On a class of time dependent quantum Dirac delta potentials

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On a class of time dependent quantum Dirac delta potentials

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Abstract

In this work we attack analytically a class of one dimensional time-dependent potentials based on a $\delta(x)$ potential whose intensity varies on time as $v(t)$, that is, $V(x,t) = v(t)\,\delta(x)$. Although this model in principle may appear too simple, a kind of toy model, it turns out to be very difficult to handle without any kind of approximation. To make progress, apart from the local potential $V(x,t)$ already mentioned, a type of non-local potentials are considered. To clarify the general results, various examples, with different initial conditions, are studied in detail.

Resumen

En este trabajo se aborda de manera analítica un tipo de potenciales unidimensionales independientes del tiempo basados en potenciales tipo $\delta(x)$, cuyas intensidades varían con el tiempo como $v(t)$, de modo que $V(x,t) = v(t)\,\delta(x)$. Aunque puedan parecer modelos de juguete demasiados sencillos, realmente resultan muy complicados de analizar de forma exacta. Para avanzar en el conocimiento de los problemas dependientes del tiempo, además de estudiar los potenciales locales $V(x,t)$ se consideran un tipo especial de potenciales no locales, analizando varios ejemplos con diferentes condiciones iniciales.

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Contents

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I. INTRODUCTION

The one-dimensional Schrödinger equation, whose expression is

$$-\frac{\hbar^2}{2m}\frac{\partial^2\phi(x,t)}{\partial x^2}+V(x,t)\phi(x,t)=i\hbar\frac{\partial\phi(x,t)}{\partial t}, \tag{1}$$

has been studied by many researchers and a lot of applications have been found. From the mathematical point of view, it is a simple linear partial differential equation of second order in two variables, similar to the heat equation, a fact which has interesting consequences. The Schrödinger equation can predict the behaviour of quantum particles, whose properties are peculiar, because they can not be described by a classic way. However, implementation of time-dependent potentials is problematic: calculations become very difficult, and performing an exact resolution or obtaining analytical solutions are rather troublesome enterprises.

The time evolution of the wave function satisfying (1), where the non relativistic Hermitian Hamiltonian $H=\frac{p^2}{2m}+V(x,t)$ is implicit, is given by the so-called evolution operator $U(t,t_0)$ [1], such that

$$\phi(x,t)=U(t,t_0)\,\phi(x,t_0), \tag{2}$$

where $\phi(x,t_0)$ is the wave function at the initial instant t_0. This evolution operator is unitary, so the norm of the wave function is conserved when the system evolves. It also verifies three interesting properties:

$$\begin{aligned} U(t_0,t_0) &= I, & (3)\\ U(t_3,t_1) &= U(t_3,t_2)U(t_2,t_1), & (4)\\ U(t,t_0) &= U^{-1}(t_0,t)=U^{\dagger}(t_0,t). & (5)\end{aligned}$$

The general equation for obtaining the evolution operator is

$$U(t,t_0)=I-\frac{i}{\hbar}\int_{t_0}^{t}dt'H(t')U(t',t_0) \tag{6}$$

In the case of a time independent Hamiltonian H, it commutes with the evolution operator and (6) can be integrated easily. The form of the evolution operator is very simple:

$$U(t,t_0)=e^{-i(t-t_0)H/\hbar}. \tag{7}$$

However, obtaining the evolution operator of a time-dependent Hamiltonian is not so easy, and the time evolution of the wave function is not trivial. This happens because, in general, the time-dependent Hamiltonian does not commute with the evolution operator, and (6) is difficult to integrate. This non commutativity is reflected in the Baker-Campbell-Hausdorff (BCH) formula, which says that the product of the exponentials of two operators A and B is given by

$$e^Ae^B=e^{\eta(A,b)}, \tag{8}$$

where

$$\eta(A,b)=\sum_{m\geq1}\frac{(-1)^{m-1}}{m}\sum_{\substack{p_i,q_i\\ p_i+q_i\geq1}}\frac{\left[A^{p_1}B^{q_1}A^{p_2}B^{q_2}...A^{p_m}B^{q_m}\right]}{\left(\sum_j(p_j+q_j)\right)p_1!q_1!p_2!q_2!...p_m!q_m!}, \tag{9}$$

being the expression inside the last sum the following

$$[CDE...J]=[...\left[[C,D],E\right],...,J]\,. \tag{10}$$

It is interesting to show that the first terms of (9) are:

$$\eta(A,b)=A+B+\frac{1}{2}[A,B]+\frac{1}{12}\left[[A,B],B\right]+\frac{1}{12}\left[[B,A],A\right]+\cdots\,. \tag{11}$$

If the operators A and B are such that $[A, B] = C$, and $[C, A] = [C, B] = 0$, then the BCH formula simplifies to

$$e^A e^B = e^{A+B+[A,B]/2}. \tag{12}$$

For obtaining the evolution operator with time-dependent Hamiltonians, we can use Neumann's iterative method in (6):

$$U(t, t_0) = I + \sum_{n=1}^{\infty} \left(-\frac{i}{\hbar}\right)^n \int_{t_0}^{t} dt_1 H(t_1) \int_{t_0}^{t_1} dt_2 H(t_2) \int_{t_0}^{t_2} dt_3 H(t_3) ... \int_{t_0}^{t_{n-1}} dt_n H(t_n). \tag{13}$$

If the system is conservative, we can recover (7) from (13).

The usual method for obtaining the time evolution of a wave function is the perturbation method [2]. Here, we start with a time independent Hamiltonian H_0, so its eigenstates $\{|\varphi_n\rangle\}$ in ket notations are known and stationary. Then, at $t = 0$ we apply a perturbation to the system, and the Hamiltonian becomes

$$H(t) = H_0 + \lambda W(t), \tag{14}$$

where λ is a real dimensionless parameter much smaller than 1 and $W(t)$ is an observable of the same order of magnitude as H_0, and null for $t < 0$. We can express the final wave function in the eigenstates basis as

$$|\phi(t)\rangle = \sum_n c_n(t) |\varphi_n\rangle, \tag{15}$$

where $c_n(t) = \langle \varphi_n | \phi(t) \rangle$, and where we have omitted the spatial dependence.

If $\lambda W(t) \neq 0$, then we can write

$$c_n(t) = b_n(t)\, e^{-iE_n t/\hbar}, \tag{16}$$

where E_n are the elements of the diagonal matrix $\langle \varphi_n | H_0 | \varphi_k \rangle$. If we write the coefficients $b_n(t)$ as a power series expansion in λ

$$b_n(t) = b_n^{(0)}(t) + \lambda\, b_n^{(1)}(t) + \lambda^2\, b_n^{(2)}(t) + \cdots, \tag{17}$$

the first terms can be explicitly obtained as

$$b_n^{(0)}(t) = \delta_{ni}, \qquad b_n^{(1)}(t) = \frac{1}{i\hbar} \int_0^t e^{i\omega_{ni} t'}\, W_{ni}(t') dt', \tag{18}$$

where ω_{nk} are the Bohr angular frequencies $\omega_{nk} = (E_n - E_k)/\hbar$, and $W_{nk}(t)$ are the matrix elements of $W(t)$: $W_{nk}(t) = \langle \varphi_n | W(t) | \varphi_k \rangle$. The index i is associated to the initial state $\varphi_i(x)$ we started from. It is also possible to calculate the transition probability between the initial state φ_i and final one φ_f as follows:

$$P_{i\to f}(t) = \frac{1}{\hbar^2} \left| \int_0^t e^{i\omega_{fi} t'} W_{fi}(t')\, dt' \right|^2. \tag{19}$$

It is important to highlight that these solutions are approximate. In general, obtaining the exact solution is not possible with this method.

Nowadays, one of the fields of study is using a non dependent-time potential, but with movable boundary conditions. In [3] and [4], two different analytical procedures of these problems can be observed. Both go for the study of the infinite unidimensional square well without potential. The temporal dependence is caused by the movement of one of the boundaries. In [3], Schrödinger equation is solved by differential equation theory. Thanks to it, solutions for movable barriers with different time dependence can be obtained. In [4], supersymmetric (SUSY) transformations are used for transforming the system to another with a Pöschl-Teller potential, which is solvable with special functions.

The most common investigations of the Schrödinger equation with time-dependent potentials are based on the use of a spatial dependence with the form of a Dirac delta function, accompanied by a temporal coefficient. This type of potentials ease the resolution of the equation, on account of the special properties of the distribution. Besides, this potential can describe many physical phenomenons. Some examples are the scattering of massless particles by time-dependent oscillating barriers [5]; the transmission of particles through an oscillating barrier [6]; or the description of a Bose-Einstein condensate in an optic trap forming a unidimensional box, with a laser acting as the Dirac delta [7]. This last problem complicates the resolution, and a numerical analysis is required.

This type of problems can also include some changes in the spatial dependence. For example, the analysis of the unidimensional semiclassical transfer problem of two nucleons approaching with uniform speed [8], which uses two Dirac deltas; or the study of the propagators associated to potentials with a finite number of Dirac deltas or their derivatives [9].

In general, obtaining analytical solutions is complex. Therefore, researchers have developed several methods for numerically solving the equation. These methods obtain approximate solutions, but are useful when just numerical results are required. Some of these methods demand the spatial discretization of the equation, and use diverse techniques, like sympletic splitting methods for simplifying the numerical integration [10], or the Magnus expansion, developed in the context of the Lie algebra [11].

It is also interesting to mention the existence in the literature of the so-called nonlocal potentials [12]. In this case, the interaction that a particle feels in each point depends on all the values of the potential whenever it is defined. This causes the expression of the Schrödinger equation to be modified as follows

$$-\frac{\hbar^2}{2m}\frac{\partial^2\phi(x,t)}{\partial x^2}+V_0\,\omega(x,t)\int_{-\infty}^{\infty}\omega^*(u,t)\phi(u,t)du=i\hbar\frac{\partial\phi(x,t)}{\partial t}, \tag{20}$$

where V_0 indicates the strength of the nonlocal potential, and $\omega(x,t)$ its shape. In particular, if we choose

$$V_0\,\omega(x,t)\omega^*(u,t)=V(x,t)\,\delta(x-u), \tag{21}$$

then equation (20) turns into (1), and the non-local potential becomes local.

