk aversion parameter γ and the elasticity of variance parameter, β.

4 A numerical illustration

In this section, we consider a numerical application to illustrate the dynamic behavior of the optimal fund, the optimal contribution rate and the optimal portfolio strategy, for several values of the elasticity parameter β. In order to compute the simulations from the SDE's and buildd the figures, the package Sim.DiffProc of the R environment has been used.

We consider a portfolio with a riskless asset and one risky asset, $n = 1$, and we denote $S_1 = S$. We select the technical rate of interest leading a spread method of funding. The values of the parameters given below are used in both scenarios, the underfunded and the overfunded cases.

- The planning horizon is $T = 10$ years;
- ages on entering the plan and retirement are $a = 25$ and $d = 65$, respectively; M is uniform;
- benefits are deterministic, with $\mu(t) = 0.015$; initial benefits are set to $P_0 = 10$;
- the risky asset has $b = 0.02$ and $\sigma = 0.1$; the initial price is $s_0 = 50$
- the risk free rate of interest is $r = \delta = 0.01$ (note that $b > \sqrt{2}r$); this implies a Sharpe ratio $\theta = 0.1$; by Proposition 2.1 the initial values of the actuarial functions are $AL_0 = 214.028$ and $NC_0 = 11.070$;
- the amortization effort is $k = 0.018$;

- the parameter in the objective function is $\alpha = 1$.

4.1 Underfunded plan

As explained in previous sections, the manager's aim is to keep the fund as close as possible to the actuarial liability at the terminal date T.

To the previous data, we set the initial value of the fund assets and the values of β.

- The initial fund assets are $F_0 = 200$; this implies $x_0 = -14.028$ (underfunded case);
- the elasticity of the variance parameter is set to three different values: $\beta = 0, -0.25, -0.5$.

Figure 2 shows a realization of the optimal surplus $X^* = F^* - AL$ and the expected optimal surplus $\mathbb{E}X^*$. Observe that the sufficient condition $r < k + \theta^\top\theta$ of Remark 3.1 holds, assuring convergence of the expected optimal surplus to 0. The convergence is stronger as β decreases. Starting from an initial unfunded actuarial liability value of $-X(0) = 14.028$, the expected unfunded actuarial liability is reduced at the end of the plan to 11.89, 6.61 and 0.07 , depending on the elasticity parameter $\beta = 0, -0.25, -0.5$, respectively.

Figure 3 shows the evolution of the optimal supplementary cost $SC^* = C^* - NC$ and its expected value $\mathbb{E}SC^*$. We observe that $C^*(t)$ and $\mathbb{E}C^*(t)$ get closer to $NC(t)$ as t approaches T, and faster for smaller values of β.

Figure 4 represents the proportion of the optimal fund invested in the risky asset, λ^*/F^*. Investments start high and show a decreasing trend. The more negative β is, the more aggressive the investment policy. For instance, borrowing is necessary during the three first years when $\beta = -0.5$.

4.2 Overfunded plan

We consider the following data for the overfunded case, which complete the general data given at the beginning of this section.

- The initial fund is set at $F_0 = 220$; this implies $x_0 = 5.972$ (overfunded):

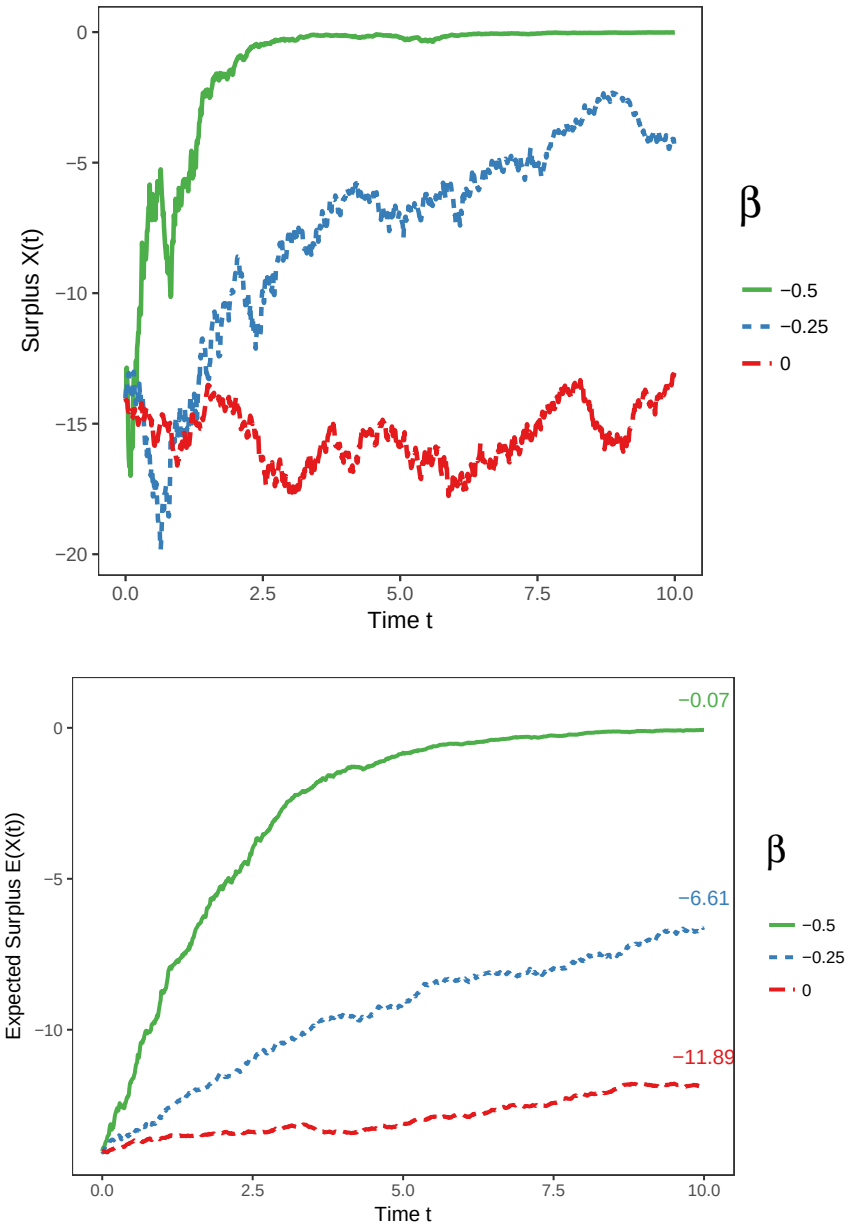

Figure 2: Evolution of surplus and expected surplus over time in the underfunded case. $\beta = 0, -0.25, -0.5$

