MODAL REDUCTION PRINCIPLES ACROSS RELATIONAL SEMANTICS

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Abstract

Sahlqvist theory is an important result in the model theory of modal logic, since it identifies a class of formulas which have effectively computable first order correspondents. Recently, this theory has been generalised to a larger set of logics by using their algebraic semantics. This fact has allowed researchers to define inequalities of formulas and to determine under which conditions these inequalities have effectively computable first order correspondents, that is, under which conditions they are Sahlqvist inequalities. Actually, there are algorithms that compute first order correspondents of these inequalities, such as ALBA algorithm. This algorithm translates any Sahlqvist inequality to a first order formula, but this translation still strongly depends on semantics. In this thesis, it is proposed a methodology to obtain first order correspondents of certain inequalities, called modal reduction principles, which are easily comparable across two relational semantics: crisp and many-valued polarity-based semantics. Concretely, this thesis presents an introduction to Sahlqvist theory and polarity-based semantics and proves that the first order correspondents of modal reduction principles are pure inclusion of binary relations on both semantics.

Keywords: Correspondence theory, Sahlqvist theory, modal logic, many-valued modal logic, modal reduction principles, Kripke models, polarity-based semantics, non-distributive logics.

Resumen

La teoría de Sahlqvist es un importante resultado de la teoría de modelos de la lógica modal, ya que identifica una clase de fórmulas que tienen un correspondiente de primer order efectivamente computable. Esta teoría ha sido recientemente generalizada a un mayor conjunto de lógicas gracias a considerar la semántica algebraica de la lógica modal. Esto ha permitido definir desigualdades de fórmulas y establecer bajo qué condiciones se puede asegurar que tienen un correspondiente de primer order efectivamente computable, es decir, bajo qué condiciones son desigualdades de Sahlqvist. De hecho, se han definido algoritmos con este objetivo, como por ejemplo el algoritmo ALBA. Este algoritmo traduce cualquier desigualdad de Sahlqvist a una fórmula de primer orden, pero esta traducción todavía depende fuertemente de la semántica considerada. En este trabajo de fin de máster, se propone una metodología para obtener correspondientes de primer order de cierto tipo de desigualdades, llamadas principios de reducción modal, que sean fácilmente comparables entre sí al interpretarlas con dos semánticas relacionales distintas: la semántica de polaridad bi-valuada y multi-valuada. Concretamente, este trabajo presenta una introducción a la teoría de Sahlqvist y a la semántica de polaridad y demuestra que los correspondientes de primer orden de estas desigualdades son inclusiones de relaciones binarias en ambas semánticas.

Palabras clave: Teoría de la correspondencia, teoría de Sahlqvist, lógica modal, lógica modal multi-valuada, principios de reducción modal, modelos de Kripke, semántica de polaridad, lógicas no distributivas.

Contents

1 Introduction

This dissertation arises in the context of Sahlqvist theory. This theory is a part of the modal logic theory which identifies a class of logical formulas which have first order and effectively computable correspondents, i.e., which can be expressed in a first order language and this translation is effectively computable. This theory started out as a model-theoretic investigation, but recently, it has been reformulated in algebraic terms via duality theory, a theory which determines a direct connection between relational and algebraic semantics. If a logic is interpreted as a lattice, then its elements are collected in a partially ordered set, and we can establish an order relation among them. As a consequence, we can define inequalities of formulas as we will see in this work, and we can also identify whether these inequalities have first order and effectively computable correspondents. In that case, we will say that they are Sahlqvist inequalities. This fact has allowed researchers to generalise Sahlqvist theory and identify Sahlqvist inequalities in different languages and logics such as intuitionistic modal mu-calculus (Conradie, 2015), distributive logic (Conradie, 2012), non-distributive logic (Conradie, 2019a) and many valued logic (Britz, 2016). Also, particularly important for those non-classical logics (such as positive modal logic (Dunn, 1995) or intuitionistic and bi-intuitionistic modal logic (Rauszer, 1974) or substructural logics (Galatos, 2007)) for which more than one type of relational semantics has been defined, the algebraic approach makes it possible to develop correspondence theory simultaneously for each type of semantics of a given logic. Given these developments, a natural question to ask is whether and how we can compare the first order correspondents in various semantic settings of one and the same Sahlqvist inequality.

Our contribution to answer this question is to explore what happens with certain Sahlqvist inequalities, called modal reduction principles, when they are interpreted both in crisp and many-valued cases of a relational semantics. Modal reduction principles are inequalities of formulas that are formed by the modal operators $\Box$ and $\Diamond$ and atomic propositional variables, for instance, $\Box\Diamond\Diamond p \leq \Diamond\Box p$ is a modal reduction principles. We have selected this kind of inequalities to carry out our study because they are the simplest ones that can be used to test our hypothesis, and working on them could result in a base case of a general methodology for semantics comparison. Explicitly, our hypothesis is that the first order correspondents of these inequalities can be expressed as pure inclusions of binary relations. Indeed, we expect that, given a Sahlqvist modal reduction principle, its first order correspondent could be formulated as the same inclusion of relations in both semantics.

The relational semantics that we have chosen are the crisp and many-valued cases of polarity-based ones. Polarity-based semantics are relational semantics which interpret very naturally non-distributive logics. They are based in Formal Concept Analysis, so we will have tuples called *concepts* formed by a set of objects and a set of features of these objects, as well as possible worlds or states in Kripke semantics. We have decided to use these semantics because Sahlqvist theory is currently developed for distributive and non distributive two valued logics and for distributive many-valued logic, but it is not established for non-distributive many-valued logic, hence this study can bring some lights to this research path too. In addition, non-distributive logic and polarity-based semantics would be a good tool for those fields which require formal representations of conceptual structures such as psychology, linguistics or biology.

Moreover, the methodology that we have followed was exposed in (Conradie, 2019b) and it consists on first implementing ALBA algorithm (Conradie, 2019a) on a given Sahlqvist modal reduction principles, and then applying certain relation composition, which must be previously defined, to get a pure inclusion of them.

With all these elements, we will give a new step towards the systematic comparison of first order correspondents of modal formulas inequalities across semantics. Exactly, this thesis has three main objectives:

1. To explain the context in which we build our results, that is, to introduce Sahlqvist theory and some of its central statements.
2. To describe polarity-based semantics and to present its crisp and many-valued definition.
3. To prove that modal reduction principles can be expressed as pure inclusions of binary relations in both mentioned semantics and to illustrate that these first order correspondents are the same in crisp and many-valued case.

We will allocate a section for each one of these goals.

2 Background theory: Sahlqvist theory

Novel results presented in this thesis are built on the Sahlqvist correspondence theory. This theory builds on translations between logics, so we will start this section by discussing the translation paradigm that is widely developed in modern logic. Then, we will introduce the three main technical pillars needed to understand how Sahlqvist theory has evolved. That is, we will talk about modal logic, Kripke frames and duality theory. Finally, we will explain what is correspondence theory about, what the state of the art in Sahlqvist theory is and which of its results we will need to achieve our conclusions.

2.1 The translation paradigm in modern logic

As mentioned early on, Sahlqvist theory forms part of the translation paradigm of modern logic. From Gottlob Frege (1848—1925), who is often called the 'founder of modern logic', mathematical logic has experienced an exponential increase. Frege and his school turned classical logic into a mathematical discipline, creating a language to represent it. This allowed them and the rest of the mathematical and philosophical community to ask very relevant questions, such as what can be expressed with a given formal language or which are its expressive limits. This kind of questions revolutionized the study of logic and, during the last century, a huge amount of results and different logics have been introduced. Fuzzy logics, conditional logics, free logics, relevant logics, intuitionistic logic or hybrid logics are only some of them. On the one hand, this explosion of new logics has meant the possibility of expressing and solving problems from very different fields such as computer science, philosophy, linguistics, economics or mathematics. However, on the other hand, a new issue has shown up: how do these logics relate to each other? Can we transfer results from one to another? How can we compare them?

These questions have concerned researchers who have developed several research lines in logic which, in spite of their differences, pertain to what we can refer to as the 'translation paradigm'. María Manzano sums up these distinct approaches as follows (Manzano, 2014: 265):

1. From a *proof-theoretical* point of view, the style of comparing logics will rest upon morphisms between calculi. The 'labelled deductive systems' of Gabbay emerge.
2. From a *model-theoretical perspective*, one will presumably compare logics by defining morphisms between the structures those logics are attempting to describe, as in the correspondence theory of van Benthem.
3. From a *suprastructural* point of view, we define morphisms between categories. Among the most abstract approaches to logic, we should highlight the "general logics" of Meseguer.

According to this classification, Sahlqvist theory can be understood as a logic translation system which originally arose as a core piece of the model-theory of modal logic and then has been systematically connected with the algebraic logic theory.

2.2 Some preliminaries

In order to explore Sahlqvist theory in detail, we outline the three main theories that support it: modal logic, Kripke frames and duality theory.

2.2.1 Modal logic

Modal logic arose with Aristotle as a tool to talk about the modes of truth. However, in order to study formal properties of that kind of logics, Clarence I. Lewis (1883-1964) and other authors established a syntaxis that allows us to write modal formulas and a list of axioms that define different modal logic systems (see Table 1).

A classical way to define the language of modal logic is using a set *AtomProp* of atomic formulas, which are represented as p_i; falsity $\bot$, which is an atomic formula; the connectives $\neg$, $\vee$, $\wedge$, $\rightarrow$ and $\leftrightarrow$; the modal operator $\Diamond$; and parentheses. The modal operator $\Box$ can be defined as $\Box\varphi := \neg\Diamond\neg\varphi$, where φ is a well-formed formula of this language. Other modal languages, that we need to use in this work, have one modal operator more, $\blacklozenge$, which can be used to define another one: $\blacksquare := \neg\blacklozenge\neg\varphi$.

The intuitive interpretation of these modal operators considers $\Diamond$ as the *possibility* operator and $\Box$ as the *necessity* operator. However, if we take the intuitive interpretation of modal operators given by tense logics, we can give a similar explanation for $\blacksquare$ and $\blacklozenge$ too:

1. $\Diamond\varphi$ may mean that φ will be true at some future point;

2. $\Box\varphi$ may mean that φ will always be the case;

3. $\blacklozenge\varphi$ may mean that φ was true at some past point;

4. and $\blacksquare\varphi$ may mean that φ has always been the case.

K	$\Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi)$
Df$\Diamond$	$\Diamond\varphi \leftrightarrow \neg\Box\neg\varphi$
D	$\Box\varphi \to \Diamond\varphi$
T	$\Box\varphi \to \varphi$
B	$\varphi \to \Box\Diamond\varphi$
4	$\Box\varphi \to \Box\Box\varphi$
5	$\Diamond\varphi \to \Box\Diamond\varphi$

Table 1: Axioms for normal logics.

2.2.2 Kripke frames

Once we have a formal language which characterises different modal logics, we need a semantics to interpret those formulas. A common way of interpreting these logics is with relational semantics, based on some structures called *Kripke frames*. A Kripke frame is a structure $\mathscr{F} = (W,R)$ with W a non empty set and R a binary relation on W. If we add a valuation function $V : AtomProp \longrightarrow \mathscr{P}(W)$, we obtain a *Kripke model* $\mathscr{M} = (W,R,V)$. One way of interpreting the set W is seeing its elements as states or worlds where each formula has a specific truth value. Therefore, given a Kripke model $\mathscr{M}$, each formula φ has an interpretation $\mathscr{M}(\varphi)$ which is the set of states or worlds where φ is true. A detailed definition of interpretation of modal formulas, satisfiability and validity within this semantics can be found in (Manzano, Extensions: 305). When modal logic is interpreted under models as the ones defined above, it is said *model-theoretic modal logic*, which will be one of our start points.

2.2.3 Duality theory

Kripke frames build a semantics known as *relational semantics*, but it is not the only type of semantics that can be used to interpret a modal language. Other very well-known semantics is *algebraic semantics*. To develop an algebraic semantics we need a set of algebraic structures (that is, a nonempty set as domain, a collection of operations on that domain and a finite set of axioms that these operations satisfy) in place of relational structures like Kripke frames. For example, we can take the class of lattices. A lattice is an algebraic structure with two especial binary operations: $\vee$ ('join') and $\wedge$ ('meet'). In addition, these operations hold the commutative, associative, idempotent or absorption laws. Given a modal language $\mathscr{L}$, we can interpret it in an algebraic way picking a lattice A^+ with extra operations (to represent modal operations) and define a valuation v from $AtomProp$ to A^+, which is uniquely extendable to another function $\hat{v}$ from $\mathscr{L}$ to A^+ (we will see an example in next paragraphs). A logic interpreted on this way is called a *lattice-based logic*. Furthermore, depending on the particular characteristics of the lattice we are considering, we can obtain different types of logics:

- If A^+ is a distributive lattice, i.e., $\vee$ holds the distributivity law over $\wedge$ and on reverse, then a logic based on A^+ is a distributive logic;
- If A^+ is a non-distributive lattice, i.e., either $\vee$ does not hold the distributivity law over $\wedge$ or on reverse, then a logic based on A^+ is a non-distributive logic;
- If A^+ is expanded with other operations different from $\vee$ and $\wedge$, then a logic based on A^+ is a lattice expansion logic; and so on.

A lattice is an algebraic structure whose domain is partially ordered, so there exist a natural ordering among its elements that will be denoted as $\leq$. As a consequence, for any formal language $\mathscr{L}$ that can be interpreted over lattice expansions, we can define inequalities $\varphi \leq \psi$ where φ and ψ are both formulas of the language $\mathscr{L}$. We can interpret the inequality $\varphi \leq \psi$ in A^+ under $v : AtomProp \to A^+$ considering its extension $\hat{v} : \mathscr{L} \to A^+$. $\hat{v}(\varphi)$ and $\hat{v}(\psi)$ are elements of A^+, so we can ask whether $\hat{v}(\varphi) \leq^{A^+} \hat{v}(\psi)$ or not. If the answer is positive, we say that the inequality $\varphi \leq \psi$ is satisfied in A^+ under v; in other case, we say that it is not satisfied. We also say that $\varphi \leq \psi$ is valid on A^+ if it is satisfied in A^+ under every assignment v.