Using nonlocal potentials implies an operational change in the analysis of the Schrödinger equation. However, the results obtained must be comparable to the results obtained with their corresponding local potential, if it exists. Binding energies, radial probability densities [13], and scattering properties [14] must be similar in the case of equivalent local and nonlocal potentials.

Usually, research focus on the use of separable potentials, since calculus are simplified. Still, it is common to apply techniques of mathematical analysis [15] or the use of special functions, like Green functions [12], which appear frequently at the solutions. This type of potentials are appropriate for describing the interaction between two or more particles, leading to results more precise than its local analogous. Therefore, they can be used at the calculus of energy levels [16] as much as the study of the scattering of particles [17]. An example of nonlocal potential very common is the Yamaguchi's potential [18], which in momentum space has the form

$$V(p,p')=-k\,g(p)g(p'), \tag{22}$$

where k is a constant and $g(p)$ is a form factor. This potential is used in the analysis of the interactions of the triplet at neutron-proton systems [20], or even for studying very large nucleus [19].

The present work is divided in two fundamental parts. In the first, the form of the wave functions corresponding to the unidimensional Schrödinger equation with time dependent potentials is obtained. With the objective of simplifying the calculations, the Fourier's transform is used for working in momentum space. Afterwards, the inverse Fourier's transform is utilized for returning to the position space. The procedure is illustrated with several examples, and the square of the norm of the wave function is represented graphically. In the second part we work with nonlocal potentials, using Fourier's transforms again. The problem is solved numerically, and graphics of the solutions are obtained.

II. THE LOCAL POTENTIAL PROBLEM

Let us consider a one dimensional quantum system with a potential given by a Dirac delta placed at the origin, $\delta(x)$, which has a time dependent amplitude

$$V(x,t)=v(t)\,\delta(x), \tag{23}$$

where $v(t)$ is arbitrary. The corresponding time-dependent Schrödinger equation (1) is

$$-\frac{\hbar^2}{2m}\frac{\partial^2\phi(x,t)}{\partial x^2}+v(t)\,\delta(x)\,\phi(x,t)=i\hbar\,\frac{\partial\phi(x,t)}{\partial t} \tag{24}$$

which can be simplified by means of the following changes

$$\tilde{x}=\frac{1}{b}\,x,\quad \tilde{t}=\frac{\hbar}{2mb^2}\,t,\quad \psi(\tilde{x},\tilde{t}\,)\equiv\phi(x,t),\quad g(\tilde{t}\,)=\frac{2mb}{\hbar^2}\,v(t), \tag{25}$$

where for the time being $b>0$ is a free parameter. For the sake of simplicity, in the sequel we will omit the tilde on top of the variables, that is: $\tilde{x}\to x$, $\tilde{t}\to t$, $\psi(\tilde{x},\tilde{t}\,)\to\psi(x,t)$. We then get

$$-\frac{\partial^2\psi(x,t)}{\partial x^2}+g(t)\,\delta(x)\,\psi(x,t)=i\,\frac{\partial\psi(x,t)}{\partial t}. \tag{26}$$

In order to solve (26), we will take its Fourier transform with respect to the spatial variable x, taking into account that

$$\widetilde{\psi}(k,t)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\psi(x,t)\,e^{-ikx}\,dx,\qquad \psi(x,t)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\widetilde{\psi}(k,t)\,e^{+ikx}\,dk. \tag{27}$$

Hence, (26) becomes

$$k^2\,\widetilde{\psi}(k,t)+\frac{1}{\sqrt{2\pi}}\,g(t)\,\psi(0,t)=i\,\frac{\partial\widetilde{\psi}(k,t)}{\partial t}. \tag{28}$$

This is a rather simple non homogeneous linear equation that can be solved as follows: the solution of the homogeneous part is

$$\widetilde{\psi}(k,t)=C(k)\,e^{-ik^2t}, \tag{29}$$

and applying variation of parameters we get

$$\frac{\partial C(k,t)}{\partial t}=-i\frac{1}{\sqrt{2\pi}}\,g(t)\,\psi(0,t)\,e^{ik^2t}. \tag{30}$$

Therefore,

$$C(k,t)=C_0(k)-\frac{i}{\sqrt{2\pi}}\int_0^t g(t')\,\psi(0,t')\,e^{ik^2t'}\,dt', \tag{31}$$

and from (29)

$$\widetilde{\psi}(k,t)=C_0(k)\,e^{-ik^2t}-\frac{i}{\sqrt{2\pi}}\int_0^t g(t')\,\psi(0,t')\;e^{-ik^2(t-t')}\,dt'. \tag{32}$$

If we take $t=0$, we have $C_0(k)=\widetilde{\psi}(k,0)$, the Fourier transform of $\psi(y,0)$. Hence, from (27)

$$\widetilde{\psi}(k,t)=\frac{1}{\sqrt{2\pi}}\,e^{-ik^2t}\int_{-\infty}^{\infty}\psi(x',0)\,e^{-ikx'}\,dx'-\frac{i}{\sqrt{2\pi}}\int_0^t g(t')\,\psi(0,t')\;e^{-ik^2(t-t')}\,dt'. \tag{33}$$

This is the formal solution for the Fourier transform of the wave function, which obviously depends (i) on an initial condition $\psi(x,0)$, and (ii) on a boundary condition at the origin $\psi(0,t)$. We can work out a little bit more the previous result, taking its inverse Fourier transform:

$$\psi(x,t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}dx'\psi(x',0)\int_{-\infty}^{\infty}e^{ik(x-x')-ik^2t}\,dk-\frac{i}{2\pi}\int_0^t dt'g(t')\psi(0,t')\int_{-\infty}^{\infty}e^{ikx-ik^2(t-t')}dk. \tag{34}$$

The integrals appearing in square brackets in the last equation are known, indeed

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ik\alpha-ik^2\beta}\,dk = \frac{1-i\,\mathrm{sign}(\beta)}{\sqrt{8\pi|\beta|}}\,e^{i\alpha^2/(4\beta)} := K(\alpha,\beta), \tag{35}$$

being sign$(\cdot)$ the sign function. Remark that from here

$$\lim_{\beta\to 0} K(\alpha,\beta) = \lim_{\beta\to 0}\frac{1-i\,\mathrm{sign}(\beta)}{\sqrt{8\pi|\beta|}}\,e^{i\alpha^2/(4\beta)} = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ik\alpha}\,dk = \delta(\alpha). \tag{36}$$

We then have in (34)

$$\psi(x,t) = \int_{-\infty}^{\infty}\psi(x',0)\,K(x-x',t)\,dx' - i\int_0^t g(t_1)\,\psi(0,t_1)\,K(x,t-t_1)\,dt_1, \tag{37}$$

and the limit $t\to 0$ in (37) gives us the correct result $\psi(x,0)$.

Remark that the first term on the right hand side (RHS) of equation (37) gives us the evolution of the initial condition without any reference to the time dependent potential $g(t)\delta(x)$: it corresponds to the free evolution of the system. On the other hand, the second term on the RHS can be interpreted as the modifications that the time dependent potential induces on the free evolution of the system.

In a natural form the kernel $K(\alpha,\beta)$ introduced in (35) appears in the result (37), which we need to work out more, because the first term already depends on the initial condition $\psi(x,0)$, but the second term includes the term $\psi(0,t)$, which can not be freely chosen. Therefore, some consistency condition must be imposed in the sequel. We can also observe that this kernel $K(\alpha,\beta)$ closely resembles the well known heat kernel or Green function for the heat equation

$$K_H(x,y,t) = \frac{1}{\sqrt{4\pi\,t}}\,e^{-(x-y)^2/4t}. \tag{38}$$

If we are interested only in positive times, then $t>0$ and (37) simplifies to

$$\begin{aligned}\psi(x,t) \;&=\; \frac{1-i}{\sqrt{8\pi t}}\int_{-\infty}^{\infty}\psi(x',0)\,\exp\left(i\,\frac{(x-x')^2}{4t}\right)dx' \\ &\quad -\frac{1+i}{\sqrt{8\pi}}\int_0^t \psi(0,t_1)\,\frac{g(t_1)}{\sqrt{t-t_1}}\,\exp\left(i\,\frac{x^2}{4(t-t_1)}\right)dt_1.\end{aligned} \tag{39}$$

In particular, from (37) and (39)

$$\psi(0,t_0) = \int_{-\infty}^{\infty} dx'\psi(x',0)\,K(x',t_0) - (1+i)\int_0^{t_0} dt_1\,\frac{g(t_1)}{\sqrt{8\pi(t_0-t_1)}}\,\psi(0,t_1). \tag{40}$$

The last expression may be used as a recurrence relation. Indeed, repeatedly inserting $\psi(0,t)$ given by (40) in the second integral of the same equation, we get

$$\begin{aligned}\psi(0,t_0) = &\int_{-\infty}^{\infty} dx'\,\psi(x',0)\,K(x',t_0) \\ &+\sum_{n=1}^{\infty}(-1)^n\left[\frac{1+i}{\sqrt{8\pi}}\right]^n\int_{-\infty}^{\infty} dx'\psi(x',0)\int_0^{t_0}\frac{g(t_1)\,dt_1}{\sqrt{t_0-t_1}}\cdots\int_0^{t_{n-1}}\frac{g(t_n)dt_n}{\sqrt{t_{n-1}-t_n}}\,K(x',t_n).\end{aligned} \tag{41}$$

This result is telling us how is the evolution on time of the wave function at the origin as a function of the initial condition $\psi(x,0)$. It can be represented as follows

$$\psi(0,t_0) = \int_{-\infty}^{\infty} dx'\,\psi(x',0)\,G(x',t_0), \tag{42}$$

being

$$G(x',t_0) = K(x',t_0) + \sum_{n=1}^{\infty}(-1)^n\left[\frac{1+i}{\sqrt{8\pi}}\right]^n\int_0^{t_0}\frac{g(t_1)\,dt_1}{\sqrt{t_0-t_1}}\cdots\int_0^{t_{n-1}}\frac{g(t_n)\,dt_n}{\sqrt{t_{n-1}-t_n}}\,K(x',t_n), \tag{43}$$

where we have a finite number of iterated integrals in each term of the series.

