# Stability for Best Experienced Payoff Dynamics*

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**Abstract**

We study a family of population game dynamics under which each revising agent randomly selects a set of strategies according to a given test-set rule; tests each strategy in this set a fixed number of times, with each play of each strategy being against a newly drawn opponent; and chooses the strategy whose total payoff was highest, breaking ties according to a given tie-breaking rule. These dynamics need not respect dominance and related properties except as the number of trials become large. Strict Nash equilibria are rest points but need not be stable. We provide a variety of sufficient conditions for stability and for instability, and illustrate their use through a range of applications from the literature. *JEL* classification numbers: C72, C73.

## 1. Introduction

By assuming that agents apply simple myopic rules to update their strategies, evolutionary game dynamics provide a counterpoint to traditional approaches to prediction in games based on equilibrium knowledge assumptions. To focus on dynamics that only impose mild informational demands on the agents, one can model explicitly how agents obtain information through individual random matches and use this information to decide which strategy to play. For instance, Helbing (1992) and Schlag (1998) show that

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if agents' decisions are based on single matches rather than complete matching, proportional imitation rules lead aggregate behavior to follow the classical replicator dynamic of Taylor and Jonker (1978).

Two distinct approaches based on optimization rather than imitation have also been analyzed. In the approach closer to traditional economic modeling, agents use information from samples of opponents play to form point estimates of the population distribution of actions, and then play a best response to this estimate (Sandholm (2001), Kosfeld et al. (2002), Kreindler and Young (2013), Oyama et al. (2015)). An approach using weaker assumptions about agents' capacities was pioneered by Osborne and Rubinstein (1998) and Sethi (2000). Here a revising agent tests each candidate strategy in random matches against distinct draws of opponents, and then selects the one that earned the highest total payoff during testing. This approach does not assume that agents know the payoffs of the game they are playing, or even that they know they are playing a game; only payoff experiences count.

This paper introduces a general formulation of these *best experienced payoff* (*BEP*) *dynamics*, allowing variation in which candidate strategies agents contemplate, in the number of trials of each such strategy, and in tie-breaking rules. While the resulting dynamics have complicated functional forms, we find that they are surprisingly susceptible to analysis.

We show that if revising agents do not consider all available strategies as candidates for revision, then the rest points of BEP dynamics can include not only strictly dominated strategies, as Osborne and Rubinstein (1998) observe, but even strategies that are guaranteed to perform worse than a given alternative strategy even if opponents choose different responses to each. We also show that strictly dominant strategies are globally asymptotically stable when the number of trials of each tested strategy is large enough. Surprisingly, the latter conclusion is not obvious, but instead requires precise estimates of the likely outcomes of samples at population states in the vicinity of the equilibrium.

The remainder of the paper concerns the instability and stability of strict equilibria. Under the assumption that revising agents know the current state, strict equilibria are stable under very weak assumptions (Sandholm (2014)). But Sethi (2000) shows that under dynamics based on testing each strategy exactly once, strict equilibria need not be stable. His sufficient condition for instability in two-player games, requires that every nonequilibrium strategy $i$ supports invasion of at least two nonequilibrium strategies $j$ and $k$, in that the presence of strategy $i$ in a match makes both $j$ and $k$ outperform the equilibrium strategy.[1]

[1]Sethi (2000) shows that in games with more than two players, it is enough that every nonequilibrium strategy supports invasion by one other nonequilibrium strategy—see Section 5.1 below.

Here we obtain instability results for a considerably more general class of dynamics, and in doing so we identify two qualitatively new sources of instability. First, we show that instability can be driven by the introduction of "spoiler" strategies, whose presence in matches causes the equilibrium strategy to earn a lower payoff than does the second-best response to the equilibrium. Second, we show that instability can be generated not only by the concerted action of all opposing strategies, but by any smaller group of nonequilibrium strategies that through both support and spoiling are mutually supportive of invasion. We complement these analyses with sufficient conditions for stability of strict equilibrium, and we illustrate the wide applicability of our conditions through a range of applications.

Our analyses of local stability all originate from a simple observation. While the general formulas for best experienced payoff dynamics are daunting (see Section 2.3), the behavior of the dynamics near strict equilibria is driven by terms of at most first order in the fractions playing nonequilibrium strategies. This greatly reduces the number of terms relevant to the stability analysis, allowing us to derive our sufficient conditions by direct manipulation and by applying basic results from linear algebra. Specifically, our main instability result is an application of Perron's theorem to the dynamics' Jacobian matrices, and our stability results rely on direct bounds on the flows of agents between strategies and on basic conditions for diagonalizability.

There are some questions about BEP dynamics—the computation of all rest points for a given instance of the dynamics in a given game, and the evaluation of stability of interior rest points—that are not susceptible to the approaches we follow here. In a companion paper, Sandholm et al. (2019), we show how Gröbner bases and other tools from computational algebra, along with approximation results from linear algebra, can be used to answer such questions. We combine these techniques with exact and numerical analyses to provide a complete account of the behavior of BEP dynamics in the Centipede game (Rosenthal (1981)).

As noted above, BEP dynamics have their origins in the work of Osborne and Rubinstein (1998) and Sethi (2000). Osborne and Rubinstein (1998) introduce the notion of $S(k)$ equilibrium to describe stationary behavior among "procedurally rational" agents. This equilibrium concept corresponds to the rest points of the BEP dynamic under which agents test all strategies, subject each to $k$ trials, and break ties via uniform randomization. They present many examples, and show that the limits of $S(k)$ equilibria as $k$ grows large are Nash equilibria. Building on this work, Sethi (2000) introduces the corresponding specification of BEP dynamics, focusing on the case in which strategies are tested once. Sethi (2000) shows that both dominant strategy equilibria and strict equilibria can be repellors under these dynamics, and that dominated strategies can be played in stable

equilibria. He also provides a sufficient condition for repulsion from strict equilibria that includes one restriction on payoffs for each nonequilibrium strategy (see Section 5.1).[2]

Our analysis generalizes the work of Sethi (2000) in various respects. First, by accounting for the effects of spoilers, our sufficient condition for repulsion applies to a larger class of games than that of Sethi (2000). Second, using an analysis of eigenvalues, we obtain conditions under which a strict equilibrium is unstable but not necessarily a repellor. Third, we provide new sufficient conditions for instability, including one that only requires mutual reinforcement by small sets of invading strategies. Finally, our results hold for the general class of BEP dynamics rather than just the basic instance considered by Sethi (2000).

Procedurally rational agents and their associated equilibria have been used in a variety of applications, including trust and delegation of control (Rowthorn and Sethi (2008)), market entry (Chmura and Güth (2011)), use of common-pool resources (Cárdenas et al. (2015)), contributions to public goods (Mantilla et al. (2019)), ultimatum bargaining (Miękisz and Ramsza (2013)), and the Traveler's Dilemma (Berkemer (2008)). As we will show, the general instability and stability criteria we develop here provide a simple and unified way of deriving many of the results that these papers derive individually, as well as several new results.

Our study of best experienced payoff dynamics also contributes to a literature on the aggregate consequences of decision rules that restrict attention to small numbers of alternative strategies. For instance, Berger and Hofbauer (2006) and Hofbauer and Sandholm (2011) show that strictly dominated strategies need not be eliminated when revising agents consider limited numbers of alternatives, as under the BNN (Brown and von Neumann (1950)) and Smith (1984) dynamics: a strictly dominated strategy may achieve the second-best payoff at many states, and so may survive when agents do not always evaluate every strategy. Our analysis of dominated strategies accords with these results. Zusai (2018) introduces a general class of optimizing dynamics that converge globally to Nash equilibrium in contractive games (Hofbauer and Sandholm (2009)), providing a general argument for convergence that allows the set of candidate strategies to be random and incomplete. In a similar spirit, our analysis shows that the instability of strict equilibria is partially robust to the incompleteness of the set of candidate strategies; however, we show that smaller consideration sets must be paired with stronger restrictions on payoffs for instability to be assured.

The remainder of the paper is organized as follows. Section 2 introduces the family of best experienced payoff dynamics. Section 3 evaluates properties of their rest points.

[2]Ramsza (2005) also provides a sufficient condition for stability of strict Nash equilibria.

Section 4 presents elimination and survival results connected to dominance and related notions. Section 5 presents sufficient conditions for the stability and instability of strict equilibria. Section 6 concludes. The appendix presents basic definitions from dynamical systems, provides proofs of most of the results in the paper, and analyzes examples that are not covered by our general results.

# 2. Best experienced payoff dynamics

## 2.1 Single-population matching in symmetric games

We consider a unit-mass population of agents who are matched to play a symmetric $p$-player normal form game $G = \{S, U\}$. This game is defined by a strategy set $S = \{1, \dots, n\}$, and a payoff function $U\colon S^p \to \mathbb{R}$, where $U(i; j_1, \dots, j_{p_-})$ represents the payoff obtained by a strategy $i$ player whose opponents play strategies $j_1, \dots, j_{p_-}$. Our symmetry assumption requires that the value of $U$ not depend on the ordering of the last $p_- \equiv p - 1$ arguments. When $n = 2$, we sometimes write $U_{ij}$ instead of $U(i; j)$.

Aggregate behavior in the population is described by a *population state* $x$ in the simplex $X = \{x \in \mathbb{R}^S_+ \colon \sum_{i \in S} x_i = 1\}$, with $x_i$ representing the fraction of agents in the population using strategy $i \in S$. When describing revision protocols, we also use $X$ to represent the set of mixed strategies. The standard basis vector $e_i \in X$ represents the pure (monomorphic) state at which all agents play strategy $i$.

## 2.2 Revision protocols and evolutionary dynamics

We define evolutionary game dynamics by specifying microfoundations in terms of revision protocols.[3] At each moment in time, each agent has a strategy he uses when matched to play game $G$. Agents occasionally receive opportunities to switch strategies according to independent rate 1 Poisson processes. An agent who receives an opportunity considers switching to a new strategy, making his decision by applying a revision protocol.

Formally, a *revision protocol* is a map $\sigma\colon \mathbb{R}^{S \times S} \times X \to X^S$, where the $X$ before the arrow represents the set of population states, and the $X$ after the arrow represents the set of mixed strategies. For any payoff function $U$ and population state $x \in X$, a revision protocol returns a matrix $\sigma(U, x)$ of *conditional switch probabilities*, where $\sigma_{ij}(U, x)$ is the probability that an agent playing strategy $i \in S$ who receives a revision opportunity switches to strategy $j \in S$.

[3] See Björnerstedt and Weibull (1996), Weibull (1995), Sandholm (2010a,b), and Izquierdo et al. (2019).

Well-known results of Benaïm and Weibull (2003) show that the behavior of a large but finite population following the procedure above is closely approximated by the solution of the associated *mean dynamic*, a differential equation which describes the expected motion of the population from each state:

$$\dot{x}_i = \sum_{j \in S} x_j \sigma_{ji}(U, x) - x_i \quad \text{for all } i \in S. \tag{1}$$

Since revision opportunities are assigned to agents randomly, there is an outflow from each strategy $i$ proportional to its current level of use, accounting for the $-x_i$ term in (1). To generate inflow into $i$, an agent playing some strategy $j$ must receive a revision opportunity, and applying his revision protocol must lead him to play strategy $i$; this leads to the initial term in (1).

## 2.3 Best experienced payoff protocols and dynamics

We now introduce the classes of revision protocols and dynamics we study in this paper. A best experienced payoff protocol is defined by a triple $(\tau, \kappa, \beta)$ consisting of a *test-set rule* $\tau = (\tau_i)_{i \in S}$, a *number of trials* $\kappa$, and a *tie-breaking rule* $\beta = (\beta_i)_{i \in S}$. The triple $(\tau, \kappa, \beta)$ defines a revision protocol in the following way. When an agent currently using strategy $i \in S$ receives an opportunity to switch strategies, he draws a set of strategies $T \subseteq S$ to test according to the distribution $\tau_i$ on the power set of $S$. He then plays each strategy in $T$ in $\kappa$ random matches against members of the opposing population. He thus engages in $|T|\kappa$ random matches in total, facing newly drawn sets of $p_-$ opponents during each. The agent then selects the strategy in $T$ that earned him the highest total payoff, breaking ties according to rule $\beta$.

Proceeding more formally, let $\mathcal{S}_i = \{T \subseteq S \colon i \in T, |T| \geq 2\}$ comprise the subsets of $S$ that include strategy $i$ and at least one other strategy. We define a *test-set distribution* $\tau_i$ used by a strategy $i \in S$ player to be a probability distribution on $\mathcal{S}_i$. Osborne and Rubinstein (1998), Sethi (2000), and subsequent papers have focused on the *test-all* rule, defined by $\tau_i^{\text{all}}(S) = 1$. Here we consider a generalization of test-all that we call *test-$\alpha$*, under which the revising agent tests his current strategy and $\alpha - 1$ strategies chosen at random from the $n - 1$ that remain:

$$\tau_i^{\alpha}(T) = \binom{n-1}{\alpha-1}^{-1} \quad \text{for all } T \in \mathcal{S}_i \text{ with } |T| = \alpha. \tag{2}$$

The key consequence of using test-$\alpha$ rules is that revising agents may wind up not consid-

ering strategies that perform well at the current state, just as in the classical BNN (Brown and von Neumann (1950)) and Smith (1984) dynamics.[4] The opposite extreme from $\tau^{\text{all}}$, under which revising agents test just $\alpha = 2$ strategies, will be called test-two, and denoted $\tau^{\text{two}}$. Most of our results extend to protocols in which $\alpha$ is itself random, and some require still less structure—see Section 6 below.

A *tie-breaking rule* for strategy $i \in S$, denoted $\beta_i$, is a function that for each vector $\pi$ of realized payoffs and each set of tested strategies $T \in \mathcal{S}_i$ specifies the probability $\beta_{ij}(\pi, T)$ of adopting strategy $j \in S$.[5] Since agents are payoff maximizers, $\beta_{ij}(\pi, T)$ may only be positive if $j \in \operatorname{argmax}_{k \in T} \pi_k$. If there is a unique optimal strategy in $T$, it is chosen with probability one; in general, $\beta_{i\cdot}(\pi, T)$ is a probability distribution on $S$ whose support is contained in $\operatorname{argmax}_{k \in T} \pi_k$. In normal form games, tie-breaking rules only matter in nongeneric cases.[6] Osborne and Rubinstein (1998) and Sethi (2000) use the *uniform-if-tie* rule, defined by

$$\beta_{ij}(\pi, T) = \frac{1}{\#(\operatorname{argmax}_{k \in T} \pi_k)} \quad \text{if } j \in \operatorname*{argmax}_{k \in T} \pi_k. \tag{3}$$

The tie-breaking rules we find most natural are *stick-if-tie* rules, which always select the agent's current strategy if it is among the optimal tested strategies:

$$\beta_{ii}(\pi, T) = 1 \text{ whenever } i \in \operatorname*{argmax}_{k \in T} \pi_k. \tag{4}$$

Condition (4) completely determines $\beta_i$ under test-set rule $\tau^{\text{two}}$. One full specification for games with many strategies uses uniform tie breaking whenever the agent's current strategy is not optimal:

$$\beta_{ij}(\pi, T) = \begin{cases} 1 & \text{if } i = j \in \operatorname{argmax}_{k \in T} \pi_k, \\ \frac{1}{\left|\operatorname{argmax}_{k \in T} \pi_k\right|} & \text{if } i \notin \operatorname{argmax}_{k \in T} \pi_k \text{ and } j \in \operatorname{argmax}_{k \in T} \pi_k. \end{cases} \tag{5}$$

In applications in which the ordering on strategies is meaningful, agents may use tie-breakers that account for this ordering. For instance, the *min-if-tie* rule favors strategies with smaller indices:

$$\beta_{ij}(\pi, T) = 1 \text{ if } j = \min\left(\operatorname*{argmax}_{k \in T} \pi_k\right). \tag{6}$$

[4] For revision protocols based on random consideration sets and exact best responses, see Zusai (2018).

[5] For now the notation $\pi$ is a placeholder; it will be defined in equation (7b). The values assigned to components of $\pi$ corresponding to strategies outside of $T$ are irrelevant, and $\beta_{ij}(\pi, T) = 0$ whenever $j \notin T$.

[6] In extensive form games and in BEP dynamics with $\kappa > 1$, it is common for different strategies to earn the same payoffs, giving tie-breaking rules greater importance; see Sandholm et al. (2019).

In what follows, we denote the uniform-if-tie rule (3), the stick-if-tie rule (5), and the min-if-tie rule (6) by $\beta^{\text{unif}}$, $\beta^{\text{stick}}$, and $\beta^{\text{min}}$, respectively.

At last, given a collection $(\tau^\alpha, \kappa, \beta)$, we define the corresponding *best experienced payoff protocol* as follows:

$$\sigma_{ij}(U, x) = \sum_{T \in \mathcal{S}_i} \tau_i^\alpha(T) \left[ \sum_{m \in M_T} \left( \prod_{k \in S} x_k^{|m^{-1}(k)|} \right) \beta_{ij}(\pi^U(m), T) \right], \tag{7a}$$

$$\text{where } \pi_k^U(m) = \sum_{\ell=1}^{\kappa} U(k; m_{k,1,\ell}, \ldots, m_{k,p_-,\ell}) \text{ for all } k \in T. \tag{7b}$$

The interior sum in (7a) is taken over all *match assignments* in the set $M_T = \{m \mid m\colon T \times \{1, \ldots, p_-\} \times \{1, \ldots, \kappa\} \to S\}$; when the agent tests strategy $k$ for the $\ell$th time, $m_{k,q,\ell}$ is the strategy of the $q$th of his $p_- = p - 1$ opponents. The exponent $|m^{-1}(k)|$ is the cardinality of the inverse image of strategy $k$ under the match assignment $m$. One can verify that (7) is the formal expression for the procedure described at the start of this section. That every test of a strategy occurs against an independently drawn opponent is captured by the product in parentheses in expression (7a). This feature of BEP protocols plays a basic role in our analysis.