As it can be observed, these semantics allows us to define inequalities of logical formulas and to take advantage of order properties of a given lattice. However, we do not have to give up relational semantics, since both semantics are intimately related. The field which study this relation is the duality theory and it gives us the following useful results:

1. Every Boolean algebra with operators can be associated with a relational frame called *ultrafilter*.
2. Every Kripke frame can be associated with its complex algebra which is a Boolean algebra with operators.
3. (Jónsson-Tarski expansion of Stone representation theorem) Every Boolean algebra with operators cannonically embeds in the complex algebra of its ultrafilter frame.

That means that, given a Kripke frame for a modal language, we can consider its dual algebraic semantics too. Let us a modal language whose well-formed formulas are given by the rule:

$$\varphi ::= p \mid \bot \mid \neg\varphi \mid \varphi \vee \varphi \mid \Diamond\varphi,$$

where $p \in AtomProp$. The complex algebra of a Kripke frame $\mathscr{F} = (W, R)$ which interprets this language is a Boolean algebra with operators

$$\mathscr{F}^+ = (\mathscr{P}(W), \cup, \cap, -W, \emptyset, W, m_R)$$

where $-W$ is the set complementation relative to W and m_R is a function defined on $\mathscr{P}(W)$ such that $m_R(X) := \{w \in W \mid Rwv \ for\ all\ v \ \in \ X\}$. In this case, the natural ordering of $\mathscr{F}^+$ is the inclusion relation ($\leq^{\mathscr{F}^+} = \subseteq$). If we have both semantics, we can choose any of them to give an interpretation of modal formulas, satisfiability and validity, since for every Kripke model $\mathscr{M} = (W, R, V)$, every $w \in W$ and every $\varphi \in \mathscr{L}$:

$$\mathscr{M}, w \models \varphi \quad iff \quad \{w\} \leq^{\mathscr{F}^+} \hat{v}(\varphi)$$

As it will be shown, duality theory allows Sahlqvist theory to keep the expressive power of relational semantics and to add the order properties that algebraic semantics brings. This is especially important for this work because our results are related with the calculation of the first order correspondents of certain *inequalities* of modal formulas, so it is useful to understand how they are related with the rest of the formal system.

2.3 Sahlqvist theory

Sahlqvist theory originally arose as a core piece of the model-theory of classical modal logic and, via duality, it has been systematically connected with the algebraic theory of modal logic. The aim of this theory was to describe the shape of modal formulas which have first order correspondents and, thanks to the order-theoretic properties that the complex algebra of Kripke frames presents, its conclusions have been systematically extended much beyond modal logic.

As we will see, Sahlqvist theory gives a collection of principles that ensure the existence of first order correspondents. We will call *correspondence theory* to the theory that identify these principles according to modal formula properties, and *unified correspondence theory* to the theory that identify them according to order-theoretic properties. In this section, we will see the different scope of both theories and we will introduce an algorithm which will be useful to develop our work.

2.3.1 Correspondence theory

Correspondence theory, as Johan van Benthem writes in his paper *Correspondence Theory* (Van Benthem, 1984), is an applied theory which takes tools from model theory and universal algebra in order to explore the possibilities and the limits of calculating the first order correspondent of a given modal formula through Kripke semantics.

The existence of first order correspondents of modal formulas is a powerful result. On the one hand, modal logic is a very expressive system that allows logicians to talk about dynamic situations (as possibility and necessity, future and past). On the other hand, first order logic is a formal system extensively studied. Thus, discovering the relations between both formal systems and being able to systematically detect them, bring the opportunities of understanding what modal axioms mean, better comparing different modal logics by contrasting the first order correspondents of their axioms, and transferring metaproperties after this comparison if it applies. In order to appreciate the benefits that this theory brings, we can look at Table 2.

Modal formula	First order correspondent
$\Box p \rightarrow p$ $\Box p \rightarrow \Box\Box p$ $p \rightarrow \Box\Diamond p$	$\forall x Rxx$ $\forall xy(Rxy \rightarrow \forall z(Ryz \rightarrow Rxz))$ $\forall xy(Rxy \rightarrow Ryx)$

Table 2: Correspondents of some modal formulas (Van Benthem, 1984: 193).

In Table 2 we can appreciate the power of the correspondence theory. While given modal formulas are few meaningful for us, their correspondents are clear: the first one describes R as a reflexive relation; the second one, as a transitive relation; and the last one, as a symmetric relation. Very briefly, these correspondents mean that those modal formulas may be viewed as constraints on the relational structure that we are using, such as Kripke frame. So, for example, if the formula $\Box p \rightarrow p$ is an axiom of our modal logic, then the Kripke frame that we use to interpret that formula will be reflexive (i.e., all states will be related with themselves).

To take full advantage of this achievement, being able to obtain the correspondent of a modal formula in a systematic way would be desirable. That is exactly the aim of Sahlqvist theory, since it defines a class of modal formulas which have a computable first order correspondent.

Definition 2.1. *Given a modal language $\mathscr{L}$, a* Sahlqvist antecedent *is a formula composed by $\top$, $\bot$, a concatenation of boxes over p (the concatenation can be null and be just p) and formulas with an odd number of negation signs using $\wedge$, $\vee$ and $\Diamond$. If we have a formula ψ with an even number of negation signs and a Sahlqvist antecedent φ, we will say that $\varphi \longrightarrow \psi$ is a* Sahlqvist implication*. Finally, a* Sahlqvist formula *is a formula composed by Sahlqvist implications applying without any restriction the operators $\Box$ and $\wedge$ and applying $\vee$ only between Sahlqvist implications that do not share any proposition letters.*

For example, the modal formulas showed in Table 2 and Table 1 are all of them Sahlqvist formulas except axiom **K** and axiom **Df**$\Diamond$.

Considering the previous definition, we can give a core result of the Sahlqvist theory:

Theorem 2.2. *(The Sahlqvist Theorem). Let $\varphi(p_1, \ldots, p_n)$ be a Sahlqvist formula in the modal language $\mathscr{L}$. Then, $\varphi(p_1, \ldots, p_n)$ locally corresponds to a first-order formula $\alpha(x)$ on frames. Moreover, $\alpha(x)$ is effectively computable from $\varphi(p_1, \ldots, p_n)$ (Sahlqvist, 1975).*

The proof of this result gives an algorithm that computes the first order correspondent of a given Sahlqvist formula, the Sahlqvist-van Benthem algorithm. However, this result is strictly applicable to modal formulas, since the given definition of being 'Sahlqvist' depends too much on modal language. In 2014, Willem Conradie, Silvio Ghilardi and Alessandra Palmigiano gave a more general definition of what means being 'Sahlqvist', that is, under which situations we can ensure the effective computation of the first order correspondent of a given input. They collected their work in the paper *Unified correspondence* (Conradie, 2014) and we will give some of its keys in the following subsection.

2.3.2 Unified correspondence theory

The clue to generalise the well-known Sahlqvist theory to different logics was to identify that the properties which Sahlvist describes in Definition 2.1 encode order-theoretic properties. This means that, given any logic whose dual algebra has a natural order relation (as every lattice expansion logics), it is possible to identify which formulas are 'Sahlqvist' and to compute the first order correspondent of those which actually are. As a consequence, it is possible to compare very different logics, due to we can identify their 'Sahlqvist' formulas and compare their first-order correspondents.

In order to be able to process most logics as possible in the same way, this theory defines a general logic in terms of sequents, since, for example, the deduction-detachment theorem does not hold in every logic. So, the first step to develop a general Sahlqvist theory was to lift the notion of Sahlqvist formulas to Sahlqvist sequents. As we are considering logics with lattices as algebraic semantics, we can interpret that

$$\varphi \vdash \psi \quad iff \quad \hat{v}(\varphi) \leq \hat{v}(\psi)$$

where $\hat{v}$ is the interpretation of formulas introduced in Section 2.2.3. Thus, we can directly define the *Sahlqvist shape* over inequalities of formulas and then, easily apply order-theoretic properties.

Remark: From here, we are going to restrict our definitions to certain type of inequalities, called modal reduction principles, since they are the inequalities that we are going to consider in our results and all definitions are simpler if we do this restriction.

Let us fix a language $\mathscr{L} = \mathscr{L}(\Diamond, \Box)$ based on a lattice with operators. A *modal reduction principle* of $\mathscr{L}$ is an inequality $s \leq t$ such that both s and t are generated as follows:

$$s,t ::= \mid \top \mid \bot \mid p \mid \Diamond \mid \Box.$$

In order to define which modal reduction principles are Sahlqvist inequalities, we need to use the generation tree of s and t (see Figure 1) and to build their signed generation tree. In particular, we will consider the positive generation tree $+s$ for the left-hand side and the negative one $-t$ for the right-hand side. Moreover, we will write $\varphi \prec \psi$ if φ is a subtree of ψ.

Definition 2.3. *The* positive *(resp.* negative*)* generation tree *of any $\mathscr{L}$ term s is defined by labelling the root node of the generation tree of s with the sign + (resp. −), and the propagating the labelling on each* *remaining node assigning the same sign to its children nodes.*

Nodes in signed generation trees are positive *(resp.* negative*) if are signed* $+$ *(resp.* $-$*).*

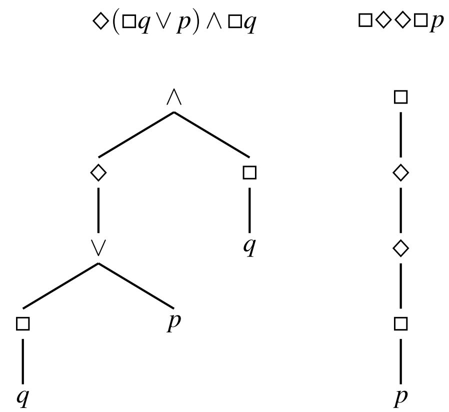


Figure 1: Some examples of generation trees.[1]

Definition 2.4. *Nodes in signed generation trees will be called* Skeleton *and* PIA *node according to the specification given in Table 3.*

A branch in a signed generation tree $*s$*, with* $* \in \{+,-\}$*, is called a* good branch *if it is the concatenation of two paths* P_1 *and* P_2*, one of which may possibly be of length* 0*, such that* P_1 *is a path from the leaf consisting (apart from variable nodes) only of PIA nodes, and* P_2 *consists (apart from variables nodes) only of Skeleton nodes.*

For any term $s(p_1 \ldots, p_n)$*, a* critical node *in a signed generation tree of* s *is a leaf node* $+p_i$*. A* critical branch *in the tree is a branch from a critical node.*

Skeleton	PIA
$+\Diamond$	$-\Diamond$
$-\Box$	$+\Box$

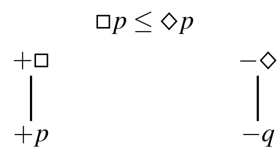
Table 3: Skeleton and PIA nodes for $\mathscr{L}$.

Clearly, the generation trees of s (resp. t) in modal reduction principles consist of a single branch from the root of the term to the single leaf.

Definition 2.5. *A modal reduction principle is a* Sahlqvist inequality *if and only if one of these two branches is good.*

For instance, if we have the model reduction principle $\Box p \leq \Diamond p$, in order to know whether it is a Sahlqvist inequality, we must build the positive generation tree of $\Box p$ and the negative generation tree of $\Diamond p$ as follows:

[1]In order to give a more illustrative example, we consider that our language includes $\wedge$ and $\vee$.



In this example, both branches are good due to they start with a PIA node (path P1) and path P2 is of length 0. Therefore, $\Box p \leq \Diamond p$ is a Sahlqvist inequality.

2.3.3 ALBA algorithm

When we have a Sahlqvist inequality, we can enter it as an input of an algorithm called ALBA (Ackermann Lemma Based Algorithm). This algorithm, defined in (Conradie, 2012) for inequalities in distributive lattices and in (Conradie, 2019a) for inequalities in non-distributive ones, returns the first order correspondent of every Sahlqvist inequality that it receives as input. However, it uses an intermediate language. In order to implement the Ackermann Rules (a fundamental part of the algorithm), it has to consider a language with an infinite set of sorted variables which range over the completely join-irreducible elements of the lattice in which the logic is based (elements which are distinct from bottom and have the property of belonging to a subset S if they are equal to $\bigvee S$), and another infinite set of sorted variables which range over the completely meet-irreducible elements of the lattice in which the logic is based (elements which are distinct from top and have the property of belonging to a subset S if they are equal to $\bigwedge S$). The former are called *nominals*, and the latter *co-nominals*.

In addition, this intermediate language needs the *adjoints* of the modal operators that we are considering. A map $f : A \longrightarrow B$ and a map $g : B \longrightarrow A$ form an *adjoint pair* if, for all $a \in A$ and $b \in B$, it holds that $f(a) \leq^{A^+} b$ if and only if $a \leq^{A^+} g(b)$. The operator $\Box$ forms an adjoint pair with $\blacklozenge$ and the operator $\Diamond$ forms an adjoint pair with $\blacksquare$. This new language $\mathscr{L}^+$ is called the expanded language of $\mathscr{L}$ and its formulas are given by the following recursive definition:

$$\varphi ::= \mathbf{j} \mid \mathbf{m} \mid \psi \mid \varphi \wedge \varphi \mid \varphi \vee \varphi \mid \Diamond \mid \Box \mid \blacklozenge \mid \blacksquare$$

where $\psi \in \mathscr{L}$, $\mathbf{j}$ is a nominal and $\mathbf{m}$ is a co-nominal.

The ALBA algorithm has in three steps. The first one consists on applied a rule called First Approximation Rule which introduce nominals and/or co-nominals to the Sahlqvist inequality that has been inserted as input. Then, the Ackermann Rule eliminates the propositional variables from the inequality, although it is possible that adjoints must be used previously in order to be able to implement this rule. Finally, a standard translation from $\mathscr{L}^+$ to a first order language is done.