Using this result in (37), we can write the complete solution of the initial value problem we are studying as follows:

$$\psi(x,t) = \int_{-\infty}^{\infty} dx'\, \psi(x',0)\, W(x,x',t), \tag{44}$$

where $\psi(x,0)$ is the initial condition and the key kernel $W(x,x',t)$ is

$$W(x,x',t) = K(x-x',t) - i \int_0^t g(t_0)\, G(x',t_0)\, K(x,t-t_0)\, dt_0. \tag{45}$$

Remark that the ingredients of this kernel $W(x,x',t)$ are:

1. The coefficient $g(t)$ of the Dirac $\delta(x)$ in (26)-(25), which is a problem data.
2. The general function $K(x,t)$, already obtained in (35).
3. The kernel $G(x',t_0)$, which must be evaluated in each case using (43), because it depends on $g(t)$.

In the sequel we will apply the results previously obtained to study some examples of time-dependent potentials with a variety of initial and boundary conditions. We will use some simple functional dependences $g(t)$ to obtain the kernel $G(x',t_0)$ and then the kernel $W(x,x',t)$ using (45). As the complete form of $G(x',t_0)$ in (43) will be too complicated, in most cases we will deal only with the first order approximation

$$G(x',t_0) \approx K(x',t_0) - \left[\frac{1+i}{\sqrt{8\pi}}\right] \int_0^{t_0} \frac{g(t_1)\, K(x',t_1)}{\sqrt{t_0-t_1}}\, dt_1, \tag{46}$$

which will be valid for small times and when $|g(t)| << 1$.

In order to get explicit results, apart from the particular time-dependence on the potential we must select some appropriate initial conditions for the wave function.

A. The free particle

We will start with a trivial case, the free particle: $g(t) = 0$. The calculations are straightforwardly done and no approximation is needed:

$$G(x',t_0) = K(x',t_0) = \frac{1-i}{\sqrt{8\pi t_0}}\, e^{i(x')^2/(4t_0)}, \qquad W(x,x',t) = K(x-x',t) = \frac{1-i}{\sqrt{8\pi t}}\, e^{i(x-x')^2/(4t)}, \tag{47}$$

because we are assuming $t, t_0 > 0$.

1. Plane wave initial condition

If we take as initial condition a plane wave $\psi(x,0) = e^{ikx}$, using (44) and (47) we get, as expected,

$$\psi(x,t) = e^{i(kx-k^2 t)}. \tag{48}$$

2. Gaussian initial condition

Let us consider now as the initial condition a gaussian wave packet

$$\psi(x,0) = \left(\frac{2}{\pi}\right)^{1/4} e^{-x^2}. \tag{49}$$

Using (47) in (44) we obtain the exact time evolution of the wave packet:

$$\psi(x,t) = \left(\frac{2}{\pi}\right)^{1/4} \frac{1-i}{\sqrt{2}} \frac{\exp(\frac{ix^2}{4t-i})}{\sqrt{4t-i}}. \tag{50}$$

It is easy to check that in the limit $t \to 0$ we recover the initial condition (49). Plots of the time evolution of the probability density $|\psi(x,t)|^2$ for this initial condition are shown in FIG. 1. The initial condition (49) can be written as the inverse Fourier transform of its Fourier transform (see for example (27)):

$$\psi(x,0) = \left(\frac{2}{\pi}\right)^{1/4} e^{-x^2} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left(\frac{1}{2\pi}\right)^{1/4} e^{-k^2/4}\, e^{+ikx}\, dk, \tag{51}$$

and its time evolution is obtained as the time evolution of each Fourier component, following (48):

$$\psi(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left(\frac{1}{2\pi}\right)^{1/4} e^{-k^2/4}\, e^{+i(kx-k^2t)}\, dk, \tag{52}$$

This effect can be understood also from the particle point of view: the particle is not subject to a potential, so, as time flows the probability of finding it at any point of the space will be similar.

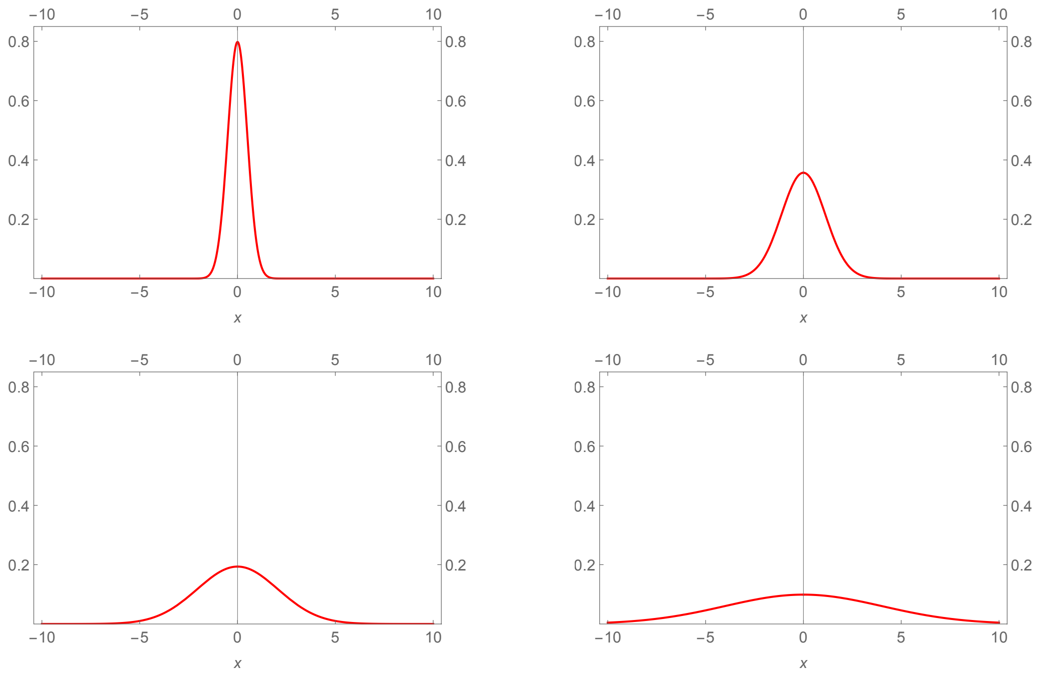

FIG. 1: Plots of the time evolution of the probability density $|\psi(x,t)|^2$ given by (50) for the gaussian initial condition (49). From left to right, and top to bottom, $t = 0$, 0.1, 0.5, and 1.

3. A spiky initial condition

Let us consider now the initial condition to be the unique bound state (with energy $E = -g^2/4$) of the time independent attractive potential (26) with $g(t) = -g$, $g > 0$. The normalized wave function is then

$$\psi(x,0) = \sqrt{\frac{g}{2}}\, e^{-g|x|/2}. \tag{53}$$

If we use again (47) in (44), we obtain the exact time evolution of this wave packet:

$$\psi(x,t) = \frac{1}{2}\sqrt{\frac{g}{2}}\, e^{ig^2 t/4}\left(e^{-gx/2} - e^{-gx/2}\operatorname{erf}\left[\frac{(1+i)(gt+ix)}{\sqrt{8t}}\right] + e^{gx/2}\operatorname{erfc}\left[\frac{(1+i)(gt-ix)}{\sqrt{8t}}\right]\right). \tag{54}$$

Taking the limit $t \to 0$ we get the initial condition (53). Plots of the time evolution of the probability density $|\psi(x,t)|^2$ for this initial condition is shown in FIG. 2.

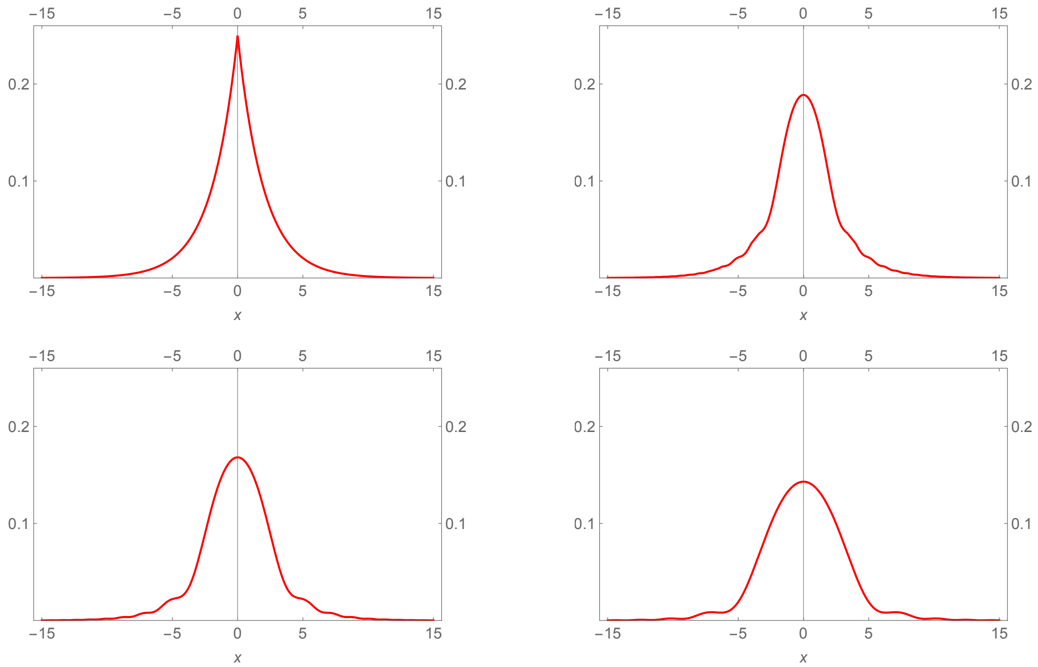

FIG. 2: Plots of the time evolution of the probability density $|\psi(x,t)|^2$ given by (54) for the initial condition (53) and $g = 1/2$. From left to right, and top to bottom, $t = 0$, 0.5, 1, and 2.

Again, the time evolution of the initial condition (53) can be understood as the time evolution of each of its Fourier components,

$$\psi(x,0) = \sqrt{\frac{g}{2}}\, e^{-g|x|/2} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \frac{2g^{3/2}}{\sqrt{\pi}}\, \frac{1}{g^2+4k^2}\, e^{+ikx}\, dk, \tag{55}$$

according to (48), and therefore we get an alternative expression for (54):

$$\psi(x,t) = \frac{\sqrt{2}\, g^{3/2}}{\pi}\int_{-\infty}^{\infty} \frac{1}{g^2+4k^2}\, e^{+i(kx-k^2 t)}\, dk. \tag{56}$$

As in the previous case, the same effect over the particle is observed.