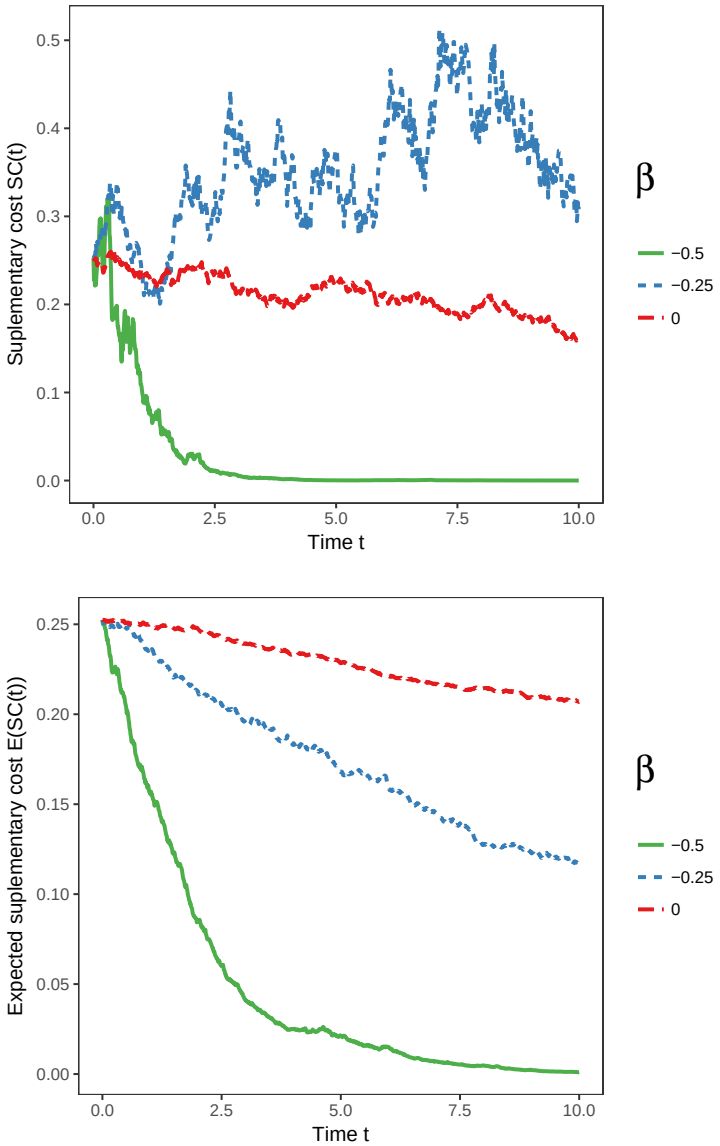

Figure 3: Evolution of the supplementary cost and the expected supplementary cost over time in the underfunded case. $\beta = 0, -0.25, -0.5$

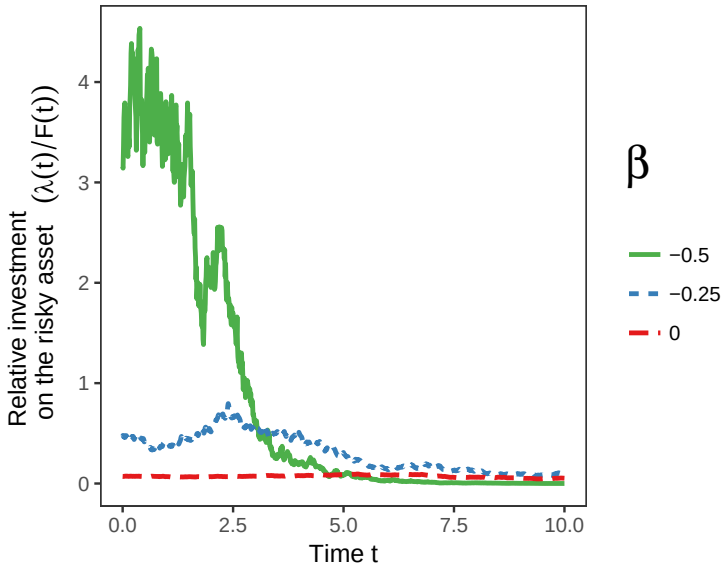

Figure 4: Evolution of the risky investment over time in the underfunded case. $\beta = 0, -0.25, -0.5$

- the relative risk aversion parameter is $\gamma = 0.5$, indicating low risk aversion of the sponsor (note that $b^2 > (1-\gamma)r^2$);
- the elasticity of the variance parameter is set to $\beta : 0, -0.1, -0.2$ (we assume lower values of β than in the underfunded case to facilitate comparison).

Figure 5 shows a realization of the optimal surplus X^* and the expected optimal surplus $\mathbb{E}X^*$. Since the aim of the sponsor is now to maximize an increasing concave utility of the surplus fund and that the drift of the SDE satisfied by the optimal surplus X^* is positive, the graphs show an increasing trend; this behavior is intensified with a decreasing β. Figure 6 shows the optimal supplementary cost, $SC(t)$, and the expected optimal contribution, $\mathbb{E}_{0,x_0,s_0}C(t)$.

Figure 7 represents the proportion of the fund invested in the risky asset, λ^*/F^*. The behavior is similar to the underfunded case. Smaller values of β require a more aggressive investment behavior, although not to the point of borrowing to invest in the risky asset, as in the underfunded case.