Following an example of how ALBA works:

	$\forall p[\Box p \leq \Diamond p]$	
iff	$\forall p \forall \mathbf{j} \forall \mathbf{m}[(\mathbf{j} \leq \Box p \& \Diamond p \leq \mathbf{m}) \Rightarrow \mathbf{j} \leq \mathbf{m}]$	first approximation
iff	$\forall p \forall \mathbf{j} \forall \mathbf{m}[(\mathbf{j} \leq \Box p \& p \leq \blacksquare \mathbf{m}) \Rightarrow \mathbf{j} \leq \mathbf{m}]$	adjunction
iff	$\forall \mathbf{j} \forall \mathbf{m}[\mathbf{j} \leq \Box \blacksquare \mathbf{m} \Rightarrow \mathbf{j} \leq \mathbf{m}]$	Ackermann's Lemma
iff	$\forall \mathbf{m}[\Box \blacksquare \mathbf{m} \leq \mathbf{m}]$	**j** completely join-generates LE

From here, we would have to apply the standard translation defined for the semantics that we were using.

The two first steps of ALBA allows us to compare logics with different syntax. However, to get the translation from the output of the second step (an inequality as the last one of the previous example) to the first order correspondent, we will need a standard translation for each semantics. Nowadays, a full ALBA definition, including the appropriate standard translation, is developed for both crisp and many-valued distributive logics and for crisp non-distributive logics (see Table 4).

Two-valued Distributive Logics (Conradie, 2012)	Many-valued Distributive Logics (Britz, 2016)
Two-valued Non-distributive Logics (Conradie, 2019a)	Many-valued Non-distributive Logics *(Developing)*

Table 4: State of the art Sahlqvist theory.

This fact is still an open issue, due to the comparison of first order correspondents across semantics is hard even when those semantics are applied to the same logic. Some first order correspondents got with these translations are extremely complex, even for simple formulas as reflexivity ($\Box p \to p$) (Conradie, 2019a). For this reason, the results presented in this thesis are quite relevant. In the following sections, we will introduce a semantics for non-distributive logics and we will explore the crisp and the many-valued case. On the one hand, we will make some contributions to the study of many-valued non-distributive logics. On the other hand, we will define a way to obtain an understandable first order correspondent for both semantics in a systematic way. Indeed, we will conclude that first orders correspondents of modal reduction principles are pure inclusion of binary relations in both semantics.

3 Polarity-based semantics for non-distributive logics

In this section we introduce the relational semantics that we will work with during the rest of this thesis. It is called *polarity-based semantics* and it is a powerful tool to intuitively understand non-distributive logics. Non-distributive logics are logics where the distributive laws do not hold, i.e.,

$$\alpha \wedge (\beta \vee \gamma) \nvdash (\alpha \wedge \beta) \vee (\alpha \wedge \gamma)$$
$$(\alpha \vee \beta) \wedge (\alpha \vee \gamma) \nvdash \alpha \vee (\beta \wedge \gamma).$$

Some well known non-distributive logics are quantum logic (Mackey, 1957) and Lambek calculus and its axiomatic extensions (Galatos, 2007). As mentioned early on, it is natural to take an algebraic interpretation of these logics, since they are defined over algebraic structures. However, it is hard to intuitively understand what non-distributive consequences, conjunctions or disjunctions mean. To this purpose, some relational semantics with an extra mathematical intuition have been developed (Conradie, 2020). One of this semantics is polarity-based semantics, which can be understood as playing the same role for non-distributive modal logics that Kripke semantics has played for classical modal logics (see Table 5).

	Classical/Distributive logics	Non-distributive logics
Relational semantics	Kripke frames	Polarity-based frames
Algebraic semantics	Boolean algebra with operators	Normal lattice expansions

Table 5: Semantics comparison between distributive and non-distributive logics.

3.1 Formal concept analysis

Polarities are triples (A,X,I) such that A and X are sets and I is a binary relation $I \subseteq A \times X$. Polarities have been very extensively studied in the context of a theory in applied mathematics known as Formal Concept Analysis (Ganter, 2005), where they have been understood as abstract representations of databases formed by objects and features. To properly understand how these objects and features are related, we will introduce some terms of Formal Concept Analysis theory in this section.

A *context* is a triple (A,X,I) where A is the set of objects, X is the set of features and $I \subseteq A \times X$ is the incidence relation which is reading as follows:

$$aIx \quad \text{iff} \quad \text{`the object } a \text{ has the feature } x\text{'}.$$

To illustrate the main idea intuitively, let us consider the following database of food items and their features:

Food	sweet	fruit	processed	dairy
mango	×	×		
lemon		×		
ice-cream	×		×	×
cheese			×	×
chocolate	×		×	

The previous table is a way to represent a given context, but it can also be represented in the format (A,X,I) of polarities introduced above. In this case, we could define A as the set {mango, lemon, ice-cream, cheese, chocolate}, X as the set {sweet, fruit, processed, dairy} and I as a function from A to X that maps each food item with its features.

Other important notions of this theory are concepts. A *concept* is a tuple formed by a set of objects called *extension* and a set of features called *intension*. Given a concept $c = (a, x)$ we will say that a is the extension of c, denoted by $[\![c]\!]$, and that x is the intension of c, denoted by $(\![c]\!)$. Given a context for a set of concepts, the extension of a concept is the collection of all objects that manifest the concept, and its intension is the collection of all features that are shared by the objects in that concept extension.

In the previous example, the extension of the concept 'dessert' is given by {mango, ice-cream, chocolate} and its intension is {sweet}. The extension of the concept 'non-veggie' is given by {ice-cream, milk} and its extension is {processed, dairy}.

As final remarks, it is important taking into account that specifying either the extension or the intension of any concept is enough to completely determine the concept itself. In addition, within a context P there may be objects (equivalently features) that are not the extension (equivalently intension) of any concept (Ganter, 2005).

3.2 Polarities

Before continuing, we are fixing the modal language and the logic that we will use in the remaining sections: Let us consider a modal language $\mathscr{L}$ with the only operators $\Box$ and $\Diamond$ and a lattice expansion $A = (L, \Diamond^A, \Box^A)$, where $\Diamond^A$ is a join-preserving operation on A with the same arity than $\Diamond$ and $\Box^A$ is a meet-preserving operation on A with the same arity than $\Box$ (in this case, both of them are unary connectives). Then, we will take the set of formulas of the lattice expansion language $\mathscr{L}_{LE}$ as those which are recursively defined over a denumerable set $AtomProp$ as follows:

$$\varphi ::= p \mid \bot \mid \top \mid \Diamond\varphi \mid \Box\varphi$$

where $p \in AtomProp$. In what follows, the basic framework is given by the *non-distributive modal logic* **L**, defined as the smallest set of sequents $\varphi \vdash \psi$ in the language $\mathscr{L}_{LE}$, containing the following axioms:

- Sequents for propositional connectives:

$$p \vdash p, \qquad \bot \vdash p, \qquad p \vdash \top$$

- Sequents for modal operators:

$$\top \vdash \Box\top \qquad \Diamond\bot \vdash \bot$$

and closed under the following inference rules:

$$\frac{\varphi \vdash \chi \quad \chi \vdash \psi}{\varphi \vdash \psi} \qquad \frac{\varphi \vdash \psi}{\varphi(\chi/p) \vdash \psi(\chi/p)}$$

$$\frac{\varphi \vdash \psi}{\Box\varphi \vdash \Box\psi} \qquad \frac{\varphi \vdash \psi}{\Diamond\varphi \vdash \Diamond\psi}$$

Next subsections allows us to give a relational semantics to this logic and to understand it.

3.2.1 Definition

A *polarity* is a structure $P = (A, X, I)$ such that A and X are sets and $I \subseteq A \times X$ is a binary relation. If P is a polarity, then we can define the following maps:

$$\begin{aligned}(\cdot)^{\uparrow} : \mathscr{P}(A) &\to \mathscr{P}(X)\\ B &\mapsto \{x \in X \mid \forall a \in A(a \in B \to aIx)\}\end{aligned}$$

$$\begin{aligned}(\cdot)^{\downarrow} : \mathscr{P}(X) &\to \mathscr{P}(A)\\ Y &\mapsto \{a \in A \mid \forall x \in X(x \in Y \to aIx)\}\end{aligned}$$

In addition, $Y \subseteq B^{\uparrow}$ if and only if $B \subseteq Y^{\downarrow}$ for all $B \in \mathscr{P}(A)$ and $Y \in \mathscr{P}(X)$.

A *formal concept* of P is a pair $c = ([\![c]\!], (\![c]\!))$ such that $[\![c]\!] \subseteq A$, $(\![c]\!) \subseteq X$ and $[\![c]\!]^{\uparrow} = (\![c]\!)$ and $(\![c]\!)^{\downarrow} = [\![c]\!]$. i.e, $[\![c]\!]^{\uparrow\downarrow} = [\![c]\!]$ and $(\![c]\!)^{\downarrow\uparrow} = (\![c]\!)$.

Let us take the set of the formal concepts of P. This set can be ordered by the partial order defined as follow: for any formal concepts c and d in P,

$$c \leq d \quad iff \quad [\![c]\!] \subseteq [\![d]\!] \quad iff \quad (\![d]\!) \subseteq (\![c]\!).$$

With this order relation, the set of formal concepts of P is a lattice with supreme and infimum which is called the *complete concept lattice* P^{+} of P.

These are some relevant results about these structures:

Proposition 3.1. *For any polarity $P = (A, X, I)$, the complete lattice P^{+} is completely join-generated by the set $\{\mathbf{a} := (a^{\uparrow\downarrow}, a^{\uparrow}) \mid a \in A\}$ and is completely meet-generated by the set $\{\mathbf{x} := (x^{\downarrow}, x^{\downarrow\uparrow}) \mid x \in X\}$* (Conradie, 2020: 4).

Theorem 3.2. (Birkhoff's representation theorem). *Any complete lattice L is isomorphic to the concept lattice P^{+} of some polarity P* (Conradie, 2020: 4).

In the same way that given a Kripe frame $\mathscr{F} = (W, R)$ and a valuation v, it holds that

$$w \models \varphi \quad iff \quad \{w\} \leq \hat{v}(\varphi),$$

where $w \in W$, φ is a formula of the language and $\hat{v}$ is the unequivocal homomorphism that extend v from the set of formulas to $\mathscr{F}^+$; given a polarity $P = (A, X, I)$ and a valuation v from $AtomProp$ to P^+, there is a homomorphism $\hat{v}$ that univocally extends v from the set of formulas L to P^+, and we can define a pairs of relations $\models \subseteq A \times L$ and $\succ \subseteq X \times L$ which verify that

$$a \models \varphi \quad iff \quad \mathbf{a} \leq \hat{v}(\varphi)$$

$$x \succ \varphi \quad iff \quad \hat{v}(\varphi) \leq \mathbf{x}$$

where a is any element of A, x is any element of X, φ is a formula of L, $\mathbf{a} = (a^{\uparrow\downarrow}, a^{\uparrow}) \in P^+$ and $\mathbf{x} = (x^{\downarrow}, x^{\downarrow\uparrow})$. Furthermore, for all **L**-formulas we have:

$a \models \bot$		aIx for all $x \in X$	$x \succ \bot$		always
$a \models \top$		always	$x \succ \top$		aIx for all $a \in A$
$a \models p$	iff	$a \in [\![\hat{v}(p)]\!]$	$x \succ p$	iff	$x \in (\![\hat{v}(p)]\!)$
$a \models \Diamond\varphi$	iff	for all $x \in X$, if $x \succ \Diamond\varphi$ then aIx	$x \succ \Diamond\varphi$	iff	for all $a \in A$, if $a \models \varphi$ then $xR_{\Diamond}a$
$a \models \Box\varphi$	iff	for all $x \in X$, if $x \succ \varphi$ then $aR_{\Box}x$	$x \succ \Box\varphi$	iff	for all $a \in A$, if $a \models \Box\varphi$ then aIx

where we have defined the accessibility relation $R_{\Diamond} \subseteq X \times A$ corresponding to the interpretation of $\Diamond$ ($\Diamond^{P^+}$) as

$$xR_{\Diamond}a \quad iff \quad \Diamond^{P^+}\mathbf{a} \leq \mathbf{x}$$

and the accessibility relation $R_{\Box} \subseteq A \times X$ corresponding to the interpretation of $\Box$ ($\Box^{P^+}$) as

$$aR_{\Box}x \quad iff \quad \mathbf{a} \leq \Box^{P^+}\mathbf{x}.$$

We can also define a relation $R_{\blacksquare} \subseteq A \times X$ by $xR_{\blacklozenge}a$ if and only if $aR_{\Box}x$, and a relation $R_{\blacksquare} \subseteq A \times X$ by $aR_{\blacksquare}x$ if and only if $xR_{\Diamond}a$. From now, when we talk about a polarity P, we will mean a usual polarity (A, X, I) enriched with $R_{\Box}$ and $R_{\Diamond}$, that is, $P = (A, X, I, R_{\Box}, R_{\Diamond})$. However, we will need to extend some definitions to $R_{\blacksquare}$ and $R_{\blacklozenge}$ since they will be needed after applying ALBA.

3.2.2 Interpretation

At this point, we already have a relational semantics for a non-distributive logic and its dual algebraic semantics. However, we do not know how to intuitively or philosophically understand this semantics yet. In (Conradie, 2020) we can find some keys towards this propose.

