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Optimal management of a stochastic pension plan

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To my mother

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Abstract

This paper analyzes the optimal management of a pension plan from the point of view of dynamic programming. The pension plan is defined benefit and stochastic. The aim of the manager is to find the fund's optimal contribution and investment strategies in a way that minimizes a financial risk.

The problem is modeled as one of stochastic optimal control. It shows how to find the optimal strategies using dynamic programming techniques when both, benefits and risky assets are geometric Brownian movements.

By means of a numerical analysis based on real data from pension funds and contributions from the IBEX 35 index, the evolution of the pension plan is studied and sensitivity analyses are carried out with respect to model parameters using the R environment.

Resumen

En este trabajo se analiza la gestión óptima de un plan de pensiones desde el punto de vista de la programación dinámica. El plan de pensiones es de prestaciones definidas y estocásticas. El objetivo del gestor es encontrar las estrategias óptimas de contribución y de inversión del fondo de manera que se minimice un riesgo financiero.

El problema se modeliza como uno de control óptimo estocástico. Se muestra cómo encontrar las estrategias óptimas mediante técnicas de programación dinámica cuando tanto las prestaciones como los activos con riesgo son movimientos Brownianos geométricos.

Mediante un análisis numérico basado en datos reales de fondos de pensiones y de cotizaciones del índice IBEX 35 se estudia la evolución del plan de pensiones y se realizan análisis de sensibilidad respecto a parámetros del modelo utilizando el entorno R.

Contents

1. Introduction

The analysis of the optimal management of pension plans from the dynamic optimization approach has become very important in recent years not only in the economic field but also in the financial field. Economically because it has been shown that the current public pension plan system is unsustainable in the long term and it is necessary to find an alternative and in the financial field since the fund managers use the financial markets to invest the fund asset of the pension plan. This fact has led to a large body of literature about the optimal management of pension plans from the dynamic optimization perspective. See Cairns (2000), Haberman *et al.* (2000), Taylor (2002), Gao (2009) and Josa-Fombellida *et al.* (2018).

The main objective of this paper is to study how a continuous-time pension plan optimization model can be analyzed as a stochastic control problem using the stochastic dynamic programming method. In order to archive this goal, we depart form Josa-Fombellida and Rincon Zapatero (2004) where a defined benefit pension plan is considered. In this model the benefits are stochastic processes correlated with the financial market and the main aim of the plan manager is to minimize both, the solvency and the contribution risks along an infinite horizon. Moreover, and as a contribution to this paper, we study the temporal evolution of the optimal solutions by a numerical illustration with real data from IBEX-35. Also the effect of some parameters of the model on the optimal investment and contribution strategies and optimal fund is analysed. The numerical analysis is realized using the statistical software R.

The paper is organized as follows:

- Section 2 explains what pension plans are, the main types of pension plans that exist, and how they work in financial markets.

- In Section 3 the stochastic optimal control problem is briefly presented and its resolution using the Hamilton-Jacobi-Bellman (HJB) equation.

- Section 4 describes the pension plan model, the financial market where the fund operates, the optimization problem, and shows its resolution through stochastic dynamic programming.

- In Section 5 a practical exercise is carried out by the numerical illustration of the model with real data of the optimal evolution of the investment, the fund and the contributions in two periods of the market: bullish and bearish. Additionally, a sensitivity analysis of the optimal policies and optimal fund with respect to some parameters is included.

- In Section 6 we explain the main conclusions.

2. The pension plans

The reality that surrounds us is very changing, which is why today's society is constantly searching for security in the future. The offer of financial products focused on saving to complete the existing public pension after retirement is increasing. The suitability of each product to the investor will depend on the aims to be achieved and its risk profile.

Among the wide range of existing financial products, one of the most popular is pension plans. A pension plan is a long-term saving financial product specially designed for retirement. It consists in generating capital that will be available when the main contingency is reached, which usually is retirement. The main objective is to complement the social security retirement pension.

The set of individuals that are part of a pension plan are the following (see Peláez Fermoso and García González (2004)):

- The manager: is the entity (company, society, entity, corporation, association or union) that promotes its creation and operation.
- Participants: they are the natural people for whom the plan is established, regardless of whether or not they make those contributions.
- Beneficiaries: they are the natural people who have the right to revive the income or capital.

2.1 Main types of pension plans

There are two main types of pension plans the defined-benefit and the defined contribution plans.

Defined benefit plans

In a defined-benefit plan, the employer guarantees that the employee receives a definite amount of benefit upon retirement, regardless of the performance of the underlying investment pool. The employer is liable for a specific flow of pension payments to the retiree, and if the fund assets in the pension plan are not sufficient to pay the benefits, the company is liable for the remainder of the payment. Depending on the promised benefits, the contributions to be made are calculated. The risk derived from the fund management is borne by the manager.

Defined contribution plans

In a defined- contribution plan, the employer makes specific plan contributions for the worker. Therefore, the participant knows in advance the contributions made but he does not know the benefit he will receive.

The final benefit received by the employee depends on the plan's investment performance. The participants will have to assume the risk.

In common language, "pension plan" often means the more traditional defined-benefit plan, with a set payout, funded and controlled entirely by the manager.

Pension plans represent a high percentage of the total volume of operations carried out in the financial markets. According to the OECD 2019 report *Pension Markets in Focus*, pension assets have been growing over the last decade, reaching USD 44.1 trillion worldwide at the end of 2018.

Some of the reasons that pension plans have become so popular and have experienced constant growth is because of the positive returns they have offered in recent decades. Furthermore, the current public pension system has been shown to be unfeasible considering the inverted population pyramid that exists in developed countries. To manage this problem, social security tries to devalue benefits and thus please private pension plans. Its importance also lies in the tax advantages it has, which means present savings.

2.2 How does a pension plan work?

The operation of a pension plan is based on the periodic or punctual contributions made by the participants. The set of all contributions generates a pool called pension fund that will be invested by the manager in the financial market based on the investment policy of the plan, following both the profitability and risk criteria.

These criteria depend on the type of plan that the investor has contracted and always maximize the profitability. Based on investments, the most frequent are:

- Fixed income plans (short or long term, public fixed income or corporate fixed income).
- Equity plans.
- Mixed plans, with different weight of fixed and variable income.

The investment profile of the participants will determine which plan is more suitable for the needs of each participant. Fixed income plans have lower risk but also lower profitability. On the other hand, equity plans have a higher theoretical risk and generate a higher potential return.

At the moment of retirement, participants will receive the money invested over the years, plus the return that the fund has been able to generate. The plan can be redeemed in two different ways: in the form of capital, that is, in a single payment or in the form of fixed income. There is also a mixed option that is a combination of both options before.

In this paper we consider only the environment where the financial market is constituted by a riskless asset and several risky assets, that is to say, a mixed plan.

The model considered in this paper is of defined benefit type. Moreover, it is analysed dynamically. Thus, at each instant of time, the manager has two instruments to increase the fund and to pay the fixed in advance benefit: the contributions realized by the participants and the investments in the financial market.