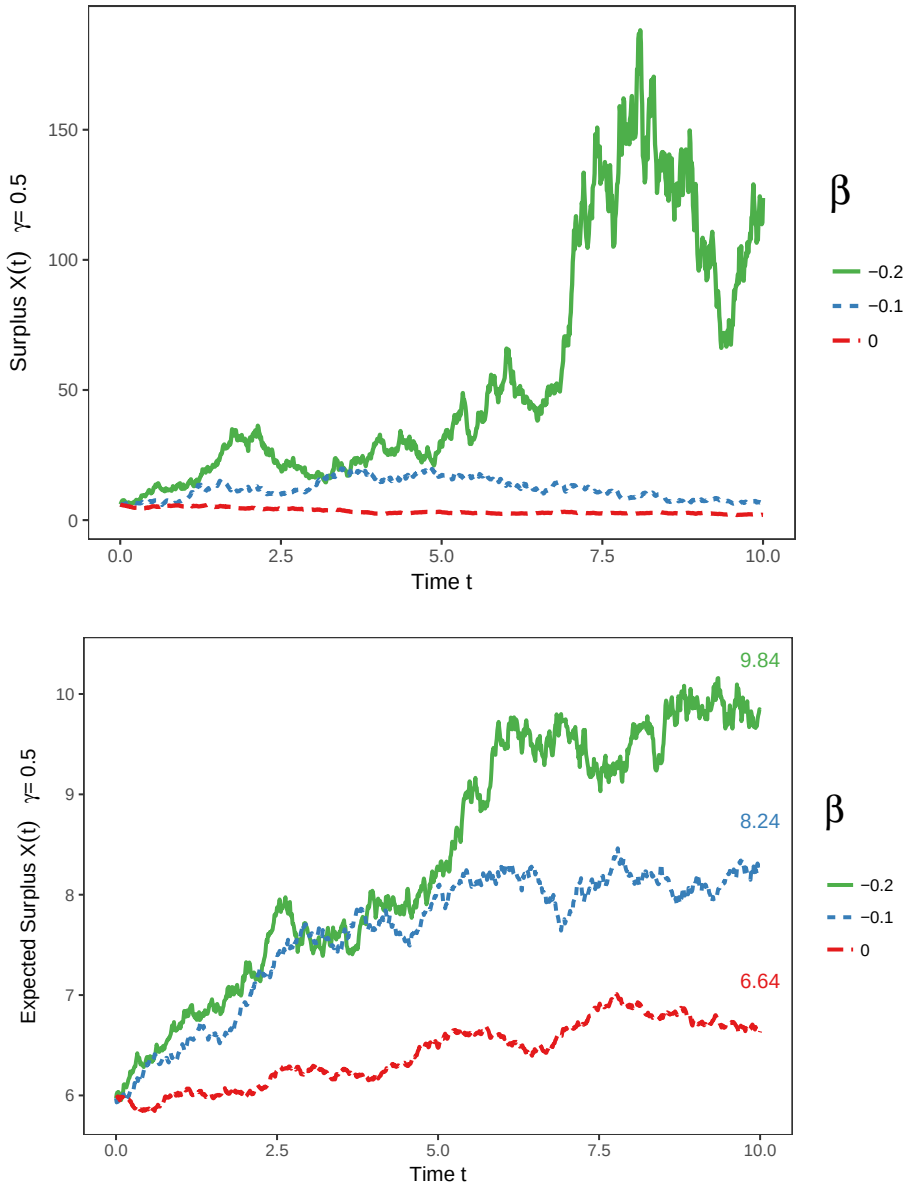

Figure 5: Evolution of surplus and expected surplus over time in the overfunded case. $\beta = 0, -0.10, -0.2$

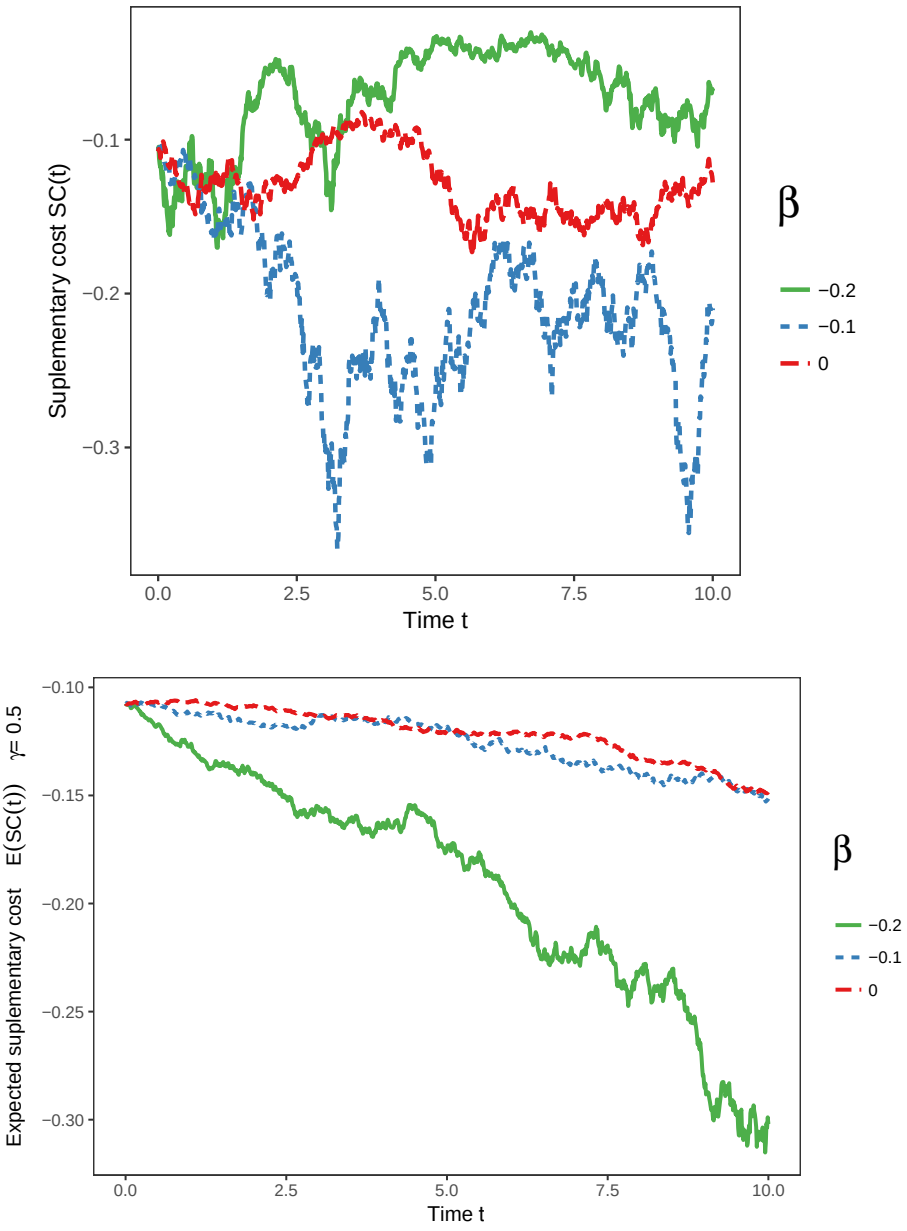

Figure 6: Evolution of the supplementary cost and the expected supplementary cost over time in the overfunded case. $\beta = 0, -0.10, -0.2$