Inserting (7) into the mean dynamic equation (1) yields the *best experienced payoff dynamic* defined by $\tau^\alpha$, $\kappa$, and $\beta$, called the *BEP*$(\tau^\alpha, \kappa, \beta)$ *dynamic* for short:

$$\dot{x}_i = \sum_{j \in S} x_j \left( \sum_{T \in \mathcal{S}_j} \tau_j^\alpha(T) \left[ \sum_{m \in M_T} \left( \prod_{k \in S} x_k^{|m^{-1}(k)|} \right) \beta_{ji}(\pi^U(m), T) \right] \right) - x_i, \tag{8}$$

with $\pi^U(m)$ defined in (7b).

When each tested strategy is played exactly once, best experienced payoff dynamics only depend on ordinal properties of payoffs. If in addition the underlying normal form game has two players, the formulas for specific instances of the dynamics are relatively simple. Using $\mathbf{1}[\cdot]$ to denote the indicator function, and writing $U_{ij}$ for $U(i; j)$, the BEP$(\tau^{\text{all}}, 1, \beta^{\text{min}})$ dynamic is expressed as

$$\dot{x}_i = \sum_{m\colon S \to S} \left( \prod_{\ell \in S} x_{m_\ell} \right) \mathbf{1}\left[ i = \min\left( \operatorname*{argmax}_{k \in S} U_{k m_k} \right) \right] - x_i. \tag{9}$$

Here the fact that an agent's choice probabilities do not depend on his current strategy leads to a particularly compact expression.

For its part, the BEP$(\tau^{\text{two}}, 1, \beta^{\text{stick}})$ dynamic is written as

$$\dot{x}_i = \frac{1}{n-1} \sum_{h \neq i} \sum_{(k,\ell) \in S \times S} x_k \, x_\ell \, (x_i \mathbf{1}[U_{ik} \geq U_{h\ell}] + x_h \mathbf{1}[U_{ik} > U_{h\ell}]) - x_i. \tag{10}$$

The first term in parentheses captures the case in which the revising agent is a strategy $i$ player who continues to play strategy $i$, while the second term captures the case in which the revising agent is a strategy $h \neq i$ player who switches to strategy $i$. The stick-if-tie rule requires that different payoff inequalities be applied in these two cases.

The definitions above generalize those of the models studied by Osborne and Rubinstein (1998) and Sethi (2000). Specifically, the rest points of the BEP($\tau^{\text{all}}, \kappa, \beta^{\text{unif}}$) dynamic are Osborne and Rubinstein's (1998) $S(k)$ equilibria (with $k = \kappa$), and Sethi's (2000) analysis concerns the corresponding BEP($\tau^{\text{all}}, 1, \beta^{\text{unif}}$) dynamic. The examples and analyses in the sections to come display both differences and commonalities in the predictions generated by different BEP dynamics.

# 3. Rest points

We start our analysis by deriving some basic properties of rest points of BEP dynamics. First we have an immediate observation about pure and strict Nash equilibria. The observation shows that BEP dynamics agree with traditional analyses of games in one basic sense, and highlights an important feature of stick-if-tie rules.

**Observation 3.1.** *Strict equilibria are rest points under all BEP dynamics. Pure equilibria are rest points under any BEP($\tau^\alpha, \kappa, \beta$) dynamic for which $\beta$ is a stick-if-tie rule* (4)*.*

In the second claim, the restriction to stick-if-tie rules ensures that indifferent agents do not switch to alternate best responses at equilibria that are pure but not strict. By definition, other tie-breaking rules do not possess this property.

Next we present a result on the limiting behavior of sequences of rest points of BEP dynamics as the number of trials grows large: for any specification of BEP dynamics, limits of such sequences are Nash equilibria.

**Proposition 3.2.** *Fix $\tau$ and $\beta$, and let $(x^\kappa)_{\kappa=1}^\infty$ be a convergent sequence of rest points of BEP($\tau^\alpha, \kappa, \beta$) dynamics for game $G$. Then the limit $x^*$ of this sequence is a Nash equilibrium of $G$.*

Proposition 3.2 generalizes Proposition 4 of Osborne and Rubinstein (1998), which concerns the BEP($\tau^{\text{all}}, \kappa, \beta^{\text{unif}}$) dynamic. The main novelty in Proposition 3.2 is its allowance for more general test set rules under which revising agents may have no optimal strategies in their test sets. To account for this, the proof of Proposition 3.2 shows that for large $\kappa$, agents rarely switch to strategies that lower their expected payoffs at the current state, and

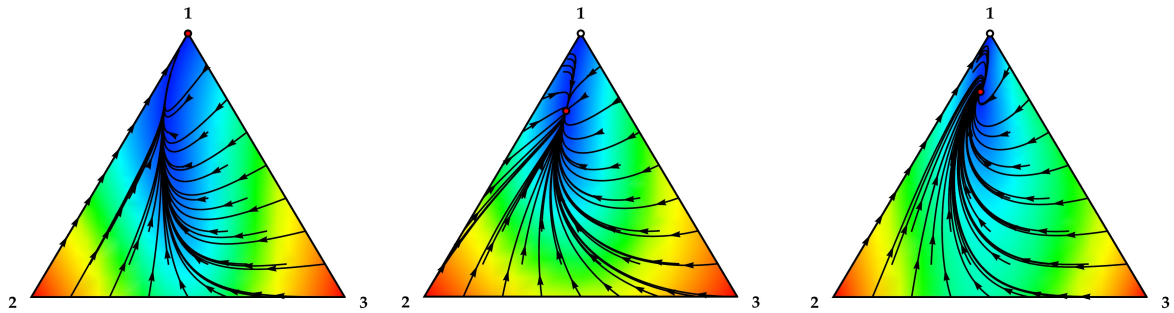
then uses this fact to argue that the mass that rest point $x^\kappa$ places on suboptimal strategies must be close to zero.

We conclude this section with an example that illustrates how predictions of play may differ among different specifications of BEP dynamics, and that motivates the stability analyses of strict equilibria in Section 5.

*Example 3.1.* Consider the following game, which can be viewed as a stylized model of Bertrand competition with a price floor:

$$U = \begin{pmatrix} 2 & 4 & 4 \\ 0 & 3 & 6 \\ 0 & 0 & 5 \end{pmatrix}.$$

Osborne and Rubinstein (1998) prove that the only rest point of BEP($\tau^{\text{all}}, 1, \beta^{\text{unif}}$) is the strict equilibrium $e_1$ (Figure 1(i)). This conclusion is sensitive both to the choice of the test-set rule $\tau^{\text{all}}$ and of the number of trials $\kappa = 1$. If agents instead test two strategies, or if they apply $\tau^{\text{all}}$ but test strategies in the test set twice, we obtain dynamics under which the strict equilibrium $e_1$ is unstable, and which instead possess stable interior rest points (Figures 1(ii) and 1(iii)).[7] ◆




(i) BEP($\tau^{\text{all}}, 1, \beta^{\text{unif}}$) (ii) BEP($\tau^{\text{two}}, 1, \beta^{\text{unif}}$) (iii) BEP($\tau^{\text{all}}, 2, \beta^{\text{unif}}$)

Figure 1: Bertrand competition under different BEP dynamics.

[7] In the figures, colors represent speed of motion: red is fastest, blue is slowest. All figures in this paper were created using EvoDyn3s (Izquierdo et al. (2018)), which also performs exact computations of rest points and exact linearization analyses.

# 4. Global elimination and convergence results

## 4.1 Overwhelmed and overwhelming strategies

Under best experienced payoff protocols, a revising agent faces different opponents when testing each of his own strategies. This leads him to make payoff comparisons that do not arise in traditional game-theoretic analyses, which are generally based on best responses to fixed beliefs.[8] Thus what is required under BEP protocols for a strategy to "always be better" than another is more than the usual dominance relation.

With this motivation, we say that strategy $i$ *overwhelms* strategy $j$ in game $G = \{S, U\}$ if $U(i; k_1, \ldots, k_{p_-}) > U(j; \ell_1, \ldots \ell_{p_-})$ for all choices of $k_1, \ldots k_{p_-} \in S$ and $\ell_1, \ldots \ell_{p_-} \in S$. This condition ensures that even if strategies $i$ and $j$ are tested against distinct sets of opponents, strategy $i$ always earns a higher payoff. Likewise, we say that strategy $s \in S$ is *overwhelming* if it overwhelms all other strategies.

Surprisingly, overwhelmed strategies need not be eliminated under BEP dynamics.

*Example 4.1.* In the game

$$U = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 2 & 2 & 0 \end{pmatrix},$$

strategy 1 is overwhelmed by strategy 2. Even so, Figure 2 shows that under the BEP($\tau^{\text{two}}, 1, \beta^{\text{unif}}$) dynamic, the unique rest point $x^* = (0.049, 0.662, 0.289)$ has strategy 1 being used by a positive fraction of the population.

The intuition behind the survival of strategy 1 is straightforward. Strategy 3 is "weakly overwhelmed" by strategy 2. But since both strategies get the same payoff against strategy 2 and since tie-breaking is uniform, strategy 3 is not eliminated. Once strategy 3 maintains a presence in the population, so does strategy 1, since an agent who tests strategy 3 against an opponent who plays strategy 3 will always choose the other strategy in his test set, be it strategy 2 or strategy 1.

It can be shown that strategy 1 is played with positive probability in any rest point of any BEP($\tau^{\text{two}}, \kappa, \beta^{\text{unif}}$) dynamic, regardless of the choice of $\kappa$. Proposition 3.2 implies that the presence of strategy 1 at any rest point will tend to 0 as the number of trials $\kappa$ goes to infinity. ◆

[8] Maxmin play in zero-sum games is an obvious exception to this rule.

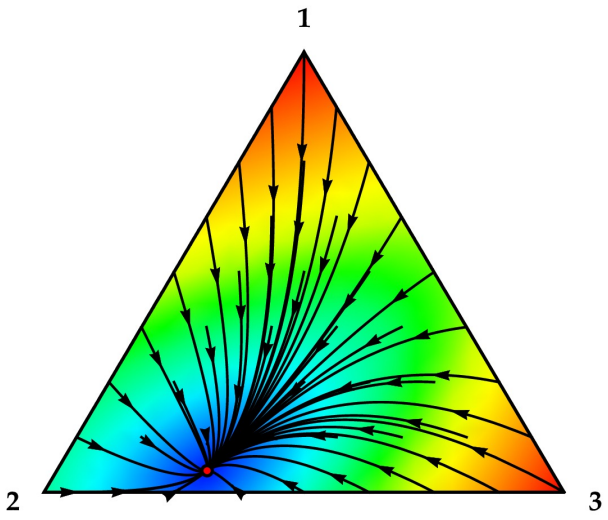



Figure 2: Survival of an overwhelmed strategy under the BEP($\tau^{\text{two}}, 1, \beta^{\text{unif}}$) dynamic.

The following simple proposition shows that overwhelmed strategies are eliminated by BEP dynamics that use the test-all rule,[9] and that overwhelming strategies are globally asymptotically stable under any test-set rule. The proof of the latter result must account for the fact that the overwhelming strategy need not be present in a revising agent's randomly-determined test set.

**Proposition 4.1.** (*i*) *Suppose that strategy $j$ is overwhelmed. Then the set $\{x \in X\colon x_j = 0\}$ is globally asymptotically stable under all BEP($\tau^{\text{all}}, \kappa, \beta$) dynamics.*

(*ii*) *Suppose that strategy $i$ is overwhelming. Then state $e_i$ is globally asymptotically stable under all BEP($\tau^{\alpha}, \kappa, \beta$) dynamics.*

*Proof.* For part (i), the $\tau^{\text{all}}$ rule ensures that revising agents always test both strategy $j$ and the strategy $i$ that overwhelms it, implying that no revising agent ever chooses $j$. It follows that $\dot{x}_j = -x_j$, which implies the result.

For part (ii), if a revising player is not playing $i$, he tests this strategy with at least probability $\frac{1}{n-1}$ (the probability it is tested under $\tau^{\text{two}}$), and if he is playing $i$, he tests it for sure. Since $i$ is overwhelming, it performs best whenever it is tested. These observations and the definition (1) of the dynamics imply that

$$\dot{x}_i \geq \left(x_i + \tfrac{1}{n-1}(1 - x_i)\right) - x_i = \tfrac{1}{n-1}(1 - x_i).$$

The last expression is positive when $x_i < 1$, implying the result. □

[9] See Osborne and Rubinstein (1998, Proposition 3) for a related result.

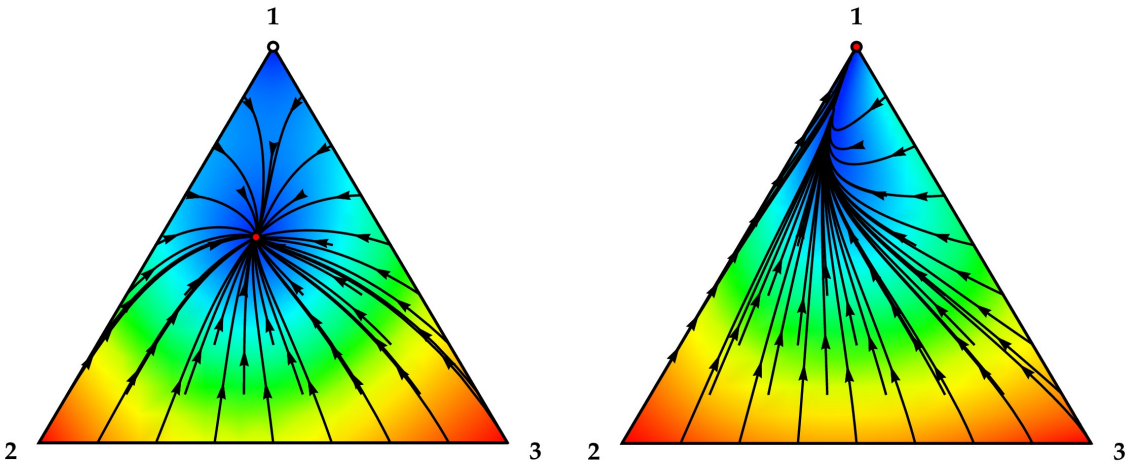
## 4.2 Dominated and dominant strategies

Because agents using BEP protocols may face different opponents when testing different strategies, domination has limited predictive power under BEP dynamics, at least when the number of trials is small.

*Example 4.2.* Consider the following public good contribution game from Osborne and Rubinstein (1998) (see also Sethi (2000) and Mantilla et al. (2019)) in which each unit of effort has a cost of $c = 4$ but confers a benefit of $b = 3$ on both players.

$$U = \begin{pmatrix} 0 & 3 & 6 \\ -1 & 2 & 5 \\ -2 & 1 & 4 \end{pmatrix}. \tag{11}$$

Clearly, choosing zero effort is a strictly dominant strategy. Game (11) is used by Osborne and Rubinstein (1998) to show that strictly dominated strategies can be used in rest points of the BEP($\tau^{\text{all}}, 1, \beta^{\text{unif}}$) dynamic, and by Sethi (2000) to show that dominant-strategy equilibria can be unstable and that rest points using strictly dominated strategies can be stable (cf. Figure 3(i)). Both of these conclusions depend on strategies being tested only once: see Figure 3(ii) and Proposition 4.3 below. We analyze the stability of equilibrium in a general class of public good contribution games in Examples 5.1 and 5.4. ◆




(i) $\kappa = 1$ (ii) $\kappa = 4$

Figure 3: A public good provision game under BEP($\tau^{\text{all}}, \kappa, \beta^{\text{unif}}$) for $\kappa = 1$ and $\kappa = 4$.

Before presenting the elimination result, we first indicate limitations on how much weight strictly dominated strategies may receive at rest points of BEP dynamics.

**Proposition 4.2.** *Suppose that strategy $i$ strictly dominates strategy $j$. Under any BEP$(\tau^\alpha, \kappa, \beta)$ dynamic, all rest points $x^*$ satisfy $x_i^* \geq x_j^*$.*

The proof of Proposition 4.2 creates pairs of match assignments in which the matches assigned to strategies $i$ and $j$ are reversed. This device allows us to take advantage of the dominance relation between $i$ and $j$, once we state the conditions for strategies $i$ and $j$ to be at rest in a form that does not refer to self-switches (e.g., decisions by $i$ players to continue playing strategy $i$). Proposition 4.2 builds on Proposition 2 of Osborne and Rubinstein (1998), which establishes the analogous comparison for weakly dominant strategies under BEP$(\tau^{\text{all}}, \kappa, \beta^{\text{unif}})$ dynamics. This conclusion for weakly dominated strategies is sensitive to the tie-breaking rule.