As it was told before, a polarity $P = (A, X, I, R_{\Box}, R_{\Diamond})$ can be understood as a database which saves information about objects (A) , features (X) and relations among them (I, $R_{\Box}$, $R_{\Diamond}$). In addition, the maps $(\cdot)^{\uparrow} : \mathscr{P}(A) \to \mathscr{P}(X)$ and $(\cdot)^{\downarrow} : \mathscr{P}(X) \to \mathscr{P}(A)$ can be understood as *concept-generating maps*, that is, given a set of objects B, the map $(\cdot)^{\uparrow}$ generates a set $B^{\uparrow}$ of all features that those objects have in common, so it uniquely determines the formal concept $(B, B^{\uparrow})$. Respectively, given a set of features Y, the map $(\cdot)^{\downarrow}$ generates a set $Y^{\downarrow}$ of all objects that share those features, so it uniquely determines the formal concept $(Y^{\downarrow}, Y)$.

Then, if we consider the logic **L** defined above, a formula φ is interpreted as a formal concept $([\![\varphi]\!], (\![\varphi]\!)) \in P^{+}$; indeed, for each object $a \in A$ and feature $x \in X$, the relations $\models$ and $\succ$ can be understood like this:

$a \models \varphi$: 'Object a is a member of concept φ' and

$x \succ \varphi$: 'Feature x describes concept φ'.

About the constants $\top$ and $\bot$ we can take this interpretation:

$\top$: 'The most generic concept, i.e., the one that allows all objects $a \in A$ as examples' and

$\bot$: 'The most restrictive concept, i.e. the one that requires its examples to have all attributes $x \in X$'.

And $\varphi \vdash \psi$ can be understood as:

$\varphi \vdash \psi$: 'Concept φ is a sub-concept of concept ψ'.

About relations, as well as relation I encodes objective information about objects and features, relations $R_{\Box}$ and $R_{\Diamond}$ encodes subjective information about them. According to this, modal operators would be interpreted as:

$aR_{\Box}x$: 'Object a has feature x according to agent i' and

$xR_{\Diamond}a$: 'Feature x describes object a according to agent i'.

Although we have not fixed $\vee$ and $\wedge$ as operators of our logic, it is interesting know how to interpret them, since they are intrinsically related with the fact that polarity-based semantics are good semantics for non-distributive logics. Therefore, we can understand them as follows:

$\varphi \wedge \psi$: 'The greatest (i.e. least restrictive) common subconcept of concept φ and concept ψ' and

$\varphi \vee \psi$: 'The least (i.e. most restrictive) common superconcept of concept φ and concept ψ'.

This meaning makes polarity-based semantics and non-distributive logics a nice framework to reasoning about psychology, sociology, linguistics, biology, chemistry and every field which requires formal representation and analysis of conceptual structures. For instance, in modern theories of grammar, lexical knowledge is organized in hierarchies of features of classes of lexical entries and formal concept analysis can be an appropriate method for generating such hierarchies of lexical information automatically (Ganter, 2005: 150).

3.2.3 Semantics for non-distributive logics

Finally, we can see with an easy example why polarity-based semantics work very well for interpreting non-distributive logics (Conradie, 2020). Let us imagine that we have the database of concepts and features given in Table 6.

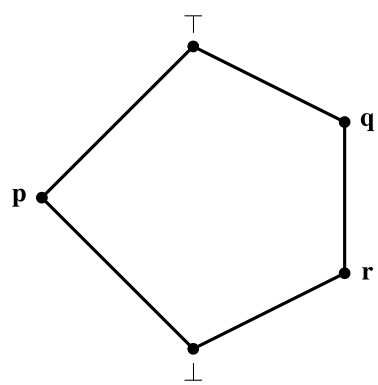
Literature	Main characters are not necessarily humans	It is told as a possible future	There is unwavering control over society
The Lord of the Rings	×		
I, Robot		×	
Nineteen Eighty-Four		×	×

Table 6: Database of concepts and features about literature.

Within this context, we can consider the following concepts:

1. Concept $p =$ 'Fantasy' $=$ (*The Lord of the Rings*, main characters are not necessarily humans) $= (a,x)$,
2. concept $q =$ 'Science fiction' $=$ (*I, Robot* and *Nineteen Eighty-Four*, it is told as a possible future) $= (bc,y)$ and
3. concept $r =$ ' Dystopia' $=$ (*Nineteen Eighty-Four*, it is told as a possible future and there is unwavering control over society) $= (c,yz)$.

And we can represent them in the following diagram:



Reading this diagram, we can observe that the greatest common subconcept between q and r is r (which makes sense since dystopia is a subgenre of science fiction) so $q \wedge r = r$; and that the least common superconcept between p and q is $\top$ (the concept formed by all objects and any feature), so $p \vee q = \top$. As a consequence, the following calculations prove that distributivity does not hold in polarities, and because of that reason, they are a good way to interpret non-distributive logics:

On the one hand, $q \wedge (r \vee p) = q \wedge \top = q$ and $(q \wedge r) \vee (q \wedge p) = r \vee \bot = r$. So, $q \wedge (r \vee p) \neq (q \wedge r) \vee (q \wedge p)$. On the other hand, $r \vee (p \wedge q) = r \vee \bot = r$ and $(r \vee p) \wedge (r \vee q) = \top \wedge q = q$. So, $r \vee (p \wedge q) \neq (r \vee p) \wedge (r \vee q)$.

3.3 Polarity-based semantics in the crisp case

One of the relational semantics that we are going to work with is crisp polarity-based semantics. In order to give an interpretation and a definition of validity in this semantics, we need to define some projections for a given relation. For any relation $T \subseteq U \times V$, and any $U' \subseteq U$ and $V' \subseteq V$, it is defined:

$$T^{(1)}[U'] := \{v \in V \mid \forall u(u \in U' \Rightarrow uTv)\} \qquad T^{(0)}[V'] := \{u \in U \mid \forall v(v \in V' \Rightarrow uTv)\}.$$

As it can be observed, given a polarity $P = (A, X, I, R_\Box, R_\Diamond)$ defined as in Section 3.2.1, for every $B \subseteq A$ and $Y \subseteq X$:

$$I^{(1)}[B] = B^\uparrow$$

$$I^{(0)}[Y] = Y^\downarrow.$$

In addition, it holds that

$$R_\blacklozenge^{(0)}[B] = R_\Box^{(1)}[B] \quad R_\blacklozenge^{(1)}[Y] = R_\Box^{(0)}[Y] \quad R_\blacksquare^{(0)}[Y] = R_\Diamond^{(1)}[Y] \quad R_\blacksquare^{(1)}[B] = R_\Diamond^{(0)}[B].$$

Having said that, the interpretation of formulas in a crisp polarity-bassed semantics is the following one:

Definition 3.3. *For any polarity $P = (A, X, I, R_\Box, R_\Diamond)$, a* valuation *on P is a map $V : AtomProp \to P^+$. For every conceptual label $p \in AtomProp$, we let $[\![p]\!] := [\![V(p)]\!]$ (resp. $(\![p]\!) := (\![V(p)]\!)$) denote the extension (resp. the intension) of the interpretation of p under V. The elements of $[\![p]\!]$ are the* members *of concept p under V; the elements of* $(\![p]\!)$ describe *concept p under V. Any valuation V on P extends homomorphically to an interpretation map of $\mathscr{L}$-formulas defined as follows:*

$$\begin{array}{rcl}
V(p) & = & ([\![p]\!], (\![p]\!)) \\
V(\top) & = & (A, A^\uparrow) \\
V(\bot) & = & (X^\downarrow, X) \\
V(\varphi \wedge \psi) & = & ([\![\varphi]\!] \cap [\![\psi]\!], ([\![\varphi]\!] \cap [\![\psi]\!])^\uparrow) \\
V(\varphi \vee \psi) & = & (((\![\varphi]\!) \cap (\![\psi]\!))^\downarrow, (\![\varphi]\!) \cap (\![\psi]\!)) \\
V(\Box\varphi) & = & (R_\Box^{(0)}[(\![\varphi]\!)], (R_\Box^{(0)}[(\![\varphi]\!)])^\uparrow) \\
V(\Diamond\varphi) & = & ((R_\Diamond^{(0)}[[\![\varphi]\!]])^\downarrow, R_\Diamond^{(0)}[[\![\varphi]\!]]).
\end{array}$$

In an expanded language of $\mathscr{L}$ with $\blacksquare$ and $\blacklozenge$, $V(\blacksquare\varphi) = (R_\blacksquare^{(0)}[(\![\varphi]\!)], (R_\blacksquare^{(0)}[(\![\varphi]\!)])^\uparrow)$ and $V(\blacklozenge\varphi) = ((R_\blacklozenge^{(0)}[[\![\varphi]\!]])^\downarrow, R_\blacklozenge^{(0)}[[\![\varphi]\!]])$.

Definition 3.4. *A* model *is a tuple $\mathscr{M} = (P, V)$. For every $\varphi \in \mathscr{L}$, we write:*

$$\begin{array}{lll}
\mathscr{M}, a \Vdash \varphi & \textit{iff} & a \in [\![\varphi]\!]_\mathscr{M} \\
\mathscr{M}, x \succ \varphi & \textit{iff} & x \in (\![\varphi]\!)_\mathscr{M}
\end{array}$$

We can find the recursive definition of the remaining formulas of the language in (Conradie, 2019b: 11). Furthermore, for the interpretation of sequents, we write:

$$\begin{array}{lll}
\mathscr{M} \models \varphi \vdash \psi & \text{iff} & \text{for all } a \in A, \text{ if } \mathscr{M}, a \Vdash \varphi, \text{ then } \mathscr{M}, a \Vdash \psi \\
& \text{iff} & \text{for all } x \in X, \text{ if } \mathscr{M}, x \succ \psi, \text{ then } \mathscr{M}, x \succ \varphi.
\end{array}$$

Finally, a sequent $\varphi \vdash \psi$ is *valid* on a polarity P (in symbols: $P \models \varphi \vdash \psi$) if $\mathscr{M} \models \varphi \vdash \psi$ for every model $\mathscr{M}$ based on $\mathscr{P}$.

3.4 Polarity-based semantics in the many-valued case

In next section, we are also going to use the many-valued polarity-based semantics. We will assume that the algebra of truth values is a Heyting algebra $\mathbf{A} = (H, \vee, \wedge, \rightarrow, 0, 1)$ that satisfies:

1. $(H, \vee, \wedge, 0, 1)$ is a distributive lattice such that $a \wedge 0 = 0$ and $a \vee 1 = 1$;
2. $a \rightarrow a = 1$;
3. $(a \rightarrow b) \wedge b = b$ and $a \wedge (a \rightarrow b) = a \wedge b$ and
4. $a \rightarrow (b \wedge c) = (a \rightarrow b) \wedge (a \rightarrow c)$ and $(a \vee b) \rightarrow c = (a \rightarrow z) \wedge (b \rightarrow c)$.

In this case, given a polarity $P = (A, X, I, R_\Box, R_\Diamond)$, all relations are defined considering the algebra of truth values $\mathbf{A}$ in this way:

$$\begin{aligned} I &: A \times X \longrightarrow \mathbf{A} \\ R_\Box &: A \times X \longrightarrow \mathbf{A} \\ R_\Diamond &: X \times A \longrightarrow \mathbf{A} \end{aligned}$$

and their projections follow this definition:

Definition 3.5. *Any* $\mathbf{A}$*-valued relation* $R : U \times W \rightarrow \mathbf{A}$ *induces maps* $R^{(0)}[-] : \mathbf{A}^W \rightarrow \mathbf{A}^U$ *and* $R^{(1)}[-] : \mathbf{A}^U \rightarrow \mathbf{A}^W$ *given by the following assignments: for every* $f : U \rightarrow \mathbf{A}$ *and every* $u : W \rightarrow \mathbf{A}$*,*

$$R^{(1)}[f] : \begin{array}{l} W \rightarrow \mathbf{A} \\ x \mapsto \bigwedge_{a \in U} (f(a) \rightarrow R(a,x)) \end{array} \qquad R^{(0)}[u] : \begin{array}{l} U \rightarrow \mathbf{A} \\ a \mapsto \bigwedge_{x \in W} (u(x) \rightarrow R(a,x)) \end{array}$$

Equivalently for $R_\blacksquare : A \times X \longrightarrow \mathbf{A}$ and $R_\blacklozenge : X \times A \longrightarrow \mathbf{A}$.

In adition, for every $\alpha \in \mathbf{A}$ and any set W, we can define $\{\alpha \backslash w\} : W \rightarrow \mathbf{A}$ by $v \mapsto \alpha$ if $v = w$ and $v \mapsto \bot^{\mathbf{A}}$ if $v \neq w$. Then, for every $f : W \longrightarrow \mathbf{A}$:

$$f = \bigvee_{w \in W} \{f(w) \backslash w\}. \tag{1}$$

As it will be seen, in this semantics we will interpret co-nominals $\mathbf{m}$ as concepts formed by functions $\{\alpha \backslash x\}$ ($\mathbf{m} := (\{\alpha \backslash x\}^\downarrow, \{\alpha \backslash x\}^{\downarrow\uparrow})$) and nominals $\mathbf{j}$ as concepts formed by functions $\{\alpha \backslash a\}$ ($\mathbf{j} := (\{\alpha \backslash a\}^{\uparrow\downarrow}, \{\alpha \backslash a\}^\uparrow)$), where x belongs to the feature set and a belongs to the object set.