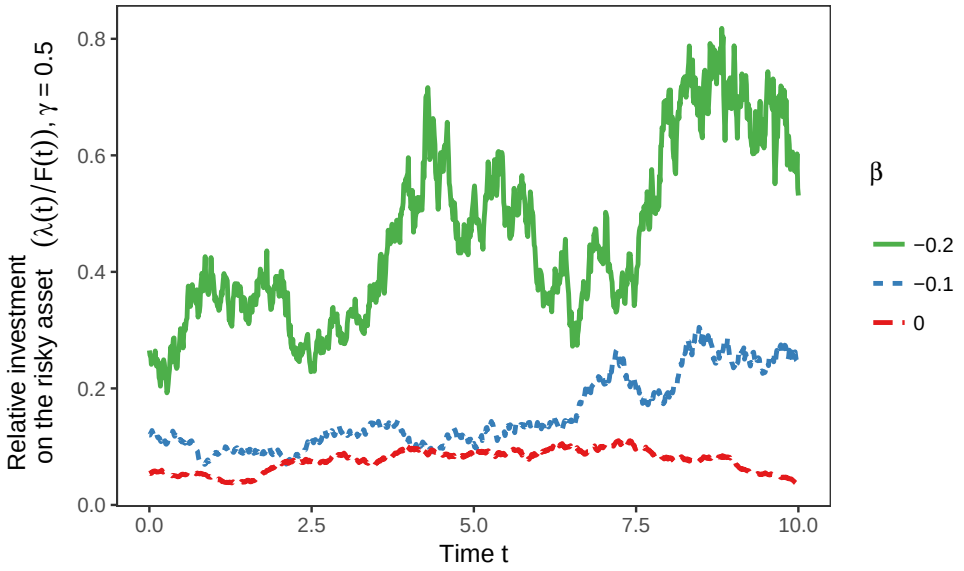

Figure 7: Evolution of the risky investment over time in the overfunded case. $\beta = 0, -0.10, -0.2$

Figures 8 and 9 show the dependence of the optimal surplus X^* and of the expected optimal surplus $\mathbb{E}X^*$, with respect to the elasticity of the variance parameter β and the risk aversion parameter γ. We observe that the expected surplus decreases with risk aversion. With low and moderate risk aversion ($\gamma = 0.5, 1$), the expected optimal fund $\mathbb{E}F^*(t)$ moves further away from the actuarial liability $AL\,(t)$ as the time t approaches the end date of the plan, $T = 10$. This distance increases as β decreases. Note that the condition in Remark 3.5 holds. With high risk aversion ($\gamma = 5, 10$), that condition does not hold. The expected optimal surplus $\mathbb{E}X^*$ is smaller than with low risk aversion, and increases as β decreases. High risk aversion does not allow the convergence of the expected optimal fund $\mathbb{E}F^*$ to the actuarial liability, AL, but the gap at the terminal date is small. For instance, with $\gamma = 10$, the initial value is $\mathbb{E}X^*(0) = 5.972$, but $\mathbb{E}X^*(10)$ is between 5.59 and 5.74, a small reduction therefore.

We find a similar behavior of the contribution and investment strategies with respect to the risk aversion parameter γ. High risk aversion implies less contributions and less aggressive investment mode than with moderate or low risk aversion.

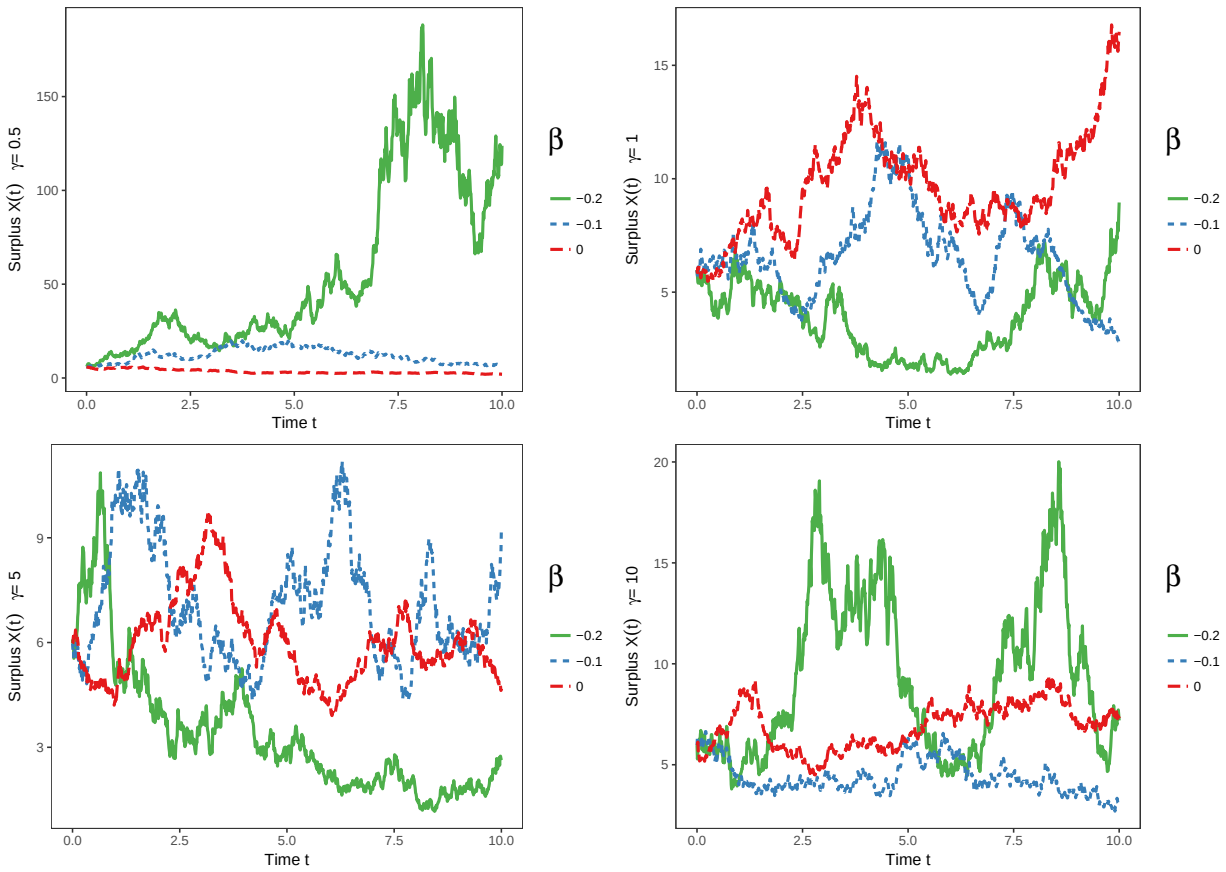

Figure 8: A comparison of the evolution of the surplus over time ($\beta = 0, -0.10, -0.2$) for several values of γ.