3. The stochastic optimal control problem

The optimal control problem in continuous time consists of optimizing (maximizing or minimizing) a functional objective that depends on the temporal variable, on a group of variables called state variables and on another group of control variables. The state variables depend on the control variables and their temporal evolution is given by the state equations, as a function of the control variables. Therefore, knowing the controls and some initial conditions on the state variables, it is assumed that the state variables are determined. In the pension model analysed in this paper, the fund and the actuarial liability are the state variables, and the contribution and the investments in the risky assets are the control variables.

The problem is one of deterministic optimal control or one of optimal control if no type of uncertainty is contemplated in the problem. On the other hand, if there is uncertainty, it is called stochastic optimal control. This uncertainty is modeled by stochastic diffusion processes, which can depend on Brownian motions, or by diffusion processes with jumps.

In a deterministic optimal control problem, the state equations are differential equations while in a stochastic control problem they are stochastic differential equations. Given some initial conditions (initial time and initial values of the state variables) and knowing the controls variables, the state variables are determined. These state variables are functions (in the deterministic case) or stochastic diffusion processes (in the stochastic case).

Given initial conditions, the aim is to find the control variables that optimize n objective functional. This depends on the initial conditions and the control variables. Obtaining the controls variables is reduced to find the relationship between them and the state variables. If this is achieved, with the state equations we have the optimal evolution of the states and controls variables.

In the deterministic case, the function objective is given in an integral form, while in the stochastic it is expressed in the form of a conditional expectation given the initial conditions.

In both, the deterministic and in the stochastic case, there are two methods of solving a control problem: the maximum principle and the dynamic programming principle. The maximum principle uses the so-called attached variables, but it is quite difficult to check the necessary and sufficient conditions. In this work we will only use the dynamic programming approach. The usual use of this solving method is the reason why the optimal control problems are also denominated dynamic programming problems in continuous time.

In the dynamic programming method, a set of optimal control problems is considered, one for each initial condition. The optimal value function, which only depends on the initial conditions (time and states) is defined as the minimum or maximum value of the objective functional in the control variables. The so-called Dynamic Programming Principle is used to obtain necessary and sufficient conditions that the value function must fulfill. This principle essentially claims that the same optimal value function works for the whole set of problems. The main result articulates that the value function is characterized by the Dynamic Programming equation or HJB equation when certain regularity conditions are satisfied. So the initial optimization problem is simplified to an optimization problem of variables, typically a nonlinear programming problem.

In the resolution of the equation for the optimal value function (the HJB equation) a differential equation appears, when we analysed a deterministic problem, and a partial differential equation, when we have a stochastic control problem. In the deterministic case this equation is of order one and in the stochastic case of order two. In both cases, at the second resolution step, the control variables no longer appear.

In this paper, the defined benefit pension plan will be modeled as a stochastic control problem where the objective is the risks minimization in the presence of a discount function. We will show as the dynamic programming method can be used.

In the next subsections we follow the references: Arnold (1974), Fleming and Soner (1993), Hernández-Lerma (1994), Josa-Fombellida (2001) and Martinez-Palacios and Venegas-Martínez (2011).

3.1 Statement of the problem

In this section, we indicate which elements are involved in a stochastic optimal control problem. The uncertainty is modelled by a d-dimensional Brownian motion W defined in a complete probabilistic space $(\Omega, \mathcal{F}, \{\mathcal{F}\}_{t\geq 0}, P)$, where $\{\mathcal{F}\}_{t\geq 0}$ is increasing, $\mathcal{F}_t \subset \mathcal{F}, \forall t \in [0,T]$. We also assume that there are *m* control variables $u = (u_1, u_2, \ldots, u_m) \in \mathcal{U} = \mathcal{U}_1 \times \mathcal{U}_2 \times \ldots \times \mathcal{U}_m$, $u : [0,T] \to \mathcal{U} \subseteq R^m$ and there are *n* state variables $X = (X_1, \ldots, X_n) : [0,T] \to X \subseteq R^n$ that satisfy the state equations:

$$dX\,(t) = f\,\big(t, X(t), u(t)\big)dt + \,\sigma\,\big(t, X(t), u(t)\big)dW(t),$$

for all $t \geq 0$, where $X(0) = x_0$. They are a system of stochastic differential equations (SDE).

The problem is to find the function u^* that minimizes the expression

$$\mathbb{E}\left\{\int_0^T e^{-\rho t}\, L\big(\,t, X(t), u(t)\big)dt + S(X(T))\right\},$$

between all control functions *u*, where ρ is a constant rate of discount, *X* satisfies the system of state SDEs and *L* and *S* are functions regular enough.

3.2 The dynamic programming method

In order to solve the problem with the dynamic programming method we consider, for each initial condition $(t,x) \in [0,T] \times R^n$, the objective functional

$$J(\,t,x;u) = \,\mathbb{E}_{tx}\left\{\int_t^T e^{-\rho(s-t)}\, L\big(\,s, X(s), u(s)\big)ds + S(X(T))\right\}.$$

So the problem is to find an optimal control u^* where $J(\,t,x;u^*) = min_u\, J(\,t,x;u)$, for all (t,x).

We define the optimal value function as $\hat{V}(t,x) = \,min_u\, J\,(t,x;u)$, for each initial condition (t,x). Thus, it is said that an admissible control $u^* \in A$ is optimal if $J(\,t,x;u^*) = \hat{V}(t,x)$ for all initial condition (t,x).

The Dynamic Programming principle states that an optimal trajectory has the property that, whatever the state and initial decisions (controls), the remaining decisions should constitute an optimal policy with respect to the state resulting from the first decision. Making use of this principle, it is possible to prove that the value function $\hat{V}$ is characterized by the denominated HJB equation:

$$-\rho V + V_t + min_{u\in U}\left\{L + fV_x + \frac{1}{2}Tr(\sigma\sigma^T V_{xx})\right\} = 0,$$

with the final condition $V(T,x) = S(x)$. That is to say, the value function is a solution of the HJB equation. And reciprocally, an admissible control u^* that minimizes the expression between brackets is an optimal control of the problem.

Next, we briefly indicate the steps to follow to solve an optimal stochastic control problem:

1. Identify the elements of the optimal control problem: state and control variables, the optimal value function for all initial conditions.

2. Write the HJB equation with the functions of the problem.

3. Find $u^* = \phi^*(t,x)$ that minimizes the expression in brackets, { }, in terms of the derivatives of *V*, which will be the candidate value function.

4. Once found, $\phi^*(t,x)$ are replaced again in HJB and we obtain a partial differential equation for the optimal function $V(t,x)$.

5. Find u^*as a function of (t,x).

6. Find the SDE of X^*, replacing the expressions of u^* in (t,x).

7. Perform a numerical illustration of the optimal control and state strategies, with true or estimated data of the parameters.