Having said that, the interpretation of formulas in a many-valued polarity-bassed semantics is the following one:

Definition 3.6. *A* conceptual $\mathbf{A}$-model *over a set AtomProp of atomic propositions is a tuple* $\mathscr{M} = (P, V)$ *such that* $P = (A, X, I, R_\Box, R_\Diamond)$ *is a polarity and* $V : AtomProp \rightarrow P^+$*. For every* $p \in AtomProp$*, let* $V(p) := ([\![p]\!], (\![p]\!))$*, where* $[\![p]\!] : A \rightarrow \mathbf{A}$ *and* $(\![p]\!) : X \rightarrow \mathbf{A}$*, and* $[\![p]\!]^\uparrow = (\![p]\!)$ *and* $(\![p]\!)^\downarrow = [\![p]\!]$*. Letting* $\mathscr{L}$ *denote the* $\{\Box, \Diamond\}$ *modal language over AtomProp, every* V *as above has a unique homomorphic extension, also denoted* $V : \mathscr{L} \rightarrow \mathbf{F}^+$*, defined as follows:*

$$
\begin{array}{rcl}
V(p) & = & ([\![p]\!], ([p])) \\
V(\top) & = & (\top^{\mathbf{A}^A}, (\top^{\mathbf{A}^A})^{\uparrow}) \\
V(\bot) & = & ((\top^{\mathbf{A}^X})^{\downarrow}, \top^{\mathbf{A}^X}) \\
V(\varphi \wedge \psi) & = & ([\![\varphi]\!] \wedge [\![\psi]\!], ([\![\varphi]\!] \wedge [\![\psi]\!])^{\uparrow}) \\
V(\varphi \vee \psi) & = & ((([\varphi]) \wedge ([\psi]))^{\downarrow}, ([\varphi]) \wedge ([\psi])) \\
V(\Box\varphi) & = & (R_{\Box}^{(0)}[([\varphi])], (R_{\Box}^{(0)}[([\varphi])])^{\uparrow}) \\
V(\Diamond\varphi) & = & ((R_{\Diamond}^{(0)}[[\![\varphi]\!]])^{\downarrow}, R_{\Diamond}^{(0)}[[\![\varphi]\!]])
\end{array}
$$

which induces α-membership relations *for each* $\alpha \in \mathbf{A}$ *(in symbols:* $\mathscr{M}, a \Vdash^{\alpha} \varphi$*), and* α-description relations *for each* $\alpha \in \mathbf{A}$ *(in symbols:* $\mathscr{M}, x \succ^{\alpha} \varphi$*) such that for every* $\varphi \in \mathscr{L}$*,*

$$\mathscr{M}, a \Vdash^{\alpha} \varphi \quad \textit{iff} \quad \alpha \leq [\![\varphi]\!](a),$$

$$\mathscr{M}, x \succ^{\alpha} \varphi \quad \textit{iff} \quad \alpha \leq ([\varphi])(x).$$

Furthermore, for the interpretation of sequents, we write:

$$
\begin{array}{rll}
\mathscr{M} \models^{\alpha} \varphi \vdash \psi & \text{iff} & \text{for all } a \in A, \text{ if } \mathscr{M}, a \Vdash^{\alpha} \varphi, \text{ then } \mathscr{M}, a \Vdash^{\alpha} \psi \\
& \text{iff} & \text{for all } x \in X, \text{ if } \mathscr{M}, x \succ^{\alpha} \psi, \text{ then } \mathscr{M}, x \succ^{\alpha} \varphi.
\end{array}
$$

Finally, a sequent $\varphi \vdash \psi$ is α*-valid* on a polarity P (in symbols: $P \models^{\alpha} \varphi \vdash \psi$) if $\mathscr{M} \models^{\alpha} \varphi \vdash \psi$ for every model $\mathscr{M}$ based on $\mathscr{P}$.

4 Results

As mentioned in the previous sections, translation systems have brought to modern logic a way to compare logics with one another. Building on Sahlqvist theory, unified correspondence proposes a methodology to systematically obtain the first order correspondent of certain formulas of any lattice expansion logics; in order to compare all of them in the first order languages of any type of relational semantic structure. However, the existence of a common ground among different logical languages, and also among different semantics of the same logics, brings about the question of whether systematic connections can be established between the first order correspondents of a given Sahlqvist modal axiom in different semantic contexts. In this section, we will give a positive partial answer to this question, by showing that the first order correspondent of any given Sahlqvist modal reduction principle can be expressed as a pure inclusion of binary relations in both crisp and many polarity-based semantics. In addition, we will briefly discuss that, given a Sahlqvist modal reduction principle, we can obtain the same pure inclusion of relations in both semantics.

4.1 First evidences

The well-known modal axiom classically corresponding to reflexivity can be written as the inequality $\Box p \leq p$. If we input this inequality in ALBA, the output is the $\mathscr{L}^+$-inequality

$$\Box \mathbf{m} \leq \mathbf{m}$$

where $\mathbf{m}$ is a co-nominal variable of $\mathscr{L}^+$. Now, if we apply the standard translation for crisp polarity-based semantics given in (Conradie, 2019a: 15), we obtain the following first order correspondent:

$$\forall a \forall x \forall m \Big[\forall x' \big[\forall a'[a'Im \to a'Ix'] \to aR_\Box x'\big] \wedge \forall a'' \big[a''Im \to a''Ix\big] \to aIx\Big] \tag{2}$$

where every a represents an object of the polarity and every x and m represent a feature. Nevertheless, in spite of formula (2) is expressed in a first order language, it is hard to identify what this condition is intuitively about. However, in (Conradie, 2019b) the first order correspondent of $\Box p \leq p$ can be equivalently represented as the following pure inclusion of binary relations[2]:

$$R_\Box \subseteq I \tag{3}$$

which facilitates both its intuitive understanding and the task of comparing it with other first order correspondents.

In (Conradie, 2019b) the authors also observed that the first order correspondent of the inequality $\Box p \leq p$ in many-valued polarity-based semantics is exactly the same first order correspondent (3). In this situation, two questions arose: can we always express the first-order correspondents of all Sahlqvist modal reduction principles as pure inclusions of relations? And if so, can we use this fact to show that the first order correspondents of Sahlqvist modal reduction principles on crisp and many-valued polarity-based semantics are verbatim the same?

In the following subsections, we will positively answer the first one and we will discuss the plausibility of the second one.

4.2 Some definitions

As we are considering Sahlqvist modal reduction principles, we will work with inequalities $\varphi \leq \psi$ such that φ or ψ may be formed by any sequence of boxes and diamonds before the atomic proposition variable. For instance, if we have the inequality $\forall p[\Box\Diamond p \leq \Box\Diamond\Diamond p]$, the output of the second step of ALBA will be

[2] This I must not be misunderstood with the relation 'identity'. It is the incidence relation of the polarity, which intuitively means that aIx if and only if the *object a* has the *feature* x.

$\forall \mathbf{m}[\Box\Diamond\blacksquare\blacksquare\mathbf{m} \leq \Box\mathbf{m}]$. If we interpret this result according to Definition 3.3, we will obtain:

$$\forall x\big(R_{\Box}^{(0)}[R_{\Diamond}^{(0)}[R_{\blacksquare}^{(0)}[I^{(1)}[R_{\blacksquare}^{(0)}[x^{\downarrow\uparrow}]]]]] \subseteq R_{\Box}^{(0)}[x^{\downarrow\uparrow}]\big).$$

In (Conradie, 2019b), there is a definition of relation composition that allows us to define compositions of relations $R,T \subseteq A \times X$ or compositions of relations $R,T \subseteq X \times A$. However, there is not a definition to also compose relation from different type, i.e., to compose diamond type relations (which are included in $X \times A$) and box type relations (which are included in $A \times X$). The following definitions sort out this problem:

Definition 4.1. *For any polarity $P = (A,X,I,R_{\Box},R_{\Diamond})$,*

1. *for all relations $R,T \subseteq X \times A$, the I-*composition *$R\,;_I T \subseteq X \times A$ is such that, for any $a \in A$ and $x \in X$,*

$$(R\,;_I T)^{(0)}[a] = R^{(0)}[I^{(0)}[T^{(0)}[a]]], \quad \textit{i.e.} \quad x(R\,;_I T)a \quad \textit{iff} \quad x \in R^{(0)}[I^{(0)}[T^{(0)}[a]]];$$

2. *for all relations $R,T \subseteq A \times X$, the I-*composition *$R\,;_I T \subseteq A \times X$ is such that, for any $a \in A$ and $x \in X$,*

$$(R\,;_I T)^{(0)}[x] = R^{(0)}[I^{(1)}[T^{(0)}[x]]], \quad \textit{i.e.} \quad a(R\,;_I T)x \quad \textit{iff} \quad a \in R^{(0)}[I^{(1)}[T^{(0)}[x]]];$$

3. *for all relations $R \subseteq A \times X$ and $T \subseteq X \times A$, the* composition *$R\,;T \subseteq A \times A$ is such that, for any $a,b \in A$,*

$$(R\,;T)^{(0)}[b] = R^{(0)}[T^{(0)}[b]], \quad \textit{i.e.} \quad a(R\,;T)b \quad \textit{iff} \quad a \in R^{(0)}[T^{(0)}[b]];$$

4. *for all relations $R \subseteq X \times A$ and $T \subseteq A \times X$, the* composition *$R\,;T \subseteq X \times X$ is such that, for any $x,y \in X$,*

$$(R\,;T)^{(0)}[y] = R^{(0)}[T^{(0)}[y]], \quad \textit{i.e.} \quad x(R\,;T)y \quad \textit{iff} \quad x \in R^{(0)}[T^{(0)}[y]].$$

5. *for all relations $R \subseteq A \times A$ and $T \subseteq A \times X$, the* composition *$R\,;T \subseteq A \times X$ is such that, for any $x \in X$ and any $a \in A$,*

$$(R\,;T)^{(0)}[x] = R^{(0)}[T^{(0)}[x]], \quad \textit{i.e.} \quad a(R\,;T)x \quad \textit{iff} \quad a \in R^{(0)}[T^{(0)}[x]].$$

6. *for all relations $R \subseteq X \times A$ and $T \subseteq A \times A$, the* composition *$R\,;T \subseteq X \times A$ is such that, for any $x \in X$ and any $a \in A$,*

$$(R\,;T)^{(0)}[a] = R^{(0)}[T^{(0)}[a]], \quad \textit{i.e.} \quad x(R\,;T)a \quad \textit{iff} \quad x \in R^{(0)}[T^{(0)}[a]].$$

7. *for all relations $R \subseteq X \times X$ and $T \subseteq X \times A$, the* composition *$R\,;T \subseteq X \times A$ is such that, for any $x \in X$ and any $a \in A$,*

$$(R\,;T)^{(0)}[a] = R^{(0)}[T^{(0)}[a]], \quad \textit{i.e.} \quad x(R\,;T)a \quad \textit{iff} \quad x \in R^{(0)}[T^{(0)}[a]].$$

8. *for all relations $R \subseteq A \times X$ and $T \subseteq X \times X$, the* composition *$R\,;T \subseteq A \times X$ is such that, for any $x \in X$ and any $a \in A$,*

$$(R\,;T)^{(0)}[x] = R^{(0)}[T^{(0)}[x]], \quad \textit{i.e.} \quad a(R\,;T)x \quad \textit{iff} \quad a \in R^{(0)}[T^{(0)}[x]].$$

The many-valued versions of these definitions are as follow:

Definition 4.2. *For any formal many-valued polarity $P = (A, X, I, R_\Box, R_\Diamond)$,*

1. *for all relations $R, T \subseteq X \times A \to \mathbf{A}$, the I-*composition *$R\,;_I T \subseteq X \times A \to \mathbf{A}$ is such that, for any $a \in A$ and $x \in X$,*

$$x(R\,;_I T)a = \bigwedge_{b \in A} \left(\bigwedge_{y \in X} (yTa \to bIy) \to xRb \right);$$

2. *for all relations $R, T \subseteq A \times X$, the I-*composition *$R\,;_I T \subseteq A \times X$ is such that, for any $a \in A$ and $x \in X$,*

$$a(R\,;_I T)x = \bigwedge_{y \in X} \left(\bigwedge_{b \in A} (bTx \to bIy) \to aRy \right);$$

3. *for all relations $R \subseteq A \times X$ and $T \subseteq X \times A$, the* composition *$R\,;T \subseteq A \times A$ is such that, for any $a, b \in A$,*

$$a(R\,;T)b = \bigwedge_{x \in X} (xTb \to aRx);$$

4. *for all relations $R \subseteq X \times A$ and $T \subseteq A \times X$, the* composition *$R\,;T \subseteq X \times X$ is such that, for any $x, y \in X$,*

$$x(R\,;T)y = \bigwedge_{a \in A} (aTy \to xRa);$$

5. *for all relations $R \subseteq A \times A$ and $T \subseteq A \times X$, the* composition *$R\,;T \subseteq A \times X$ is such that, for any $x \in X$ and any $a \in A$,*

$$a(R\,;T)x = \bigwedge_{b \in A} (bTx \to aRb);$$

6. *for all relations $R \subseteq X \times A$ and $T \subseteq A \times A$, the* composition *$R\,;T \subseteq X \times A$ is such that, for any $x \in X$ and any $a \in A$,*

$$x(R\,;T)a = \bigwedge_{b \in A} (bTa \to xRb);$$

7. *for all relations $R \subseteq X \times X$ and $T \subseteq X \times A$, the* composition *$R\,;T \subseteq X \times A$ is such that, for any $x \in X$ and any $a \in A$,*

$$x(R\,;T)a = \bigwedge_{y \in X} (yTa \to xRy);$$

8. *for all relations $R \subseteq A \times X$ and $T \subseteq X \times X$, the* composition $R;T \subseteq A \times X$ *is such that, for any $x \in X$ and any $a \in A$,*

$$a(R;T)x = \bigwedge_{y \in X} (yTx \to aRy)\,.$$

4.3 Crisp correspondents of modal reduction principles on polarity-based frames

Proposition 4.3. *For any language $\mathcal{L}$, the crisp first order correspondent of any Sahlqvist modal reduction principle of $\mathcal{L}$ on polarity-based $\mathcal{L}$-frames can be represented as* pure inclusions of binary relations.