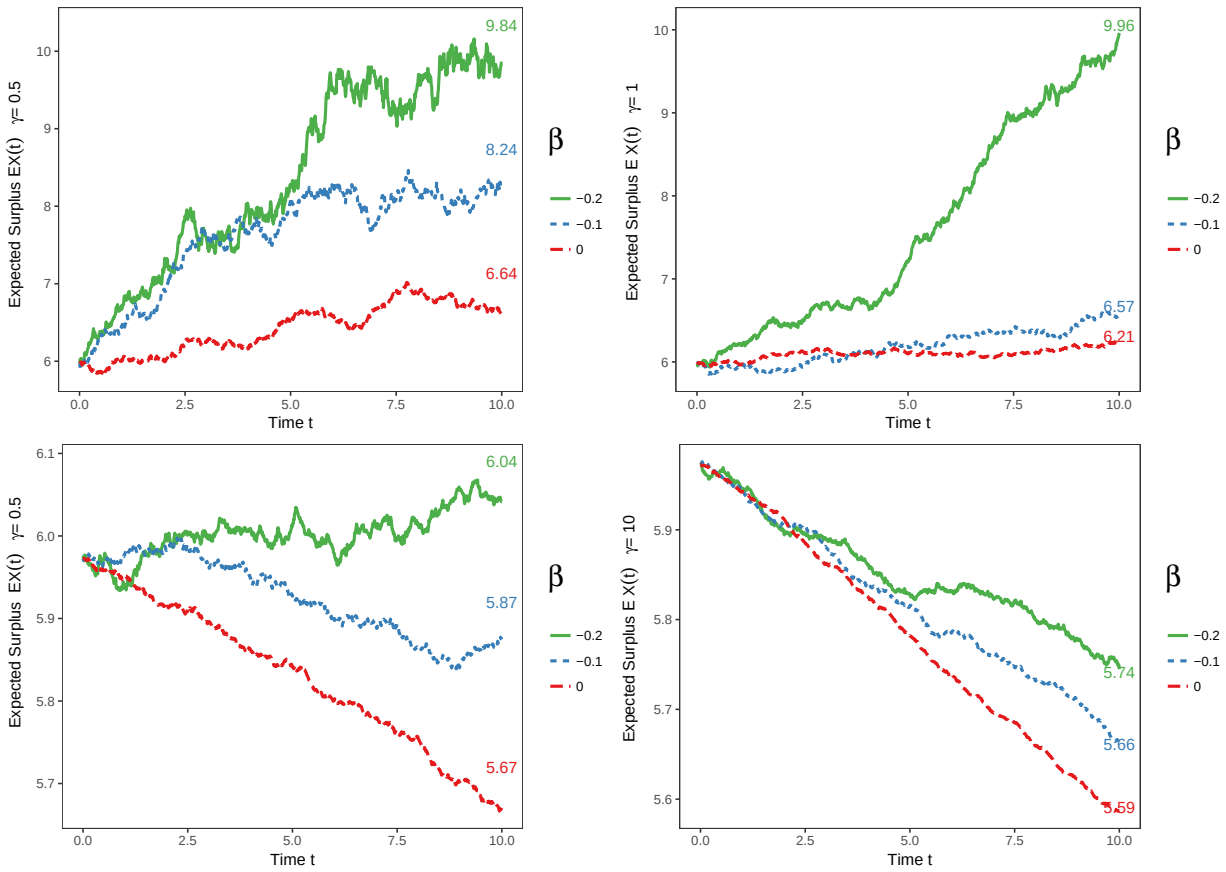

Figure 9: A comparison of the evolution of the expected surplus over time ($\beta = 0, -0.10, -0.2$) for several values of γ.

5 Conclusions

The management of an aggregated defined benefit pension plan in the presence of risky assets modeled by CEV processes has been analyzed by means of dynamic programming techniques. We have considered the minimization of the solvency risk when the plan is underfunded, and the maximization of the constant relative risk aversion utility function of the surplus, when the plan is overfunded, at the end of the planning horizon. The optimal solutions in both cases are investment rules which are proportional to the difference between the actuarial liability and the fund assets, taking into account the risk attitude of the manager of the pension plan. The factor of proportionality depends on the asset prices and on the elasticity of variance, as well as on the amortization rate chosen by the sponsor. In the particular case in which the risky assets are uncorrelated, we obtain the plausible result that the amount invested in each risky asset depends only on its own price.

A numerical illustration, carried out in the case of only one risky asset, shows how the optimal solution responds to changes in the elasticity of the variance parameter. In the underfunded case, with a suitable selection of the amortization rate, the gap between the fund assets and the actuarial liability is reduced at the terminal time. This reduction is more significant as the elasticity of variance takes on more negative values. There is also convergence of the contribution rate to the normal cost, showing the same pattern of behavior with respect to the elasticity of variance. Regarding the portfolio, the manager takes on more risky investments as the elasticity of variance decreases. This is in agreement with the fact that the reduction of the unfunded actuarial liability is harder to attain for values of elasticity close to 0. In the overfunded case, a moderate risk aversion parameter makes both the fund surplus and the contribution rate increase with time. When the risk aversion parameter increases, the expected surplus diminishes. As in the previous case, more negative values of the elasticity of variance makes the fund assets grow, for moderate risk aversion, and diminish, though at a slow rhythm, for high risk aversion. Regarding investment, the manager is more cautious for high risk aversion than for moderate or low risk aversion.

Future research will be addressed to extending the paper to a Markov regime-switching model, as in Chen and Hao (2013) or Hainaut (2014). Another interesting direction of research is to consider stochastic benefits.

A Appendix

Proof of Proposition 2.1. We follow the proof of Proposition 2.2 in Xu *et al.* (2007). If Assumption 1 holds, then

$$AL\,(t) = \int_a^d e^{-\delta(d-u)} P_0 e^{\int_0^{t+d-u}\mu(v)dv} M(u)\,du = P(t)\int_a^d e^{-\delta(d-u)} e^{\int_t^{t+d-u}\mu(v)dv} M(u)\,du = P(t)h(t)$$

that is (3), and

$$NC\,(t) = \int_a^d e^{-\delta(d-u)} P_0 e^{\int_0^{t+d-u}\mu(v)dv} M'(u)\,du = P(t)\int_a^d e^{\int_t^{t+d-u}(\mu(v)-\delta)dv} M'(u)\,du.$$

Integrating by parts, this last term is

$$\begin{aligned}\int_a^d e^{\int_t^{t+d-u}(\mu(v)-\delta)dv} M'(u)\,du &= 1 + \int_a^d (\mu(t+d-u)-\delta) e^{\int_t^{t+d-u}(\mu(v)-\delta)dv} M'(u)\,du \\ &= 1 + h'(t) + (\mu(t)-\delta)h(t).\end{aligned}$$

Thus, by (3), (4) holds. □

Proof of Theorem 3.1. For the problem (5), (12), (13), the HJB system is given by the system (14). If there is a smooth solution V of the equation (14), strictly convex, then the minimizer value of the investment rate is given by

$$\widehat{\mathbf{\Lambda}}(V_x, V_{xx}, V_{xs}) = -(\mathcal{S}^\beta \sigma\sigma^\top \mathcal{S}^\beta)^{-1}(\mathbf{b} - r\mathbf{1})\frac{V_x}{V_{xx}} - \mathcal{S}V_{xs}\frac{1}{V_{xx}}. \tag{29}$$