The pension model that will be presented in Chapter 4 corresponds to a stochastic control problem with an infinitive horizon, $T = \infty$, but autonomous. This means that drift and diffusion functions of the state equations are time independent:

$f(t,x,u) = f(x,u)$ and $\sigma(t,x,u) = \sigma(x,u)$, for all (t,x,u). With this structure, it is possible to check that both the value function and the optimal control are time independent, $V(t,x) = V(x)$ and $\phi^*(t,x) = \phi^*(x)$, for every t. The final condition must be substituted by the transversality condition

$$lim_{T\to\infty} e^{-\rho T}\mathbb{E}\hat{V}(X(T)) = 0,$$

for all initial condition x_0.

4. The pension model

The model analyzed in this paper is a dynamic model for the optimal management of a defined benefit pension plan where we consider the relationships between the actuarial functions stochastic. The objective of the fund manager will be to minimize both contribution and solvency risks. The mathematical perspective to consider this problem is as one of continuous-time dynamic programming that is solved with dynamic programming techniques.

The model is taken from the paper Josa-Fombellida and Rincón-Zapatero (2004). The novelty of this work with respect to that paper is the use of real data through simulation, the programming of the model in the R statistical software and the sensitivity analysis in bullish and bearish periods and with respect to other parameters as the weight of two risks in the aim of the manager.

The pension plan studied is this paper is a defined benefit pension plan of aggregate type where active participants coexist with retired participants. The promised liabilities (benefits) to the participants of the plan at the age of retirement are established in advance by the manager. The pension plan has an infinite planning horizon. The plan manager will invest the fund in the financial market taking into consideration that it has the contributions that the participants will make periodically are used to maintain the fund within adequate levels in order to face the payments of the retired participants.

The actuarial valuation to estimate the main components of the plan are done at each instant of time. The main elements intervening in the funding process and essential hypotheses allowing its temporary evolution to be determined are as follows:

- *F(t)*: Value of the fund assets fund at time *t*.
- *P(t)*: Benefits promised to the participants at time *t*.
- *C(t)*: Contribution rate of the manager at time *t* needed to accrue the amount of the defined benefits at the moment of retirement.
- *AL(t)*: Actuarial liability at time *t*, or total liabilities of the manager. It is the ideal value of *F(t)*.

- *NC(t)*: Normal cost for all participants at time *t*. It is the ideal value of *C(t)*.
- *UAL(t)*: Unfunded actuarial liability at time *t*. It is the difference between the liability and the fund, *UAL = AL − F*.
- *X(t)*: Fund surplus at time *t*. It is the difference between the fund and the liability, X = *F – AL* = - *UAL*.
- *SC(t)*: Supplementary cost at time *t*. It is the difference between the contribution and the normal cost, *SC = C −NC*.
- *M(u)*: Distribution function of workers aged u, *u ∈[a, d]*.
- *δ*: Technical rate of interest of valuation of the plan. Fixed in advance by the manager.

We will model the possibility of the existence of disturbances that affect the evolution of the benefits, and hence the evolution of the normal cost and the actuarial liability, by a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, where $\mathcal{F} = \{\mathcal{F}\}_{t\geq 0}$ is a complete and right continuous filtration generated by the one-dimensional Brownian motion $\{B(t)\}_{t\geq 0}$ and $\mathbb{P}$ is a probability measure on Ω. One of the most general hypotheses about the behaviour of *P* is to suppose that *P* is a path-continuous scalar Itô process defined on $(\Omega, \mathcal{F}, \mathbb{P})$ as

$$P(t) = P_0 + \int_0^t \mu\,(P(s), s)ds + \int_0^t \eta(P(s), s)dB(s), \quad t \geq 0,$$

where $P_0 = P(0)$ represents the initial liabilities. It can be expressed as the SDE

$$dP(t) = \mu\,(P(t), t)dt + \eta(P(t), t)dB(t), \qquad t \geq 0.$$

We also define the functions *AL* and *NC* as:

$$AL(t) = \int_a^d e^{-\delta(d-x)}\, M(x)\mathbb{E}(P(t+d-x)|\mathcal{F}_t)\, dx$$

$$NC(t) = \int_a^d e^{-\delta(d-x)}\, m(x)\mathbb{E}(P(t+d-x)|\mathcal{F}_t)\, dx$$

The value $M(x)$ represents the percentage of the actuarial value of the future benefits accumulated until the age x with the associate density function m. The support of *m* is the fixed interval $[a, d]$, hence $m(x) = 0$, if $x \leq a$ or $x \geq d$. An interesting particular case is when *M* is the uniform distribution function. We are supposing that all the participants enter the plan at the age a, whereas the common age of retirement is d. So, the actuarial liability *AL* is the total expected value of the promised benefits accumulated according to the distribution function *M*, and the normal cost *NC* is the total expected value of the promised benefits accumulated according to the density function *m*.

In order to carry out a detailed analysis we need a concrete specification of P. In the case of certainty, P would be modeled as an exponential growth, see Bowers *et al*. (1986). In the stochastic case, we consider the benefits outgo as a geometric Brownian motion:

$$dP(t) = \mu P(t)dt + \eta P(t)dB(t), \quad t \geq 0, \tag{1}$$

where $\mu \in \mathbb{R}$ is the instantaneous growth rate of the benefit and $\eta \in \mathbb{R}_+$ is the instantaneous volatility of the benefit. We assume that the initial condition $P(0) = P_0$ is a constant that represents the initial liabilities (note that P_0 could be a random variable). The expected benefit is the exponential function $\mathbb{E}P(t) = P_0\, e^{\mu t}, t \geq 0$. Note that in the particular case where $\eta = 0$, *P* is this same exponential function.

Hence, we are supposing that the benefits increase or decrease on average at a constant exponential rate. By equation (1), the actuarial functions are $AL(t) = \psi_{AL} P(t)$ and $NC(t) = \psi_{NC} P(t)$, where

$$\psi_{AL} = \int_a^d e^{(\mu-\delta)(d-x)}\, M(x)\, dx,$$

$$\psi_{NC} = \int_a^d e^{(\mu-\delta)(d-x)}\, m(x)\, dx.$$

Moreover, $\psi_{NC} = 1 + (\mu - \delta)\psi_{AL}$ and the relation between all the functions is $(\mu - \delta)AL(t) + NC(t) - P(t) = 0$, for every $t \geq 0$.

As a consequence, *AL* and *NC* are also geometric Brownian motions. For instance, *AL* satisfies:

$$dAL(t) = \mu AL(t)dt + \eta AL(t)dB(t), \quad t \geq 0,$$

with the initial condition $AL(0) = \psi_{AL} P_0 = AL_0$.