*Example 4.3.* Consider a game $U$ in which all payoffs are 0, and suppose that test sets are determined using a test-$\alpha$ rule. Under uniform tie-breaking, the unique rest point is $x^* = (\frac{1}{n}, \ldots, \frac{1}{n})$, agreeing with the conclusion of the proposition. But under any stick-if-tie rule (4), all states are rest points, and under min-if-tie (6), only state $e_1$ is a rest point. ◆

With a large enough number of trials, the empirical distributions of opponents' play that an agent faces when testing each of his strategies should each be close to the current population state. This should lead the agent not to choose a dominated strategy when the dominating strategy is available. We next develop this intuition into a global stability result. Paralleling Proposition 4.1, the elimination result for dominated strategies is only proved for the $\tau^{\text{all}}$ rule, while the selection result for dominant strategies is proved for general test-set rules.

**Proposition 4.3.** (*i*) *Suppose that strategy $j$ is strictly dominated. Then for each $\varepsilon > 0$, the set $\{x \in X : x_j \leq \varepsilon\}$ is globally asymptotically stable under BEP$(\tau^{\text{all}}, \kappa, \beta)$ dynamics for all large enough numbers of trials $\kappa$.*

(*ii*) *Suppose that strategy $s$ is strictly dominant. Then state $e_s$ is globally asymptotically stable under BEP$(\tau^\alpha, \kappa, \beta)$ dynamics for all large enough numbers of trials $\kappa$.*

The proof of part (i) shows that for $\kappa$ large enough, the proportion of agents playing a strictly dominated strategy decreases monotonically until it is used by less than an $\varepsilon$ fraction of the population. For a strictly dominant strategy $s$, Proposition 5.7 below shows that state $e_s$ is locally asymptotically stable once $\kappa$ is large enough. Combining these two facts is not enough to conclude that $e_s$ is globally stable for large enough numbers of trials, since in principle, the basins of attraction of $e_s$ could become arbitrarily small as $\kappa$ grows large. To complete the proof of part (ii), we use results from large deviations theory (Dembo and Zeitouni (1998)) to obtain precise upper bounds on the probabilities

of unrepresentative results during testing when $x_s$ is small. Doing so enables us to show that the rate of inflow into strategy $s$ due to revising players choosing it always exceeds the rate of outflow of $-x_s$.

# 5. Instability and stability of strict equilibria

Sethi (2000) shows that a strict symmetric equilibrium $s$ of a normal form game can be stable or unstable under BEP($\tau^{\text{all}}, 1, \beta^{\text{unif}}$) dynamics, and provides a sufficient condition for instability. In this section, we introduce new sufficient conditions for instability and stability of strict equilibrium for general BEP($\tau, \kappa, \beta$) dynamics.

All of our analyses build on a simple observation about the behavior of BEP dynamics near strict equilibria, versions of which underlie all existing stability analyses. If $s$ is a strict equilibrium, the polynomial form of the dynamics (8) implies that in a neighborhood of the state $x_s$, match assignments $m$ (see (7)) can be ranked in probability by the number of matches against opponents who do not use strategy $s$. If all opponents use $s$, then the strict equilibrium $s$ earns the best experienced payoff. Thus in generic cases, local stability can be determined from the consequences of match assignments in which exactly one match out of the $\alpha p\_\kappa$ in total is against an opponent who plays a strategy besides $s$. The analyses to follow come down to accounting for the consequences of such assignments.

With this motivation, we introduce notation needed to state the results to come:

$$\begin{aligned} u_{i|\boldsymbol{s}} &= U(i; s, s, ..., s), \qquad & v^{\kappa}_{i|\boldsymbol{s}} &= \kappa u_{i|\boldsymbol{s}}, \\ u_{i|j,\boldsymbol{s}} &= U(i; j, s, ..., s), \qquad & v^{\kappa}_{i|j,\boldsymbol{s}} &= (\kappa - 1)u_{i|\boldsymbol{s}} + u_{i|j,\boldsymbol{s}}. \end{aligned} \tag{12}$$

In words, $u_{i|\boldsymbol{s}}$ is the payoff to playing $i$ when all opponents play the equilibrium strategy $s$, and $u_{i|j,\boldsymbol{s}}$ is the payoff to playing $i$ when all but one opponent plays the equilibrium strategy $s$, with the remaining opponent playing $j$. Likewise, $v^{\kappa}_{i|\boldsymbol{s}}$ is the total payoff in $\kappa$ trials from always playing $i$ against opponents playing $s$, and $v^{\kappa}_{i|j,\boldsymbol{s}}$ is the total payoff to $i$ in $\kappa$ trials when all but one opponent in all trials play $s$, with the remaining opponent playing $j$.

Our instability and stability conditions are based on two kinds of inequalities, each representing a distinct contribution to the destabilization of strict equilibrium $s$. We call strategy $j \neq s$ a *supporter* of invasion by strategy $i \neq s$ if

$$v^{\kappa}_{i|j,\boldsymbol{s}} > v^{\kappa}_{s|\boldsymbol{s}}. \tag{13}$$

Thus $j$ supports invasion by strategy $i$ if over the course of $\kappa$ trials, strategy $i$ earns a higher

total payoff than $s$ if $i$ has one match that includes an opponent playing $j$, with all other opponents of $i$ and $s$ being $s$ players. In short, a supporter of invasion works by increasing the payoffs of some invading strategy.

We call strategy $j \neq s$ a *spoiler* that benefits strategy $t \neq s$ if

$$v^{\kappa}_{s|j,\boldsymbol{s}} < v^{\kappa}_{t|\boldsymbol{s}}. \tag{14}$$

Thus $j$ is a spoiler if when one of strategy $s$'s matches includes an opponent playing $j$ (with all remaining opponents playing $s$), the total payoff that $s$ earns in its matches is lower than that of another strategy that only faces $s$. Put differently, a spoiler works by decreasing the payoffs of the strict equilibrium strategy. If a spoiler works to its own benefit, in that $t = j$ in (14), then we describe $j$ as *spiteful*. In this case, $j$ enables itself to outperform $s$ by reducing the payoff that $s$ achieves. The fact that spiteful behavior can influence equilibrium outcomes in finite-player strategic interactions is well known in other contexts; see Schaffer (1988), Bergin and Bernhardt (2004), and the references therein.

## 5.1 Repulsion

To start, we extend work of Sethi (2000) providing a sufficient condition for a strict equilibrium $e_s$ to be a repellor under BEP dynamics, meaning that all solutions of (8) from initial conditions near $e_s$ lead away from $e_s$.

**Proposition 5.1.** *Let $s$ be a strict equilibrium, let $t \in \operatorname{argmax}_{i \neq s} u_{i|\boldsymbol{s}}$, and consider any BEP$(\tau^{\alpha}, \kappa, \beta)$ dynamic. Then state $e_s$ is a repellor if*

$$p_{-}\kappa \left( \sum_{i \neq s} \mathbf{1}[v^{\kappa}_{s|\boldsymbol{s}} < v^{\kappa}_{i|j,\boldsymbol{s}}] + \mathbf{1}[v^{\kappa}_{s|j,\boldsymbol{s}} < v^{\kappa}_{t|\boldsymbol{s}}] \right) > 1 \ \textit{for all } j \neq s. \tag{15}$$

**Corollary 5.2.** *If we define*

$$\bar{\kappa} = \min_{j \neq s} \max \left\{ \max_{i \neq s} \left\lceil \frac{u_{i|j,\boldsymbol{s}} - u_{s|\boldsymbol{s}}}{u_{s|\boldsymbol{s}} - u_{i|\boldsymbol{s}}} \right\rceil, \left\lceil \frac{u_{t|\boldsymbol{s}} - u_{s|j,\boldsymbol{s}}}{u_{s|\boldsymbol{s}} - u_{t|\boldsymbol{s}}} \right\rceil \right\}, \tag{16}$$

*then state $e_s$ is a repellor if $p = 2$ and $\kappa \in \{2, \ldots, \bar{\kappa}\}$, or if $p > 2$ and $\kappa \in \{1, \ldots, \bar{\kappa}\}$.*

The sufficient condition (15) for $e_s$ to be a repellor comprises an inequality for each strategy $j \neq s$. The condition adds up the number of strategies $i$ that $j$ supports as an invader, and adds an additional unit if $j$ is a spoiler. If $p = 2$ and $\kappa = 1$, a total of 2 implies

repulsion; in all other cases, a total of 1 is enough. Corollary 5.2 is obtained by rewriting (15) in terms of the payoffs of the original normal form game.

Sethi's (2000) conditions for an *inferior* and a *twice inferior* strict equilibrium only consider the initial sum in (15).[10] Proposition 5.1 generalizes the instability results in Sethi (2000) for the BEP($\tau^{\text{all}}, 1, \beta^{\text{unif}}$) case to other BEP dynamics and to any number of trials, and also identifies spoilers as a novel source of instability.

Following Sethi's (2000) analysis, the proof of Proposition 5.1 argues that under condition (15), the growth rate $\dot{x}_s$ of strategy $s$ is negative in a neighborhood of the equilibrium $e_s$. As suggested above, the fact that few agents use other strategies means that terms in $\dot{x}_s$ that are of more than linear order in such strategies are of negligible magnitude, letting us focus on revision opportunities and match assignments with exactly one agent playing a strategy other than $s$. By accounting for all of the ways that supporters and spoilers can reduce $\dot{x}_s$, we show that condition (15) leads to a lower bound on $\dot{x}_s$ that is linear in $x_s$.

*Example 5.1.* Consider a $p$-player public good contribution game in which the allowable contributions are $i \in \{0, 1, ..., n-1\}$, $n \geq 2$, and where each unit of contribution costs the contributing player $c$ and provides a benefit of $b < c$ to all players. Payoffs are thus given by

$$U(i; j_1, \ldots, j_{p_-}) = b \sum_{q=1}^{p_-} j_q + (b-c)i, \tag{17}$$

Since $c > b$, contributing zero units is a strictly dominant strategy.

Evidently $u_{i|j,\mathbf{0}} = bj + (b-c)i$ and $v^{\kappa}_{i|j,\mathbf{0}} = bj + \kappa(b-c)i$. Thus if $\kappa < \frac{b}{c-b}$, then $v^{\kappa}_{i|i,\mathbf{0}} > 0 = v^{\kappa}_{0|\mathbf{0}}$ for all $i > 0$, so the sufficient condition for repulsion (15) holds when $p_-\kappa \geq 2$, that is, when there are more than two players or at least two trials.

With two players and one trial, observe that if $\frac{c}{b} < \frac{3}{2}$, then $u_{i|j,\mathbf{0}} > 0 = u_{0|\mathbf{0}}$ when $j \geq i > 0$ and when $(i, j) = (2, 1)$. Thus if $n \geq 3$, then for each $j \neq 0$ the sum in (15) is at least 2, again implying that equilibrium $e_0$ is a repellor under any BEP($\tau^{\alpha}, 1, \beta$) dynamic.

For the case of the BEP($\tau^{\text{all}}, 1, \beta^{\text{unif}}$) dynamic, the conclusions about repulsion above can be obtained using Sethi's (2000) inferiority conditions. The novelty from applying Proposition 5.1 here is that repulsion is established for dynamics based on any test-$\alpha$ rule (2), any number of trials $\kappa$, and any tiebreaking rule $\beta$. ◆

While stability in Example 5.1 relies on support of profitable invasions by other strategies, Proposition 5.1 also shows that BEP dynamics can select against strict equilibria

[10] Strict equilibrium $s$ is *inferior* if $\sum_{i \neq s} \mathbf{1}[u_{i|j,\mathbf{s}} > u_{s|\mathbf{s}}] > 0$ for all $j \neq s$, and *twice inferior* if this sum always exceeds 1. Sethi (2000) proves that under the BEP($\tau^{\text{all}}, 1, \beta^{\text{unif}}$) dynamic, an inferior equilibrium is unstable if $p > 2$, and a twice inferior equilibrium is unstable if $p \geq 2$.

that instead admit spiteful deviations. For a simple result to this effect, we call strict equilibrium $s$ *uniformly susceptible to spite* if

$$u_{j|\mathbf{s}} \geq u_{s|j,\mathbf{s}} \text{ for all } j \neq s. \tag{18}$$

In words, (18) says that each strategy $j$ does better as a unilateral deviation from equilibrium $s$ than $s$ does in a match that includes a single $j$ player.

**Corollary 5.3.** *Suppose $G$ has more than two players. If the strict equilibrium $s$ of $G$ is uniformly susceptible to spite, then state $e_s$ is a repellor under all BEP$(\tau^\alpha, 1, \beta)$ dynamics.*

*Proof.* By condition (18), $u_{s|j,\mathbf{s}} \leq u_{j|\mathbf{s}} \leq u_{t|\mathbf{s}} < u_{s|\mathbf{s}}$, implying that $\bar{\kappa} \geq 1$ in Corollary 5.2. □

*Example 5.2.* Consider a $p > 2$ player, $n = 2$ strategy public good contribution game in which contributing is a strictly dominant strategy: specifically, contributing costs the contributor $c$ but benefits all players $b > c$. If strategy 1 represents contributing to the public good and strategy 2 represents not contributing, then payoffs are given by

$$U(1; j_1, \ldots, j_{p-}) = |\{q\colon\ j_q = 1\}|\, b + (b - c),$$
$$U(2; j_1, \ldots, j_{p-}) = |\{q\colon\ j_q = 1\}|\, b.$$

Clearly, strategy 1 is strictly dominant. But since $u_{2|\mathbf{1}} = (p-1)b > (p-1)b - c = u_{1|2,\mathbf{1}}$, Corollary 5.3 implies that state $e_1$ is a repellor when $\kappa = 1$. In fact, Proposition 5.1 implies that $e_1$ is a repellor for numbers of trials $\kappa$ up to

$$\left\lceil \frac{u_{2|\mathbf{1}} - u_{1|2,\mathbf{1}}}{u_{1|\mathbf{1}} - u_{2|\mathbf{1}}} \right\rceil = \left\lceil \frac{(p-1)b - ((p-1)b - c)}{(pb - c) - ((p-1)b)} \right\rceil = \left\lceil \frac{c}{b-c} \right\rceil. \quad \blacklozenge$$

## 5.2 Instability

In order to provide a condition ensuring that a strict equilibrium repels solutions from all initial conditions, Proposition 5.1 imposes conditions on support and spoiling by all alternative strategies. If our primary interest is in instability rather than repulsion—in particular, if it is enough that solutions from most (rather than all) nearby initial conditions move away from the equilibrium—then intuition suggests that restrictions involving smaller numbers of strategies whose presences are mutually reinforcing might be sufficient for this conclusion.

Proposition 5.4 verifies that the intuition above is correct, providing conditions under which invasions by a mutually reinforcing group of strategies will succeed.

**Proposition 5.4.** *Let $s$ be a strict equilibrium, let $S_2 = \operatorname{argmax}_{i \neq s} u_{i|\boldsymbol{s}}$, and let $t \in S_2$. Under any BEP$(\tau^\alpha, \kappa, \beta)$ dynamic, state $e_s$ is linearly unstable if either of these conditions hold:*

(*i*) *For some nonempty $J \subseteq S \setminus \{s\}$,*

$$p_- \kappa \frac{\alpha - 1}{n - 1} \sum_{j \in J} \mathbf{1}[v^\kappa_{i|j,\boldsymbol{s}} > v^\kappa_{s|\boldsymbol{s}}] > 1 \quad \textit{for all } i \in J. \tag{19}$$

(*ii*) *For some nonempty $J \subseteq S \setminus \{s\}$,*

$$p_- \kappa \frac{\alpha - 1}{n - 1} \left( \sum_{i \in J} \mathbf{1}[v^\kappa_{i|j,\boldsymbol{s}} > v^\kappa_{s|\boldsymbol{s}}] + \mathbf{1}[S_2 \subseteq J]\, \mathbf{1}[v^\kappa_{s|j,\boldsymbol{s}} < v^\kappa_{t|\boldsymbol{s}}] \right) > 1 \quad \textit{for all } j \in J; \tag{20}$$

**Corollary 5.5.** *Strict equilibrium $e_s$ is linearly unstable under any BEP$(\tau^\alpha, \kappa, \beta)$ dynamic if*

$$p_- \kappa \frac{\alpha - 1}{n - 1} \left( \mathbf{1}[v^\kappa_{j|j,\boldsymbol{s}} > v^\kappa_{s|\boldsymbol{s}}] + \mathbf{1}[S_2 = \{j\}]\, \mathbf{1}[v^\kappa_{s|j,\boldsymbol{s}} < v^\kappa_{j|\boldsymbol{s}}] \right) > 1 \quad \textit{for some } j \neq s. \tag{21}$$

Proposition 5.4 tells us that given any set $J \subseteq S \setminus \{s\}$, we can establish that $e_s$ is unstable by showing that each invading strategy $j \in J$ provides an adequate combination of support of invasion and spoiling in favor of strategies that are themselves in $J$. Sufficient condition (19) focuses on the lowest total benefit obtained by any of the invading strategies from the presence of other strategies in the invading group. Sufficient condition (20) focuses instead on the lowest total benefit provided by an invading strategy to other members of the group, accounting both for support and for spoiling. Corollary 5.5 highlights the case of a lone invader.

The proof of Proposition 5.4 uses Perron's theorem to show that under condition (19) or (20), the Jacobian matrix of any BEP$(\tau^\alpha, \kappa, \beta)$ dynamic at equilibrium state $e_s$ has a positive eigenvalue, implying instability. Specifically, consider the "inflow Jacobian" of the dynamic, by which we mean the contribution to the Jacobian of the inflow terms in (1). (The "outflow Jacobian" is $-I$, the negative of the identity matrix.) The conditions of the proposition each imply that the submatrix of the inflow Jacobian corresponding to the set of invading strategies $J$ has Perron eigenvalue greater than 1.[11] Then the monotonicity of the Perron eigenvalue of a nonnegative matrix in the values of its components implies that the full inflow Jacobian, which is a nonnegative matrix, also has Perron eigenvalue greater than 1. Therefore, the complete Jacobian matrix has a positive eigenvalue; this eigenvalue is associated with a nonnegative eigenvector that represents the proportions of strategies in a self-reinforcing invasion.

[11] To be more precise, this is true even after a dimension reduction that eliminates the $x_s$ component of the state: see equations (54), (55), and (52).

Corollary 5.6 provides a number of sufficient conditions for instability stated directly in terms of the payoff matrix of the underlying normal form game. The conditions are sufficient when revising agents test more than half of the available strategies—more precisely, under the assumption that $\alpha \geq \lceil \frac{n}{2} \rceil + 1$. All conditions are easily derived from condition (20). The (a) conditions only reflect support of invasions, while the (b) conditions also account for spoiling.

**Corollary 5.6.** *Let $s$ be a strict equilibrium. The following conditions are sufficient for linear instability of state $e_s$ under BEP$(\tau^\alpha, \kappa, \beta)$ dynamics with $\alpha \geq \lceil \frac{n}{2} \rceil + 1$.*

*(i.a)* $\kappa \geq 2$ *and* $\kappa \leq \max_{i \neq s} \left\lceil \frac{u_{i|i,s} - u_{s|s}}{u_{s|s} - u_{i|s}} \right\rceil$.

*(i.b)* $\kappa \geq 2$, $S_2 = \{j\}$, *and* $\kappa \leq \left\lceil \frac{u_{j|s} - u_{s|j,s}}{u_{s|s} - u_{j|s}} \right\rceil$.

*(ii.a)* $\kappa = 1$, $p > 2$, *and* $\max_{i \neq s} u_{i|i,s} > u_{s|s}$.

*(ii.b)* $\kappa = 1$, $p > 2$, $S_2 = \{j\}$, *and* $u_{j|s} > u_{s|j,s}$

*(iii.a)* $\kappa = 1$, $p = 2$, *and* $\min\{u_{ii}, u_{ij}, u_{ji}, u_{jj}\} > u_{ss}$ *for some pair of distinct strategies* $i, j \neq s$.

*(iii.b)* $\kappa = 1$, $p = 2$, $S_2 = \{j\}$, $u_{j|s} > u_{s|j,s}$, *and* $u_{j|j,s} > u_{s|s}$

*Example 5.3.* The Traveler's Dilemma (Basu (1994)) is a normal form analogue of the Centipede game (Rosenthal (1981)) in which the unique rationalizable strategy earns the players far less than many other strategy profiles. Payoffs in the $n$-strategy Traveler's Dilemma are

$$U(i,j) = \begin{cases} i+2 & \text{if } i < j, \\ i & \text{if } i = j, \\ j-2 & \text{if } i > j, \end{cases}$$

or, in matrix form,

$$\text{(22)} \qquad \begin{pmatrix} 1 & 3 & 3 & 3 & \cdots & 3 \\ -1 & 2 & 4 & 4 & \cdots & 4 \\ -1 & 0 & 3 & 5 & \cdots & 5 \\ -1 & 0 & 1 & 4 & \ddots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \ddots & n+1 \\ -1 & 0 & 1 & \dots & n-3 & n \end{pmatrix}$$

Strategy 1 is the unique Nash equilibrium and the unique rationalizable strategy. Berkemer (2008) shows that under the BEP$(\tau^{\text{all}}, \kappa, \beta^{\text{unif}})$ dynamic, state $e_1$ is stable when is $n$ odd and $\kappa > \frac{n+1}{2}$, and that it is unstable when $n$ is odd and $\kappa = \frac{n+1}{2}$.

We consider the stability of $e_1$ under any BEP$(\tau^\alpha, \kappa, \beta)$ dynamic with $\alpha \geq \lceil \frac{n}{2} \rceil + 1$. If $\kappa = 1$,

Corollary 5.6(iii.a) implies that $e_1$ is unstable if $n \geq 5$: choosing $i = n-1$ and $j = n$ leads to $\min\{u_{ii}, u_{ij}, u_{ji}, u_{jj}\} = n-3 > 1 = u_{11}$. If instead $2 \leq \kappa \leq \lfloor \frac{n}{2} \rfloor$, then Corollary 5.6(i.a) implies that $e_1$ is unstable for $2 \leq \kappa \leq \lfloor \frac{n}{2} \rfloor$: this follows because $\max_{i \neq 1} \lceil \frac{u_{ii}-u_{11}}{u_{11}-u_{i1}} \rceil = \lceil \frac{n-1}{1-(-1)} \rceil = \lceil \frac{n-1}{2} \rceil = \lfloor \frac{n}{2} \rfloor$. In Section 5.4 we consider stability in Traveler's Dilemmas with larger numbers of trials. ◆

## 5.3 Stability

We now provide some sufficient conditions for a strict equilibrium to be stable. One can write down more general conditions for stability, but the ones we focus on are simple, and enough to complete the analyses of some of our earlier examples.

Proposition 5.7 is our first sufficient condition. It requires that no strategy supports invasion of the strict equilibrium or is a spoiler against the strict equilibrium. The proof of the proposition shows that under this condition, the growth rate of $x_s$ must be positive at states close to $e_s$.

**Proposition 5.7.** *Let $s$ be a strict equilibrium. Then state $e_s$ is asymptotically stable under any BEP$(\tau^\alpha, \kappa, \beta)$ dynamic if*

$$v^\kappa_{s|\boldsymbol{s}} > v^\kappa_{i|j,\boldsymbol{s}} \;\; \textit{and} \;\; v^\kappa_{s|j,\boldsymbol{s}} > v^\kappa_{i|\boldsymbol{s}} \;\; \textit{for all } i, j \neq s. \tag{23}$$

The immediate Corollary 5.8 provides sufficient conditions for stability in terms of payoffs in the normal form game and the number of trials $\kappa$.

**Corollary 5.8.** *State $e_s$ is asymptotically stable under any BEP$(\tau^\alpha, \kappa, \beta)$ dynamic if*

$$u_{s|\boldsymbol{s}} > u_{i|j,\boldsymbol{s}} \;\; \textit{and} \;\; u_{s|j,\boldsymbol{s}} > u_{i|s} \;\; \textit{for all } i, j \neq s. \tag{24}$$

*or if*

$$\begin{aligned} &\kappa \geq \underline{\kappa}^1 \equiv \max_{i \neq s} \left\lfloor \frac{\max_{j \neq s} u_{i|j,\boldsymbol{s}} - u_{s|\boldsymbol{s}}}{u_{s|\boldsymbol{s}} - u_{i|\boldsymbol{s}}} \right\rfloor + 2 \;\; \textit{and} \\ &\kappa \geq \underline{\kappa}^2 \equiv \max_{i \neq s} \left\lfloor \frac{u_{i|\boldsymbol{s}} - \min_{j \neq s} u_{s|j,\boldsymbol{s}}}{u_{s|\boldsymbol{s}} - u_{i|\boldsymbol{s}}} \right\rfloor + 2 = \left\lfloor \frac{u_{\boldsymbol{s}|\boldsymbol{s}} - \min_{j \neq s} u_{s|j,\boldsymbol{s}}}{u_{s|\boldsymbol{s}} - \max_{i \neq s} u_{i|\boldsymbol{s}}} \right\rfloor + 1. \end{aligned} \tag{25}$$

*In particular, any strict equilibrium of any game is asymptotically stable under any BEP$(\tau, \kappa, \beta)$ dynamics with $\kappa$ large enough.*

The prospects for stability of a strict equilibrium $s$ are best when agents use the test-all rule $\tau^{\text{all}}$, which ensures that agents currently experimenting with other strategies

always consider strategy $s$ itself during subsequent testing and revision. Because of this, considerably fewer inequalities from (23) must hold for stability to be ensured: compare the quantifier in condition (26) to that in (23).

**Proposition 5.9.** *Let $s$ be a strict symmetric Nash equilibrium. Then state $e_s$ is asymptotically stable under any BEP$(\tau^{\text{all}}, \kappa, \beta)$ dynamic if*

$$v^{\kappa}_{s|\boldsymbol{s}} > v^{\kappa}_{i|j,\boldsymbol{s}} \text{ and } v^{\kappa}_{s|j,\boldsymbol{s}} > v^{\kappa}_{i|\boldsymbol{s}} \text{ for all } i, j \neq s \text{ with } i \geq j. \tag{26}$$

**Corollary 5.10.** *State $e_s$ is asymptotically stable under any BEP$(\tau^{\text{all}}, \kappa, \beta)$ dynamic if*

$$\kappa \geq \underline{\kappa}^1_{\text{all}} \equiv \max_{i \neq s} \left\lfloor \frac{\max_{j \neq s, j \leq i} u_{i|j,\boldsymbol{s}} - u_{s|\boldsymbol{s}}}{u_{s|\boldsymbol{s}} - u_{i|\boldsymbol{s}}} \right\rfloor + 2 \text{ and } \kappa \geq \underline{\kappa}^2 \tag{27}$$

Of course, stability only requires conditions (26) and (27) to hold after some relabelling of the strategies.

Condition (26) requires that no strategy $j \neq s$ supports or acts as a spoiler in favor of itself or any higher-numbered strategy. For intuition about why this condition is sufficient, suppose for convenience that $s = 1$. Condition (26) with $i = n$ ensures that strategy $n$ cannot invade on its own. Condition (26) with $i = n - 1$ says that strategy $n - 1$ only benefits from the presence of strategy $n$, and so it in turn cannot invade. Continuing in this vein shows that no strategy down through strategy 2 is able to invade. Formally, the proof of Proposition 5.9 shows that under condition (26), the Jacobian of BEP$(\tau^{\text{all}}, \kappa, \beta)$ dynamics at state $e_s$ is upper triangular with all diagonal elements equal to $-1$, which implies asymptotic stability.

*Example 5.4.* In Example 5.1, we showed that in the public good contribution game (17), when either $p > 2$ or $\kappa \geq 2$, the zero-contribution equilibrium $e_0$ is a repellor under BEP$(\tau^{\alpha}, \kappa, \beta)$ dynamics when the number of trials $\kappa$ is less than $\frac{b}{c-b}$. Proposition 5.9 implies that if instead $\kappa > \frac{b}{c-b}$, then equilibrium $e_0$ is asymptotically stable under BEP$(\tau^{\text{all}}, \kappa, \beta)$ dynamics. Recall that payoffs from $\kappa$ trials in game (17) satisfy $v^{\kappa}_{i|j,\boldsymbol{0}} = bj + \kappa(b - c)i$. It follows easily from this that $v^{\kappa}_{0|\boldsymbol{0}} > v^{\kappa}_{i|j,\boldsymbol{0}}$ for all $i \geq j > 0$, and that $v^{\kappa}_{0|j,\boldsymbol{0}} > 0 > v^{\kappa}_{i|\boldsymbol{0}}$ for all $i, j \neq 0$, and so that stability condition (26) holds. ◆

We conclude this section with a simple stability result for two-player normal form games under BEP$(\tau^{\text{all}}, 1, \beta)$ dynamics.

**Proposition 5.11.** *Let $s$ be a strict Nash equilibrium of the two-player symmetric game $G = \{S, U\}$.*

(*i*) *If $U_{ss} > U_{ij}$ for all $i, j \neq s$, then the growth rate of strategy $s$ under BEP$(\tau^{\text{all}}, 1, \beta)$ dynamics is nonnegative at all states in $X$, so state $e_s$ is Lyapunov stable.*

(*ii*) *If in addition to (i) we have* $U_{sj} > \min\{U_{ij}, U_{is}\}$ *for all* $i, j \neq s$*, then the growth rate of strategy* $s$ *is positive whenever* $x_s \in (0,1)$*, so state* $e_s$ *is almost globally asymptotically stable.*

(*iii*) *If in addition to (i) strategy* $s$ *is strictly dominant, then state* $e_s$ *is globally asymptotically stable.*

*The strict inequalities in* (*ii*) *can be replaced by weak inequalities under* $\beta^{\text{unif}}$ *or* $\beta^{\text{stick}}$*, and all inequalities can be replaced by weak inequalities under* $\beta^{\text{min}}$ *if* $s = 1$*.*

*Example 5.5.* Studying a family of two-player public good contribution games, Mantilla et al. (2019) show that if full contribution is a dominant strategy, it is globally stable under the BEP($\tau^{\text{all}}, 1, \beta^{\text{unif}}$) dynamic. Analogously, suppose that in the public good contribution game (17) from Example 5.1, there are two players, and that the per-player benefit $b$ of contributing exceeds the contribution cost $c$. Then choosing the maximum contribution is strictly dominant for each player, and each player's payoffs are maximized when both choose this contribution level. Thus Proposition 5.11(iii) implies that the maximal contribution is globally asymptotically stable under BEP($\tau^{\text{all}}, 1, \beta$) dynamics. In fact, the proof of Proposition 5.11(iii) builds directly on the analysis of Mantilla et al. (2019).[12] ◆

As another application of our results, we now show that the stability of a strict equilibrium need not be monotonic in the number of trials $\kappa$.

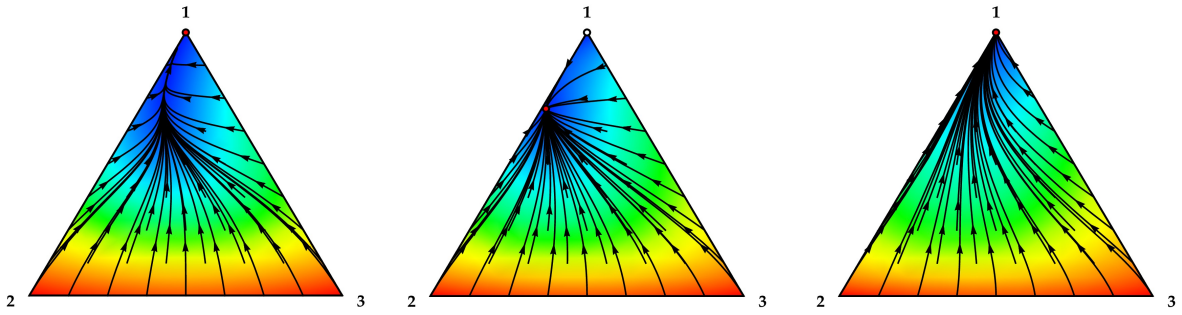
*Example 5.6.* Consider evolution under the BEP($\tau^{\text{all}}, \kappa, \beta^{\text{unif}}$) dynamic in the game below:

$$U = \begin{pmatrix} 6 & 1 & 1 \\ 4 & 0 & 0 \\ 2 & 0 & 0 \end{pmatrix}$$

Proposition 5.11(iii) implies that strict equilibrium $e_1$ is globally asymptotically stable when $\kappa = 1$ (Figure 4(i)), Corollary 5.2 implies that $e_1$ is unstable for $2 \leq \kappa \leq \lceil \frac{4-1}{6-4} \rceil = 2$ (Figure 4(ii)), and Corollary 5.8 implies that $e_1$ is asymptotically stable when $\kappa$ is at least $\lfloor \frac{4-1}{6-4} \rfloor + 2 = 3$ (Figure 4(iii)).

The intuition behind this nonmonotonicity is as follows: When $\kappa = 1$, the presence of a small number of agents playing the spiteful strategy 2 does not place it in enough tests of strategy 1 to render $e_1$ unstable. If we increase $\kappa$ to 2, the appearance of strategy 2 in a test of strategy 1 becomes twice as likely, which is sufficient to make $e_1$ unstable. But once $\kappa = 3$, strategy 2 is very unlikely to appear in more than one-third of the trials in which strategy 1 is tested, and this again is not enough to render $e_1$ unstable. ◆

[12] We thank an anonymous referee for suggesting this example.




(i) $\kappa = 1$ (ii) $\kappa = 2$ (iii) $\kappa = 3$

Figure 4: Nonmonotonicity of stability under BEP($\tau^{\text{all}}, \kappa, \beta^{\text{unif}}$) dynamics.

## 5.4 Examples

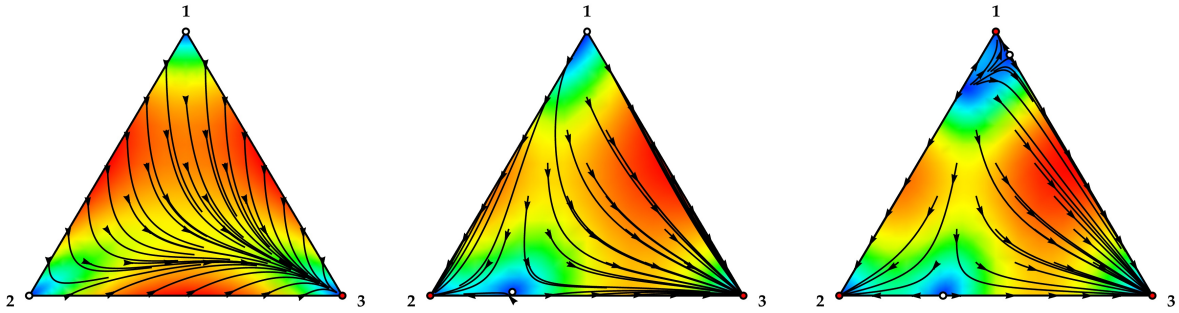
The foregoing results are enough to establish the stability and instability of strict equilibria in many examples. In instances where they are not enough—for instance, in cases where the details of the tie-breaking rules matter—one can directly compute the Jacobian of the dynamics (8) at the equilibrium to assess stability (see Lemma A.3 in Appendix A.3). Sometimes, as in Section 5.2, one can use Perron's theorem to establish instability without actually calculating the eigenvalues of the Jacobian; alternatively, as in Section 5.3, one can show without calculation all eigenvalues of the Jacobian are negative. When these approaches are not applicable, one can explicitly compute the eigenvalues in order to establish stability or instability. We illustrate these ideas in the examples that follow.

*Example 5.7.* In Example 5.3, we used Corollary 5.6 to show that in the Traveler's Dilemma (22) with $n \geq 5$, the strict equilibrium $e_1$ is unstable under BEP($\tau^\alpha, \kappa, \beta$) dynamics with $\alpha \geq \lceil \frac{n}{2} \rceil + 1$ and $1 \leq \kappa \leq \lfloor \frac{n}{2} \rfloor$. On the other hand, Corollary 5.8 implies that $e_1$ is asymptotically stable under any BEP($\tau^\alpha, \kappa, \beta$) dynamic with $\kappa \geq \lfloor \frac{(n+1)-1}{3-1} \rfloor + 2 = \lfloor \frac{n}{2} \rfloor + 2$. Under test-all, Corollary 5.10 gives us the improved lower bound $\kappa \geq \lfloor \frac{n-1}{3-1} \rfloor + 2 = \lceil \frac{n}{2} \rceil + 1$. Thus for test-all, the only undecided cases are those with $\kappa = \frac{n+1}{2}$ (and hence $n$ odd). What is special about this case is that the payoff from $\kappa$ rounds of playing strategy 1 against itself is the same as the payoff from playing strategy $n$ $\kappa - 1$ times against strategy 1 and once against itself. (That is, $v^\kappa_{nn} = \frac{n-1}{2}(-1) + n = \frac{n+1}{2} = v^\kappa_{11}$.) Because of this, stability depends on the tie-breaking rule $\beta$: rules that favor the incumbent strategy 1 will support stability compared to those that do not. In Appendix A.3 we use linearization to show that when $\kappa = \frac{n+1}{2}$, $e_1$ is asymptotically stable under $\beta^{\text{stick}}$ and $\beta^{\text{min}}$, but unstable under $\beta^{\text{unif}}$ for $n \geq 5$. ◆

*Example 5.8.* Consider BEP($\tau^{\text{all}}, \kappa, \beta^{\text{unif}}$) dynamics in 123 Coordination:

$$U = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}. \tag{28}$$

All pure strategies of this game are strict equilibria. Since $U_{33}$ is the highest payoff and $U_{3j} \geq U_{i3}$ for $i, j \neq 3$, Proposition 5.11(ii) implies that when $\kappa = 1$, the equilibrium state $e_3$ is almost globally asymptotically stable, and so that states $e_1$ and $e_2$ are unstable (Figure 5(i)). Corollary 5.8 implies that states $e_3$ and $e_2$ are asymptotically stable when $\kappa \geq 2$, and that state $e_1$ is asymptotically stable when $\kappa \geq \lfloor \frac{3-1}{1-0} \rfloor + 2 = 4$ (Figure 5(iii)). Corollary 5.6 implies that $e_1$ is unstable when $\kappa = \lceil \frac{3-1}{1-0} \rceil = 2$. In Appendix A.3 we use linearization to show that for $\kappa = 3$, $e_1$ is unstable under $\beta^{\text{unif}}$, but asymptotically stable under $\beta^{\text{stick}}$ and $\beta^{\text{min}}$. ◆




(i) $\kappa = 1$ (ii) $\kappa = 2$ (iii) $\kappa = 4$

Figure 5: 123 Coordination under BEP($\tau^{\text{all}}, \kappa, \beta^{\text{unif}}$), for different values of $\kappa$.

# 6. Concluding remarks

Building on the work of Osborne and Rubinstein (1998) and Sethi (2000), we defined the family of best experienced payoff dynamics, under which revising agents test each strategy from a random candidate set $\kappa$ times and choose the strategy from the set that earned the highest total payoffs. We developed results on the elimination of overwhelmed and dominated strategies for large numbers of trials, introduced sufficient conditions for the instability and stability of strict equilibrium, and illustrated the use of these results in a range of applications.

We conclude with a few comments on the generality of the analysis. While we have defined the dynamics (8) using a fixed test-set size $\alpha$ and a fixed number of trials $\kappa$ of each tested strategy, neither of these assumptions is necessary for our results. We call the test set rule $\tau$ *symmetric* if feasible test sets with the same cardinality have the same probability

under $\tau$. One such example is the rule *test-q*, $q \in (0, 1]$, under which each alternate strategy is chosen independently with probability $q$ to be in the test set. We can likewise define dynamics under which the number of trials $\kappa$ of each tested strategy is drawn from a finite-support distribution on the positive integers. What makes these extensions tractable is the fact that symmetric test-set rules and random numbers of trials generate convex combinations of the original dynamics (8). It turns out that the analyses leading to our main results are "closed under convex combinations", so that their conclusions extend immediately to the class of dynamics defined here.[13]

Some of the techniques employed in this paper to establish stability and instability of equilibrium have broader applicability. BEP dynamics are mean dynamics of the form

$$\dot{x}_i = \sum_{j \in S} x_j \sigma_{ji}(U, x) - x_i \quad \text{for all } i \in S. \tag{1}$$

where the conditional switch rates $\sigma_{ji}(U, x)$ are polynomials with nonnegative coefficients. Thus the inflow Jacobian of (1) (i.e., the Jacobian corresponding to the initial sum in (1)) also has nonnegative coefficients. In Section 5.2, this fact allowed us to establish sufficient conditions for instability using Perron's theorem. Clearly, the same approach is applicable for other dynamics based on random sampling, including, for example, the sampling best response dynamics of Oyama et al. (2015). This phenomenon is actually quite general: in Appendix A.2, we argue that linearizations at pure rest points of dynamics of form (1) *always* have this property, and so are always candidates for analysis using Perron's theorem. Likewise, the methods from computational algebra that we employ in our companion paper, Sandholm et al. (2019), are applicable beyond the class of best experienced payoff dynamics studied here.

# A. Appendix

## A.1 Background from dynamical systems

Consider a $C^1$ differential equation $\dot{x} = V(x)$ defined on $X$ whose forward solutions $(x(t))_{t \geq 0}$ do not leave $X$. State $x^*$ is a *rest point* if $V(x^*) = 0$, so that the unique solution starting from $x^*$ is stationary.

Let $Y \subset X$ be closed and connected. The set $Y$ is *Lyapunov stable* if for every neighborhood $O \subset X$ of $Y$, there exists a neighborhood $O' \subset X$ of $Y$ such that every forward

[13] A number of our results do not even require symmetry of the test-set rule, but instead hold under any test-set rule that puts a lower bound on the probabilities with which each strategy is tested. These include the results in Sections 3, 4 (except Proposition 4.2) and 5.3 in which a specific test-set rule is not specified.

solution that starts in $O'$ is contained in $O$. If $Y$ is not Lyapunov stable it is *unstable*, and it is *repelling* if there is a neighborhood $O \subset X$ of $Y$ such that solutions from all initial conditions in $O \setminus Y$ leave $O$. A minimal repelling set is called a *repellor*.

A rest point $x^*$ is *linearly unstable* if the Jacobian $DV(x^*)$ of $V$ at $x^*$ has an eigenvector in the tangent space $TX = \{z \in \mathbb{R}^n \colon \sum_i z_i = 0\}$ whose eigenvalue has positive real part. If in addition $x^*$ is hyperbolic, in that all eigenvalues corresponding to eigenvectors in $TX$ have nonzero real part, then the Hartman-Grobman theorem implies that there is a neighborhood of $O$ of $x^*$ such that solutions starting from almost all initial conditions in $O$ move away from $x^*$ at an exponential rate.

The set $Y$ is *attracting* if there is a neighborhood $O \subset X$ of $Y$ such that all solutions that start in $O$ converge to $Y$. A set that is Lyapunov stable and attracting is *asymptotically stable*; a minimal asymptotically stable set is called an *attractor*. In this case, the maximal (relatively) open set of states from which solutions converge to $Y$ is called the *basin* of $Y$. If the basin of $Y$ contains $\text{int}(X)$, we call $Y$ *almost globally asymptotically stable*; if it is $X$ itself, we call $Y$ *globally asymptotically stable*.

The Lipschitz function $L \colon O \to [c, \infty)$ is a *strict Lyapunov function* for set $Y \subset O$ if $L^{-1}(c) = Y$, and if its time derivative $\dot{L}(x) \equiv \nabla L(x)' V(x)$ is negative on $O \setminus Y$. Standard results imply that if such a function exists, then $Y$ is asymptotically stable.[14] If $L$ is a strict Lyapunov function for $Y$ with domain $X$, then $Y$ is globally asymptotically stable.

## A.2 Proofs

*Proof of Proposition 3.2.*

To start, recall that in the normal form game $G = \{S, U\}$ the expected payoff to strategy $i$ at state $x \in X$ takes the usual multilinear form:

$$\mathcal{U}_i(x) = \sum_{(j_1,\ldots,j_{p-1}) \in S^{p-1}} \left( \prod_{\ell=1}^{p-1} x_{j_\ell} \right) \mathcal{U}(i|j_1, \ldots, j_{p-1}).$$

Next we state a slight generalization of the weak law of large numbers.

**Lemma A.1.** *For each $k$, let $(X_i^k)_{i=1}^\infty$ be i.i.d. with $\mathbb{E}(X_1^k) = \mu^k$ and $\text{Var}(X_1^k) \leq \bar{\sigma}^2 < \infty$. If $\lim_{k\to\infty} \mu^k = \mu$, then $\bar{X}_k^k \equiv \frac{1}{k} \sum_{i=1}^k X_i^k \xrightarrow{p} \mu$ as $k \to \infty$.*

*Proof.* Let $c > 0$ be arbitrary. Then for some $\underline{k}$, $|\mu_k - \mu| < \frac{c}{2}$ for all $k \geq \underline{k}$. Chebyshev's inequality and the triangle inequality imply that for such $k$, $\text{Var}(\bar{X}_k^k) \geq \mathbb{P}(|\bar{X}_k^k - \mu^k| \geq c)\, c^2 \geq \mathbb{P}(|\bar{X}_k^k - \mu| \geq \frac{c}{2})\, c^2$, and so $\mathbb{P}(|\bar{X}_k^k - \mu| \geq \frac{c}{2}) \leq \frac{\bar{\sigma}^2}{k\, c^2}$, which vanishes as $k$ grows large. □

[14] See, e.g., Sandholm (2010b, Appendix 7.B).

Taking a subsequence if necessary, let $(x^\kappa)_{\kappa=1}^\infty$ be a sequence of BEP$(\tau,\kappa,\beta)$ equilibria that converges to $x^*$. For each $\kappa$, let $(J_{i,j}^{\kappa,m})$ with $i \in S$, $j \in \{1,\dots,p-1\}$, $m \in \{1,\dots,\kappa\}$ be a collection of i.i.d. *multinomial*$(x^\kappa)$ random variables whose realizations represent the strategies of opponents faced during testing. An agent's payoff during his $m$th test of strategy $i \in T$ is $U(i|J_{i,1}^{\kappa,m},\dots,J_{i,p-1}^{\kappa,m})$, a random variable with expectation $\mathcal{U}_i(x^\kappa)$. Thus Lemma A.1 implies that the realized average payoff during $\kappa$ tests of action $i$ converges in probability to $\mathcal{U}_i(x^*)$:

$$\frac{1}{\kappa}\sum_{m=1}^{\kappa} U(i|J_{i,1}^{\kappa,m},\dots,J_{i,p-1}^{\kappa,m}) \xrightarrow{p} \mathcal{U}_i(x^*) \text{ as } \kappa\to\infty. \tag{29}$$

Next, let $\sigma^\kappa$ be a BEP$(\tau^\alpha,\kappa,\beta)$ protocol as defined in (7), and for $i,j \in T \subseteq S$, let

$$\sigma_{ij}^\kappa(U,x|T) = \sum_{m\in M_T}\left(\prod_{k\in S} x_k^{|m^{-1}(k)|}\right)\beta_{ij}(\pi^U(m),T) \tag{30}$$

be the probability under (7) that a revising agent playing strategy $i$ who uses test set $T$ chooses strategy $j$ at state $x$. Then (29) implies that if $j \notin \operatorname{argmax}_{k\in T}\mathcal{U}_k(x^*)$, then $\sigma_{ij}^\kappa(U,x^\kappa|T)$ vanishes as $\kappa$ grows large. It follows that if $\mathcal{U}_i(x^*) > \mathcal{U}_j(x^*)$, then

$$\sigma_{ij}^\kappa(U,x^\kappa) = \sum_{T\in\mathcal{S}_i}\tau_i^\alpha(T)\,\sigma_{ij}^\kappa(U,x^\kappa|T) \to 0 \text{ as } \kappa\to\infty.$$

Thus if $S^*$ is the set of optimal strategies at $x^*$ and $i \in S^*$, we have

$$\lim_{\kappa\to\infty}\sum_{k\in S^*}\sigma_{ik}^\kappa(U,x^\kappa) = 1. \tag{31}$$

Since $x^\kappa$ is a rest point of the BEP$(\tau^\alpha,\kappa,\beta)$ dynamic (8), the total growth rate of strategies in $S^*$ at state $x^\kappa$ is zero. Thus

$$\begin{aligned} 0 &= \sum_{k\in S^*}\left(\sum_{j\in S} x_j^\kappa\sigma_{jk}^\kappa(U,x^\kappa) - x_k^\kappa\right) \\ &= \sum_{i\in S^*} x_i^\kappa\left(\sum_{k\in S^*}\sigma_{ik}^\kappa(U,x^\kappa)\right) + \sum_{j\notin S^*} x_j^\kappa\left[\sum_{k\in S^*}\sigma_{jk}^\kappa(U,x^\kappa)\right] - \sum_{k\in S^*} x_k^\kappa. \end{aligned} \tag{32}$$

Equation (31) says that the sum in parentheses in (32) converges to 1, implying that in the limit the first and third terms of (32) cancel. The definition (7) of $\sigma^\kappa$ implies that for each $j \notin S^*$, the sum in brackets converges to a limit greater than $\frac{1}{n}$ under any test-$\alpha$ rule.

Altogether, this implies that

$$\sum_{j \notin S^*} x_j^* = \lim_{\kappa \to \infty} \sum_{j \notin S^*} x_j^\kappa = 0,$$

Since $x^*$ puts no mass on suboptimal strategies, it is a Nash equilibrium. □

*Proof of Proposition 4.2.* The inflow into strategy $i$ at state $x$ under (8) is

$$\sum_{h \in S} x_h \sigma_{hi}(U, x) = \sum_{h \in S} x_h \left( \sum_{T \in \mathcal{S}_{hi}} \tau_h^\alpha(T) \sum_{m \in M_T} P(m|x)\, \beta_{hi}(\pi^U(m), T) \right), \text{ where} \tag{33a}$$

$$P(m|x) = \prod_{k \in S} x_k^{|m^{-1}(k)|} \text{ and} \tag{33b}$$

$$\pi_k^U(m) = \sum_{\ell=1}^{\kappa} U(k; m_{k,1,\ell}, \ldots, m_{k,p_-,\ell}) \tag{33c}$$

are the probability of match assignment $m \in M_T$ and the total payoff to strategy $k$ during the relevant matches in $m$, respectively. The second sum in (33a) is taken over $\mathcal{S}_{hi}$, the set of subsets of $S$ containing strategies $h$ and $i$. The notations $\mathcal{S}_{hj}$ and $\mathcal{S}_{hij}$ used below are defined similarly.

Fix a test set $T \in \mathcal{S}_{hij}$. For a match assignment $m \in M_T$ for this test set, define the match assignment $\tilde{m}$ by

$$\tilde{m}_{k,q,\ell} = \begin{cases} m_{j,q,\ell} & \text{if } k = i, \\ m_{i,q,\ell} & \text{if } k = j, \\ m_{k,q,\ell} & \text{otherwise.} \end{cases} \tag{34}$$

That is, $\tilde{m}$ switches the strategies faced under $m$ when testing $i$ and $j$. Then by construction and by strict dominance,

$$U(i; m_{i,\cdot,\ell}) = U(i; \tilde{m}_{j,\cdot,\ell}) > U(j; \tilde{m}_{j,\cdot,\ell}),$$

where $m_{i,\cdot,\ell} = (m_{i,1,\ell}, \ldots, m_{i,p-1,\ell})$. (33c) then implies that $\pi_i^U(m) > \pi_j^U(\tilde{m})$, which in turn implies that

$$\beta_{hi}(\pi^U(m), T) \geq \beta_{hj}(\pi^U(\tilde{m}), T) \text{ for all } h \in T. \tag{35}$$

Also, since $m$ and $\tilde{m}$ are permutations of one another, (33b) implies that $P(m|x) = P(\tilde{m}|x)$.

Combining this fact with (35) and considering all $T$ in $\mathcal{S}_{hij}$, we have

$$\sum_{T\in\mathcal{S}_{hij}} \tau_h^\alpha(T) \sum_{m\in M_T} P(m|x)\,\beta_{hi}(\pi^U(m),T) \geq \sum_{T\in\mathcal{S}_{hij}} \tau_h^\alpha(T) \sum_{\tilde{m}\in M_T} P(\tilde{m}|x)\,\beta_{hj}(\pi^U(\tilde{m}),T). \tag{36}$$

Now consider a test set $T$ in $\mathcal{S}_{hi}\setminus\mathcal{S}_{hij}$. Let $\tilde{T} = T\cup\{j\}\setminus\{i\}$, and for each $m\in T$ define $\tilde{m}\in\tilde{T}$ by (34) (noting that the middle case of (34) never occurs). Then the fact that $\tau_h^\alpha(T) = \tau_h^\alpha(\tilde{T})$ and a minor variation on the argument above shows that

$$\sum_{T\in\mathcal{S}_{hi}\setminus\mathcal{S}_{hij}} \tau_h^\alpha(T) \sum_{m\in M_T} P(m|x)\,\beta_{hi}(\pi^U(m),T) \geq \sum_{\tilde{T}\in\mathcal{S}_{hj}\setminus\mathcal{S}_{hij}} \tau_h^\alpha(\tilde{T}) \sum_{\tilde{m}\in M_{\tilde{T}}} P(\tilde{m}|x)\,\beta_{hj}(\pi^U(\tilde{m}),\tilde{T}). \tag{37}$$

Combining (36) and (37) with (33a) and (33b), we conclude that

$$\sigma_{hi}(U,x) \geq \sigma_{hj}(U,x) \text{ for all } x\in X \text{ and } h\neq i,j. \tag{38}$$

Using similar arguments, one can also show that

$$\sigma_{jh}(U,x) \geq \sigma_{ih}(U,x) \text{ for all } x\in X \text{ and } h\neq i,j, \text{ and} \tag{39}$$

$$\sigma_{ji}(U,x) \geq \sigma_{ij}(U,x) \text{ for all } x\in X. \tag{40}$$

Now let $y$ be a rest point of (8). Then equating inflows and ouflows shows that

$$y_j\sigma_{ji}(U,y) + \sum_{k\neq i,j} y_k\sigma_{ki}(U,y) = y_i\sigma_{ij}(U,y) + y_i\sum_{k\neq i,j}\sigma_{ik}(U,y) \text{ and}$$
$$y_i\sigma_{ij}(U,y) + \sum_{k\neq i,j} y_k\sigma_{kj}(U,y) = y_j\sigma_{ji}(U,y) + y_j\sum_{k\neq i,j}\sigma_{jk}(U,y).$$

Subtracting and rearranging yields

$$\sum_{k\neq i,j} y_k\Big(\sigma_{ki}(U,y) - \sigma_{kj}(U,y)\Big) = 2\Big(y_i\sigma_{ij}(U,y) - y_j\sigma_{ji}(U,y)\Big) + \left(y_i\sum_{k\neq i,j}\sigma_{ik}(U,y) - y_j\sum_{k\neq i,j}\sigma_{jk}(U,y)\right). \tag{41}$$

(38) implies that the left-hand side of (41) is nonnegative, which along with (39) and (40) implies that $y_i \geq y_j$. □

*Proof of Proposition 4.3.* For part (i), suppose that strategy $j$ is strictly dominated by strategy $i$: $\mathcal{U}_i(x) > \mathcal{U}_j(x)$ for all $x\in X$. The weak law of large numbers implies that as $\kappa$ grows large,

the realized average payoffs during $\kappa$ tests of each $k$ at state $x$ converges in probability to $\mathcal{U}_k(x)$ (cf. (29)). Moreover, since the variance in realized payoffs can be bounded over $x \in X$, the proof of the weak law (cf. Lemma A.1) implies that for each $\varepsilon > 0$ there is a $\underline{\kappa}$ such that for all $x \in X$ and $\kappa \geq \underline{\kappa}$, the probability is at most $\varepsilon$ that the total realized payoff from $\kappa$ trials of $j$ exceeds the total realized payoff from $\kappa$ trials of $i$. Hence, letting $\sigma^\kappa$ denote a BEP($\tau^{\text{all}}, \kappa, \beta$) protocol (7), we have that the probability $\sigma^\kappa_{kj}(U, x)$ that a revising $k$ player chooses $j$ is at most $\varepsilon$ when $\kappa \geq \underline{\kappa}$. Thus when $\kappa \geq \underline{\kappa}$, definition (1) of the dynamic implies that

$$\dot{x}_j = \sum_{k \in S} x_k \sigma_{kj}(U, x) - x_j \leq \varepsilon - x_j.$$

The last expression is negative when $x_j > \varepsilon$, implying that $L(x) = \max\{x_j, \varepsilon\}$ is a strict global Lyapunov function for the set $\{x \in X \colon x_j \leq \varepsilon\}$.

For part (ii), let $i$ be the strictly dominant strategy. Mimicking the argument above shows that for each $\varepsilon > 0$ there is a $\underline{\kappa}$ such that for all $x \in X$ and $\kappa \geq \underline{\kappa}$, the probability is at least $1 - \varepsilon$ that the total realized payoff from $\kappa$ trials of strategy $i$ exceeds the total realized payoff from $\kappa$ trials of strategy $j$ for all $j \neq i$. Since under any test-$\alpha$ rule, strategy $i$ is in the test set with probability greater than $\frac{1}{n}$, and with probability 1 if the revising agent is currently playing it, we find that for $\kappa \geq \underline{\kappa}$,

$$\begin{aligned} \dot{x}_i &= \sum_{k \in S} x_k \sigma_{ki}(U, x) - x_i \\ &> \sum_{k \neq i} \tfrac{1}{n} x_k (1 - \varepsilon) + x_i (1 - \varepsilon) - x_i \\ &= \tfrac{1}{n}(1 - x_i)(1 - \varepsilon) - \varepsilon x_i \\ &= \tfrac{1}{n}\left[(1 - \varepsilon) - (1 + (n-1)\varepsilon) x_i\right], \end{aligned} \tag{42}$$

and so $\dot{x}_i$ is positive whenever $x_i < 1 - \frac{n\varepsilon}{1+(n-1)\varepsilon}$.

Since $e_i$ is a strict equilibrium, there is a $b > 0$ such that $\mathcal{U}_i(x) > \mathcal{U}_j(y)$ for all $j \neq i$ whenever $x_i$ and $y_i$ exceed $1 - b$. The proof of part (ii) follows from the previous argument, with $\varepsilon > 0$ chosen small enough that $\frac{n\varepsilon}{1+(n-1)\varepsilon} \leq \frac{b}{3}$, and the following lemma.

**Lemma A.2.** *Under any BEP($\tau^\alpha, \kappa, \beta$) dynamic with $\kappa$ sufficiently large and at any state $x \in X$ with $x_i \in [1 - \frac{1}{3}b, 1)$, we have $\dot{x}_i > 0$.*

*Proof.* By the choice of $b$, a revising agent with strategy $i$ in his test set will choose strategy $i$ if when testing each strategy $j$ in his test set, the fraction of those encountered playing strategies other than $i$ is at most $b$. Then a necessary condition for his choosing some

other strategy is that there be a strategy in his test set for which this fraction exceeds $b$. Thus, letting $Q_b^\kappa(x)$ denote the probability that when testing a given strategy, the fraction of those of those encountered playing strategies other than $i$ exceeds $b$, defining $\sigma_{ji}^\kappa(U, x|T)$ as in (30), and using the fact that the probability of the union of events is at most the sum of the events' probabilities, we have

$$1 - \sigma_{ji}^\kappa(U, x|T) \leq nQ_b^\kappa(x). \tag{43}$$

Now let $x_{-i} = \sum_{j \neq i} x_j$, and suppose that

$$Q_b^\kappa(x) \leq \tfrac{1}{n^2} x_{-i}. \tag{44}$$

If (44) holds, then by equations (8) and (43), the growth rate of strategy $i$ satisfies

$$\dot{x}_i \geq \sum_{j \neq i} \tfrac{1}{n} x_j (1 - nQ_b^\kappa(x)) + x_i(1 - nQ_b^\kappa(x)) - x_i \geq \tfrac{1}{n} x_{-i} \left(1 - \tfrac{1}{n} x_{-i} - x_i\right).$$

The last expression is positive when $x_i \in (0, 1)$.

To show that (44) holds when $x_{-i} \in (0, \frac{1}{3}b]$, we use a large deviations bound. The proof of Cramér's theorem, along with the fact that the Cramér transforms of binomial distributions are relative entropy functions (see Dembo and Zeitouni (1998, Theorem 2.2.3 and Exercise 2.2.23)), imply that

$$Q_b^\kappa(x) \leq 2 \exp\left(-\kappa \left(b \log\left(\tfrac{b}{x_{-i}}\right) + (1-b) \log\left(\tfrac{1-b}{1-x_{-i}}\right)\right)\right).$$

Some rearranging shows that to establish (44), it is enough to prove that

$$2n^2 (x_{-i})^{\kappa b - 1} b^{-\kappa b} \exp\left(-\kappa(1-b) \log\left(\tfrac{1-b}{1-x_{-i}}\right)\right) < 1. \tag{45}$$

Since $x_{-i} \in (0, \frac{1}{3}b]$, the the LHS of (45) is at most

$$\begin{aligned} 2n^2 \left(\frac{b}{3}\right)^{\kappa b - 1} b^{-\kappa b} \exp(-\kappa(1-b)\log(1-b)) &\leq 2n^2 b^{-1} \left(\frac{1}{3}\right)^{\kappa b - 1} \exp(\kappa b) \\ &= 6n^2 b^{-1} \left(\frac{\mathrm{e}}{3}\right)^{\kappa b} \to 0 \quad \text{as } \kappa \to \infty. \end{aligned}$$

The inequality follows from the fact that the function $f(z) = -(1-z)\log(1-z)$ satisfies $f(b) \leq b$ (since $f(0) = 0$, $f'(0) = 1$, $f''(z) < 0$ on $[0, 1)$ ). □

We prove the results from Section 5.3 before returning to those from Sections 5.1 and

5.2.

*Proof of Proposition 5.7.* Write the law of motion (1) for strategy $s$ as

$$\dot{x}_s = \sum_{j \neq s} x_j \sigma_{js}(U, x) - x_s \sum_{j \neq s} \sigma_{sj}(U, x), \tag{46}$$

where the conditional switch rates are given by

$$\sigma_{ij}(U, x) = \sum_{T \in \mathcal{S}_i} \tau_i^{\alpha}(T) \left[ \sum_{m \in M_T} \left( \prod_{k \in S} x_k^{|m^{-1}(k)|} \right) \beta_{ij}(\pi^U(m), T) \right] \tag{7a}$$

and (7b).

Write $x_{-s} = \sum_{j \neq s} x_j$. To understand the behavior of the dynamics in a neighborhood of $e_s$, we can focus on terms of the polynomial (46) that are of order no greater than 1 in $x_{-s}$. In particular, we need only look at terms of (46) in which there is at most one $x_j$, $j \neq s$ of order 1, with the remaining terms for nonequilibrium strategies having order 0. In the initial sum in (46), because of the initial $x_j$, the only such terms (i) have agents only playing $s$ in the match assignment $m$ (implying that the revising agent chooses $s$). In the last expression of (46), there are two kinds of such terms: (i) as above; (ii) those that have exactly one opponent playing $k \neq s$ in a match in $m$. Defining $\mathcal{S}_{js}$ as in the proof of Proposition 4.2, we can express (46) as follows:

$$\begin{aligned} \dot{x}_s = &\sum_{j \neq s} x_j \sum_{T \in \mathcal{S}_{js}} \tau_j^{\alpha}(T) x_s^{|T|p_- \kappa} \beta_{js}(\pi_{\boldsymbol{s}}^U, T) - \sum_{j \neq s} \sum_{T \in \mathcal{S}_s} \tau_s^{\alpha}(T) x_s^{|T|p_- \kappa + 1} \beta_{sj}(\pi_{\boldsymbol{s}}^U, T) \\ &- \sum_{j \neq s} \sum_{T \in \mathcal{S}_s} \tau_s^{\alpha}(T) \left[ \sum_{k \neq s} p_- \kappa x_s^{|T|p_- \kappa} x_k \sum_{\ell \in T} \beta_{sj}(\pi_{\boldsymbol{s},k@\ell}^U, T) \right] + O((x_{-s})^2), \end{aligned} \tag{47}$$

where

$$\begin{aligned} &(\pi_{\boldsymbol{s}}^U)_h = v_{h|\boldsymbol{s}}^{\kappa} = \kappa u_{h|\boldsymbol{s}} \text{ and} \\ &(\pi_{\boldsymbol{s},k@\ell}^U)_h = \begin{cases} v_{h|\boldsymbol{s}}^{\kappa} = \kappa u_{h|\boldsymbol{s}} & \text{if } h \neq \ell, \\ v_{h|k,\boldsymbol{s}}^{\kappa} = (\kappa - 1) u_{h|\boldsymbol{s}} + u_{h|k,\boldsymbol{s}} & \text{if } h = \ell. \end{cases} \end{aligned} \tag{48}$$

The first two sums in (47) correspond to case (i) above. Since $s$ is uniquely optimal in $\pi_{\boldsymbol{s}}^U$ we can rewrite (47) as

$$\dot{x}_s = \sum_{j\neq s} x_j \sum_{T\in\mathcal{S}_{js}} \tau_j^\alpha(T) x_s^{|T|p_-\kappa} - \sum_{T\in\mathcal{S}_s}\sum_{j\neq s}\tau_s^\alpha(T)\left[\sum_{k\neq s} p_-\kappa x_s^{|T|p_-\kappa} x_k \sum_{\ell\in T}\beta_{sj}(\pi^U_{\boldsymbol{s},k@\ell}, T)\right] + O((x_{-s})^2). \tag{49}$$

The first term on the right-hand side of (49) is at least $\underline{\tau}\, x_{-s} x_s^{|T|p_-\kappa}$, where $\underline{\tau} = \min_{j\neq s}\sum_{T\in\mathcal{S}_{js}}\tau_j^\alpha(T) > 0$. To prove that $\dot{x}_s > 0$ when $x_{-s} \neq 0$ is small, we show that the second term in (49) is 0 by showing that $\beta_{ss}(\pi^{U,\kappa}_{\boldsymbol{s},k@\ell}, T) = 1$ for all $T \in \mathcal{S}_s$, $k \neq s$, and $\ell \in T$. For this to be true, it is sufficient that $v^\kappa_{s|\boldsymbol{s}} > v^\kappa_{\ell|k,\boldsymbol{s}}$ for all $k, \ell \neq s$ (to cover cases in which $\ell \neq s$) and $v^\kappa_{s|k,\boldsymbol{s}} > v^\kappa_{h|\boldsymbol{s}}$ (for $\ell = s$). (The first cases also require $v^\kappa_{s|\boldsymbol{s}} > v^\kappa_{h|\boldsymbol{s}}$ for $h \neq s$; this is true because $s$ is a strict equilibrium.) These are precisely the conditions assumed in (23). Thus $\dot{x}_s > 0$ when $x_{-s} \neq 0$ is small, proving the proposition. □

*Proof of Proposition 5.9.* Write the law of motion $\dot{x} = V(x)$ (1) as

$$V_i(x) = \sum_{j\in S} x_j \sigma_{ji}(U, x) - x_i, \tag{50}$$

where the conditional switch rates are given by (7a) and (7b). Write $x_{-s} = \sum_{j\neq s} x_j$. Assume from this point forward that $i \neq s$.

As in the previous proof, we can focus on terms of the sum in (50) that are of order no greater than 1 in $x_{-s}$. There are two sorts of such terms: (i) those that have agents only playing $s$ in the match assignment $m$; (ii) those in which the revising agent plays $s$ and that have exactly one opponent playing $k \neq s$ in a match in $m$. Noting that in the first case, a revising agent playing $s$ will continue to play $s$ (since $s$ will be in his test set), we can express (50) as follows:

$$V_i(x) = \sum_{j\neq s}\sum_{T\in\mathcal{S}_j}\tau_j^\alpha(T) x_s^{|T|p_-\kappa} x_j \beta_{ji}(\pi^U_{\boldsymbol{s}}, T) + \sum_{T\in\mathcal{S}_{si}}\tau_s^\alpha(T)\left[\sum_{k\neq s} p_-\kappa x_s^{|T|p_-\kappa} x_k \sum_{\ell\in T}\beta_{si}(\pi^U_{\boldsymbol{s},k@\ell}, T)\right] - x_i + O((x_{-s})^2), \tag{51}$$

with $\pi^U_{\boldsymbol{s},k@\ell}$ defined in (48).