The system (14) obtained, once we have substituted this value for $\widehat{\mathbf{\Lambda}}$, is

$$\begin{aligned}&V_t + \mathbf{b}^\top \mathcal{S}V_s + (r-k)xV_x + \frac{1}{2}\operatorname{tr}\{\mathcal{S}^{\beta+1}\sigma\sigma^\top \mathcal{S}^{\beta+1}V_{ss}\} - \frac{1}{2}(\mathbf{b}-r\mathbf{1})^\top(\mathcal{S}^\beta\sigma\sigma^\top\mathcal{S}^\beta)^{-1}(\mathbf{b}-r\mathbf{1})\frac{V_x^2}{V_{xx}} \\ &- (\mathbf{b}-r\mathbf{1})^\top \mathcal{S}V_{xs}\frac{V_x}{V_{xx}} - \frac{1}{2}V_{xs}^\top \mathcal{S}^{\beta+1}\sigma\sigma^\top\mathcal{S}^{\beta+1}V_{xs}\frac{1}{V_{xx}} = 0,\end{aligned} \tag{30}$$

with the final condition $V(T,x,s)=\alpha x^2$. The structure of this equation suggests the solution $V(t,x,s)=g(t,s)x^2$. Imposing the solution in (30), we obtain the system of PDEs (17) for the function g. Substituting in (29) we obtain (15). Finally, applying a verification Theorem in Fleming and Soner (2006), Ch. 3, Th. 8.1, we deduce that V is the value function and $\mathbf{\Lambda}^*$, given in (15), is the optimal control. □

Proof of Theorem 3.2. It is analogous to the proof of Theorem 3.1, but here $V(t,x,s)=g(t,s)x^{1-\gamma}$ and $V(T,x,s)=\frac{x^{1-\gamma}}{1-\gamma}$. □

Proof of Proposition 3.1. We apply Theorem 3.1. First we write equation (17) more explicitly

$$g_t+\sum_{i=1}^{n}(2r-b_i)s_ig_{s_i}+2(r-k)g+\frac{1}{2}\sum_{i=1}^{n}\sum_{j=1,j\neq i}^{n}c_{ij}s_j^{\beta+1}s_i^{\beta+1}g_{s_is_j}+\frac{1}{2}\sum_{i=1}^{n}c_{ii}s_i^{2\beta+2}g_{s_is_i}$$
$$-\sum_{i=1}^{n}\sum_{j=1}^{n}(b_i-r)s_i^{-\beta}\psi_{ij}s_j^{-\beta}(b_j-r)g-\sum_{i=1}^{n}\sum_{j=1,j\neq i}^{n}c_{ij}s_i^{\beta+1}s_j^{\beta+1}g_{s_i}g_{s_j}\frac{1}{g}-\sum_{i=1}^{n}s_i^{2\beta+2}c_{ii}g_{s_i}^2\frac{1}{g}=0, \tag{31}$$

where $c_{ij}=(\sigma\sigma^\top)_{ij}=\sum_{k=1}^{n}\sigma_{ik}\sigma_{jk}$ and $\psi_{ij}=((\sigma\sigma^\top)^{-1})_{ij}$, for all $i,j=1,\ldots,n$. Next we consider the transformation and change of variables: $g(t,s)=1/f(t,y)$, where $y_i=s_i^{-2\beta}$, for all $i=1,\ldots,n$. In terms of f, the system of PDEs (31) can be simplified to

$$f_t-2\beta\sum_{i=1}^{n}(2r-b_i)y_if_{y_i}+2(k-r)f+2\beta^2\sum_{i=1}^{n}\sum_{j=1,j\neq i}^{n}c_{ij}\sqrt{y_iy_j}(-2f_{y_i}f_{y_j}+f_{y_iy_j})$$
$$+\beta\sum_{i=1}^{n}c_{ii}((2\beta+1)f_{y_i}+2\beta y_if_{y_iy_i})+\sum_{i=1}^{n}\sum_{j=1}^{n}(b_i-r)\sqrt{y_iy_j}(b_j-r)f\psi_{ij} \tag{32}$$
$$+4\beta^2\sum_{i=1}^{n}\sum_{j=1,j\neq i}^{n}c_{ij}\sqrt{y_iy_j}f_{y_i}f_{y_j}\frac{1}{f^2}=0,$$

with the final condition $f(T,y)=1/\alpha$, for all y.

Under Assumption 3, we have $c_{ij}=0$, for $i\neq j$, $c_{ii}=\sigma_i^2$, $\psi_{ij}=0$, for $i\neq j$ and $\psi_{ii}=\sigma_i^{-2}$. This leads (32) to the linear system:

$$f_t+\Big(\sum_{i=1}^{n}\theta_i^2y_i+2(k-r)\Big)f+\beta\sum_{i=1}^{n}((2\beta+1)\sigma_i^2+2(b_i-2r)y_i)f_{y_i}+2\beta^2\sum_{i=1}^{n}\sigma_i^2y_if_{y_iy_i}=0, \tag{33}$$

with the final condition $f(T,y)=1/\alpha$, for all y. Inserting in (33) the function

$$f(t,y_1,\ldots,y_n)=A(t)e^{\sum_{i=1}^{n}B_i(t)y_i}$$

we then get

$$\dot{B}_i(t) + \theta_i^2 + 2\beta(b_i - 2r)B_i(t) + 2\beta^2\sigma_i^2 B_i(t)^2 = 0, \quad B_i(T) = 0, \quad i = 1, 2, \ldots, n, \tag{34}$$

$$\frac{\dot{A}(t)}{A(t)} + 2(k - r) + \beta(2\beta + 1)\sum_{i=1}^{n} \sigma_i^2 B_i(t) = 0, \quad A(T) = 1/\alpha. \tag{35}$$

Taking into account $g = f^{-1}$ and $g_{s_i} = 2\beta s_i^{-2\beta-1} f^{-2} f_{y_i} = 2\beta s_i^{-2\beta-1} B_i f^{-1}$, from (15), the optimal investments in terms of A, B_i are

$$\lambda_i^* = -\left(\frac{\theta_i}{\sigma_i} + 2\beta B_i(t)\right) s_i^{-2\beta} x, \quad i = 1, \ldots, n,$$

which leads to (23).