4.1 The financial market

We consider two types of assets, one without risk and another with risk. We will call the free-risk asset bond $S^0(t)$ and its evolution:

$$dS^0(t) = rS^0(t)dt, \qquad S^0(0) = 1, \qquad (2)$$

where r > 0 is the constant risk-free rate of interest. The risky assets evolution is given by the SDE system:

$$dS^i(t) = S^i(t)(b_i\, dt + \sum_{j=1}^{n} \sigma_{ij} dW_j(t)), \qquad S^i(0) = s_i, \qquad (3)$$

where $b_i > 0$ is the mean rate of return of the risky asset, and $\sigma_{ij} > 0$ are the uncertainty parameters.

We need to extend the probabilistic space where all uncertainty is collected. For it, consider $(W_0, W_1, \dots, W_n)$ an n+1 dimensional standard Brownian motion defined on the complete probabilistic space $(\Omega, G, \mathbb{P})$, where G is the filtration generated by it. In this space, we assume that one dimensional Brownian motions W_i are correlated with the Brownian motion *B*. If we denote the correlations by q_i, then the relation $B(t) = \sqrt{1 - q^T q} W_0 + q^T W$ holds, where $q = (q_1, \dots, q_n)^T$ and $W = (W_1, \dots, W_n)^T$. Then, in terms of *W*, the evolution of *AL* is given by

$$dAL(t) = \mu\, AL(t)dt + \eta\, AL(t) \sqrt{1 - q^T q}\, dW_0(t) + q^T\, dW(t), \quad t \geq 0, \qquad (4)$$

with the initial condition $AL(0) = AL_0$.

We assume that $b_i > r$, for each $i = 1, \dots, n$ so the manager has incentives to invest in assets with risk. $\lambda_i(t)$ denotes the quantity of the fund invested by the manager in the asset i, for $0 \leq i \leq n$. The reminder $F(t) - \sum_{i=1}^{n} \lambda_i(t)$ is invested in the risk-free bond. A negative value of λ_i means that the manager is selling short the corresponding stock and if λ_i is greater than the value of the fund $F(t)$, then the manager is borrowing money at rate r to invest in the stocks.

4.2 The fund wealth

The manager invests the fund in the financial market. Taking into consideration what has been described above, we can obtain that the fund evolution is given by the SDE:

$$dF(t) = \sum_{i=1}^{n} \lambda_i(t) \frac{dS^i(t)}{S^i(t)} + \left(F(t) - \sum_{i=1}^{n} \lambda_i(t)\right) \frac{dS^0(t)}{S^0(t)} + \big(C(t) - P(t)\big)dt. \quad (5)$$

By substituting (2) and (3) in (5), we obtain the SDE that determines the fund evolution:

$$dF(t) = \left(rF(t) + \sum_{i=1}^{n} \lambda_i(t)(b_i - r) + SC(t) + (\mu - \delta)AL(t) \right) dt$$

$$+ \sum_{i=1}^{n} \sum_{j=1}^{n} \lambda_i(t)\, \sigma_{ij} dW_j(t), \quad (6)$$

with initial condition $F(0) = F_0$.

4.3 The optimization problem and the optimal strategies

The manager wishes to minimize a convex combination of the contribution rate risk and the solvency risk in an infinite horizon. Following Haberman *et al.* (2000) and Josa-Fombellida and Rincón-Zapatero (2004), at every instant of time t, we define the solvency risk as the quadratic deviation of the fund assets $F(t)$ with respect to its ideal value $AL(t)$, that is to say, $\mathbb{E}_{F_0, AL_0} UAL(t)^2$. Analogously, we define the contribution rate risk as $\mathbb{E}_{F_0, AL_0} SC(t)^2$. We have denoted by $\mathbb{E}_{F_0, AL_0}$ the conditional expectation given the initial condition (F_0, AL_0). Then the functional objective is

$$J\big((F_0, AL_0); SC\big) = \mathbb{E}_{F_0, AL_0} \int_0^{\infty} e^{-\rho t} \left(kSC^2(t) + (1-k)\big(AL(t) - F(t)\big)^2 \right) dt. \quad (7)$$

The parameter $k \in [0, 1]$ is a weighting factor reflecting the relative importance for the manager to the two different types of risks. When $k = 0$ we are minimizing the solvency risk only, when $k = 1$ we are minimizing the contribution rate risk only, and when $k = 0.5$ we are minimizing both risks with the same priority. The expression (7) implies that the fund manager gives the same importance to over and under deviations of the fund's assets *F* and contributions *C* from their respective targets *AL* and *NC*.

Thus, we are considering a stochastic control problem, (7) subject to (4) and (6), with two state variables, *F* and *AL*, and two control variables *SC* and *λ*. In order to solve it by the dynamic programming approach, we consider the value function that is defined as

$$\hat{V}(F, AL) = \ min_{(SC,\lambda)\in A_{F,AL}} \left\{ J\left((F, AL); \left(SC, \bar{\lambda}\right)\right) : \ s.t.(4), (6)\right\},$$

where $A_{F,AL}$ denotes the set of admissible controls given the initial condition (F, AL). The value function solves the HJB as has been explained in Chapter 3.

The optimal investment strategy can be obtained by means of the HJB equation. Following Josa-Fombellida and Rincón-Zapatero (2004), in order to get the stability of the fund to the long term by means an spread method of the fund, we impose the assumption to the technical rate of actualization

$$\delta = r + \ \eta \ q^T\theta, \tag{8}$$

where $\theta = \sigma^{-1}(b - r\bar{1})$ is the market price of risk or Sharpe ratio. Note that if there is no correlation between benefit and financial market, $q_i = 0$, for all *i*, or if the benefit is deterministic, $\eta = 0$, then $\delta = r$. We have denoted $b = (b_1, \dots, b_n)^T$, $\bar{1} = (1, \dots, 1)^T$ and $\sigma = \left(\sigma_{ij}\right)$.

If the inequality

$$2\mu + \eta^2 < \ \rho$$

is satisfied, then the optimal investment strategy in the risky assets and the optimal rate of contribution can be obtained with the method of the dynamic programming (see the aforementioned paper) and are given by:

$$\lambda^* = \ \Sigma^{-1}(b - r\bar{1})UAL + \ \eta\sigma^{-T}qAL, \tag{9}$$

$$C^* = NC + \frac{\beta_{FF}}{k} \ UAL, \tag{10}$$

where $\Sigma = \sigma\sigma^T$ and β_{FF} is the unique positive solution to the equation

$$\beta_{FF}^2 + k(\ \rho - 2r + \ \theta^T\theta)\beta_{FF} - k(1 - k) = 0.$$

Substituting (8), (9) and (10) in (6), we obtain that the optimal fund wealth is given by the SDE:

$$dF^*(t) = \left(\left(r - \theta^T\theta - \frac{\beta_{FF}}{\kappa}\right)F^*(t) + \left(-r + \theta^T\theta + \frac{\beta_{FF}}{\kappa} + \mu\right)AL(t)\right)dt$$

$$+\big(-\theta^T F^*(t) + (\theta^T + \eta\, q^T)AL(t)\big)dW(t) \qquad (11)$$

with $F(0) = F_0$ and $AL(0) = AL_0$.

Note that (11) together with (4) allows to numerically analyse the optimal fund evolution. From it, by (9) and (10), we can obtain the optimal risky investment evolution and the contribution rate evolution. They are some of our purposes in the following chapter.

5. Numerical illustration

In this section we perform simulations to examine the behaviour of the difference between F and AL and perform a sensitivity analysis of different parameters for two different scenarios, a bull market period and a bear period. To carry out this simulation, the Sim.DiffProc library of the statistical software R has been used.

The volatility and expected return parameters of risky assets have been estimated using the daily closing prices of the IBEX 35 in different periods of time. The data has been extracted from Bolsamania and Yahoo Finance from where it can be downloaded in .csv format. For the bullish period we have the data from 01-01-2014 to 01-06-2015 and for the bearish period we have the data from 01-01-2014 to 01-01-2015. They are represented graphically in Figure 5.1.

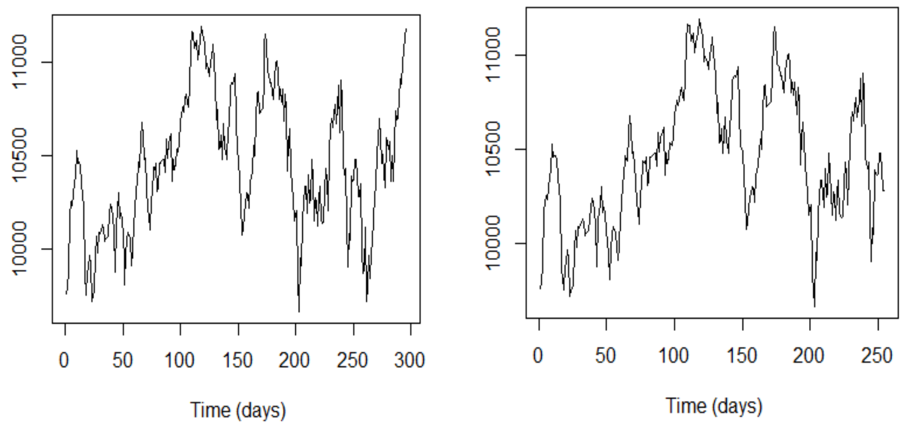

Fig.5.1: Closing prices for bullish and bearish periods

We follow Mosiño *et al.* (2019) for the estimation of parameters. First, we use logarithms to obtain the rate of return where $s(t)$ is the closing value of the IBEX 35 at time t.

$$y(t) = log\left(\frac{s(t+1)}{s(t)}\right)$$

Next, we calculate the mean and variance of the return logarithms and we obtain the estimators $\hat{\sigma}$ and $\hat{b}$. With the indicated steps, for the bullish period we obtain an estimated variability σ_{bull}= 0.167 and an average rate of return $b_{bull} =$ 0.115 and, for the bear period we obtain estimated variability σ_{bear}= 0.184 and an average rate of return $b_{bear} =$ 0.0683.

We assume the following base model:

- The planned time horizon T, despite being infinite, will be limited to 10 years in order to be able to make graphical representations.

- We assume that the rate of discount is $\rho = 0.04$.

- The age of entry to the plan is $a = 25$ years and the age of exit from the plan is $d = 65$ years.

- We consider that the benefits have a mean return $\mu = 0.018$ and standard deviation $\eta = 0.05$

- Although other functions may be considered, we assume, that the distribution function M is uniform in $[a, d]$, that is to say $M(X) = \frac{x-a}{d-a}$ for $a \leq x \leq d$.

- We consider that there is one risky asset (the bull period) where the main rate of return is $b =$ 0.115 and volatility deviation $\sigma = 0.167$. This implies a Sharpe ratio of θ= 0.52994 which has a correlation coefficient with benefits of $q = 0.5$.

- The risk-free rate of interest is equal to $r = 0.0265$.

- The risk preference parameter would be $k = 0.5$, that is, we consider that the manager gives equal value to the contribution rate risk and to the solvency risk to be minimized.

The parameters b and σ have been estimated from daily variation of the IBEX 35 index from 01-01-2014 to 01-06-2015, that is, the bull period. The rest of parameters have been taken from Josa-Fombellida and Navas (2020).

Inspire by these data, initial values for the actuarial liability and the fund are set, respectively to $AL(0) = 100$ and $F(0) = 87.1$, so we consider that at time $t = 0$ the actuarial liability is not totally covered by the fund. The pension plan is called underfunded plan.

The objective in both periods is to maintain the wealth level of the fund as close as possible to actuarial liability (its ideal value), or the difference between the fund and the actuarial liability close to 0, in order to fulfill the payment of the benefits.

5.1 Baseline model

As a illustration, Figure 5.2 shows one of the possible trajectories of the risky asset and the expected value of the risky asset after having performed a thousand simulations. Giving an initial value of $S(0) = 50$ we can observe a growing trend over time.

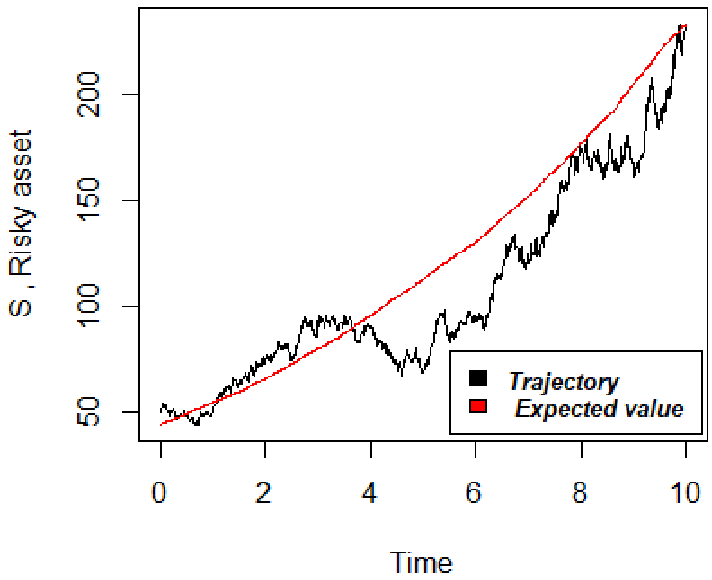

Fig. 5.2: Evolution of the risky asset and its expected value in the bullish period.

Figure 5.3 shows the temporal evolution of the expected optimal surplus $\mathbb{E}X(t)$ that is, the difference between the evolution of the fund and the actuarial liability process $\mathbb{E}\big(F(t) - AL(t)\big)$ over a time horizon of 10 years. As in the previous case, the expected value has been obtained through a thousand realizations. It can be seen how the surplus is bounded by the value 0 and the expected value of the fund approaches to the expected actuarial liability in the long term.

To better understand this situation, we look at Fig. 5.4. This figure shows the time evolution of the expected optimal investment relative to fund size $\mathbb{E}\left(\lambda^{*}/F\right)(t) = \mathbb{E}_{F0,AL0}(\lambda^{*}/F)(t)$. In this case, the global trend is decreasing starting at the highest level and stabilizing in the long run.

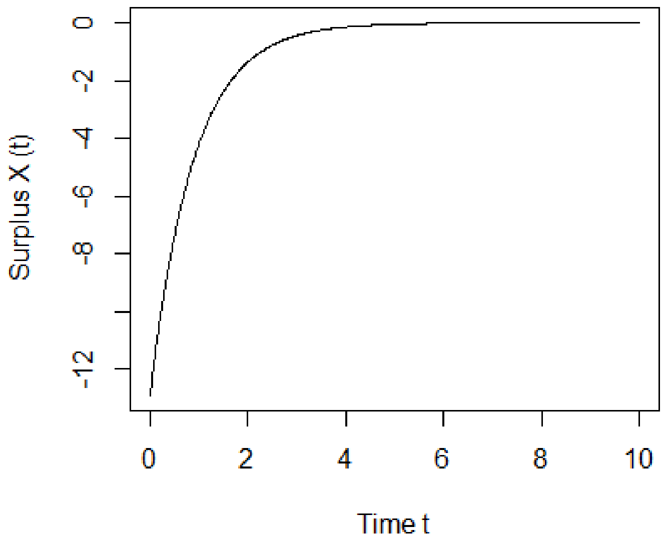

Fig. 5.3 Evolution of the expected surplus $\mathbb{E}$X(t) in the bullish period

The investment is more attractive in the initial stages, which is when a higher return is needed, since the funds that we have at the beginning do not reach the actuarial obligations. As the fund approaches actuarial liability, it is not necessary to take as much risk so the investment in the risky asset decreases. Note that the relative investment has values between 0 and 1, that is, it is not necessary to borrow or short selling money.

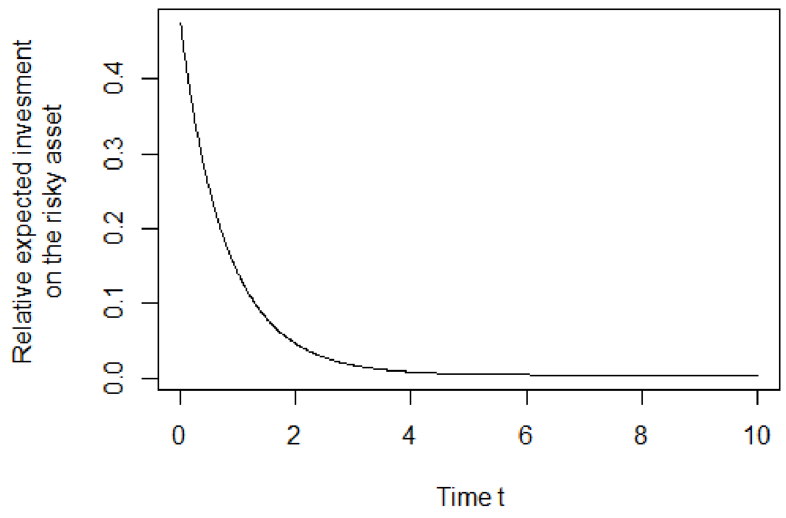

Fig. 5.4 Evolution of the expected investment in the bullish period

Regarding the contributions, we will make a graphic analysis of the control variable $SC^*(t) = C^*(t) - NC(t)$. Fig. 5.5 shows the evolution of the expected supplementary cost. We observe that $\mathbb{E}C^*(t)$ get closer to $\mathbb{E}\,NC(t)$ as t approaches T.

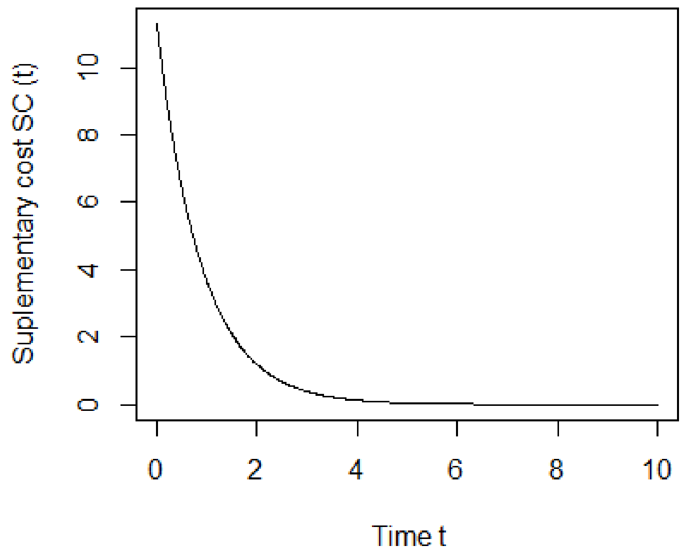

Fig. 5.5 Evolution of the expected suplementary cost in the bullish period

5.2 Bullish – Bearish comparison

In this section, a comparative analysis is carried out on the evolutions of surplus, investment and contributions in two different periods of the market: bullish and bearish. Unlike the bullish period, in the bearish period investing in risky assets is not so attractive, making it more difficult to achieve a balanced budget. In Figure 5.6 the temporal evolution of the expected optimal surplus $\mathbb{E}X(t)$ is represented in both periods. Taking into account that a Sharpe Ratio greater than 0.3 for a period of time indicates an bullish trend, while a value less than 0.3 is an indicator of a bearish trend, in the estimation of our data we have for the upward period a Shape Ratio θ_{bull} = 0.53 and for the bearish period, θ_{bear} = 0.23.

We can observe how the bullish period approaches the ideal value more quickly than the bearish period.

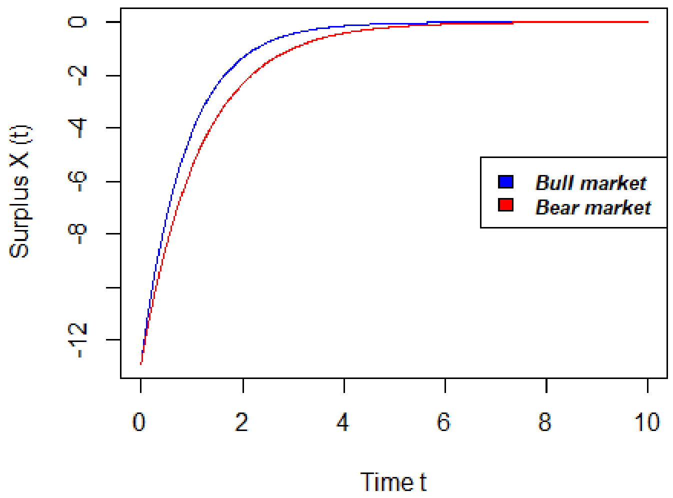

Fig. 5.6 Evolution of the expected surplus $\mathbb{E}X(t)$ in the bullish and bearish periods.

Regarding the evolution of the expected investment, we can see in Figure 5.7 how investment in risky asset is lower in the bearish period than in the bullish period. This is because the expected returns of risky assets in the market during a bearish period are very soft or may even be negative in some cases.

In both cases, there is no indebtedness and the proportion invested in assets with risk remains in values between 0 and 1.

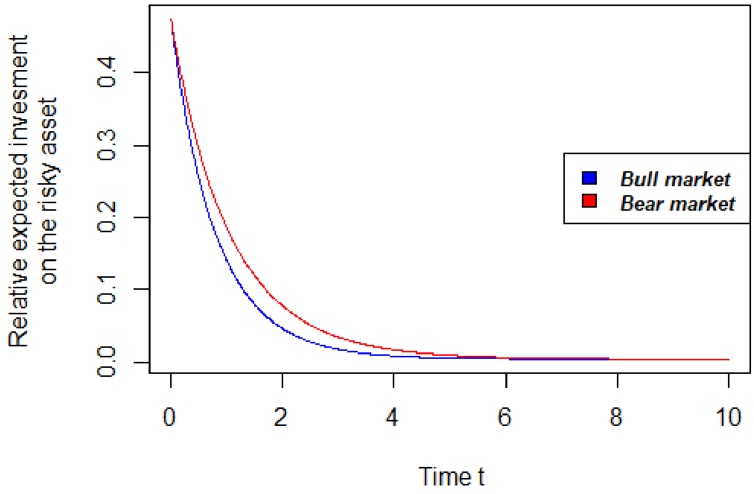

Fig. 5.7 Evolution of the expected investment in the bullish and bearish periods.

In the case of contributions, a decreasing trend can be observed in both cases, that is, as time goes by, the contributions that the participants must make are less because the fund will be profitable and that will imply that it is not necessary contribute with an amount as high as the initial one. As we can see in Figure 5.8, in the bearish case, this decrease is slower than in the bullish case.

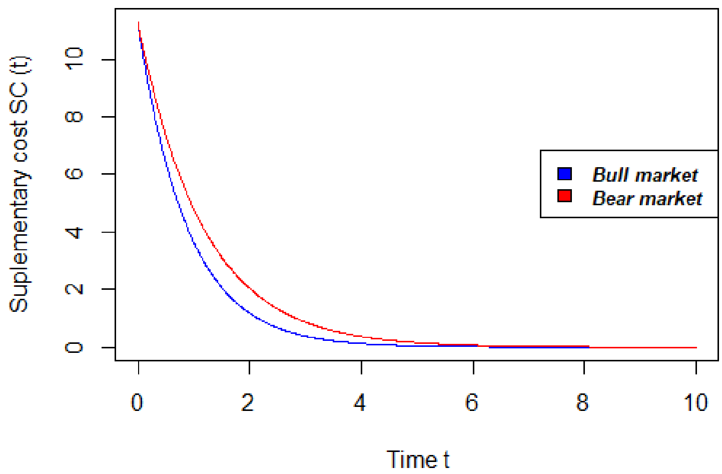

Fig. 5.8 Evolution of the expected supplementary cost in the bullish and bearish periods.

5.3 Sensitivity analysis on parameter k

In this section we study a sensitivity analysis on k. The parameter k, $0 \leq k \leq 1$, is a weighting factor reflecting the relative importance for the employer of the two different types of risks. For example, if we initially take as risk preference k= 0.5, we will say that the manager gives equal value to the contribution rate risk and the solvency risk to be minimized.

A sensitivity analysis of the surplus $X(t)$ with respect to the risk preference parameter k shows a similar behaviour in the approach to the ideal values than in the baseline case, but it is faster with small k. In Fig 5.9 we see how in the case of k = 0, that is, when the objective is to minimize the solvency risk, the fund's approach to AL is very fast, while when we have the case of k = 1 the approach is much slower.

For a fixed t, as the parameter k decreases, that is, as the importance of the weight of the solvency risk increases, the surplus more quickly approaches zero. For instance, for T = 2 years, the values of the expected surplus according to the parameter $k = 1$ is -7.226, while for $k = 0.75$ the expected surplus is -3.12108 and for $k = 0.25$ the expected value is -0.3199.

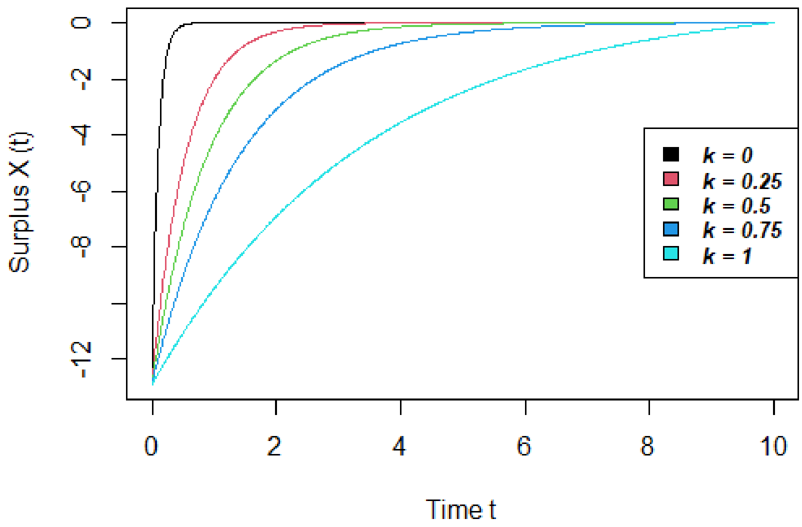

Fig. 5.9 Evolution of the expected surplus for different parameters of k

The opposite property would be fulfilled for the risk of contribution, as k decreases, so does SC. In this case, in contributions with respect to the risk preference parameter k, we see that the decreasing trend is faster for lower values of k. See Fig 5.10.

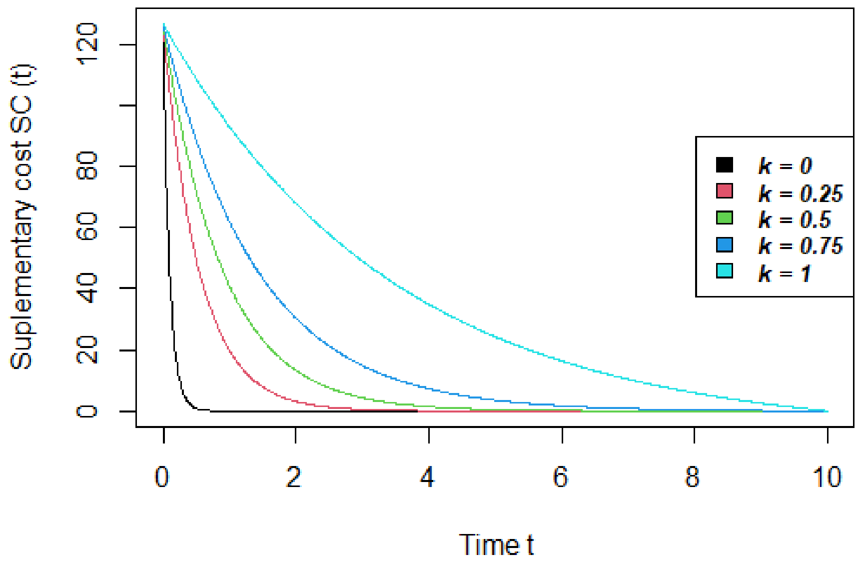

Fig. 5.10 Evolution of the expected supplementary cost for different parameters of k

Regarding relative expected investment on the risky asset, Fig. 5.11, we see how a k close to one pleases investment in risky assets over the longer period of time, although the trend is always decreasing. On the other hand, a value of k close to 0 means that the investment in risky assets is high in the first moments of the life of the fund but then it is very close to zero since it minimizes the solvency risk.

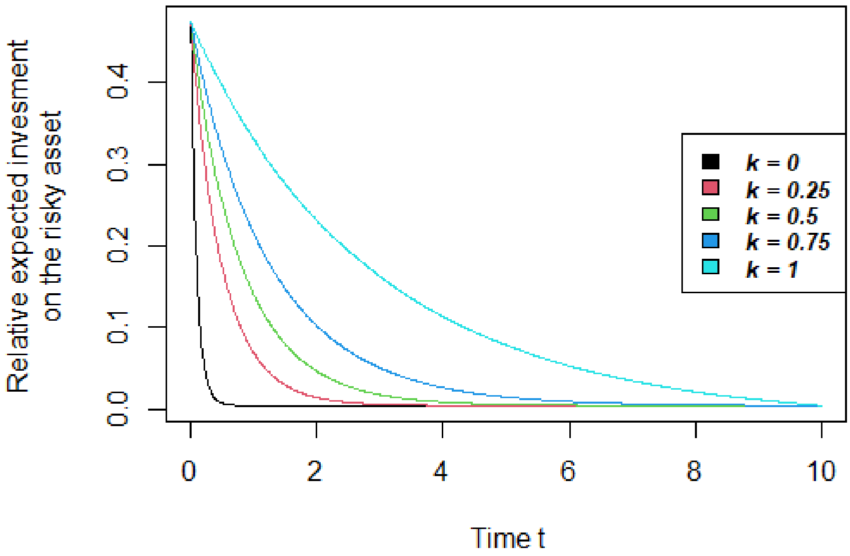

Fig. 5.11 Evolution of the expected investment on the risky asset for different parameters of k

6. Conclusions

In this paper, we have analysed by means of dynamic programming techniques the management of a defined benefit pension plan where the benefits are stochastic. Assuming an underfunded pension plan, the main objective of the manager is to minimize both, solvency and contribution rate risks.

The numerical illustrations based in real data from IBEX 35 have allowed to check the stability properties of the optimal fund and the optimal strategies, as well as to realize a sensitivity analysis of the optimal solutions with respect to parameters, as the time and the weight of each risk in the joint objective, in two different scenarios, bullish and bearish period.

The paper shows that the expected value of the fund converges faster in the long run to the expected value of the actuarial liability when we are in the bullish period than when we are in the bearish period. The same happens with the rest of the variables analyzed, the ideal expected value of the investment and the contributions is closer to their respective ideal value more quickly when we are in the bullish period.

The statistical software R has been useful to study the dynamic behaviour of diffusion stochastic processes along the time and how it is influenced by variations of parameter values. It is a good alternative to other as Mathematica, Maple, Matlab or Python.

As further research we propose:

- To build an interface that allow to obtain graphical and numerical results to a user dynamically in parameters and real data. Shiny of R can be a good option.
- To analyse other economic, financial or insurance dynamic models.
- To provide visualization of stochastic diffusion processes appearing in other problems.

References

1. Arnold, L., 1974. Stochastic Differential Equations. Theory and Applications. John Wiley and Sons, New York.

2. Cairns, A.J.G., 2000. Some notes on the dynamics and optimal control of stochastic pension fund models in continuous time. Astin Bulletin 30, 19-55.

3. Carpintero Gutiérrez, E., 2020. Optimización dinámica de carteras de inversión en tiempo continuo. Trabajo Fin de Grado. Grado en Estadística, Universidad de Valladolid.

4. Devolder, P., Janssen, J., Manca, R., 2012. Stochastic Methods for Pension Funds. Wiley, London.

5. Fleming, W.H., Soner, H.M., 1993. Controlled Markov Processes and Viscosity Solutions. Springer Verlag, New York.

6. Gao, J., 2009. Optimal portfolios for DC pension plans under a CEV model. Insurance Mathematics and Economic. 44, 479–490.

7. Haberman, S., Butt, Z., Megaloudi, C., 2000. Contribution and solvency risk in a defined benefit pension scheme. Insurance: Mathematics and Economics 27, 237-259.

8. Hernández-Lerma, O., 1994. Lectures on continuous-time Markov control processes. Instituto de Publicaciones de la UNAM. México.

9. Josa-Fombellida, R., 2001. Control optimo estocástico. Aplicaciones a la Economía y a la Matemática Financiera. Dpto. de Estadística e Investigación Operativa. Facultad de Ciencias.

10. Josa-Fombellida, R., Rincón-Zapatero, J.P., 2004. Optimal risk management in defined benefit stochastic pension funds. Insurance: Mathematics and Economics 34, 489-503.

11. Josa-Fombellida, R., López-Casado, P., Rincón-Zapatero, J.P., 2018. Portfolio optimization in a defined benefit pension plan where the risky assets are processes with constant elasticity of variance. Insurance: Mathematics and Economics 82, 73-86.

12. Josa-Fombellida, R., Navas, J., 2020. Time consistent pension funding in a defined benefit pension plan with non-constant discounting. Insurance: Mathematics and Economics 94, 142-153.

13. Martínez-Palacios, M.T.V., Venegas-Martínez, F., 2011. Control óptimo estocástico en la enseñanza de la economía matemática. Educación Matemática, vol.23, n.3, pp. 147-181.

14. Mosiño, A., Salomón-Núñez, L.A., Moreno-Okuno, A.T., 2019. Estudio empírico sobre el tipo de cambio MXN/USD: movimiento browniano geométrico versus proceso varianza-gamma. EconoQuantum, vol.16, n.1, pp. 33-56.

15. Peláez Fermoso, F.J., García Gonzalez, A., 2004.Envejecimiento demográfico y equilibrio financiero del sistema público de pensiones.

16. Taylor, G., 2002. Stochastic control of funding systems. Insurance: Mathematics and Economics 30, 323-350.

17. https://es.finance.yahoo.com/quote/%5EIBEX/

18. http://www.oecd.org/daf/fin/private-pensions/globalpensionstatistics.htm

19.https://www.bbva.es/finanzas-vistazo/ef/planes-de-pensiones/como-funciona-plan-pensiones.html