Since $(e_s)_k = 0$ for $k \neq s$, it follows from (51) that

$$\frac{\partial V_i}{\partial x_s}(e_s) = 0. \tag{52}$$

Focusing on the case of $\tau^{\text{all}}$, note that $\beta_{ji}(\pi^U_{\boldsymbol{s}}, S) = 0$ for $i \neq s$, implying that the first term

in (51) is zero, and note that $\beta_{si}(\pi^U_{\boldsymbol{s},k@\ell}, S)$ can only be positive if $\ell = i$ or $\ell = s$. Combining these two facts with (51), we find that under $\tau^{\text{all}}$,

$$V_i(x) = p_- \kappa x_s^{|T|p_- \kappa} \sum_{k \neq s} x_k \left( \beta_{si}(\pi^U_{\boldsymbol{s},k@i}, S) + \beta_{si}(\pi^U_{\boldsymbol{s},k@s}, S) \right) - x_i + O((x_{-s})^2),$$

and hence

$$\frac{\partial V_i}{\partial x_j}(e_s) = p_- \kappa \left( \beta_{si}(\pi^U_{\boldsymbol{s},j@i}, S) + \beta_{si}(\pi^U_{\boldsymbol{s},j@s}, S) \right) - \mathbf{1}[j = i] \quad \text{for } j \neq s. \tag{53}$$

Assumption (26) ensures the terms in parentheses in (53) are 0 when $i \neq s$ is not less than $j$. Thus $\frac{\partial V_i}{\partial x_j}(e_s) = 0$ when $i \neq s$ is greater than $j$, and $\frac{\partial V_i}{\partial x_i}(e_s) = -1$.

To complete the proof, we change the state space of the dynamic $\dot{x} = V(x)$ from the simplex $X$ to the set $Y = \{y \in \mathbb{R}^{n-1}_+ \colon \sum_i y_i \leq 1\}$ by leaving off the $s$th component of both $x$ and $V(x)$ but retaining the labels of the remaining coordinates: that is, $x \mapsto (y_1, \dots, y_{s-1}, y_{s+1}, \dots, y_n)' = (x_1, \dots, x_{s-1}, x_{s+1}, \dots, x_n)'$. The transformed dynamic can be expressed as $\dot{y} = W(y)$, where

$$W_i(y) = V_i\left(y_1, \dots, y_{s-1}, 1 - \sum\nolimits_{i \neq s} y_i, y_{s+1}, \dots, y_n\right). \tag{54}$$

and the equilibrium $x = e_s$ is sent to $y = \mathbf{0}$. Thus the Jacobian of the transformed dynamic at the transformed rest point $\mathbf{0}$ has components

$$\frac{\partial W_i}{\partial y_j}(\mathbf{0}) = \frac{\partial V_i}{\partial x_j}(e_s) - \frac{\partial V_i}{\partial x_s}(e_s), \quad i, j \neq s. \tag{55}$$

Combining (55) with our previous arguments shows that $\frac{\partial W_i}{\partial y_j}(\mathbf{0}) = 0$ when $j \neq i$ is less than $i$ and that $\frac{\partial W_i}{\partial y_i}(\mathbf{0}) = -1$. Thus $DW(\mathbf{0})$ is an upper diagonal matrix with all diagonal elements equal to $-1$. Therefore all eigenvalues of $DW(\mathbf{0})$ are $-1$, and so $\mathbf{0}$ is asymptotically stable under $\dot{y} = W(y)$, which implies in turn that $e_s$ is asymptotically stable under $\dot{x} = V(x)$. □

*Proof of Proposition 5.11.* For part (i), note that since $U_{ss} > U_{ij}$ for all $i, j \neq s$, any match assignment $r$ in which the revising agent tests $s$ against an opponent playing $s$ will lead to him to choose $s$. Since such $r$ have total probability $x_s$, we have

$$\dot{x}_s = \sum_{j \in S} x_j \sigma_{js}(U, x) - x_s \geq \left( \sum_{j \in S} x_j \right) x_s - x_s = 0.$$

Turning to part (ii), let $x$ be a state with $x_s \in (0, 1)$. Then there is a $j^* \neq s$ such that

$x_{j^*} > 0$. By assumption, for each $i \neq s$, either $U_{sj^*} > U_{is}$ or $U_{sj^*} > U_{ij^*}$. Let $R \subset T \setminus \{s\}$ be the set of strategies $i$ for which the first inequality holds. Consider the match result $r \in S^n$ defined by $r_i = s$ for $i \in R$ and $r_i = j^*$ for $i \notin R$. Then match result $r$ leads a revising agent to choose $s$. The definition of $\tau^{\text{all}}$ and (7) then imply that

$$\begin{aligned}
\dot{x}_s &= \sum_{j \in S} x_j \sigma_{js}(U, x) - x_s \\
&\geq \left( \sum_{j \in S} x_j \right) \left( x_s + x_{j^*}^{n-|R|} (x_s)^{|R|} \right) - x_s \\
&= x_{j^*}^{n-|R|} (x_s)^{|R|} \\
&> 0,
\end{aligned}$$

completing the proof.

For part (iii), following the proof of (i), we obtain

$$\dot{x}_s = \sum_{j \in S} x_j \sigma_{js}(U, x) - x_s \geq \left( \sum_{j \in S} x_j \right) \left( x_s + \sum_{j \neq s} x_j^n \right) - x_s = \sum_{j \neq s} x_j^n.$$

where the terms $x_j^n$ correspond to the probability of match assignments such that the revising agent tests each of his $n$ strategies against an agent playing strategy $j \neq s$, leading to the choice of the dominant strategy $s$. It follows then that $\dot{x}_s > 0$ if $x_s < 1$, proving the result. □

*Proof of Proposition 5.1.* Writing

$$\dot{x}_s = \sum_{j \neq s} x_j \sigma_{js}(U, x) - x_s \sum_{j \neq s} \sigma_{sj}(U, x)$$

and using the logic from the proof of Proposition 5.7, we express $\dot{x}_s$ as follows:

$$\begin{aligned}
V_s(x) = &\sum_{j \neq s} x_j \sum_{T \in \mathcal{S}_{js}} \tau_j^\alpha(T) x_s^{|T|p_- \kappa} \beta_{js}(\pi_{\boldsymbol{s}}^U, T) - \sum_{j \neq s} \sum_{T \in \mathcal{S}_s} \tau_s^\alpha(T) x_s^{|T|p_- \kappa + 1} \beta_{sj}(\pi_{\boldsymbol{s}}^U, T) \\
&- \sum_{j \neq s} \sum_{T \in \mathcal{S}_s} \tau_s^\alpha(T) \left[ \sum_{k \neq s} p_- \kappa x_s^{|T|p_- \kappa} x_k \sum_{\ell \in T} \beta_{sj}(\pi_{\boldsymbol{s},k@\ell}^U, T) \right] + O((x_{-s})^2).
\end{aligned} \tag{56}$$

The first two terms correspond to cases in which all agents in the match assignment play $s$. Since after such an assignment the revising agent chooses $s$, (56) simplifies to

$$\dot{x}_s = \sum_{j\neq s} x_j \sum_{T\in\mathcal{S}_{js}} \tau_j^\alpha(T) x_s^{|T|p_-\kappa} - \sum_{T\in\mathcal{S}_s}\sum_{j\neq s}\tau_s^\alpha(T)\left[\sum_{k\neq s} p_-\kappa x_s^{|T|p_-\kappa} x_k \sum_{\ell\in T}\beta_{sj}(\pi^U_{\boldsymbol{s},k@\ell},T)\right] + O((x_{-s})^2). \tag{57}$$

Define $\mathcal{S}_s^\alpha = \{T\subseteq S\colon s\in T,\ |T|=\alpha\}$ and define $\mathcal{S}_{js}^\alpha$ analogously. Then using the facts that $\tau_j^\alpha(T) = \binom{n-1}{\alpha-1}^{-1}$ and $|\mathcal{S}_{js}^\alpha| = \binom{n-2}{\alpha-2}$, we obtain

$$\begin{aligned}\dot{x}_s &= \frac{\alpha-1}{n-1}x_s^{|T|p_-\kappa}x_{-s} - \sum_{T\in\mathcal{S}_s^\alpha}\sum_{j\neq s}\tbinom{n-1}{\alpha-1}^{-1}\left[\sum_{k\neq s} p_-\kappa x_s^{|T|p_-\kappa}x_k\sum_{\ell\in T}\beta_{sj}(\pi^U_{\boldsymbol{s},k@\ell},T)\right] + O((x_{-s})^2)\\ &\le \frac{\alpha-1}{n-1}x_s^{|T|p_-\kappa}x_{-s} - \sum_{\ell\neq s}\sum_{T\in\mathcal{S}_{s\ell}^\alpha}\tbinom{n-1}{\alpha-1}^{-1}\sum_{k\neq s} p_-\kappa x_s^{|T|p_-\kappa}x_k\beta_{s\ell}(\pi^U_{\boldsymbol{s},k@\ell},T)\\ &\quad - \sum_{T\in\mathcal{S}_s^\alpha:\,T\cap S_2\neq\varnothing}\tbinom{n-1}{\alpha-1}^{-1}\sum_{k\neq s} p_-\kappa x_s^{|T|p_-\kappa}x_k\sum_{t\in T\cap S_2}\beta_{st}(\pi^U_{\boldsymbol{s},k@s},T) + O((x_{-s})^2).\end{aligned} \tag{58}$$

The upper bound (58) is obtained by considering two cases: (i) for some $\ell\neq s$ in $T$, $\ell$ is matched once against some $k\neq s$; and (ii) test set $T$ contains at least one second-best strategy $t\in S_2$, and when tested $s$ is matched once against some $k\neq s$. In case (i) we also use the fact that changing a test result for strategy $\ell\neq s$ cannot cause strategy $j\neq\ell,s$ to have the best test result.

The third term in (58) (with the minus sign excluded) is smallest when $S_2=\{t\}$ is a singleton, so that its initial sum is over $T$ in $\mathcal{S}_{st}^\alpha$. Thus, again using the fact that $|\mathcal{S}_{sj}^\alpha| = \binom{n-2}{\alpha-2}$, we obtain

$$\dot{x}_s \le \frac{\alpha-1}{n-1}x_s^{|T|p_-\kappa}\left(x_{-s} - p_-\kappa\left(\sum_{k\neq s}x_k\left[\sum_{\ell\neq s}\mathbf{1}[v_{s|\boldsymbol{s}} < v_{\ell|k,\boldsymbol{s}}] + \mathbf{1}[v_{s|k,\boldsymbol{s}} < v_{t|\boldsymbol{s}}]\right]\right)\right) + O((x_{-s})^2). \tag{59}$$

The assumption of the proposition is that for each $k\neq s$ the expression in brackets exceeds $\frac{1}{p_-\kappa}$. We conclude that for some $c>0$, $\dot{x}_s \le -cx_{-s} + O((x_{-s})^2)$, proving the proposition. □

*Proof of Corollary 5.2.* Rewrite condition (15) as

$$p_-\kappa\left(\sum_{i\neq s}\mathbf{1}\left[\kappa-1 < \frac{u_{i|j,\boldsymbol{s}}-u_{s|\boldsymbol{s}}}{u_{s|\boldsymbol{s}}-u_{i|\boldsymbol{s}}}\right] + \mathbf{1}\left[\kappa-1 < \frac{u_{s|\boldsymbol{s}}-u_{s|j,\boldsymbol{s}}}{u_{s|\boldsymbol{s}}-u_{t|\boldsymbol{s}}}\right]\right) > 1 \ \text{ for all } j\neq s, \tag{60}$$

Definition (16) of $\bar{\kappa}$ ensures that this inequality holds for $\kappa\le\bar{\kappa}$, completing the proof of the proposition. □

*Proof of Proposition 5.4.* Let $\dot{x} = V(x)$ be the law of motion for the BEP dynamic, and define $V^+(x)$ by $V(x) = V^+(x) - \text{diag}(x)$. Let $J \subseteq S \setminus \{s\}$. Taking the derivative of (51) at $x = e_s$ and dropping some nonnegative terms, we obtain for all $i, j \neq s$ that

$$\begin{aligned}\frac{\partial V_i^+}{\partial x_j}(e_s) &= \sum_{T \in \mathcal{S}_j} \tau_j^\alpha(T)\beta_{ji}(\pi_{\boldsymbol{s}}^U, T) + \sum_{T \in \mathcal{S}_{si}} \tau_s^\alpha(T) p_- \kappa \sum_{\ell \in T} \beta_{si}(\pi_{\boldsymbol{s},j@\ell}^U, T) \\ &\geq p_- \kappa \sum_{T \in \mathcal{S}_s} \tau_s^\alpha(T) \left(\beta_{si}(\pi_{\boldsymbol{s},j@i}^U, T) + \beta_{si}(\pi_{\boldsymbol{s},j@s}^U, T)\right) \qquad (61) \\ &\geq 0.\end{aligned}$$

To prove part (i), we use the facts that $\tau_j^\alpha(T) = \binom{n-1}{\alpha-1}^{-1}$ and $|\mathcal{S}_{js}^\alpha| = \binom{n-2}{\alpha-2}$ to bound the submatrix row sums of $DV^+(x)$:

$$\sum_{j \in J} \frac{\partial V_i^+}{\partial x_j}(e_s) \geq p_- \kappa \frac{\alpha - 1}{n-1} \sum_{j \in J} \mathbf{1}[v_{i|j,\boldsymbol{s}}^\kappa > v_{s|\boldsymbol{s}}^\kappa] \quad \text{for all } i \neq s. \qquad (62)$$

Now define the change of coordinates $W$ of $V$ as in (54), and define $W^+(x)$ by $W(x) = W^+(x) - \text{diag}(x)$. Equations (52) and (55) imply that

$$\frac{\partial W_i^+}{\partial y_j}(\mathbf{0}) = \frac{\partial V_i^+}{\partial x_j}(e_s) \geq 0 \quad \text{for all } i, j \neq s.$$

Equation (62) and condition (19) from the proposition imply that (after rearranging indices) the square block of $DW^+(\mathbf{0})$ corresponding to $J$ has minimum row sum greater than 1. Therefore, since all entries of $DW^+(\mathbf{0})$ are nonnegative, Horn and Johnson (1985, Theorem 8.1.22) implies that the matrix in $\mathbb{R}^{(n-1)\times(n-1)}$ consisting of block $DW^+(\mathbf{0})$ and three zero blocks has spectral radius greater than 1, and the monotonicity of the spectral radius in matrix entries (Horn and Johnson (1985, Theorem 8.1.18)) then implies that $DW^+(\mathbf{0})$ itself has spectral radius greater than 1. A standard extension of Perron's Theorem (Horn and Johnson (1985, Theorem 8.3.1)) implies that the spectral radius of $DW^+(\mathbf{0})$ is an eigenvalue of $DW^+(x)$ corresponding to a nonnegative eigenvector. We thus conclude that $DW(\mathbf{0}) = DW^+(\mathbf{0}) - I$ has a positive eigenvalue corresponding to a nonnegative eigenvector. We conclude that $\mathbf{0}$ is unstable under $\dot{y} = W(y)$, and hence that $e_s$ is unstable under $\dot{x} = V(x)$.

To prove part (ii), we must bound the submatrix column sum of $DV^+(e_s)$. To do so note as in the proof of Proposition 5.1 that the probability under test-$\alpha$ of choosing a test set in $\mathcal{S}_s$ that contains at least one element $t$ of $S_2$ is smallest when $S_2$ is a singleton, in which case the probability is $\frac{\alpha-1}{n-1}$. If such a test set is chosen, the realized payoffs are $\pi_{\boldsymbol{s},j@s}^U$, and $v_{s|j,\boldsymbol{s}}^\kappa < v_{t|\boldsymbol{s}}^\kappa$, a revising agent will chose some strategy in $S_2$. From this and equation (61) it

follows that

$$\sum_{i\in J}\frac{\partial V_i^+}{\partial x_j}(e_s) \geq p_-\kappa\left(\frac{\alpha-1}{n-1}\sum_{i\in J}\mathbf{1}[v^\kappa_{i|j,\boldsymbol{s}} > v^\kappa_{s|\boldsymbol{s}}] + \sum_{T\in\mathcal{S}_s}\tau_s^\alpha(T)\sum_{i\in J}\beta_{si}(\pi^U_{\boldsymbol{s},j@s},T)\right)$$

$$\geq p_-\kappa\left(\frac{\alpha-1}{n-1}\sum_{i\in J}\mathbf{1}[v^\kappa_{i|j,\boldsymbol{s}} > v^\kappa_{s|\boldsymbol{s}}] + \mathbf{1}[S_2\subseteq J]\left(\frac{\alpha-1}{n-1}\mathbf{1}[v^\kappa_{s|j,\boldsymbol{s}} < v^\kappa_{t|\boldsymbol{s}}]\right)\right)$$

$$= p_-\kappa\,\frac{\alpha-1}{n-1}\left(\sum_{i\in J}\mathbf{1}[v^\kappa_{i|j,\boldsymbol{s}} > v^\kappa_{s|\boldsymbol{s}}] + \mathbf{1}[S_2\subseteq J]\,\mathbf{1}[v^\kappa_{s|j,\boldsymbol{s}} < v^\kappa_{t|\boldsymbol{s}}]\right) \quad \text{for all } j\neq s. \tag{63}$$

Equation (63) and condition (20) from the proposition imply that the square block of $DW^+(\mathbf{0})$ corresponding to $J$ has minimum row sum greater than 1. The reminder of the proof is identical to that of part (i). □

*Proof of Corollary 5.6.* For $\alpha \geq \lceil\frac{n}{2}\rceil + 1$ we have $\frac{\alpha-1}{n-1} > \frac{1}{2}$, and then condition (20) in Proposition 5.4 implies that for $\kappa > 1$, and for $\kappa = 1$ if $p_- > 1$, a sufficient condition for instability is either that there is some strategy $i \neq s$ with $v^\kappa_{i|i,\boldsymbol{s}} > v^\kappa_{s|\boldsymbol{s}}$, or that $S_2 = \{j\}$ and $v_{j|\boldsymbol{s}} > v_{\boldsymbol{s}|j,\boldsymbol{s}}$. Parts (i) and (ii) follow directly from these conditions and the relation (12) between total payoffs and matrix payoffs. Part (iii.a) follows from (20) from setting $J = \{i, j\}$, and part (iii.b) in the second case follows from (20) from setting $J = \{j\}$ when $S_2 = \{j\}$. □

*Argument that components of the inflow Jacobian of the reduced dynamics at a pure rest point are nonnegative* (*Section 6*). At pure rest point $e_s$, after dimension reduction (see (54) and (55)), the partial derivatives of the inflow terms of system (1) are

$$\frac{\partial W_i^+}{\partial x_j}(\mathbf{0}) = \frac{\partial V_i^+}{\partial x_j}(e_s) - \frac{\partial V_i^+}{\partial x_s}(e_s) = \left(\sigma_{ji}(U,e_s) + \frac{\partial\sigma_{si}}{\partial x_j}(U,e_s)\right) - \left(\sigma_{si}(U,e_s) + \frac{\partial\sigma_{si}}{\partial x_s}(U,e_s)\right). \tag{64}$$

Since $e_s$ is a rest point, we must have $\sigma_{si}(U,e_s) = 0$, and hence $\frac{\partial\sigma_{si}}{\partial x_j}(U,e_s) \geq 0$ (since $(e_s)_j = 0$) and $\frac{\partial\sigma_{si}}{\partial x_s}(U,e_s) \leq 0$ (since $(e_s)_s = 1$). Thus (64) is nonnegative.

## A.3 Analyses of examples

We describe the Jacobian at rest point $e_s$ after dropping the $s$th coordinate of the state, changing from $\dot{x} = V(x)$ to $\dot{y} = W(y)$ as in equation (54), so that the rest point $e_s$ is sent to $y = \mathbf{0}$ and the Jacobian of $W$ is described by (55). To specify the Jacobian of $W$ at $\mathbf{0}$, we let $S_2 = \operatorname{argmax}_{k\neq s} u_{k|\boldsymbol{s}}$ be the set of strategies that earn the second-best payoff at $e_s$.

**Lemma A.3.**

(*i*) *The Jacobian DW*(**0**) *for BEP*($\tau^{\text{all}}, \kappa, \beta^{\min}$) *dynamics has components*

$$\frac{\partial W_i}{\partial y_j}(\mathbf{0}) = \begin{cases} p_-\kappa\, \mathbf{1}[v^\kappa_{i|j,\boldsymbol{s}} > v^\kappa_{s|\boldsymbol{s}}] - \mathbf{1}[j=i] & \text{if } i \neq \min S_2 \text{ and } i > s, \\ p_-\kappa\, \mathbf{1}[v^\kappa_{i|j,\boldsymbol{s}} \geq v^\kappa_{s|\boldsymbol{s}}] - \mathbf{1}[j=i] & \text{if } i \neq \min S_2 \text{ and } i < s, \\ p_-\kappa \left(\mathbf{1}[v^\kappa_{i|j,\boldsymbol{s}} > v^\kappa_{s|\boldsymbol{s}}] + \mathbf{1}[v^\kappa_{i|\boldsymbol{s}} > v^\kappa_{s|j,\boldsymbol{s}}]\right) - \mathbf{1}[j=i] & \text{if } i = \min S_2 \text{ and } i > s, \\ p_-\kappa \left(\mathbf{1}[v^\kappa_{i|j,\boldsymbol{s}} \geq v^\kappa_{s|\boldsymbol{s}}] + \mathbf{1}[v^\kappa_{i|\boldsymbol{s}} \geq v^\kappa_{s|j,\boldsymbol{s}}]\right) - \mathbf{1}[j=i] & \text{if } i = \min S_2 \text{ and } i < s. \end{cases} \tag{65}$$

(*ii*) *The Jacobian DW*(**0**) *for the BEP*($\tau^{\text{all}}, \kappa, \beta^{\text{stick}}$) *dynamic has components*

$$\frac{\partial W_i}{\partial y_j}(\mathbf{0}) = \begin{cases} p_-\kappa\, \mathbf{1}[v^\kappa_{i|j,\boldsymbol{s}} > v^\kappa_{s|\boldsymbol{s}}] - \mathbf{1}[j=i] & \text{if } i \notin S_2, \\ p_-\kappa \left(\mathbf{1}[v^\kappa_{i|j,\boldsymbol{s}} > v^\kappa_{s|\boldsymbol{s}}] + \frac{1}{|S_2|}\mathbf{1}[v^\kappa_{i|\boldsymbol{s}} > v^\kappa_{s|j,\boldsymbol{s}}]\right) - \mathbf{1}[j=i] & \text{if } i \in S_2. \end{cases}$$

(*iii*) *The Jacobian DW*(**0**) *for the BEP*($\tau^{\text{all}}, \kappa, \beta^{\text{unif}}$) *dynamic has components*

$$\frac{\partial W_i}{\partial y_j}(\mathbf{0}) = \begin{cases} p_-\kappa \left(\mathbf{1}[v^\kappa_{i|j,\boldsymbol{s}} > v^\kappa_{s|\boldsymbol{s}}] + \frac{1}{2}\mathbf{1}[v^\kappa_{i|j,\boldsymbol{s}} = v^\kappa_{s|\boldsymbol{s}}]\right) - \mathbf{1}[j=i] & \text{if } i \notin S_2, \\ p_-\kappa \Big(\mathbf{1}[v^\kappa_{i|j,\boldsymbol{s}} > v^\kappa_{s|\boldsymbol{s}}] + \frac{1}{2}\mathbf{1}[v^\kappa_{i|j,\boldsymbol{s}} = v^\kappa_{s|\boldsymbol{s}}] \\ \qquad + \frac{1}{|S_2|}\mathbf{1}[v^\kappa_{i|\boldsymbol{s}} > v^\kappa_{s|j,\boldsymbol{s}}] + \frac{1}{|S_2+1|}\mathbf{1}[v^\kappa_{i|\boldsymbol{s}} = v^\kappa_{s|j,\boldsymbol{s}}]\Big) - \mathbf{1}[j=i] & \text{if } i \in S_2. \end{cases}$$

*Proof.* We only prove part (i); the proofs of parts (ii) and (iii) are similar. The BEP($\tau^{\text{all}}, \kappa, \beta^{\min}$) dynamic is expressed as

$$\dot{x}_i = V_i(x) = \sum_{m \in M_S} \left(\prod_{k \in S} x_k^{|m^{-1}(k)|}\right) \mathbf{1}\left[i = \min\left(\operatorname*{argmax}_{k \in S} \pi^U_k(m)\right)\right] - x_i, \tag{66}$$

where $M_S = \{m \mid m\colon S \times \{1, \ldots, p_-\} \times \{1, \ldots, \kappa\} \to S\}$.

Now suppose that $i \neq s$, and consider the partial derivatives of $V_i$ at the equilibrium state $e_1$. Since the match assignment in which all opponents play $s$ leads the indicator function in (66) to equal 0, the only summands in (66) whose partial derivatives at state $x = e_s$ can be nonzero are those in which $|m^{-1}(s)| = np_-\kappa - 1$. There are two types of match assignments that can lead to nonzero derivatives in (66). In the first, one of the $p_-\kappa$ opponents when strategy $i$ is tested plays strategy $j \neq s$, and the remaining opponents play $s$. Strategy $i$ is best for such match assignments if $v^\kappa_{i|j,\boldsymbol{s}} > v^\kappa_{s|\boldsymbol{s}}$, so in this case, these match assignments contribute $p_-\kappa$ (i.e., the corresponding number of summands in (66)) to $\frac{\partial V_i}{\partial x_j}(e_s)$. This contribution also occurs if $v^\kappa_{i|j,\boldsymbol{s}} = v^\kappa_{s|\boldsymbol{s}}$ and $i < s$, in which case $i$ beats $s$ under the $\beta^{\min}$ tiebreaker. In the second type of match assignment, $i = \min S_2$, one of the $p_-\kappa$

opponents when strategy $s$ is tested plays strategy $j \neq s$, and the remaining opponents play $s$. Strategy $i$ is best for such match assignments if $v^\kappa_{t|\boldsymbol{s}} > v^\kappa_{s|j,\boldsymbol{s}}$ (or if $v^\kappa_{t|\boldsymbol{s}} = v^\kappa_{s|j,\boldsymbol{s}}$ and $i < s$), contributing $p_-\kappa$ to $\frac{\partial V_i}{\partial x_j}(e_s)$ in this case. Accounting for these and for the $-x_i$ term in (66) and then using (55), we obtain (65). □

*Analysis of Example 5.7.*

Here we analyze the stability of the strict equilibrium state $e_1$ of the Traveler's Dilemma (22) under BEP($\tau^{\text{all}}, \kappa, \beta$) dynamics with $n$ odd and $\kappa = \frac{n+1}{2}$. For $\kappa = \frac{n+1}{2}$ and $1 < j \leq i$, the only total payoff $v^\kappa_{ij} = (\kappa-1)u_{i1} + u_{ij}$ which is not smaller than $v^\kappa_{11} = \frac{n+1}{2}$ is $v^\kappa_{nn} = \frac{n-1}{2}(-1) + n = \frac{n+1}{2} = v^\kappa_{11}$. According to Lemma A.3, this leads to an upper triangular Jacobian with all diagonal elements equal to $-1$ except possibly the last one, $\frac{\partial W_n}{\partial y_n}(\mathbf{0})$. Since the diagonal elements are the eigenvalues of the matrix, the stability of $e_1$ is determined by the sign of $\frac{\partial W_n}{\partial y_n}(\mathbf{0})$. For $\beta^{\text{unif}}$ we obtain $\frac{\partial W_n}{\partial y_n}(\mathbf{0}) = \frac{n-3}{4}$, so $e_1$ is unstable for odd $n \geq 5$. For $\beta^{\text{stick}}$ and $\beta^{\text{min}}$ we obtain $\frac{\partial W_n}{\partial y_n}(\mathbf{0}) = -1$, so $e_1$ is asymptotically stable.

*Analysis of Example 5.8*

Here we analyze the stability of the strict equilibrium state $e_1$ of game (28) under BEP($\tau^{\text{all}}, \kappa, \beta$) dynamics for $\kappa = 3$. The matrix $v^3$, with elements $v^\kappa_{ij} = 2u_{i1} + u_{ij}$, is

$$v^3 = 2 \cdot \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix} = \begin{pmatrix} 3 & 2 & 2 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

By Lemma A.3 and the fact that $v^3_{11} = 3$, the Jacobian for the BEP($\tau^{\text{all}}, 3, \beta^{\text{unif}}$) dynamic is

$$DW(\mathbf{0}) = 3 \cdot \begin{pmatrix} 0 & 0 \\ 0 & \frac{1}{2}\mathbf{1}[3=3] \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & \frac{1}{2} \end{pmatrix},$$

so the strict equilibrium $e_1$ is unstable. For the BEP($\tau^{\text{all}}, 3, \beta^{\text{min}}$) and BEP($\tau^{\text{all}}, 3, \beta^{\text{stick}}$) dynamics, the Jacobian is

$$DW(\mathbf{0}) = 3 \cdot \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix},$$

so $e_1$ is asymptotically stable.

# References


Basu, K. (1994). The traveler's dilemma: Paradoxes of rationality in game theory. *American Economic Review Papers and Proceedings*, 84:391–395.

Benaïm, M. and Weibull, J. W. (2003). Deterministic approximation of stochastic evolution in games. *Econometrica*, 71:873–903.

Berger, U. and Hofbauer, J. (2006). Irrational behavior in the Brown-von Neumann-Nash dynamics. *Games and Economic Behavior*, 56:1–6.

Bergin, J. and Bernhardt, D. (2004). Comparative learning dynamics. *International Economic Review*, 45:431–465.

Berkemer, R. (2008). Disputable advantage of experience in the travelers' dilemma. Unpublished manuscript, Technical University of Denmark. Abstract in *International Conference on Economic Science with Heterogeneous Interacting Agents, Warsaw, 2008*.

Björnerstedt, J. and Weibull, J. W. (1996). Nash equilibrium and evolution by imitation. In Arrow, K. J. et al., editors, *The Rational Foundations of Economic Behavior*, pages 155–181. St. Martin's Press, New York.

Brown, G. W. and von Neumann, J. (1950). Solutions of games by differential equations. In Kuhn, H. W. and Tucker, A. W., editors, *Contributions to the Theory of Games I*, volume 24 of *Annals of Mathematics Studies*, pages 73–79. Princeton University Press, Princeton.

Cárdenas, J. C., Mantilla, C., and Sethi, R. (2015). Stable sampling equilibrium in common pool resource games. *Games*, 6:299–317.

Chmura, T. and Güth, W. (2011). The minority of three-game: An experimental and theoretical analysis. *Games*, 2:333–354.

Dembo, A. and Zeitouni, O. (1998). *Large Deviations Techniques and Applications*. Springer, New York, second edition.

Helbing, D. (1992). A mathematical model for behavioral changes by pair interactions. In Haag, G., Mueller, U., and Troitzsch, K. G., editors, *Economic Evolution and Demographic Change: Formal Models in Social Sciences*, pages 330–348. Springer, Berlin.

Hofbauer, J. and Sandholm, W. H. (2009). Stable games and their dynamics. *Journal of Economic Theory*, 144:1665–1693.

Hofbauer, J. and Sandholm, W. H. (2011). Survival of dominated strategies under evolutionary dynamics. *Theoretical Economics*, 6:341–377.

Horn, R. A. and Johnson, C. R. (1985). *Matrix Analysis*. Cambridge University Press, Cambridge.

Izquierdo, L. R., Izquierdo, S. S., and Sandholm, W. H. (2018). *EvoDyn-3s*: A Mathematica computable document to analyze evolutionary dynamics in 3-strategy games. *SoftwareX*, 7:226–233.

Izquierdo, L. R., Izquierdo, S. S., and Sandholm, W. H. (2019). An introduction to ABED: Agent-based simulation of evolutionary game dynamics. *Games and Economic Behavior*, 118:434 – 462.

Kosfeld, M., Droste, E., and Voorneveld, M. (2002). A myopic adjustment process leading to best reply matching. *Journal of Economic Theory*, 40:270–298.

Kreindler, G. E. and Young, H. P. (2013). Fast convergence in evolutionary equilibrium selection. *Games and Economic Behavior*, 80:39–67.

Mantilla, C., Sethi, R., and Cárdenas, J. C. (2019). Efficiency and stability of sampling equilibrium in public good games. *Journal of Public Economic Theory*, forthcoming.

Miękisz, J. and Ramsza, M. (2013). Sampling dynamics of a symmetric ultimatum game. *Dynamic Games and Applications*, 3:374–386.

Osborne, M. J. and Rubinstein, A. (1998). Games with procedurally rational players. *American Economic Review*, 88:834–847.

Oyama, D., Sandholm, W. H., and Tercieux, O. (2015). Sampling best response dynamics and deterministic equilibrium selection. *Theoretical Economics*, 10:243–281.

Ramsza, M. (2005). Stability of pure strategy sampling equilibria. *International Journal of Game Theory*, 33:515–521.

Rosenthal, R. W. (1981). Games of perfect information, predatory pricing and the chain-store paradox. *Journal of Economic Theory*, 25:92–100.

Rowthorn, R. and Sethi, R. (2008). Procedural rationality and equilibrium trust. *The Economic Journal*, 118:889–905.

Sandholm, W. H. (2001). Almost global convergence to $p$-dominant equilibrium. *International Journal of Game Theory*, 30:107–116.

Sandholm, W. H. (2010a). Pairwise comparison dynamics and evolutionary foundations for Nash equilibrium. *Games*, 1:3–17.

Sandholm, W. H. (2010b). *Population Games and Evolutionary Dynamics*. MIT Press, Cambridge.

Sandholm, W. H. (2014). Local stability of strict equilibrium under evolutionary game dynamics. *Journal of Dynamics and Games*, 1:485–495.

Sandholm, W. H., Izquierdo, S. S., and Izquierdo, L. R. (2019). Best experienced payoff dynamics and cooperation in the Centipede game. *Theoretical Economics*, in press.

Schaffer, M. E. (1988). Evolutionarily stable strategies for a finite population and a variable contest size. *Journal of Theoretical Biology*, 132:469–478.

Schlag, K. H. (1998). Why imitate, and if so, how? A boundedly rational approach to multi-armed bandits. *Journal of Economic Theory*, 78:130–156.

Sethi, R. (2000). Stability of equilibria in games with procedurally rational players. *Games and Economic Behavior*, 32:85–104.

Smith, M. J. (1984). The stability of a dynamic model of traffic assignment—an application of a method of Lyapunov. *Transportation Science*, 18:245–252.

Taylor, P. D. and Jonker, L. (1978). Evolutionarily stable strategies and game dynamics. *Mathematical Biosciences*, 40:145–156.

Weibull, J. W. (1995). *Evolutionary Game Theory*. MIT Press, Cambridge.

Zusai, D. (2018). Gains in evolutionary dynamics: a unifying approach to stability for contractive games and ESS. Unpublished manuscript, Temple University.