Next we explicitly obtain function $B_i(t)$, for a fixed index $i \in \{1, \ldots, n\}$. The differential Riccati equation (34) can be rewritten as

$$\dot{B}_i(t) = p_i B_i(t)^2 + q_i B_i(t) + r_i, \quad B_i(T) = 0,$$

where $p_i = -2\beta^2\sigma_i^2$, $q_i = -2\beta(b_i - 2r)$ and $r_i = -\theta_i^2$, $i = 1, \ldots, n$. The discriminant of the equation $p_i m^2 + q_i m + r_i = 0$ is $\Delta_i = q_i^2 - 4p_i r_i = 4\beta^2(2r^2 - b_i^2)$. Integrating with respect to time t, we obtain

$$\int \frac{1}{p_i B_i(t)^2 + q_i B_i(t) + r_i} dB_i(t) = t + K_i,$$

where K_i is a constant. Three cases appear, depending on the sign of Δ_i: $b_i > \sqrt{2}r$, $b_i = \sqrt{2}r$ and $b_i < \sqrt{2}r$.

Case $b_i > \sqrt{2}r$. This integral is equal to

$$\begin{aligned} t + K_i = \int \frac{1}{p_i B_i(t)^2 + q_i B_i(t) + r_i} dB_i(t) &= \frac{1}{p_i\left(\frac{r_i}{p_i} - \frac{q_i^2}{4p_i^2}\right)} \int \frac{1}{\left(\frac{B_i(t) + \frac{q_i}{2p_i}}{\sqrt{\frac{r_i}{p_i} - \frac{q_i^2}{4p_i^2}}}\right)^2 + 1} dB_i(t) \\ &= \frac{1}{p_i\sqrt{\frac{r_i}{p_i} - \frac{q_i^2}{4p_i^2}}} \arctan\left(\frac{B_i(t) + \frac{q_i}{2p_i}}{\sqrt{\frac{r_i}{p_i} - \frac{q_i^2}{4p_i^2}}}\right), \end{aligned} \tag{36}$$

where constant K_i is determined by condition $B_i(T) = 0$. Note that the expressions are well defined because the discriminant Δ_i is negative, by hypothesis. We obtain

$$K_i = \frac{\arctan\left(\frac{\frac{q_i}{2p_i}}{\sqrt{\frac{r_i}{p_i} - \frac{q_i^2}{4p_i^2}}}\right)}{p_i\sqrt{\frac{r_i}{p_i} - \frac{q_i^2}{4p_i^2}}} - T,$$

and then, from (36),

$$B_i(t) = \sqrt{\frac{r_i}{p_i} - \frac{q_i^2}{4p_i^2}}\tan\left(-p_i\sqrt{\frac{r_i}{p_i} - \frac{q_i^2}{4p_i^2}}(T-t) + \arctan\left(\frac{\frac{q_i}{2p_i}}{\sqrt{\frac{r_i}{p_i} - \frac{q_i^2}{4p_i^2}}}\right)\right) - \frac{q_i}{2p_i},$$

which, in terms of the parameters of the model, is (24).

Case $b_i = \sqrt{2}r$. This integral is equal to

$$\begin{aligned} t + K_i' = \int \frac{1}{p_i B_i(t)^2 + q_i B_i(t) + r_i} dB_i(t) &= \frac{1}{p_i}\int \frac{1}{(B_i(t) - m)^2} dB_i(t) \\ &= \frac{-1}{p_i}\frac{1}{B_i(t) - m}, \end{aligned} \tag{37}$$

where $m = -\frac{q_i}{p_i}$ and constant K_i' is determined by condition $B_i(T) = 0$. We obtain

$$K_i' = \frac{-2}{q_i} - T,$$

and then, from (37)

$$B_i(t) = \frac{-\frac{q_i^2}{2}(T-t)}{p_i q_i (T-t) + 2p_i},$$

which, in terms of the parameters of the model, is (25) because $b_i = \sqrt{2}r$.

Case $b_i < \sqrt{2}r$. This integral is equal to

$$\begin{aligned} t + K_i'' &= \int \frac{1}{p_i B_i(t)^2 + q_i B_i(t) + r_i} dB_i(t) \\ &= \frac{1}{p_i(m^+ - m^-)}\int \left(\frac{1}{B_i(t) - m^+} - \frac{1}{B_i(t) - m^-}\right) dB_i(t) \\ &= \ln\left(\left(\frac{B_i(t) - m^+}{B_i(t) - m^-}\right)^{\frac{1}{p_i(m^+ - m^-)}}\right), \end{aligned} \tag{38}$$

where $m_-^+ = \frac{-q_i {}_-^+ \sqrt{\Delta_i}}{2p_i}$ and the constant K_i'' is determined by condition $B_i(T) = 0$. We obtain

$$K_i'' = \ln\left(\left(\frac{m^+}{m^-}\right)^{\frac{1}{p_i(m^+ - m^-)}}\right) - T,$$

and then, from (38),

$$B_i(t) = \frac{m^- m^+ \left(1 - e^{-p_i(m^+ - m^-)(T-t)}\right))}{m^- - m^+ e^{-p_i(m^+ - m^-)(T-t)}},$$

which, in terms of the parameters of the model, is (26), (27).

Finally, we check that $V_{xx} > 0$. As $V_{xx}(t,x,s) = 2A(t)^{-1}\exp\{\sum_{i=1}^n B_i(t)s_i^{-2\beta}\} = 2\exp\{...\}K$, where K is a constant satisfying $A(T) = \exp\{...\}K = 1/\alpha$. See below. $V_{xx} > 0$ iff $K > 0$. This is true because $1/\alpha > 0$. □

Obtaining the function $A(t)$. In order to determine the value function, we need to obtain the function $A(t)$. The differential equation (35) can be rewritten as

$$\frac{\dot{A(t)}}{A(t)} = \frac{d}{dt}\ln A(t) = m + \sum_{i=1}^n d_i B_i(t), \quad A(T) = 1/\alpha,$$

where $m = 2(r-k)$ and $d_i = -\beta(2\beta+1)\sigma_i^2$, $i = 1,\ldots,n$. The solution is

$$A(t) = \exp\left\{mt + \sum_{i=1}^n d_i\phi_i(t)\right\}K,$$

where $\phi_i(t)$ is a primitive function of $B_i(t)$, that is to say $\phi_i'(t) = B_i(t)$, and K is a constant determined by condition $A(T) = 1/\alpha$. Thus $K = \frac{1}{\alpha}e^{-mT-\sum_{i=1}^n d_i\phi_i(T)}$. Each term $\phi_i(t)$, $i = 1,\ldots,n$, can be explicitly obtained, but dependent on each case.