B. Impulsive time dependence

An interesting potential may be an impulse extremely localized in space and time, which may be simulated by taking

$$g(t) = g\,\delta(t). \tag{57}$$

To obtain the solution of such a singular problem it is better to go directly to equation (30) than to the approximations. Indeed, the solution of (37) is

$$\psi(x,t) = \frac{1-i}{\sqrt{8\pi t}}\int_{-\infty}^{\infty} \psi(x',0)\, e^{i(x-x')^2/(4t)}\, dx' - ig\,\psi(0,0)\,\frac{1-i}{\sqrt{8\pi t}}\, e^{ix^2/(4t)}\, H(t), \tag{58}$$

where $H(t)$ is the Heaviside step function. Let us consider now some special initial conditions.

As a first initial condition let us consider a plane wave $\psi(x,0)=e^{ikx}$, which introduced in (58) gives

$$\psi(x,t) = e^{i(kx-k^2t)} - ig\,\frac{1-i}{\sqrt{8\pi t}}\,e^{ix^2/(4t)}\,H(t), \tag{59}$$

which seems to be a small modification on the result (48). Nevertheless, this solution is pathological, and in the limit $t\to 0^+$, using (36) gives

$$\psi(x,0^+) = e^{ikx} - ig\,\delta(x), \tag{60}$$

meaning that indeed the value of $\psi(0,0)$ is not well defined, and therefore the term $\delta(t)\,\psi(0,t)$ in (30) is meaningless. The conclusion is that the potential (57) is an ill posed problem from the mathematical point of view.

For any other initial condition, either gaussian or whatever, the same problem will appear, because the last term on the RHS of equation (58) has the same pathological behavior as the shown in (60).

C. Smooth time dependence

Let us consider now a smooth time dependent amplitude, for example of cosine form

$$g(t) = g\,\cos(\omega t). \tag{61}$$

Unfortunately, no hope of finding an exact solution exists in this case, because the formal expressions are too complicated to be found analytically, but we can use some approximations. First, we will use (46) instead of (43), and we get

$$\begin{aligned} G(x',t_0) &\approx \frac{1-i}{\sqrt{8\pi t_0}}\,e^{i(x')^2/(4t_0)} - \frac{g}{4\pi}\int_0^{t_0}\frac{\cos(\omega t_1)}{\sqrt{(t_0-t_1)t_1}}\,e^{i(x')^2/(4t_1)}\,dt_1 \\ &= \frac{1-i}{\sqrt{8\pi t_0}}\,e^{i(x')^2/(4t_0)} - \frac{g}{4\pi}\left[\int_0^{t_0}\frac{e^{i(x')^2/(4t_1)}}{\sqrt{(t_0-t_1)t_1}}\,dt_1 - \frac{\omega^2}{2!}\int_0^{t_0}\frac{t_1^2\,e^{i(x')^2/(4t_1)}}{\sqrt{(t_0-t_1)t_1}}\,dt_1 + \cdots\right]. \end{aligned} \tag{62}$$

The integrals can be done explicitly and are expressed in terms of Fresnel integrals. For the time being, we keep only the lowest order approximation (we can say that we are in the adiabatic approximation, $\omega\approx 0$):

$$G(x',t_0) = \frac{1-i}{\sqrt{8\pi t_0}}\,e^{i(x')^2/(4t_0)} - \frac{g}{4}\left[1-(1-i)\,C\!\left(\frac{|x'|}{\sqrt{2\pi\,t_0}}\right) - (1+i)\,S\!\left(\frac{|x'|}{\sqrt{2\pi\,t_0}}\right)\right], \tag{63}$$

where $S(\cdot)$ and $C(\cdot)$ are Fresnel integrals. Now, we take this approximate result in (45)

$$\begin{aligned} W(x,x',t) = {}& \frac{1-i}{\sqrt{8\pi t}}\,e^{i(x-x')^2/(4t)} - \frac{g(1+i)}{\sqrt{8\pi}}\int_0^t\frac{\cos(\omega t_0)}{\sqrt{(t-t_0)t_0}}\,e^{i(x')^2/(4t_0)}\,e^{ix^2/(4(t-t_0))}\,dt_0 \\ &+\frac{g^2(1+i)}{4\sqrt{8\pi}}\int_0^t\left\{\left[1-(1-i)\,C\!\left(\frac{|x'|}{\sqrt{2\pi\,t_0}}\right)-(1+i)\,S\!\left(\frac{|x'|}{\sqrt{2\pi\,t_0}}\right)\right]\right\}\frac{\cos(\omega t_0)}{\sqrt{t-t_0}}\,e^{ix^2/(4(t-t_0))}\,dt_0. \end{aligned} \tag{64}$$

Considering only the first order terms in g, as well as the zero order approximation in the time dependence, we have

$$W(x,x',t) \approx \frac{1-i}{\sqrt{8\pi t}}\,e^{i(x-x')^2/(4t)} - \frac{g(1+i)}{\sqrt{8\pi}}\int_0^t\frac{e^{i(x')^2/(4t_0)}\,e^{ix^2/(4(t-t_0))}}{\sqrt{(t-t_0)t_0}}\,dt_0, \tag{65}$$

an expression that we have not been able to deal with.

In case of using a different smooth time dependent amplitude, for example a decaying exponential $g(t) = ge^{-b^2t}$, the type of difficulties we are going to find are completely similar to the cosine analyzed before. The main consequence we can extract from all this work is that although the theoretical results seemed encouraging, when facing concrete examples the practical difficulties are too big to allow a reasonable approximation. New approaches to the local time dependent problems will be necessary. In the next section we turn our efforts towards non local time dependent problems.

III. THE NONLOCAL POTENTIAL PROBLEM

As we already mentioned in the Introduction, there is an alternative approach to solving some quantum problems using a nonlocal version of the potential term in the Schrödinger equation (20), which can be rewritten as follows using dimensionless variables as in (26)

$$-\psi''(x,t)+g(t)\omega(x)\lambda(t)=i\dot{\psi}(x,t), \tag{66}$$

where

$$\lambda(t)=\int_{-\infty}^{\infty}\omega^*(u)\psi(u,t)\,du=\int_{-\infty}^{\infty}\tilde{\omega}^*(k)\tilde{\psi}(k,t)\,dk, \tag{67}$$

where on the integral in the RHS we have the Fourier transforms of the wave function $\psi(x,t)$ and the conjugate shape potential $\omega^*(x)$, with respect to the variable x.

Using again the Fourier transform on x in (66), we get the following differential equation:

$$\frac{\partial}{\partial t}\left[e^{ik^2t}\tilde{\psi}(k,t)\right]=-ie^{ik^2t}g(t)\lambda(t)\,\tilde{\omega}(k). \tag{68}$$

Using as initial condition the function

$$\tilde{\psi}(k,0)=\tilde{\psi}_0(k) \tag{69}$$

equation (68) can be solved as:

$$\tilde{\psi}(k,t)=e^{-ik^2t}\tilde{\psi}_0(k)-i\tilde{\omega}(k)\int_0^t e^{-ik^2(t-\tau)}\,g(\tau)\,\lambda(\tau)\,d\tau. \tag{70}$$

Now, using (70) on the RHS of equation (67), we obtain the following *compatibility condition* for the function $\lambda(t)$:

$$\lambda(t)=\int_{-\infty}^{\infty}dk\;\tilde{\omega}^*(k)\left[e^{-ik^2t}\tilde{\psi}_0(k)-i\tilde{\omega}(k)\int_0^t e^{-ik^2(t-\tau)}g(\tau)\lambda(\tau)\,d\tau\right]. \tag{71}$$

For small times and small perturbations ($|g|<<1$), the last term in the RHS of (71) can be neglected, and we can approximate $\lambda(t)$ as:

$$\lambda(t)\approx e^{-ik^2t}\int_{-\infty}^{\infty}dk\,\tilde{\omega}^*(k)\,\tilde{\psi}_0(k). \tag{72}$$

Now we are going to consider several examples. In this case we will organize them first according to the initial condition, and then we will consider several potentials $\omega(x)g(t)$.

A. Plane wave initial condition

We choose as initial condition a plane wave function with the form: $\psi(x,0)=e^{ipx}$, whose Fourier transform is:

$$\tilde{\psi}_0(k)=\sqrt{2\pi}\,\delta(k+p). \tag{73}$$

1. *Null potential*

In this example, we want to use a null potential:

$$V(x,t)=0. \tag{74}$$

In this case $g(t) = 0$ and we can choose $\omega(x) = \delta(x)$, so the Fourier transform is very simple:

$$\tilde{\omega}(k) = \frac{1}{\sqrt{2\pi}}. \tag{75}$$

If we replace all those ingredients in (70), the second term in the RHS nullifies, and we obtain:

$$\tilde{\psi}(k,t) = \sqrt{2\pi}\, e^{-ik^2 t}\, \delta\,(k+p). \tag{76}$$

The inverse Fourier transform of this wave function is very simple, and we recover the usual temporal evolution of a plane wave:

$$\psi(x,t) = e^{i(px - p^2 t)}. \tag{77}$$

2. *Cosine time dependence*

In this case, we want to use an equivalent local potential with shape $\omega(x) = \delta(x)$, so its Fourier transform is like (75), and $g(t) = g\,\cos(\omega\, t)$, such that the equivalent local potential is

$$V(x,t) = g\,\delta(x)\,\cos(\omega\, t). \tag{78}$$

Putting (75) and (73) in (72), we have:

$$\lambda(t) = e^{-ip^2 t}. \tag{79}$$

If now we replace in (70), we obtain:

$$\tilde{\psi}(k,t) = e^{-ik^2 t}\sqrt{2\pi}\,\delta(k+p) - \frac{ig}{\sqrt{2\pi}}\,\frac{ie^{-ik^2 t}(k^2-p^2) + e^{-ip^2 t}(-i(k^2-p^2)\cos(\omega\, t) - \omega\sin(\omega\, t))}{(k^2-p^2)^2 - \omega^2}. \tag{80}$$

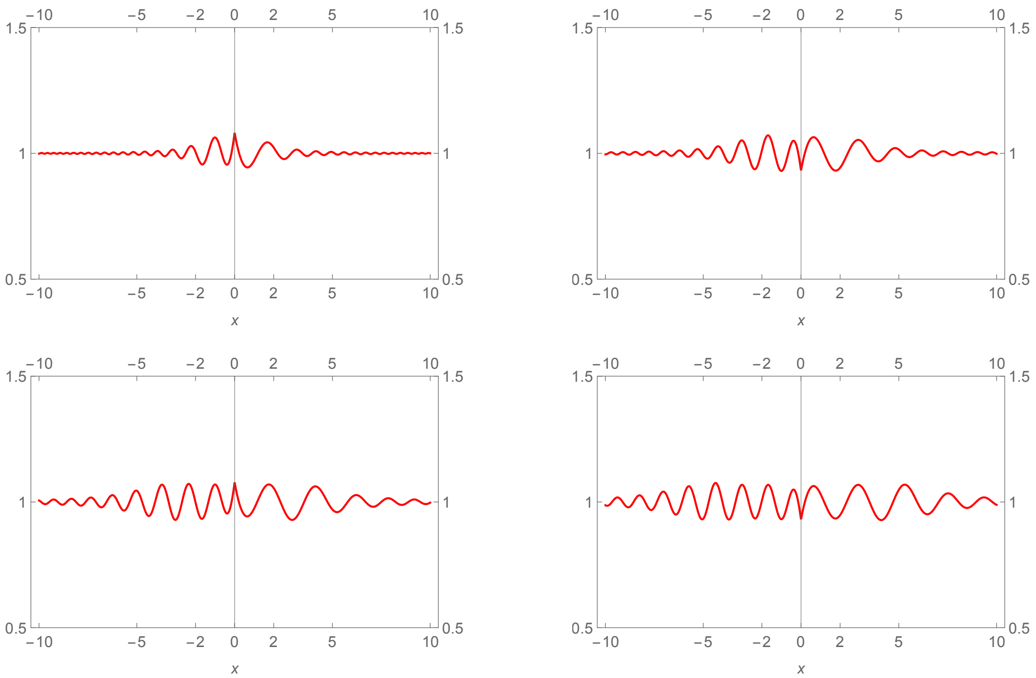

FIG. 3: Representation of the square of the norm of the inverse Fourier transform of (80) with a plane wave initial condition for a time dependent cosine nonlocal potential, with $\omega = 4\pi$, $g = 0.5$ and $p = 1$, for $t = 0.25, 0.5, 0.75, 1$.

The inverse Fourier transform of this wave function can not be done analytically, but it can be done numerically and graphically. Plots of the time evolution of the probability density $|\psi(x,t)|^2$ for this potential are shown in FIG. 3. It shows that, when the time progresses, the system stops being a plane wave in certain interval of the space, which grows as time increases.

3. Exponential time dependence

In this case, we want to use the equivalent local potential

$$V(x,t) = g\,\delta(x)\,e^{-b^2\,t}. \tag{81}$$

Then, $\omega(x) = \delta(x)$, so the Fourier transform is like (75). Also, we choose

$$g(t) = g\,e^{-b^2\,t}. \tag{82}$$

The function $\lambda(t)$ has the same form of (79), because it only depends on the initial condition.

If we replace (75), (82), (73) and (79) in (70), we obtain:

$$\tilde{\psi}(k,t) = e^{-ik^2 t}\sqrt{2\pi}\,\delta(k+p) - \frac{ig}{\sqrt{2\pi}}\,\frac{e^{-ik^2 t} - e^{-(b^2+ip^2)t}}{b^2 - i(k^2-p^2)}. \tag{83}$$

The inverse Fourier transform of this wave function can not be done analytically, but it can be done numerically. In this way, we can obtain a graphical representation.

Plots of the time evolution of the probability density $|\psi(x,t)|^2$ for this potential are shown in FIG. 4. It seems that the plane wave encounters the delta potential, and one part is reflected and the other is transmitted. The system stabilizes when the time increases.

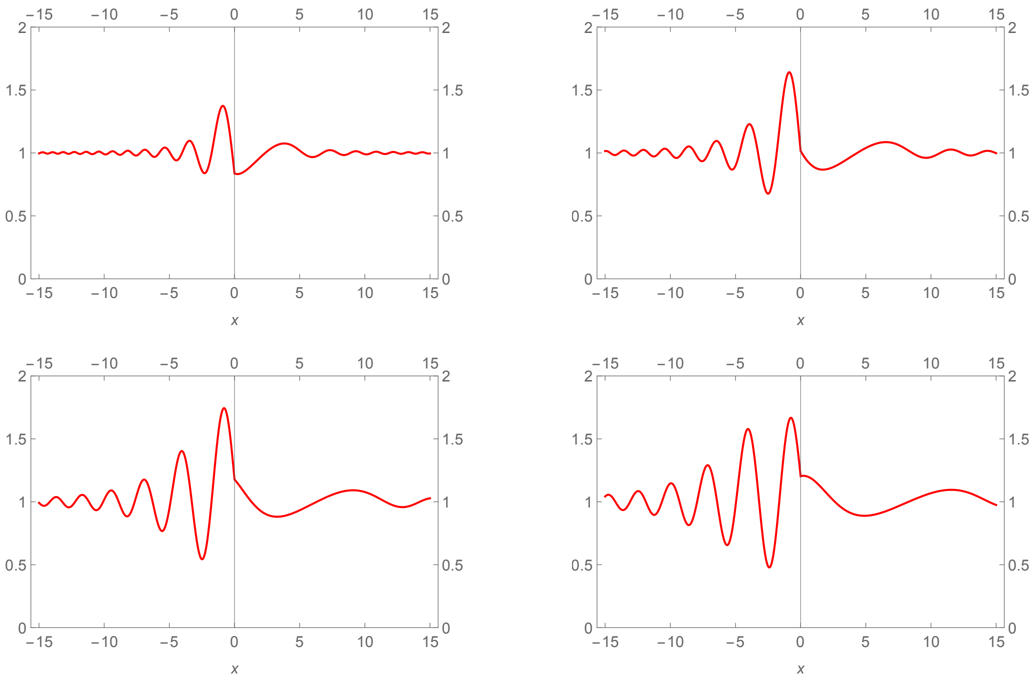

FIG. 4: Representation of the square of the norm of the inverse Fourier transform of (83) with a plane wave initial condition for a time dependent exponential nonlocal potential, with $b = 0.1$, $g = 0.5$ and $p = 1$, for $t = 1, 2, 3, 4$.

B. Gaussian initial condition

Now we choose as initial condition a Gaussian function like (49), whose Fourier transform is:

$$\tilde{\psi}_0(k) = \frac{e^{-\frac{k^2}{4}}}{(2\pi)^{1/4}}. \tag{84}$$

Let us consider now several potential shapes.

1. Cosine time dependence

This case was considered before: $\omega(x) = \delta(x)$ and $g(t) = g\cos(\omega t)$, such that the equivalent local potential is (78). Putting (75) and (84) in (72), we obtain:

$$\lambda(t) = \frac{(2/\pi)^{1/4}}{\sqrt{1+4it}}. \tag{85}$$

If we replace in (70), we obtain:

$$\begin{aligned}\tilde{\psi}(k,t) = \frac{e^{-ik^2 t}e^{-\frac{k^2}{4}}}{(2\pi)^{1/4}} - \frac{i}{\sqrt{2\pi}}\left\{\frac{ie^{\frac{(k^2(-1-4it)-\omega)}{4}} g\pi^{1/4}}{2^{7/4}\sqrt{\omega^2-k^4}}\left(e^{\omega/2}\sqrt{k^2+\omega}\left[\mathrm{erf}\left(\frac{\sqrt{\omega-k^2}}{2}\right)\right.\right.\right. \\ \left.\left.\left. -\mathrm{erf}\left(\frac{\sqrt{(\omega-k^2)(1+4it)}}{2}\right)\right] + \sqrt{\omega-k^2}\left[\mathrm{erfi}\left(\frac{\sqrt{\omega+k^2}}{2}\right) - \mathrm{erfi}\left(\frac{\sqrt{(\omega+k^2)(1+4it)}}{2}\right)\right]\right)\right\},\end{aligned} \tag{86}$$

with $\mathrm{erf}(x)$ and $\mathrm{erfi}(x)$ the error function and the imaginary error function respectively. Plots of the time evolution of the probability density $|\psi(x,t)|^2$ obtained numerically are shown in FIG. 5. Observe that when the time progresses, the system is affected by the oscillating delta potential, which changes from a barrier to a well alternatively. The form of the potential affects the system in $x = 0$, which provokes the change of the shape of the rest of the wave function.

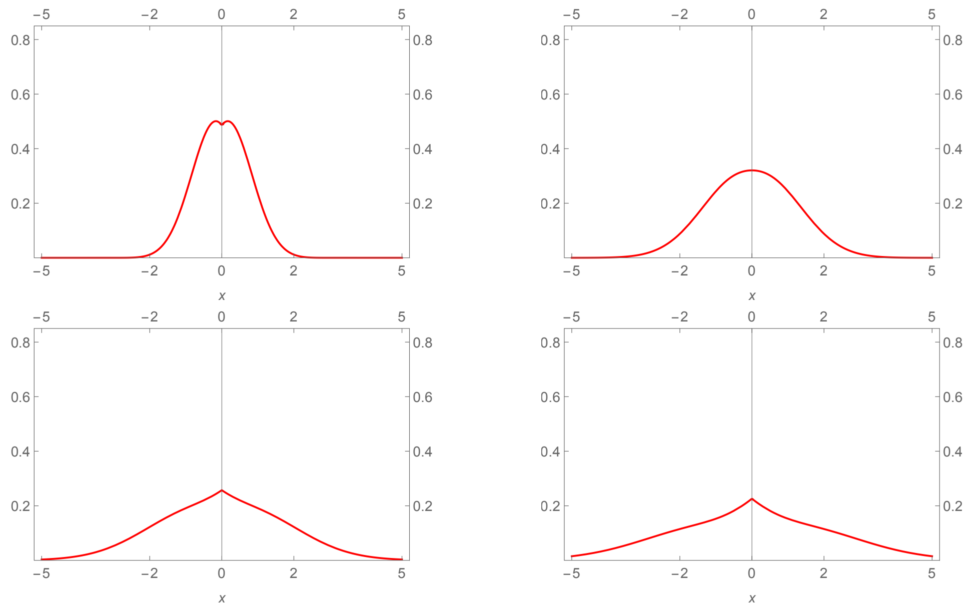

FIG. 5: Representation of $|\psi(x,t)|^2$ from (86) with a Gaussian initial condition for a time dependent cosine nonlocal potential, with $\omega = \pi$ and $g = 0.5$, for $t = 0.25, 0.5, 0.75, 1$.

2. Exponential time dependence

In this case, we want to use an equivalent local potential with the shape (81). The function $\lambda(t)$ has the expression (85), because it only depends on the initial condition (49). Replacing in (70), we obtain:

$$\begin{aligned}\tilde{\psi}(k,t) &= \frac{e^{-ik^2 t}\, e^{-\frac{k^2}{4}}}{(2\pi)^{1/4}} \\ &\quad -g\left(\frac{\pi}{2}\right)^{1/4}\frac{1+i}{2\sqrt{2\pi}}\, e^{-i(b^2+k^2(4t-i))/4}\, \frac{\mathrm{erf}\left(\frac{(-1)^{3/4}\sqrt{b^2-ik^2}}{2}\right) - \mathrm{erf}\left(\frac{(-1)^{3/4}\sqrt{(b^2-ik^2)(1+4it)}}{2}\right)}{\sqrt{b^2-ik^2}}.\end{aligned} \tag{87}$$

The inverse Fourier transform of this wave function can not be done analytically, but it can be done numerically graphically. Plots of the time evolution of the probability density $|\psi(x,t)|^2$ are shown in FIG. 6 and FIG. 7. If $g > 0$, the barrier affects the system, diminishing the probability density at $x = 0$, where the potential barrier is located. This phenomenon is maintained through all the process, even if the intensity of the barrier is reduced. If $g < 0$, the well provokes that the maximum of the probability density will be located at $x = 0$ for any time. However, the decrease of the intensity of the well mixed with the natural tendency of the system causes a reduction of the value of the maximum.

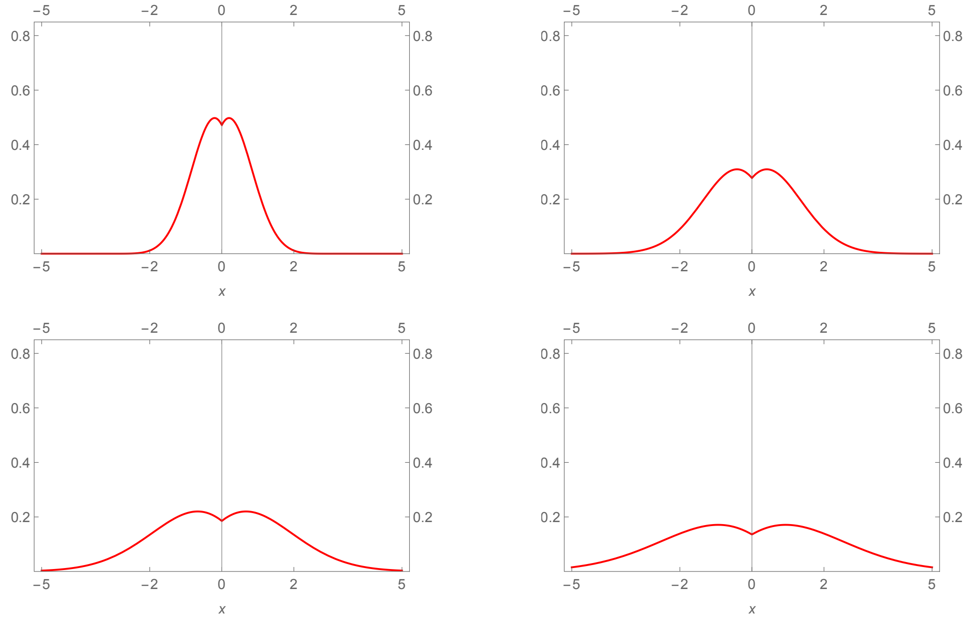

FIG. 6: Representation of $|\psi(x,t)|^2$ from (87) with a Gaussian initial condition for a time dependent exponential nonlocal potential, with $b = 0.1$ and $g = 0.5$, for $t = 0.25, 0.5, 0.75, 1$.

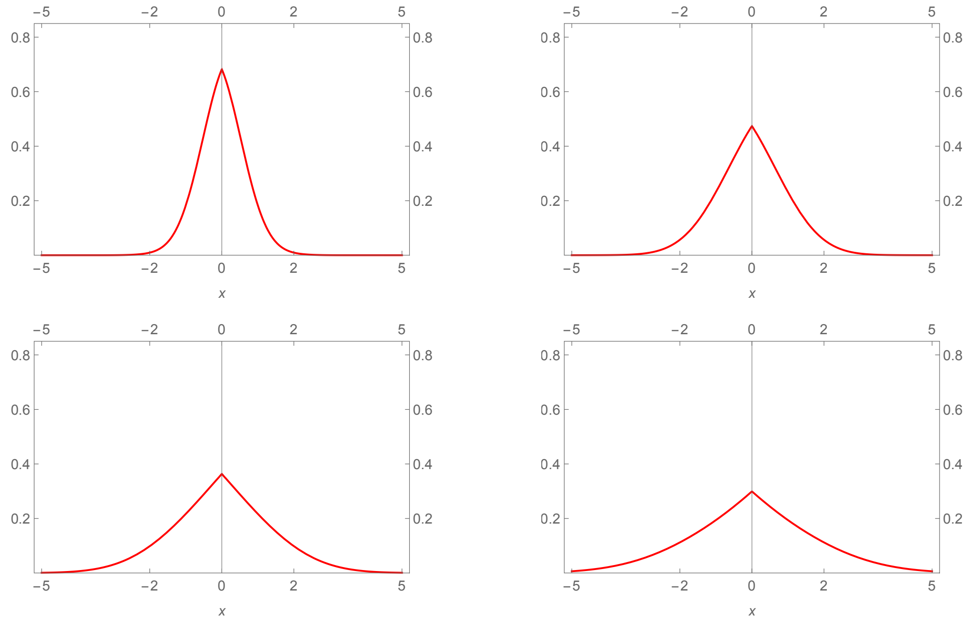

FIG. 7: Representation of $|\psi(x,t)|^2$ from (87) with a Gaussian initial condition for a time dependent exponential nonlocal potential, with $b = 0.1$ and $g = -0.5$, for $t = 0.25, 0.5, 0.75, 1$.

C. Spiky initial condition

Let us consider now a spiky initial condition like (53) and several potential shapes.

1. Time independent potential

First, we want to use an equivalent local potential $V(x,t) = -g\delta(x)$, with $g > 0$. In this case, $\omega(x) = \delta(x)$, so the Fourier transform has the form of (75). Also, $g(t) = -g$, that is, the potential is time independent. Therefore, equation (72) gives us in this case the following result:

$$\lambda(t) = \sqrt{\frac{g}{2}}\, e^{\frac{ig^2 t}{4}} \left(1 - (1+i)C\left(\frac{g\sqrt{t}}{\sqrt{2\pi}}\right) - (1-i)S\left(\frac{g\sqrt{t}}{\sqrt{2\pi}}\right)\right), \tag{88}$$

with $S(x)$ and $C(x)$ the Fresnel integrals. If we replace those results in (70), we obtain:

$$\begin{aligned}\tilde{\psi}(k,t) \;=\; & \frac{2g^{3/2}}{\sqrt{\pi}}\frac{e^{-ik^2 t}}{g^2+4k^2} \\ & +\frac{g^{3/2}}{2\sqrt{\pi}}\frac{e^{\frac{i(g^2-4k^2)t}{4}}(e^{ik^2 t}-1)}{k^2}\left(1-(1+i)\,C\left(\frac{g\sqrt{t}}{\sqrt{2\pi}}\right)-(1-i)\,S\left(\frac{g\sqrt{t}}{\sqrt{2\pi}}\right)\right),\end{aligned} \tag{89}$$

with $S(x)$ and $C(x)$ the Fresnel integrals. The inverse Fourier transform of this wave function must be done numerically, and we obtain the graphical representations shown in FIG. 8. As expected, the bound state of the potential does not vary with time.

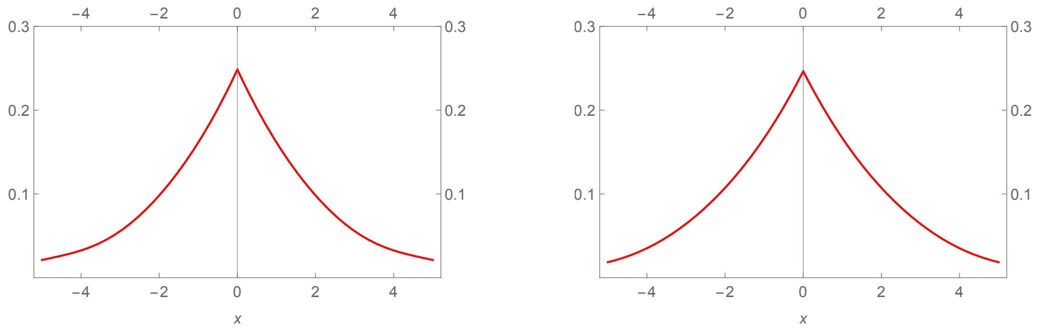

FIG. 8: Representation of $|\psi(x,t)|^2$ from (89) with the eigenstate of the Dirac delta well as initial condition for a time independent nonlocal potential, with $g = 0.5$, for $t = 1, 2$.

2. Cosine time dependence

We consider again the equivalent local potential $V(x,t) = -g\delta(x)\cos(\omega\, t)$, with $\omega(x) = \delta(x)$ and $g(t) = -g\cos(\omega\, t)$. The function $\lambda(t)$ has the same form of (88), because it only depends on the initial condition. From (70), we obtain:

$$\begin{aligned}\tilde{\psi}(k,t) = & \frac{2g^{3/2}\,e^{-ik^2 t}}{(g^2+4k^2)\sqrt{\pi}} + \frac{i}{\sqrt{2\pi}}\frac{g^{3/2}}{\sqrt{2}((g^2+4k^2)^2-16\omega^2)}\left\{\frac{e^{-ik^2\,t}}{\sqrt{k^4-\omega^2}}\right. \\ & \times\left[-g\sqrt{k^2+\omega}(g^2+4(k^2+\omega))\,\mathrm{erf}\left((-1)^{3/4}\sqrt{t(k^2-\omega)}\right)\right. \\ & \left.+\sqrt{k^2-\omega}\left(4i(g^2+4k^2)\sqrt{k^2+\omega}\left(1-e^{\frac{i(g^2+4k^2)t}{4}}\cos(\omega\, t)\right)-g(g^2+4k^2-4\omega)\,\mathrm{erf}\left((-1)^{3/4}\sqrt{t(k^2+\omega)}\right)\right)\right] \\ & \left.+e^{\frac{ig^2 t}{4}}\left[-16\,\omega\sin(\omega\, t)+(4+4i)\left(iC\left(\frac{g\sqrt{t}}{\sqrt{2\pi}}\right)+S\left(\frac{g\sqrt{t}}{\sqrt{2\pi}}\right)\right)((g^2+4k^2)\cos(\omega\, t)-4i\omega\sin(\omega\, t))\right]\right\}.\end{aligned} \tag{90}$$

The inverse Fourier transform of this complicated wave function can be obtained numerically. Plots of the time evolution of the probability density $|\psi(x,t)|^2$ for this potential are shown in FIG. 9 and FIG. 10. The alternating change of the shape of the potential from a well to a barrier provokes changes in the probability density at the point $x = 0$ that affects all the system, and a maximum and a minimum alternate at $x = 0$. When the frequency of the oscillations is increased, the shape of the potential changes faster, so the evolution can be observed better.

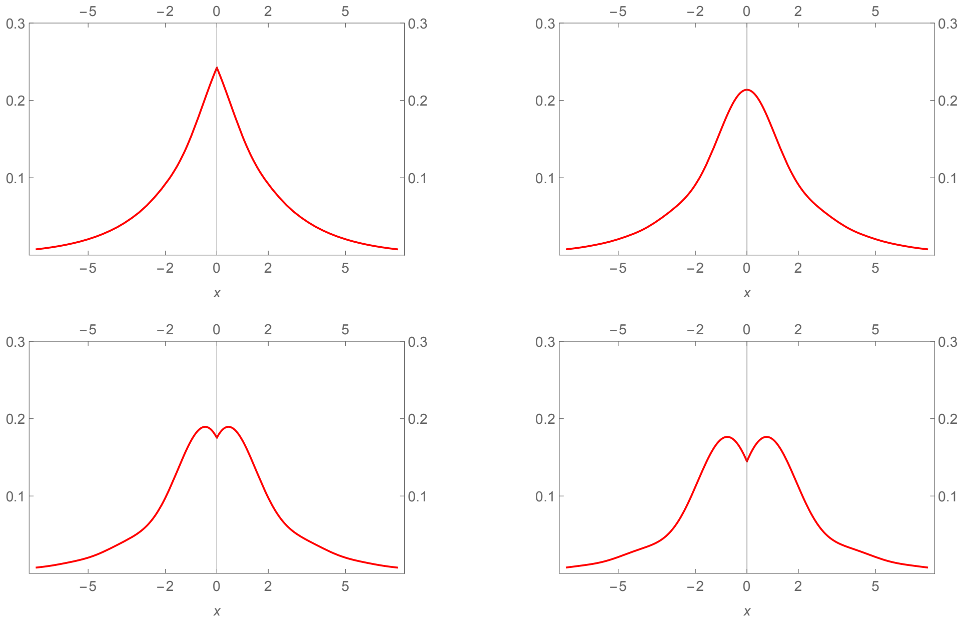

FIG. 9: Representation of $|\psi(x,t)|^2$ from (90) with the eigenstate of the Dirac delta well as initial condition for a time dependent cosine nonlocal potential, with $\omega = \pi$ and $g = 0.5$, for $t = 0.25, 0.5, 0.75, 1$.

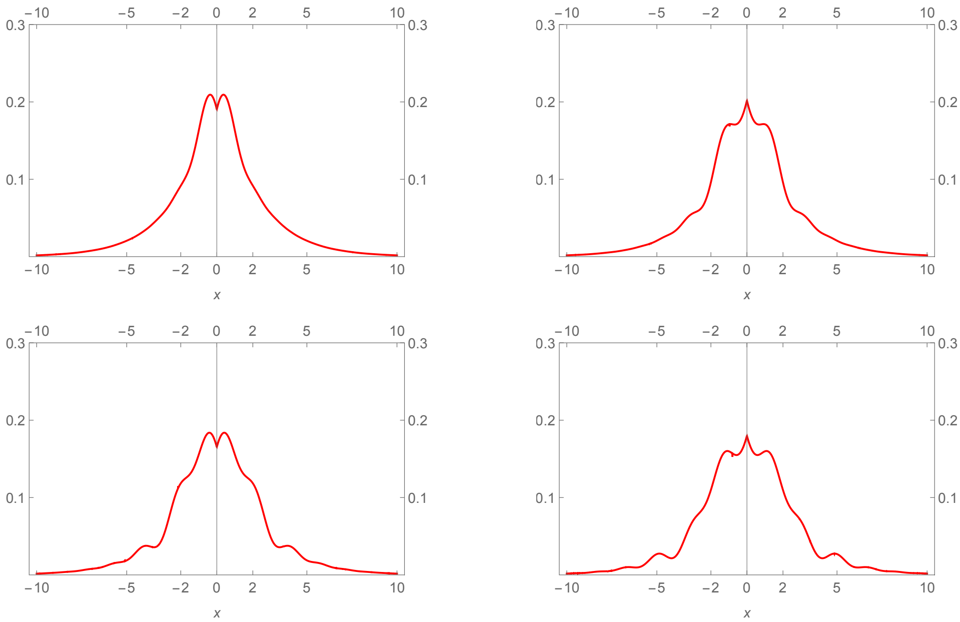

FIG. 10: Representation of $|\psi(x,t)|^2$ from (90) with the eigenstate of the Dirac delta well as initial condition for a time dependent cosine nonlocal potential, with $\omega = 4\pi$ and $g = 0.5$, for $t = 0.25, 0.5, 0.75, 1$.

3. Exponential time dependence

As a final example we use again an equivalent local potential with shape $V(x,t) = -g\delta(x)e^{-b^2\,t}$, with $\omega(x) = \delta(x)$ and $g(t) = -g\,e^{-b^2\,t}$. The form of $\lambda(t)$ is already known to be (88), because it only depends on the initial condition. From (70) we obtain:

$$\tilde{\psi}(k,t) = e^{-ik^2t}\frac{2g^{3/2}}{(g^2+4k^2)\sqrt{\pi}} + \frac{i}{\sqrt{2\pi}}\frac{(1+i)g^{3/2}}{4ib^2+g^2+4k^2}\Bigg\{(-\frac{ie^{-ik^2t}g\,\mathrm{erf}\left(\sqrt{(b^2-ik^2)t}\right)}{\sqrt{b^2-ik^2}} \tag{91}$$
$$+2(-1)^{1/4}e^{-b^2t+\frac{ig^2t}{4}}\left[(1+i)\,C\left(\frac{g\sqrt{t}}{\sqrt{2\pi}}\right)+(1-i)\,S\left(\frac{g\sqrt{t}}{\sqrt{2\pi}}\right)-\mathrm{sign}(g)\right]+2(-1)^{1/4}e^{-ik^2t}\mathrm{sign}(g)\Bigg\}.$$

The inverse Fourier transform of this wave function is done numerically and plots of the time evolution of the probability density $|\psi(x,t)|^2$ are shown in FIG. 11 and FIG. 12. The location of the maximum is maintained, but the shape evolves to a bell form, due to the decrease of the intensity of the well. If we increase the parameter b, the intensity of the potential reduces faster and the probability density adopts the bell form quickest.

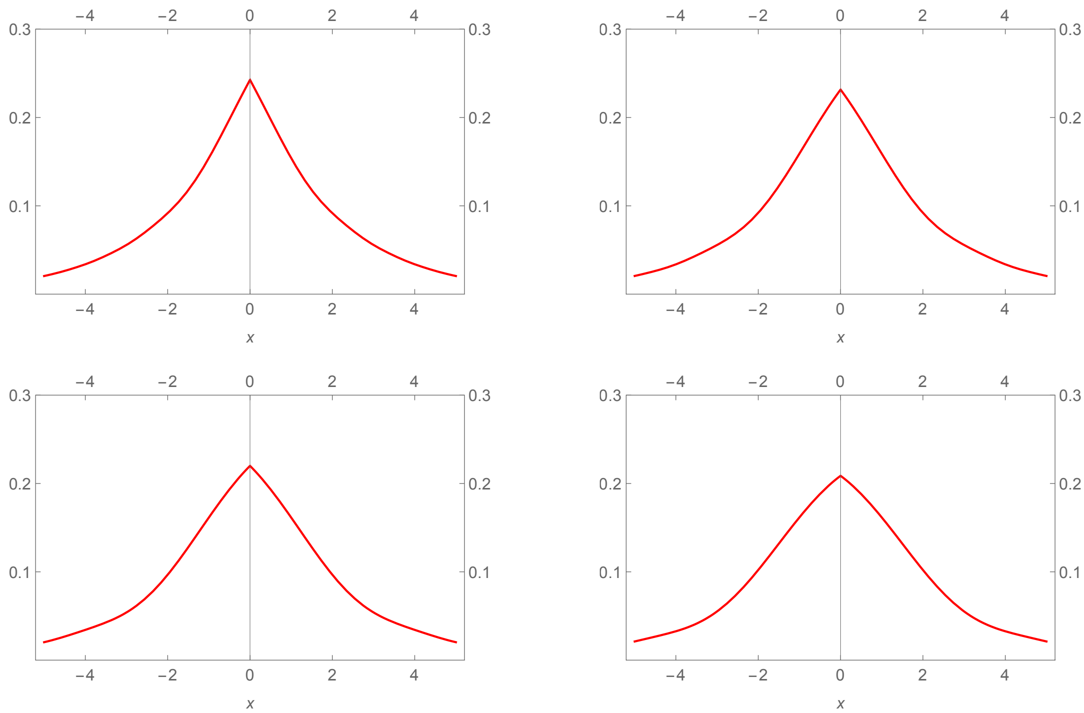

FIG. 11: Representation of $|\psi(x,t)|^2$ from (91) with the eigenstate of the Dirac delta well as initial condition for a time dependent exponential nonlocal potential, with $b = 1$ and $g = 0.5$, for $t = 0.25, 0.5, 0.75, 1$.