Proof. Given a Sahlqvist modal reduction principle $s(p) \leq t(p)$, we will have a Sahlqvist inequality (see Definition 2.5) of one of these two shapes:

a) If the good branch is on the left side, $s(p)$ will be a possibly empty concatenation of diamonds (denoted as $\Diamond_1^{n_s} := \Diamond_1 \dots \Diamond_{n_s}$), followed by a possibly empty concatenation of boxes (denoted as $\Box_1^{m_s} := \Box_1 \dots \Box_{m_s}$), followed by p. In contrast, $t(p)$ will be a possibly empty concatenation of boxes ($\Box_1^{m_t}$) followed by $\chi(p)$, where $\chi(p)$ can be p or a concatenation of concatenations of boxes and diamonds which starts by a concatenation of diamonds, that is, $\chi(p) = p$ or $\chi(p) = \Diamond \dots \Diamond \Box \dots \Box \Diamond \dots \Diamond \dots p$. It could finish on a concatenation of diamond or of boxes independently. Therefore, in this case the Sahlqvist modal reduction principle would be

$$\Diamond_1^{n_s} \Box_1^{m_s} p \leq \Box_1^{m_t} \chi(p).$$

b) If the good branch is on the right side, $t(p)$ will be a possible empty concatenation of boxes ($\Box_1^{m_t}$), followed by a possibly empty concatenation of diamonds ($\Diamond_1^{n_t}$), followed by p. On contrast, $s(p)$ will be a possibly empty concatenation of diamonds ($\Diamond_1^{n_s}$) followed by $\chi(p)$, where $\chi(p)$ can be p or a concatenation of concatenations of boxes and diamonds which starts by a concatenation of boxes, that is, $\chi(p) = p$ or $\chi(p) = \Box \dots \Box \Diamond \dots \Diamond \Box \dots \Box \dots p$. It could finish on a concatenation of diamond or of boxes independently. Therefore, in this case the Sahlqvist modal reduction principle would be

$$\Diamond_1^{n_s} \chi(p) \leq \Box_1^{m_t} \Diamond_1^{n_t} p.$$

The output of ALBA in the case a) would be:

$$\forall \mathbf{j}[\blacklozenge_1^{m_t} \Diamond_1^{n_s} \mathbf{j} \leq \chi(\blacklozenge_1^{m_s} \mathbf{j})] \tag{4}$$

where $\blacklozenge_1^{m_t}$ is the left adjoint of $\Box_1^{m_t}$ and $\blacklozenge_1^{m_s}$ is the left adjoint of $\Box_1^{m_s}$ (see (Conradie, 2019a)). When interpreting the condition above on a given polarity $P = (A, X, I, R_\Box, R_\Diamond)$, $\mathbf{j}$ ranges over the formal concepts $(a^{\uparrow\downarrow}, a^{\uparrow})$ for $a \in A$. In this case, the condition above can be rewritten according to Section 3.2.1, where it was shown that a concept is less or equal to another if and only if the intension of the latter is included in the intension of the former, as follows:

$$\forall a\big(([\chi(\blacklozenge_1^{m_s}[a^{\uparrow\downarrow}])]) \subseteq ([\blacklozenge_1^{m_t}\Diamond_1^{n_s}[a^{\uparrow\downarrow}]])\big) \tag{5}$$

The output of ALBA in the case b) would be:

$$\forall \mathbf{m}[\chi(\blacksquare_1^{n_t}\mathbf{m}) \leq \blacksquare_1^{n_s}\Box_1^{m_t}\mathbf{m}] \tag{6}$$

where $\blacksquare_1^{n_t}$ is the right adjoint of $\Diamond_1^{n_t}$ and $\blacksquare_1^{n_s}$ is the right adjoint of $\Diamond_1^{n_s}$ (see (Conradie, 2019a)). When interpreting the condition above on a given polarity $P = (A, X, I, R_\Box, R_\Diamond)$, $\mathbf{m}$ ranges over the formal concepts $(x^{\downarrow\uparrow}, x^{\downarrow})$ for $x \in X$. In this case, the condition above can be rewritten according to Section 3.2.1, where it was shown that a concept is less or equal to another if and only if the extension of the former is included in the extension of the latter, as follows:

$$\forall x\big([\![\chi(\blacksquare_1^{n_t}[x^{\downarrow\uparrow}])]\!] \subseteq [\![\blacksquare_1^{n_s}\Box_1^{m_t}[x^{\downarrow\uparrow}]]\!]\big) \tag{7}$$

Let us associate binary relations on P with the shapes described in the previous formulas:

1. For any formula $\varphi(p)$ formed by a possible empty finite concatenations of $\Diamond$ and $\blacklozenge$ followed by p, let us define the diamond type relation $R_\varphi \subseteq X \times A$ by induction on φ as follows:
   If $\varphi := p$, then $R_\varphi := J \subseteq X \times A$, where xJa iff aIx;
   if $\varphi := \Diamond\varphi'$, then $R_\varphi := R_\Diamond ;_I R_{\varphi'}$;
   if $\varphi := \blacklozenge\varphi'$, then $R_\varphi := R_\blacklozenge ;_I R_{\varphi'}$.
2. For any formula $\psi(p)$ formed by a possible empty finite concatenations of $\Box$ and $\blacksquare$ followed by p, let us define the box type relation $R_\psi \subseteq A \times X$ by induction on ψ as follows:
   If $\psi := p$, then $R_\psi := I \subseteq A \times X$;
   if $\psi := \Box\psi'$, then $R_\psi := R_\Box ;_I R_{\psi'}$;
   if $\psi := \blacksquare\psi'$, then $R_\psi := R_\blacksquare ;_I R_{\psi'}$.

Given these definitions, we can already interpret the right side of (5) according to the valuations given in Definition 3.3:

$$\begin{aligned}&([\blacklozenge_1^{m_t}\Diamond_1^{n_s}[a^{\uparrow\downarrow}]]) =\\ &R^{(0)}_{\blacklozenge_1}[(\ldots([R^{(0)}_{\blacklozenge_{m_t}}[([R^{(0)}_{\Diamond_1}[\ldots(R^{(0)}_{\Diamond_{n_s}}[a^{\uparrow\downarrow}])^{\downarrow}\ldots]])^{\downarrow}])^{\downarrow}])^{\downarrow}\ldots] =\\ &(R_{\blacklozenge_1};_I\ldots;_I R_{\blacklozenge_{m_t}};_I R_{\Diamond_1};_I\ldots;_I R_{\Diamond_{n_s}})^{(0)}[a^{\uparrow\downarrow}]\end{aligned} \tag{8}$$

Then, $\chi(p) = \Diamond\ldots\Diamond\Box\ldots\Box\Diamond\ldots\Diamond\Box\ldots\Box p$, i.e., $\chi(p)$ is a concatenation of concatenations of boxes and diamonds, starting by a concatenation of diamonds and ending by a concatenation of boxes. We can associate each concatenation of diamonds with a diamond type relation $R_{\Diamond\ldots\Diamond} \subseteq X \times A$ defined as in point 1. Equivalently, we also can associate each concatenation of boxes with a box type relation $R_{\Box\ldots\Box} \subseteq A \times X$ defined as in point 2. As the left side of (5) is $(\![\chi(\blacklozenge_1^{m_s}[a^{\uparrow\downarrow}])]\!)$, next to last box of χ there will be another concatenation of diamonds $\blacklozenge_1^{m_s}$ that can be associated with the diamond relation $R_{\blacklozenge_1^{m_s}}$, following the definition of point 1. Therefore, the fact that the left side of (5) can be interpreted as a composition between diamond and box type relation (see Definition 4.1):

$$\begin{aligned}
&(\![\chi(\blacklozenge_1^{m_s}[a^{\uparrow\downarrow}])]\!) = \\
&R^{(0)}_{\Diamond\ldots\Diamond}[[R^{(0)}_{\Box\ldots\Box}[\ldots[R^{(0)}_{\Diamond\ldots\Diamond}[R^{(0)}_{\Box\ldots\Box}[R^{(0)}_{\blacklozenge_1^{m_s}}[a^{\uparrow\downarrow}]]]]]]] = \\
&(R_{\Diamond\ldots\Diamond};R_{\Box\ldots\Box};\ldots;R_{\Diamond\ldots\Diamond};R_{\Box\ldots\Box};R_{\blacklozenge_1^{m_s}})^{(0)}[a^{\uparrow\downarrow}]
\end{aligned} \tag{9}$$

which is another diamond type relation, i.e., it is included in $X \times A$.

If $\chi(p) = \Diamond\ldots\Diamond\Box\ldots\Box\Diamond\ldots\Diamond p$, i.e, if $\chi(p)$ is a concatenation of concatenations of boxes and diamonds, starting and ending by a concatenation of diamonds, we can directly associate the last concatenation of diamonds of χ with $\blacklozenge_1^{m_s}$ and obtain a diamond type relation as before:

$$\begin{aligned}
&(\![\chi(\blacklozenge_1^{m_s}[a^{\uparrow\downarrow}])]\!) = \\
&R^{(0)}_{\Diamond\ldots\Diamond}[[R^{(0)}_{\Box\ldots\Box}[\ldots[R^{(0)}_{\Diamond\ldots\Diamond}[R^{(0)}_{\blacklozenge_1^{m_s}}[a^{\uparrow\downarrow}]]]]]] = \\
&(R_{\Diamond\ldots\Diamond};R_{\Box\ldots\Box};\ldots;R_{\Diamond\ldots\Diamond\blacklozenge_1^{m_s}})^{(0)}[a^{\uparrow\downarrow}]
\end{aligned} \tag{10}$$

In both cases, we can rewrite (5) as:

$$\forall a\big(R^{(0)}_{\Diamond\ type}[a] \subseteq (R_{\blacklozenge_1};_I\ldots;_I R_{\blacklozenge_{m_t}};_I R_{\Diamond_1};_I\ldots;_I R_{\Diamond_{n_s}})^{(0)}[a]\big)$$

which is equivalent to the following pure inclusion of binary relations of type $X \times A$:

$$R_{\Diamond\ type} \subseteq R_{\blacklozenge_1};_I\ldots;_I R_{\blacklozenge_{m_t}};_I R_{\Diamond_1};_I\ldots;_I R_{\Diamond_{n_s}}$$

In case b), we can process very similarly. The right side of (7) can be interpreted as:

$$(R_{\blacksquare_1};_I\ldots;_I R_{\blacksquare_{n_s}};_I R_{\Box_1};_I\ldots;_I R_{\Box_{m_t}})^{(0)}[x^{\downarrow\uparrow}] \tag{11}$$

The left side will be a box type relation, since χ starts with a concatenation of boxes and after the last concatenation (of boxes or diamonds) there is another concatenation of boxes ($\blacksquare_1^{n_t}$), so we can apply the same methodology than in case a). Therefore, we can rewrite (7) as:

$$\forall x\big(R^{(0)}_{\Box\ type}[x] \subseteq (R_{\blacksquare_1};_I\ldots;_I R_{\blacksquare_{n_s}};_I R_{\Box_1};_I\ldots;_I R_{\Box_{m_t}})^{(0)}[x]\big)$$

which is equivalent to the following pure inclusion of binary relations of type $X \times A$:

$$R_{\Box\ type} \subseteq R_{\blacksquare_1} ;_I \ldots ;_I R_{\blacksquare_{n_s}} ;_I R_{\Box_1} ;_I \ldots ;_I R_{\Box_{m_t}}$$

□

Example 4.4. *The modal reduction principle $\Box\Diamond p \leq \Box\Diamond\Diamond p$ is Sahlqvist of shape b), that is, the good branch is on the right side.*

$$\begin{array}{lll}
 & \forall p[\Box\Diamond p \leq \Box\Diamond\Diamond p] & \\
\textit{iff} & \forall \mathbf{m}[\Box\Diamond\blacksquare\blacksquare\mathbf{m} \leq \Box\mathbf{m}] & \textit{ALBA output} \\
\textit{i.e.} & \forall x\left(R_\Box^{(0)}[R_\Diamond^{(0)}[R_\blacksquare^{(0)}[I^{(1)}[R_\blacksquare^{(0)}[x^{\downarrow\uparrow}]]]] \subseteq R_\Box^{(0)}[x^{\downarrow\uparrow}]\right) & \textit{Definition 3.3} \\
\textit{iff} & \forall x\left(R_\Box^{(0)}[R_\Diamond^{(0)}[R_\blacksquare^{(0)}[I^{(1)}[R_\blacksquare^{(0)}[x]]]]] \subseteq R_\Box^{(0)}[x]\right) & x^{\downarrow\uparrow} = x \\
\textit{iff} & \forall x\forall a\left(a \in R_\Box^{(0)}[R_\Diamond^{(0)}[R_\blacksquare^{(0)}[I^{(1)}[R_\blacksquare^{(0)}[x]]]]] \Rightarrow a \in R_\Box^{(0)}[x]\right) & \textit{Definition } R^{(0)} \\
\textit{iff} & \forall x\forall a\,(a(R_\Box ; R_\Diamond ; (R_\blacksquare ;_I R_\blacksquare))x \Rightarrow aR_\Box x) & \textit{Definition 4.1} \\
\textit{iff} & R_\Box ; R_\Diamond ; (R_\blacksquare ;_I R_\blacksquare) \subseteq R_\Box. &
\end{array}$$

4.4 Many-valued correspondents of modal reduction principles on polarity-based frames

In order to prove that Sahlqvist modal reduction principles can be represented as pure inclusions of binary relations in the many-valued case too, we need to apply a more complex strategy than in the crisp case. This fact is because some good properties which hold in the crisp case do not hold in the many-valued setting. For example, we can not proceed as in Example 4.4, since Definition 4.2 does not guarantee the second-two-last equivalence. To solve this issue, we will have to first eliminate α and, only then, to apply Definition 4.2 and achieve the required pure relational representation. We will be able to removing the parameter α thanks to the following definitions and propositions, due to we will establish an equivalency between an inequality that depends on α and an inequality that depends just on 1. However, in order to apply this equivalency, we would need one of the branches to be $\mathbf{j}$, $\Diamond j$, $\mathbf{m}$ or $\Box\mathbf{m}$[3], which is perfectly reachable but we need to add an extra adjunction step, as we will see.

Considering that we start with the same type of inequalities than in the previous subsection, we can consider their possible shape as it was described in the proof of Proposition 4.3, that is, we can have Sahlqvist modal reduction principles of shape a) or of shape b).