$$\begin{aligned}\phi_i(t) &= \frac{1}{-p_i}\ln\left(\cos\left(-p_i\sqrt{\frac{r_i}{p_i} - \frac{q_i^2}{4p_i^2}}(T-t) + \arctan\left(\frac{\frac{q_i}{2p_i}}{\sqrt{\frac{r_i}{p_i} - \frac{q_i^2}{4p_i^2}}}\right)\right)\right) - \frac{q_i}{2p_i}t \\ &= \frac{1}{2\beta^2\sigma_i^2}\ln\left(\cos\left(\beta\sqrt{b_i^2 - 2r^2}(T-t) + \arctan\left(\frac{b_i^2 - 2r}{b_i^2 - 2r^2}\right)\right)\right) - \frac{b_i - 2r}{2\beta\sigma_i^2}t,\end{aligned}$$

when $b_i > \sqrt{2}r$,

$$\begin{aligned}\phi_i(t) &= \frac{(2-\sqrt{2})^2 r^2 (T-t)}{2\sigma_i^2\left(\beta(2-\sqrt{2})r(T-t) + 1\right)} \\ &= \frac{1}{2\sigma_i^2\beta^2}\left(\ln\left((2-\sqrt{2})r(T-t) + \frac{1}{\beta}\right) + (2-\sqrt{2})r\beta t\right),\end{aligned}$$

when $b_i = \sqrt{2}r$, and

$$\begin{aligned}\phi_i(t) &= \frac{m^- m^+ \left(1 - e^{-2\beta\sqrt{2r^2-b_i^2}(T-t)}\right))}{m^- - m^+ e^{-2\beta\sqrt{2r^2-b_i^2}(T-t)}} \\ &= \frac{m^- - m^+}{2\beta\sqrt{2r^2-b_i^2}} \ln\left(m^- e^{2\beta\sqrt{2r^2-b_i^2}T} - m^+ e^{2\beta\sqrt{2r^2-b_i^2}t}\right) + m^+ t,\end{aligned}$$

when $b_i < \sqrt{2}r$. □

Proof of Proposition 3.2. We apply Theorem 3.2. First we write equation (20) more explicitly

$$\begin{aligned}&g_t + \frac{1}{\gamma}\sum_{i=1}^n (b_i - (1-\gamma)r)s_i g_{s_i} + (1-\gamma)(r-k)g + \frac{1}{2}\sum_{i=1}^n \sum_{j=1, j\neq i}^n c_{ij} s_j^{\beta+1} s_i^{\beta+1} g_{s_i s_j} \\ &+ \frac{1}{2}\sum_{i=1}^n c_{ii} s_i^{2\beta+2} g_{s_i s_i} + \frac{1}{2}\frac{1-\gamma}{\gamma}\sum_{i=1}^n \sum_{j=1}^n (b_i - r)s_i^{-\beta}\psi_{ij} s_j^{-\beta}(b_j - r)g \\ &+ \frac{1}{2}\frac{1-\gamma}{\gamma}\sum_{i=1}^n \sum_{j=1, j\neq i}^n c_{ij} s_i^{\beta+1} s_j^{\beta+1} g_{s_i} g_{s_j}\frac{1}{g} + \frac{1}{2}\frac{1-\gamma}{\gamma}\sum_{i=1}^n s_i^{2\beta+2} c_{ii} g_{s_i}^2 \frac{1}{g} = 0,\end{aligned} \tag{39}$$

where $c_{ij} = (\sigma\sigma^\top)_{ij} = \sum_{k=1}^n \sigma_{ik}\sigma_{jk}$ and $\psi_{ij} = ((\sigma\sigma^\top)^{-1})_{ij}$, for all $i, j = 1, \ldots, n$. Next, we consider the transformation and change of variables: $g(t,s) = 1/f(t,y)$, where $y_i = s_i^{-2\beta}$, for all $i = 1, \ldots, n$. In terms of f, under Assumption 3, the system of PDEs (39) is

$$\begin{aligned}&f_t - (1-\gamma)\Big(\frac{1}{2\gamma}\sum_{i=1}^n \theta_i^2 y_i + r - k\Big)f + \beta\sum_{i=1}^n ((2\beta+1)\sigma_i^2 - \frac{2}{\gamma}(b_i - (1-\gamma)r)y_i)f_{y_i} \\ &+ 2\beta^2\sum_{i=1}^n \sigma_i^2 y_i f_{y_i y_i} - 4\beta^2\left(1 + \frac{1}{2}\frac{1-\gamma}{\gamma}\right)f^{-1}\left(\sum_{i=1}^n \sigma_i^2 y_i f_{y_i}^2\right) = 0,\end{aligned} \tag{40}$$

with the final condition $f(T,y) = 1-\gamma$, for all y. Though this system is non-linear, we once more insert the function $f(t, y_1, \ldots, y_n) = A(t)e^{\sum_{i=1}^n B_i(t)y_i}$ in (40) and then we get

$$\dot{B}_i(t) - \frac{1-\gamma}{2\gamma}\theta_i^2 - 2\frac{\beta}{\gamma}(b_i - (1-\gamma)r)B_i(t) - 2\frac{\beta^2}{\gamma}\sigma_i^2 B_i(t)^2 = 0, \quad B_i(T) = 0, \tag{41}$$

$$\frac{\dot{A}(t)}{A(t)} + (1-\gamma)(k-r) + \beta(2\beta+1)\sum_{i=1}^n \sigma_i^2 B_i(t) = 0, \quad A(T) = 1-\gamma. \tag{42}$$

for $i = 1, 2, \ldots, n$. Taking into account $g = f^{-1}$ and $g_{s_i} = 2\beta s_i^{-2\beta-1} f^{-2} f_{y_i} = 2\beta s_i^{-2\beta-1} B_i f^{-1}$, from (19), the optimal investments in terms of A, B_i are

$$\lambda_i^* = \frac{1}{\gamma}\left(\frac{\theta_i}{\sigma_i} + 2\beta B_i(t)\right)s_i^{-2\beta} x, \quad i = 1, \ldots, n,$$

which leads to (28).

Next, we explicitly obtain function $B_i(t)$, for a fixed index $i \in \{1, \ldots, n\}$. The differential Riccati equation (41) can be rewritten as

$$\dot{B}_i(t) = p_i B_i(t)^2 + q_i B_i(t) + r_i, \quad B_i(T) = 0,$$

where $p_i = 2\frac{\beta^2}{\gamma}\sigma_i^2$, $q_i = 2\frac{\beta}{\gamma}(b_i - (1-\gamma)r)$ and $r_i = \frac{1-\gamma}{2\gamma}\theta_i^2$, $i = 1, \ldots, n$. The discriminant of the equation $p_i m^2 + q_i m + r_i = 0$ is $\Delta_i = q_i^2 - 4p_i r_i = 4\frac{\beta^2}{\gamma}(b_i^2 - (1-\gamma)r^2)$. Now, the three cases that appear, depending on the sign of Δ_i, are: $b_i^2 < (1-\gamma)r^2$ (negative), $b_i^2 = (1-\gamma)r^2$ (zero) and $b_i^2 > (1-\gamma)r^2$ (positive). The rest of the development is identical to the proof of Proposition 3.1. □

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