IV. FINAL CONCLUSIONS

In this work, we obtain solutions to a model of the Schrödinger equation, where we use a position and time dependent local potential of the form $V(x,t) = g(t)\,\delta(x)$. For its resolution, we simplify the equations eliminating the constants, with the use of some changes of variables. Then, we consider two different methods. First, we use some properties of the Fourier's transform to obtain expressions for solving the equation. Some of these are in the form of an infinite sum of integrals, so, for the sake of solving the problem, we work only with approximations. In the case of the free particle ($g(t) = 0$),

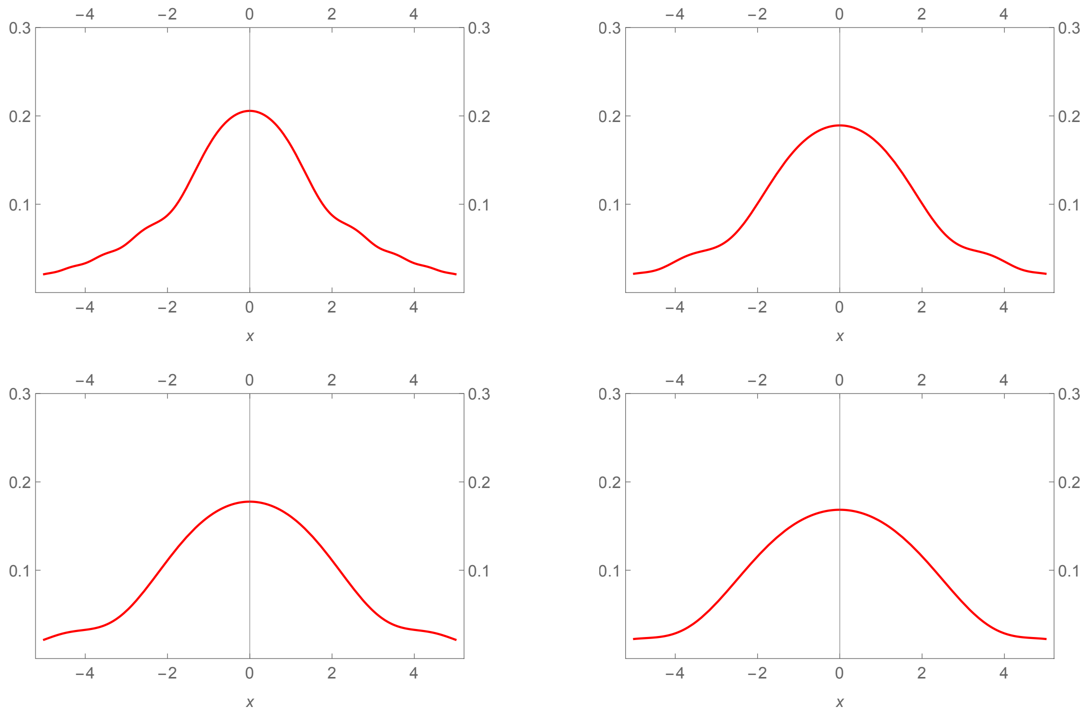

FIG. 12: Representation of $|\psi(x,t)|^2$ from (91) with the eigenstate of the Dirac delta well as initial condition for a time dependent exponential nonlocal potential, with $b = 10$ and $g = 0.5$, for $t = 0.25, 0.5, 0.75, 1$.

we obtain solutions for some initial conditions, and we observed that the evolution of the probability density agrees with the well-known behaviour (FIG. 1 and FIG. 2). This points out that the theoretical expressions are correct. Then, we work with an impulsive time-dependence (57), but we arrive to some pathological solutions which indicate that this problem is ill posed from the mathematical point of view. Unfortunately, when we try other more complex time-dependent potentials, we arrive to some difficult expressions which we have not be able to deal with, even with first order approximations. More work in this direction will be necessary in the future in order to make relevant progress.

To deal with more interesting potentials we use an alternative approach, which consists in the employment of nonlocal potentials. Then, Schrödinger equation can be rewritten as (66) with (67). In this case, we use the Fourier's transform for working in the frequency domain; only at the end we return to the usual variables. The solution depends on an initial condition and on a boundary condition at the origin. Choosing the spatial dependence as a Dirac's delta allows us to work with the usual local potentials in its nonlocal form. Surprisingly, even if we can not obtain exact expressions, with this in appearance more complicated method we can use numerical methods to get graphical representations of the solutions.

For solving the problem we have to impose an initial condition. Firstly, we use a plane wave. Applying a null potential allows us to obtain the exact solution, which is the usual expression of a travelling plane wave. Adding a non-null delta potential with a time-dependence alters the behaviour, and the plane wave distorts (FIG. 3 and FIG. 4).

Afterwards, we consider a Gaussian initial condition (49). A cosine time dependence provokes the alternating change of the shape of the potential from an initial barrier to a well. A consequence of this is a modification of the behaviour of the point at $x = 0$: when the potential is a barrier, it becomes a minimum (the probability density decreases), and when the potential changes to a well, it appears a maximum (the probability density increases) (FIG. 5). An exponential time dependence modulating a potential barrier causes the appearance of a minimum (FIG. 6), and if it is modulating a potential well the maximum of the Gaussian is kept (FIG. 7).

Finally, we study the effects over a spiky initial condition (53). If we use the time independent potential, we observe that the shape of the wavefunction is conserved (FIG. 8), as expected from a bound state. Adding a time-dependence causes the loss of the bound state, which will evolve to another conformation. A cosine time-dependence causes the same effect as in the Gaussian example: and alternating minimum and maximum at the point $x = 0$ (FIG. 9 and FIG. 10). An exponential time-dependence maintains the maximum at $x = 0$, but the decrease of the intensity of the well results in the progressive fading of the peak, acquiring a more rounded appearance but maintaining the maximum (FIG. 11 and FIG. 12).

We can conclude saying that all the systems have two contributions to their shape: one that only depends of the initial condition and which corresponds to the natural evolution of the system when it is immersed in the time independent potential; and a second that is associated to the specific time-dependent coefficient of the delta.

It will be clear by now that the problems attacked in the present work were not at all simple, and obtaining results became a very hard task, that still needs additional efforts. Work in this direction is in progress.

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[1] A. Galindo, P. Pascual, *Quantum Mechanics*, Springer (1978).

[2] C. Cohen-Tannoudji, B. Diu, F. Laloe, *Quantum Mechanics*, Wiley (1992).

[3] M.L. Glasser, J. Mateo, J. Negro, and L.M. Nieto, *Quantum infinite square well with an oscillating wall*, Chaos, Solitons and Fractals **41** (2009) 2067-2074.

[4] A. Contreras-Astorga, V. Hussin, *Infinite Square-Well, Trigonometric Pöschl-Teller and Other Potential Wells with a Moving Barrier*, en “Integrability, Supersymmetry and Coherent States”, Ş. Kuru, J. Negro and L.M. Nieto (Eds), en CRM Series in Mathematical Physics, Springer (2019) 285-299.

[5] C. González-Santander, F. Domínguez-Adame, C.H. Fuentevilla, and E. Diez, *Scattering of massless Dirac particles by oscillating barriers in one dimension*, Phys. Lett. A **378** (2014) 927-930.

[6] D.F. Martinez and L.E. Reichl, *Transmission properties of the oscillating δ-function potential*, Phys. Rev. B **64** (2001) 245315.

[7] S. Ki Baek, S. Do Yi, and M. Kim, *Particle in a box with a time-dependent δ-function potential*, Phys. Rev. A **94** (2016) 052124.

[8] G. Breit, *A Soluble Semiclassical Particle Transfer Problem*, Ann. Phys. **34** (1965) 377-399.

[9] F. Erman, M. Gadella, and H. Uncu, *The Propagators for δ and δ' Potentials with Time-Dependent Strengths*, Front. Phys. **8** (2020) 65.

[10] S. Blanes, F. Casas, A. Murua *Symplectic time-average propagators for the Schrödinger equation with a time-dependent Hamiltonian*, J. Chem. Phys. **146** (2017) 114109.

[11] P. Bader, A. Iserles, K. Kropielnicka, P. Singh *Efficient methods for linear Schrödinger equation in the semiclassical regime with time-dependent potential*, Proc. R. Soc. A **472** (2016) 20150733.

[12] W.H. Nichols *Separable Nonlocal Potentials in Quantum Mechanics*, Am. J. Phys. **33** 474 (1965).

[13] Woon-young So, Byung-taik Kim *Calculations of Bound States for Nonlocal Potentials*, J. Korean Phys. Soc. **30** (1997) 175.

[14] C.S. Shastry, M. Singh *Analytical Studies in Scattering by Non-Local Separable Potentials*, Prog. Theor.Phys., Vol. 51, No, 1 (1974).

[15] M. Bertero, G. Talenti, G.A. Viano *Scattering and Bound State Solutions for a Class of Nonlocal Potentials (S-wave)*, Commun. Math. Phys. **6** (1967) 128-150.

[16] C. González-Santander, F. Domínguez-Adame *Non-local separable solutions of two interacting particles in a harmonic trap*, Phys. Lett. A **375** (2011) 314-317.

[17] R. Augusiak *Nonrelativistic quantum scattering from nonlocal separable potentials: the eigenchannel approach*, Ann. Phys. **14** (2005) 398-407.

[18] Y. Yamaguchi *Two nucleon problem when the potential is nonlocal but separable. 1.*, Phys. Rev. **95** (1954) 1628

[19] M.K. Sundaresan *Separable Nonlocal Potential and Nuclear Saturation*, Phys. Rev. **105**, (1957) 1075; Erratum Phys. Rev. **106** (1957) 1371.

[20] W. van Dijk, M.W. Kermode, S.A. Moszkowski *Solvable separable nonlocal potential models*, J. Phys. A: Math. Gen. **31** (1998) 95719577.