As it was seen before, in the case a), ALBA returns:

$$\forall \mathbf{j}[\blacklozenge_1^{m_t} \Diamond_1^{n_s} \mathbf{j} \leq \chi(\blacklozenge_1^{m_s} \mathbf{j})] \tag{12}$$

and, in case b), it returns:

[3] In this dissertation, we will consider that one of the branch is specifically $\mathbf{j}$ or $\mathbf{m}$ to simplify some proofs.

$$\forall \mathbf{m}[\chi(\blacksquare_1^{n_t}\mathbf{m}) \leq \blacksquare_1^{n_s}\Box_1^{m_t}\mathbf{m}] \tag{13}$$

However, they should be transformed via adjunction to

$$\forall \mathbf{j}[\mathbf{j} \leq \blacksquare_1^{n_s}\Box_1^{m_t}\chi(\blacklozenge_1^{m_s}\mathbf{j})] \tag{14}$$

and

$$\forall \mathbf{m}[\blacklozenge_1^{m_t}\Diamond_1^{n_s}\chi(\blacksquare_1^{n_t}\mathbf{m}) \leq \mathbf{m}] \tag{15}$$

respectively, to allow us to develop a systematic methodology to reduce those outputs to a pure inclusion of binary relations.

Then, we need to add some definitions and propositions.

Definition 4.5. *For any many-valued frame based on the polarity $P = (A, X, I, R_\Box, R_\Diamond)$:*

1. *Let $\varphi = \varphi(\mathbf{j})$ be a $\mathcal{L}^+$-formula built up from a given nominal $\mathbf{j}$ using box and diamond operators. For any $\alpha \in \mathbf{A}$, and any $a, b \in A$ and $x \in X$, we let*

$$G_\varphi(\alpha, x, a) = ([\varphi])(x)[\mathbf{j} := (\{\alpha/a\}^{\uparrow\downarrow}, \{\alpha/a\}^{\uparrow})];$$

$$G_\varphi(\alpha, b, a) = [\![\varphi]\!](b)[\mathbf{j} := (\{\alpha/a\}^{\uparrow\downarrow}, \{\alpha/a\}^{\uparrow})].$$

2. *Let $\psi = \psi(\mathbf{m})$ be a $\mathcal{L}^+$-formula built up from a given co-nominal $\mathbf{m}$ using box and diamond operators. For any $\alpha \in \mathbf{A}$, and any $a \in A$ and $x, y \in X$, we let*

$$F_\psi(\alpha, a, x) = [\![\psi]\!](a)[\mathbf{m} := (\{\alpha/x\}^{\downarrow}, \{\alpha/x\}^{\downarrow\uparrow})];$$

$$F_\psi(\alpha, y, x) = ([\psi])(y)[\mathbf{m} := (\{\alpha/x\}^{\downarrow}, \{\alpha/x\}^{\downarrow\uparrow})].$$

Proposition 4.6. *For any many-valued frame based on the polarity $P = (A, X, I, R_\Box, R_\Diamond)$, any $\alpha \in \mathbf{A}$, and $a \in A$ and $x \in X$,*

1. *$\alpha \to G_\mathbf{j}(1, x, a) = G_\mathbf{j}(\alpha, x, a)$;*
2. *$\alpha \to F_\mathbf{m}(1, a, x) = F_\mathbf{m}(\alpha, a, x)$.*

Proof. We only show the first item, the second one being proved similarly.

On the one hand, applying the definitions of Section 3.4 we get that

$$G_{\mathbf{j}}(\alpha,x,a) = ([\mathbf{j}])(x)[\mathbf{m} := (\{\alpha/a\}^{\uparrow\downarrow}, \{\alpha/a\}^{\uparrow})] = \{\alpha/a\}^{\uparrow}(x) = \bigwedge_{b\in A}(\{\alpha/a\}(b) \to bIx) = \alpha \to aIx.$$

On the other hand, if we apply same definitions to $G_{\mathbf{j}}(1,x,)$, we obtain that $G_{\mathbf{j}}(1,x,a) = 1 \to aIx$ which is equal to aIx, since $1 \to c = c$ for every element c of a Heyting algebra (Esakia, 2019). ◻

Proposition 4.7. *Let $\psi = \psi(\mathbf{m})$ and $\varphi = \varphi(\mathbf{j})$ be $\mathscr{L}^{+}$-formulas built up from a given co-nominal $\mathbf{m}$ and a given nominal $\mathbf{j}$ respectively, using box and diamond operators. For any many-valued polarity $P = (A,X,I,R_{\Box},R_{\Diamond})$:*

1. *$G_{\varphi}(\alpha,x,a) \leq \alpha \to G_{\varphi}(1,x,a)$ for any $\alpha \in \mathbf{A}$, $a \in A$ and $x \in X$.*
2. *$F_{\psi}(\alpha,a,x) \leq \alpha \to F_{\psi}(1,a,x)$ for any $\alpha \in \mathbf{A}$, and any $a \in A$ and $x \in X$.*

Proof. We only show the first item, the second one being proved similarly.

Let us prove that $G_{\varphi}(\alpha,x,a) \leq \alpha \to G_{\varphi}(1,x,a)$ by induction on φ. The base case is $\varphi := \mathbf{j}$ and it is proved in Proposition 4.7, since if $G_{\mathbf{j}}(\alpha,x,a) = \alpha \to G_{\mathbf{j}}(1,x,a)$ then $G_{\mathbf{j}}(\alpha,x,a) \leq \alpha \to G_{\mathbf{j}}(1,x,a)$.

As to the induction step, we proceed by cases. If $\varphi := \Box\varphi'(\mathbf{j})$ then, if we apply Definition 4.5 then we get:

$$\begin{aligned} G_{\varphi}(\alpha,x,a) &= ([\Box\varphi'(\mathbf{j})])(x)[\mathbf{j} := (\{\alpha/a\}^{\uparrow\downarrow}, \{\alpha/a\}^{\uparrow})] \\ &= \bigwedge_{b\in A}([\![\Box\varphi'(\mathbf{j})]\!](b) \to bIx) \\ &= \bigwedge_{b\in A}(\bigwedge_{y\in X}(([\varphi'(\mathbf{j})])(y) \to bR_{\Box}y) \to bIx) \\ &= \bigwedge_{b\in A}(\bigwedge_{y\in X}(G_{\varphi'}(\alpha,y,a) \to bR_{\Box}y) \to bIx) \end{aligned} \tag{16}$$

Hence, by applying induction hypothesis on the end of (16), we get:

$$G_{\varphi}(\alpha,x,a) \leq \bigwedge_{b\in A}(\bigwedge_{y\in X}((\alpha \to G_{\varphi'}(1,y,a)) \to bR_{\Box}y) \to bIx) \tag{17}$$

Moreover,

$$
\begin{aligned}
\alpha \to G_{\varphi}(1,x,a) &= \alpha \to \bigwedge_{b\in A}(\bigwedge_{y\in X}(G_{\varphi'}(1,y,a) \to bR_{\Box}y) \to bIx) \\
&= \bigwedge_{b\in A}(\alpha \to (\bigwedge_{y\in X}(G_{\varphi'}(1,y,a) \to bR_{\Box}y) \to bIx)) \\
&= \bigwedge_{b\in A}((\alpha \wedge \bigwedge_{y\in X}(G_{\varphi'}(1,y,a) \to bR_{\Box}y)) \to bIx) \\
&= \bigwedge_{b\in A}(\bigwedge_{y\in X}(\alpha \wedge (G_{\varphi'}(1,y,a) \to bR_{\Box}y)) \to bIx)
\end{aligned} \tag{18}
$$

Hence, to finish the proof of the claim in this case, it is enough to show that (17) is less or equal to the end of (18). It is equivalent to prove that for any $b,a \in A$ any $x \in X$ and any $\alpha \in \mathbf{A}$, $\bigwedge_{y\in X}((\alpha \to G_{\varphi'}(1,y,a)) \to bR_{\Box}y) \to bIx \leq \bigwedge_{y\in X}(\alpha \wedge (G_{\varphi'}(1,y,a) \to bR_{\Box}y)) \to bIx$. Because of antitonicity of $\to$ in the first coordinate, if $\alpha \leq \gamma$ then for all γ, we have $\beta \to \gamma \leq \alpha \to \gamma$, so in this case, if $\bigwedge_{y\in X}(\alpha \wedge (G_{\varphi'}(1,y,a) \to bR_{\Box}y)) \leq \bigwedge_{y\in X}((\alpha \to G_{\varphi'}(1,y,a)) \to bR_{\Box}y)$, then our statement holds. This is equivalent to show that for any $y \in X$ any $a,b \in A$ and any $\alpha \in \mathbf{A}$,

$$
\alpha \wedge (G_{\varphi'}(1,y,a) \to bR_{\Box}y) \leq (\alpha \to G_{\varphi'}(1,y,a)) \to bR_{\Box}y
$$

which is valid in every Heyting algebra (Esakia, 2019).

If $\varphi := \Diamond\varphi'(\mathbf{j})$ and we apply Definition 4.5, then

$$
\begin{aligned}
G_{\varphi}(\alpha,x,a) &= ([\Diamond\varphi'(\mathbf{j})])(x)[\mathbf{j} := (\{\alpha/a\}^{\uparrow\downarrow}, \{\alpha/a\}^{\uparrow})] \\
&= \bigwedge_{b\in A}([\![\varphi'(\mathbf{j})]\!](b) \to xR_{\Diamond}b) \\
&= \bigwedge_{b\in A}(G_{\varphi'}(\alpha,b,a) \to xR_{\Diamond}b)
\end{aligned} \tag{19}
$$

Hence, by applying induction hypothesis on the end of (19), we get:

$$
G_{\varphi}(\alpha,x,a) \leq \bigwedge_{b\in A}((\alpha \wedge G_{\varphi'}(1,b,a)) \to xR_{\Diamond}b) \tag{20}
$$

Moreover,

$$
\begin{aligned}
\alpha \to G_{\varphi}(1,x,a) &= \alpha \to \bigwedge_{b\in A}(G_{\varphi'}(1,b,a) \to xR_{\Diamond}b) \\
&= \bigwedge_{b\in A}(\alpha \to (G_{\varphi'}(1,b,a) \to xR_{\Diamond}b)) \\
&= \bigwedge_{b\in A}((\alpha \wedge G_{\varphi'}(1,b,a)) \to xR_{\Diamond}b)
\end{aligned} \tag{21}
$$

As (20) is equal to the end of (21), the proof of the claim in this case is finished. The cases of $\varphi := \blacklozenge\varphi'(\mathbf{j})$

and $\varphi := \blacksquare\varphi'(\mathbf{j})$ are similar and are omitted. □

Definition 4.8. *Let $\psi = \psi(\mathbf{m})$ and $\varphi = \varphi(\mathbf{j})$ be $\mathscr{L}^+$-formulas built up from a given co-nominal $\mathbf{m}$ and a given nominal $\mathbf{j}$ respectively, using box and diamond operators. Let us define $R_\psi \subseteq A \times X$ and $R_\psi \subseteq X \times A$ by induction on ψ and on φ respectively:*

1. (a) *If $\psi := \mathbf{m}$, then $R_\psi = R_\mathbf{m} = I$.*
   (b) *If $\psi := \Box\psi'$, then $R_\psi = R_\Box ;_I R_{\psi'}$. Idem for $\psi := \blacksquare\psi$.*
   (c) *If $\psi := \Diamond\psi'$, then $R_\psi = I;(R_\Diamond ; R_{\psi'})$. Idem for $\psi := \blacklozenge\psi$.*
2. (a) *If $\varphi := \mathbf{j}$, then $R_\varphi = R_\mathbf{j} = J$.*
   (b) *If $\varphi := \Diamond\varphi'$, then $R_\varphi = R_\Diamond ;_I R_{\varphi'}$. Idem for $\varphi := \blacklozenge\varphi'$.*
   (c) *If $\psi := \Box\varphi'$, then $R_\psi = J;(R_\Box ; R_{\varphi'})$. Idem for $\varphi := \blacksquare\varphi'$.*

Proposition 4.9. *Let $\psi = \psi(\mathbf{m})$ and $\varphi = \varphi(\mathbf{j})$ be $\mathscr{L}^+$-formulas built up from a given co-nominal $\mathbf{m}$ and a given nominal $\mathbf{j}$ respectively, using box and diamond operators. For any many-valued frame based on the polarity $P = (A,X,I)$, and any $a,b \in A$ and $x,y \in X$,*

1. *$G_\varphi(1,x,a) = xR_\varphi a$ and $G_\varphi(1,b,a) = b(I;R_\varphi)a$;*
2. *$F_\psi(1,a,x) = aR_\psi x$ and $F_\psi(1,y,x) = y(J;R_\psi)x$.*

Proof. We only show the first item, the second one being proved similarly.

By induction on φ and applying Definitions 4.5, 3.5 and 4.2:

a) Base case:
   If $\varphi := \mathbf{j}$, then $R_\varphi = R_\mathbf{j} = J$:

$$G_\mathbf{j}(1,x,a) = (\![\mathbf{j}]\!)(x)[\mathbf{m} := (\{1/a\}^{\uparrow\downarrow}, \{1/a\}^{\uparrow})] = \{1/a\}^{\uparrow}(x) = \bigwedge_{b\in A}(\{1/a\}(b) \to bIx) = 1 \to aIx = aIx = xJa$$

and

$$\begin{aligned} G_\mathbf{j}(1,b,a) &= [\![\varphi]\!](b)[\mathbf{j} := \{1\backslash a\}^{\uparrow\downarrow}, \{1\backslash a\}^{\uparrow}] = (\{1\backslash a\}^{\uparrow})^{\downarrow}(b) = \bigwedge_{y\in X}(\{1\backslash a\}^{\uparrow}(y) \to bIy) \\ &= \bigwedge_{y\in X}(\bigwedge_{c\in A}(\{1\backslash a\}(c) \to cIy) \to bIy) = \bigwedge_{y\in X}((1 \to aIy) \to bIy) = \bigwedge_{y\in X}(aIy \to bIy) \\ &= \bigwedge_{y\in X}(yJa \to bIy) = b(I;J)a \end{aligned}$$

b) Inductive hypothesis:

$$G_{\varphi'}(1,x,a) = xR_{\varphi'}a \text{ and } G_{\varphi'}(1,b,a) = b(I;R_{\varphi'})a$$

c) General case:

– If $\varphi := \Box\varphi'$, then $R_\varphi = R_\Box ;_I R_{\varphi'}$:

$$G_\varphi(1,x,a) = ([\Box\varphi'])(x)[\mathbf{j} := (\{1\backslash a\}^{\uparrow\downarrow}, \{1\backslash a\}^{\uparrow})] = \bigwedge_{b\in A}([\![\varphi']\!](b)[\mathbf{j} := (\{1\backslash a\}^{\uparrow\downarrow}, \{1\backslash a\}^{\uparrow})] \to bR_\Box x)$$
$$= \bigwedge_{b\in A}(G_{\varphi'}(1,b,a) \to bR_\Box x) = \bigwedge_{b\in A}(b(I;R_{\varphi'})a \to bR_\Box x) = \bigwedge_{b\in A}(\bigwedge_{y\in X}(yR_{\varphi'}a \to bIy) \to bR_\Box x)$$
$$= x(R_\Box ;_I R_{\varphi'})a = xR_\varphi a$$

and

$$G_\varphi(1,b,a) = [\![\varphi]\!](b)[\mathbf{j} := (\{1\backslash a\}^{\uparrow\downarrow}, \{1\backslash a\}^{\uparrow})] = \bigwedge_{y\in X}(([\Box\varphi'])(y)[\mathbf{j} := (\{1\backslash a\}^{\uparrow\downarrow}, \{1\backslash a\}^{\uparrow})] \to bIy)$$
$$= \bigwedge_{y\in X}(G_\varphi(1,x,a) \to bIy) = \bigwedge_{y\in X}(xR_\varphi a \to bIy) = b(I;R_\varphi)a$$

– $\varphi := \Diamond\varphi'$, $\varphi := \blacklozenge\varphi'$ and $\varphi := \blacksquare\varphi'$ cases are similar.

□

After these developments, we have precise tools to be able to prove the main proposition of this subsection:

Proposition 4.10. *For any language $\mathcal{L}$, the first order correspondent of any Sahlqvist modal reduction principle of $\mathcal{L}$ on many-valued polarity-based $\mathcal{L}$-frames can be represented as* pure inclusions of binary relations.

Proof. As it was said before, we can have a Sahlvist modal reduction principle with shape a) or shape b).
If a), then we have the modified ALBA output (14):

$$\forall \mathbf{j}[\mathbf{j} \leq \blacksquare_1^{n_s}\Box_1^{m_t}\chi(\blacklozenge_1^{m_s}\mathbf{j})].$$

When interpreting the condition above on concept lattices arising from a given many-valued polarity $P = (A, X, I, R_\Box, R_\Diamond)$, we need to recall the definition of the order relation defined on concept lattices, and the fact that $\mathbf{j}$ ranges over the formal concepts $(\{\alpha/a\}^{\uparrow\downarrow}, \{\alpha/a\}^{\uparrow})$ for $\alpha \in \mathbf{A}$ and $a \in A$. In this case, the condition above can be rewritten according to Section 3.2.1, where it was shown that a concept is less or equal to another if and only if the intension of the latter is included in the intension of the former, as follows:

$$\forall\alpha\forall a\left(([\blacksquare_1^{n_s}\Box_1^{m_t}\chi(\blacklozenge_1^{m_s}\mathbf{j})]) \leq ([\mathbf{j}])\right)$$

and then as follows:

$$\forall\alpha\forall a\forall x\left((\llbracket\blacksquare_1^{n_s}\Box_1^{m_t}\chi(\blacklozenge_1^{m_s}\mathbf{j})\rrbracket)(x)\leq(\llbracket\mathbf{j}\rrbracket)\right)(x)$$

If $\xi(\mathbf{j}):=\blacksquare_1^{n_s}\Box_1^{m_t}\chi(\blacklozenge_1^{m_s}\mathbf{j})$, the condition above can be rewritten using the notation introduced in Definition 4.5:

$$\forall\alpha\forall a\forall x\left(G_\xi(\alpha,x,a)\leq G_{\mathbf{j}}(\alpha,x,a)\right),$$

which holds if and only if

$$\forall a\forall x\left(G_\xi(1,x,a)\leq G_{\mathbf{j}}(1,x,a)\right).$$

Let us prove this statement:

On the one hand, given any $\alpha\in\mathbf{A}$, $a\in A$ and $x\in X$, if $G_\xi(\alpha,x,a)\leq G_{\mathbf{j}}(\alpha,x,a)$, then $G_\xi(1,x,a)\leq G_{\mathbf{j}}(1,x,a)$ by instantiating $\alpha:=1$.

On the other hand, given any $\alpha\in\mathbf{A}$, $a\in A$ and $x\in X$, if $G_\xi(1,x,a)\leq G_{\mathbf{j}}(1,x,a)$, then $\alpha\to G_\xi(1,x,a)\leq\alpha\to G_{\mathbf{j}}(1,x,a)$ because of the monotonicity of $\to$ in its second coordinate. For Proposition 4.7, $G_\xi(\alpha,x,a)\leq\alpha\to G_\xi(1,x,a)$ and, for Proposition 4.6, $\alpha\to G_{\mathbf{j}}(1,x,a)\leq G_{\mathbf{j}}(\alpha,x,a)$. Hence, $G_\xi(\alpha,x,a)\leq\alpha\to G_\xi(1,x,a)\leq\alpha\to G_{\mathbf{j}}(1,x,a)\leq G_{\mathbf{j}}(\alpha,x,a)$ and $G_\xi(\alpha,x,a)\leq G_{\mathbf{j}}(\alpha,x,a)$ as required.

Therefore, we have that the output of ALBA in case a) can be expressed as $\forall a\forall x\left(G_\xi(1,x,a)\leq G_{\mathbf{j}}(1,x,a)\right)$, which, by Proposition 4.9, can be rewritten as

$$\forall a\forall x\left(xR_\xi a\leq xJa\right),$$

and hence as

$$R_\xi\leq J,$$

that is, as a pure inclusion of binary relations.

If b), the discussion is similar, but taking into account that $\mathbf{m}$ ranges over the formal concepts $(\{\alpha/x\}^{\downarrow},\{\alpha/x\}^{\downarrow\uparrow})$ for $\alpha\in\mathbf{A}$ and $x\in X$. In this case, the condition (15) can be rewritten according to Section 3.2.1, where it was shown that a concept is less or equal to another if and only if the extension of the former is included in the extension of the latter. □

Example 4.11. *The modal reduction principle $\Box\Diamond p\leq\Box\Diamond\Diamond p$ is Sahlqvist of shape b), that is, the good branch is on the right side.*

$$
\begin{array}{lll}
& \forall p[\Box\Diamond p \leq \Box\Diamond\Diamond p] & \\
\textit{iff} & \forall \mathbf{m}[\Box\Diamond\blacksquare\blacksquare\mathbf{m} \leq \Box\mathbf{m}] & \textit{ALBA output} \\
\textit{iff} & \forall \mathbf{m}[\blacklozenge\Box\Diamond\blacksquare\blacksquare\mathbf{m} \leq \mathbf{m}] & \textit{Adjunction} \\
\textit{iff} & \forall \mathbf{m}\,([\![\blacklozenge\Box\Diamond\blacksquare\blacksquare\mathbf{m}]\!] \leq [\![\mathbf{m}]\!]) & \textit{Comparison of extensions} \\
\textit{i.e.} & \forall\alpha\forall x\left(I^{(0)}[R^{(0)}_{\blacklozenge}[R^{(0)}_{\Box}[R^{(0)}_{\Diamond}[R^{(0)}_{\blacksquare}[I^{(1)}[R^{(0)}_{\blacksquare}[\{\alpha\backslash x\}^{\downarrow\uparrow}]]]]]] \leq I^{(0)}[\{\alpha\backslash x\}]\right) & \textit{Definition 3.6} \\
\textit{iff} & \forall\alpha\forall x\left(I^{(0)}[R^{(0)}_{\blacklozenge}[R^{(0)}_{\Box}[R^{(0)}_{\Diamond}[R^{(0)}_{\blacksquare}[I^{(1)}[R^{(0)}_{\blacksquare}[\{\alpha\backslash x\}]]]]]] \leq I^{(0)}[\{\alpha\backslash x\}]\right) & \{\alpha\backslash x\}^{\downarrow\uparrow} = \{\alpha\backslash x\} \\
\textit{iff} & \forall\alpha\forall x\forall a\left(I^{(0)}[R^{(0)}_{\blacklozenge}[R^{(0)}_{\Box}[R^{(0)}_{\Diamond}[R^{(0)}_{\blacksquare}[I^{(1)}[R^{(0)}_{\blacksquare}[\{\alpha\backslash x\}]]]]]](a) \leq I^{(0)}[\{\alpha\backslash x\}](a)\right) & \\
\textit{iff} & \forall\alpha\forall x\forall a(\bigwedge_{y\in X}(\bigwedge_{b\in A}(\bigwedge_{z\in X}(\bigwedge_{c\in A}(\bigwedge_{w\in X}(\bigwedge_{d\in A}(\alpha\to dR_{\blacksquare}x)\to dIw)\to cR_{\blacksquare}w) & \textit{Definition 3.5} \\
& \to zR_{\Diamond}c)\to bR_{\Box}z)\to yR_{\blacklozenge}b)\to aIy \leq \alpha\to aIx) & \\
\textit{iff} & \forall\alpha\forall x\forall a(\bigwedge_{y\in X}(\bigwedge_{b\in A}(\bigwedge_{z\in X}(\bigwedge_{c\in A}(\bigwedge_{w\in X}(\bigwedge_{d\in A}(dR_{\blacksquare}x\to dIw)\to cR_{\blacksquare}w)\to zR_{\Diamond}c) & \textit{Propositions 4.6 and 4.7} \\
& \to bR_{\Box}z)\to yR_{\blacklozenge}b)\to aIy \leq aIx) & \\
\textit{iff} & \forall a\forall x\,(a(I;(R_{\blacklozenge};(R_{\Box};(R_{\Diamond};(R_{\blacksquare};_I R_{\blacksquare})))))x \leq aIx) & \textit{Definition 4.2} \\
\textit{iff} & I;(R_{\blacklozenge};(R_{\Box};(R_{\Diamond};(R_{\blacksquare};_I R_{\blacksquare})))) \subseteq I &
\end{array}
$$

4.5 Discussion

Results presented in this section show that, given any lattice expansion logic and any modal reduction principles, we can express the information encoded on that inequality as a pure inclusion of binary relations. This is strongly relevant because we do not obtain just a first order translation of the modal reduction principle, but a first order translation that is easily understandable and comparable.

In addition, we prove the same result with two different semantics, what can be a first step towards achieving this conclusion to a larger collection of semantics. Actually, in (Conradie, 2019b) a direct relation between Kripke frames and polarity-based frames is showed, so our thesis is probably extendable to Kripke semantics too.

Another interesting remark is that, in the many-valued case, the parameter α is deleted almost from the beginning of the translation and it does not have influence on the final output. After eliminating α from the inequality, the methodology used in both crisp and many-valued case is exactly the same, so we can conclude that, if this methodology is applied to the same $\mathscr{L}^+$-inequality, the first order correspondent obtained is verbatim the same in both semantics.

Considering all of the above, these results can be a practical step towards the systematic comparison of first order correspondents across semantics.

5 Conclusions

The aim of this thesis was to prove that the first order correspondent of any Sahlqvist modal reduction principle can be expressed as a pure inclusion of binary relations. In order to achieve this result, three main goals were proposed.

First, we have introduced what Sahlqvist theory is about and what its state of the art is. As we showed, it arose from model-theoretic modal logic, although it was generalised thanks to transforming its results into algebraic results, via duality theory. This reformulation from relational to algebraic perspective stimulated researchers into using order properties of the associated algebras and, as a consequence, the definition of algorithms like ALBA, which receives a Sahlqvist inequality of any lattice-based logic and returns its first order correspondence. This algorithm has been defined for both crisp and many-valued distributive logics and crisp non-distributivity logics.

Secondly, we have introduced two relational semantics, crisp and many-valued polarity-based semantics, which give a good interpretation of non-distributive logics. This is especially interesting because ALBA algorithm is not developed for many-valued non-distributive logics yet, so our results are also a progress towards the systematical generation of first order correspondents of Sahlqvist inequalities for this type of logics.

Finally, we presented some evidences that support our hypotheses and some definitions that have allowed us to achieve our final goal: proving that the first order correspondents of Sahlqvist modal reduction principles are pure inclusion of binary relation in both crisp and many-valued polarity-based semantics. Furthermore, we discussed that, apparently, not only these first order correspondents are inclusions of binary relations, but starting from the same ALBA output, Sahlqvist modal reduction principles have the same first order correspondents in both semantics.

This dissertation is highly relevant because comparison of first order correspondents is still an open problem due to its current dependence on semantics. For this reason, having a methodology to obtain understandable first order correspondents across semantics is a significant step towards an effective and systematic comparison across logics. From this point of view, some future steps could be to extend these results to a larger set of formulas and to study how these conclusions are related with their homologous in Kripke and other semantics.

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Curso: 2019-2020

Título del trabajo: Modal reduction principles across relational semantics

1. Sé que copiar es una forma de deshonestidad académica.
2. He leído el documento sobre cómo ser intelectualmente íntegro, estoy familiarizado con sus contenidos y he evitado todas las formas de plagio allí recogidas.
3. Cuando utilizo las palabras de otros, lo indico mediante el uso de comillas.
4. He referenciado todas las citas e igualmente el resto de ideas tomadas de otros.
5. No he plagiado mi propio trabajo.
6. No permitiré a otros que plagien mi trabajo.

Fecha: 30 de julio de 2020

Firma: