ARTICLE TEMPLATE

Robust H_∞ controller design for uncertain 2-D continuous systems with interval time-varying delays

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ABSTRACT
The design of delay-dependent robust H_∞ controllers is solved here for a class of uncertain 2-D continuous systems: those with interval time-varying delays and norm-bounded parameter uncertainties. By constructing a novel augmented Lyapunov-Krasovskii functional and then using the Wirtinger inequality, a new delay-dependent stability condition is developed, that uses the known lower and upper bounds of the time-varying delays to develop less conservative solutions that previous results in the literature. This condition is then applied to H_∞ performance analysis and robust H_∞ controller design, using linear matrix inequalities (LMIs). Two numerical examples are presented that illustrate the effectiveness of the proposed method.

KEYWORDS
2-D continuous systems; interval time-varying delays; Wirtinger inequality; delay-dependent; robust H_∞ control; LMIs

1. Introduction

Most real-world physical systems have, by nature, multidimensional characteristics, so many researchers are currently studying two-dimensional (2-D) systems. These studies on 2-D systems can generally be easily extended to other multidimensional systems, which is not the case for 1-D results. 2-D linear state-space models were introduced in the 1970s (Fornasini & Marchesini, 1976, 1978; Givone & Roesser, 1972), and are being applied to various science and engineering problems, in digital data filtering, in image processing (Roesser, 1975), in thermal engineering (Kaczorek, 1985), etc. These applications prompted theoretical developments, concerning stability analysis, stabilization, filter design for 2-D systems, etc, in both the discrete and continuous frameworks (See, for example, Alfidi & Hmamed (2007); Badie et al. (2018a); Benzaouia et al. (2011b); Dhawan & Kar (2007); Du et al. (2001); Hmamed et al. (2008, 2013) and references therein).

The phenomena of time delays are considered here, as they are an inherent part of a wide variety of dynamic systems, such as nuclear reactors, aircrafts, chemical processes, etc. The presence of time delays are known to lead to complex dynamic behaviors, such

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as oscillations, instabilities or degraded performance (Boukas & Liu, 2001; Fridman & Shaked, 2002). Thus, analyzing stability and designing controllers that are adequate for systems with time delays is receiving significant attention. It should be mentioned that the stability criteria can be classified into two categories, namely, delay-independent (Benzaouia et al., 2011a; Bokharaie & Mason, 2014; Paszke et al., 2004; Souza et al., 2009) and delay-dependent (El Aiss et al., 2017; Kwon et al., 2016; Sun et al., 2010). Delay-independent stability conditions do not take the delay size into consideration, so they are conservative for many systems, specially if the delay is small. Therefore, delay-dependent stability criteria are being studied: for 2-D delayed systems, Yao et al. (2013) developed some delay-dependent stability criteria for uncertain 2-D state-delayed systems in the Fornasini–Marchesini second model by using Lyapunov function methods and free weighting matrices techniques. By employing a delay decomposition approach Hmamed et al. (2016) presented some delay-dependent stability criteria for a class of continuous 2-D delayed systems, which improves the existing results in Benhayoun et al. (2013). Recently, by the use of the auxiliary function-based integral/summation inequalities (Park et al., 2016) some delay-dependent stability criteria for 2-D delayed systems in discrete and continuous-time have been presented by Badie et al. (2018b,c). It must be pointed out that in practice delays are not constant, so stability, control and filtering for systems with varying delays is currently a hot topic. In the 2-D context some results have already been reported that consider varying delays: for instance, El-Kasri et al. (2013) solved delay-dependent robust H_∞ filtering for uncertain 2-D continuous systems described by Roesser model with time-varying delays. In Ghous & Xiang (2016a), a free-weighting matrix approach was proposed to investigate the robust stability and H_∞ control problems of uncertain 2-D continuous systems with time-varying delays. Recently, Le & Trinh (2017) has proposed new delay-dependent conditions that ensure the exponential stability for a class of 2-D linear continuous-time systems with time-varying delay. However, the results in (El-Kasri et al., 2013; Ghous & Xiang, 2016a; Le & Trinh, 2017) assume that the lower bounds of the delays are zero, but most engineering systems with delays have non-zero lower bounds: this means that they are interval delays. Thus, existing delay-dependent stability criteria for 2-D continuous systems with varying delays would generally be conservative in the presence of interval delays. Removing these conservative limitations is becoming very important. This motivates the present study: a delay-dependent stability criterion is developed that considers explicitly interval delays.

In addition, H_∞ control is considered here: this research area is nowadays rather popular, as it deals with robustness in a practical way. It has been studied in detail for different types of systems during the last decades. For the problem at hand (2-D systems with delays), we refer the reader to (Badie et al., 2019; Ghous & Xiang, 2015, 2016a,b; Ghous et al., 2017); for example, H_∞ control of 2-D continuous nonlinear systems with time-varying delays has been solved in Ghous & Xiang (2015). In Ghous & Xiang (2016b), the H_∞ control problem of 2-D continuous switched systems with time-varying delays has been studied. In Ghous et al. (2017), the stability analysis and H_∞ control problem of 2-D continuous-time Markovian jump systems with partially unknown transition probabilities have been studied.

Thus, this paper consider the stability analysis and H_∞ control for 2-D uncertain continuous systems with interval time-varying delays and norm-bounded parameter uncertainties. By constructing a Lyapunov-Krasovskii functional, using the Wirtinger inequality and the reciprocal convex combination technique, an approach is derived for analyzing stability, which can achieve less conservative results than those in (El-Kasri et al., 2013; Ghous & Xiang, 2016a). Then, the H_∞ performance analysis for the uncertain 2-D continuous systems with delays is proposed. As a result, a robust controller is designed in terms of linear matrix inequalities (LMIs). Two numerical examples illustrate the effectiveness and the merits of the proposed approach.

Notations Throughout the paper, $\mathbb{R}^n$ denotes the n-dimensional real Euclidean space, $\mathbb{R}^{n\times m}$ denotes the set of $n \times m$ real matrices. I and 0 represent the identity matrix and zero matrix respectively. $||.||$ denotes the Euclidean norm. The superscripts T and -1 stand for the matrix transpose and inverse, respectively. $P > 0$ denotes a real symmetric and positive definite matrix. $(*)$ are terms induced by symmetry in symmetric matrices. $diag\{...\}$ denotes a block diagonal matrix. $sym(M)$ is the shorthand notation for $M+M^T$. The $\mathcal{L}_2$ norm of a 2-D signal $\omega(t_1,t_2)$ is

$$||w||_2 = \sqrt{\int_0^\infty \int_0^\infty w^T(t_1,t_2)w(t_1,t_2)dt_1dt_2},$$

where $w(t_1,t_2)$ is in $\mathcal{L}_2\{[0,\infty),[0,\infty)\}$ or, as shorthand, in $\mathcal{L}_2$ if $||w||_2 < \infty$.

2. Problem Statement and Preliminaries

This paper considers the following class of 2-D continuous Roesser-like model with varying delays:

$$\begin{aligned}\begin{bmatrix} \frac{\partial x^h(t_1,t_2)}{\partial t_1} \\ \frac{\partial x^v(t_1,t_2)}{\partial t_2} \end{bmatrix} &= \hat{A}\begin{bmatrix} x^h(t_1,t_2) \\ x^v(t_1,t_2) \end{bmatrix} + \hat{A}_d\begin{bmatrix} x^h(t_1-h(t_1),t_2) \\ x^v(t_1,t_2-d(t_2)) \end{bmatrix} + \hat{B}w(t_1,t_2) + Eu(t_1,t_2), \\ z(t_1,t_2) &= C\begin{bmatrix} x^h(t_1,t_2) \\ x^v(t_1,t_2) \end{bmatrix} + C_d\begin{bmatrix} x^h(t_1-h(t_1),t_2) \\ x^v(t_1,t_2-d(t_2)) \end{bmatrix} + Dw(t_1,t_2) + Fu(t_1,t_2),\end{aligned} \tag{1}$$

with

$$\begin{aligned} \hat{A} &= A + \Delta A, \quad \hat{A}_d = A_d + \Delta A_d, \quad \hat{B} = B + \Delta B, \\ A &= \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \quad A_d = \begin{bmatrix} A_{d11} & A_{d12} \\ A_{d21} & A_{d22} \end{bmatrix}, \quad B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}, \quad E = \begin{bmatrix} E_1 \\ E_2 \end{bmatrix}, \\ C &= \begin{bmatrix} C_1 & C_2 \end{bmatrix}, \quad C_d = \begin{bmatrix} C_{d1} & C_{d2} \end{bmatrix}, \end{aligned}$$

where $x^h(t_1,t_2) \in \mathbb{R}^{n_h}$, is the horizontal state vector, $x^v(t_1,t_2) \in \mathbb{R}^{n_v}$ is the vertical state vector, $w(t_1,t_2) \in \mathbb{R}^{n_w}$ is the disturbance input, that belongs to $\mathcal{L}_2\{[0,\infty),[0,\infty)\}$, and $z(t_1,t_2) \in \mathbb{R}^{n_z}$ is the measured output. A_{11}, A_{12}, A_{21}, A_{22}, A_{d11}, A_{d12}, A_{d21}, A_{d22}, B_1, B_2, E_1, E_2, C_1, C_2, C_{d1}, C_{d2}, D, and F are assumed to be constant matrices with appropriate dimensions. ΔA, ΔA_d, and ΔB are uncertain matrices of the following form:

$$\begin{bmatrix} \Delta A & \Delta A_d & \Delta B \end{bmatrix} = G\mathcal{F}(t_1,t_2)\begin{bmatrix} H_1 & H_2 & H_3 \end{bmatrix}, \tag{2}$$

where

$$G = \begin{bmatrix} G_1 \\ G_2 \end{bmatrix}, \quad H_1 = \begin{bmatrix} H_{11} & H_{12} \end{bmatrix}, \quad H_2 = \begin{bmatrix} H_{21} & H_{22} \end{bmatrix},$$

and H_3 are known real constant matrices, and $\mathcal{F}(t_1, t_2)$ is an unknown continuous matrix satisfying:

$$\mathcal{F}^T(t_1, t_2)\mathcal{F}(t_1, t_2) \leq I. \tag{3}$$

$h(t_1)$ and $d(t_2)$ are time-varying continuous differential functions, that represent the varying state delays along horizontal direction and vertical direction, respectively, satisfying:

$$\begin{cases} h_1 \leq h(t_1) \leq h_2, & \dot{h}(t_1) \leq \mu_1 \leq 1, & h_{12} = h_2 - h_1, \\ d_1 \leq d(t_2) \leq d_2, & \dot{d}(t_2) \leq \mu_2 \leq 1, & d_{12} = d_2 - d_1. \end{cases} \tag{4}$$

where h_1, h_2, d_1, d_2, μ_1 and μ_2 are positive scalars.

The initial conditions are given by:

$$\begin{cases} x^h(\theta, t_2) = \phi_\theta(t_2), & -h_2 \leq \theta \leq 0, \quad 0 \leq t_2 \leq T_2, \\ x^h(\theta, t_2) = 0, & -h_2 \leq \theta \leq 0, \quad t_2 \geq T_2, \\ x^v(t_1, \delta) = \varphi_\delta(t_1), & -d_2 \leq \delta \leq 0, \quad 0 \leq t_1 \leq T_1, \\ x^v(t_1, \delta) = 0, & -d_2 \leq \delta \leq 0, \quad t_1 \geq T_1, \end{cases} \tag{5}$$

where $T_1 < \infty$ and $T_2 < \infty$ are positive constants, $\phi_\theta(t_2)$ and $\varphi_\delta(t_1)$ are given continuous vectors.

Remark 1. The term uncertainty refers to the differences between models and real systems. The polytopic and norm-bounded uncertainties are the most used representations. In the present paper, we consider the problems of robust stability and H_∞ control for uncertain 2-D continuous systems with interval time-varying delays and norm bounded parameter uncertainties, where the uncertain system is represented by a nominal model at the center of the hyper ellipsoid of uncertainty in the parameter space.

Remark 2. When the lower bounds h_1 and d_1 are zero and $C_d = 0$, system (1) becomes the system studied in Ghous & Xiang (2016a). Therefore, system (1) is more general than the one considered in Ghous & Xiang (2016a).

The uncertain matrices ΔA, ΔA_d and ΔB are said to be admissible if both (2) and (3) hold.

When $w(t_1, t_2) = 0$ and $u(t_1, t_2) = 0$ system (1) becomes the free system:

$$\begin{bmatrix} \frac{\partial x^h(t_1,t_2)}{\partial t_1} \\ \frac{\partial x^v(t_1,t_2)}{\partial t_2} \end{bmatrix} = \hat{A} \begin{bmatrix} x^h(t_1, t_2) \\ x^v(t_1, t_2) \end{bmatrix} + \hat{A}_d \begin{bmatrix} x^h(t_1 - h_1, t_2) \\ x^v(t_1, t_2 - h_2) \end{bmatrix}. \tag{6}$$

Definition 2.1. (Ghous & Xiang, 2016a) The 2-D continuous system (6) with boundary conditions (5) is said to be asymptotically stable if

$$\lim_{(t_1+t_2)\to\infty} sup||x(t_1,t_2)|| = 0, \tag{7}$$

where

$$x(t_1,t_2) = \left[\ x^{hT}(t_1,t_2)\ \ x^{vT}(t_1,t_2)\ \right]^T.$$

Definition 2.2. (Hmamed et al., 2010) Let $V(t_1,t_2) = V^h(t_1,t_2) + V^v(t_1,t_2)$ be a Lyapunov functional of the system (6): then, its unidirectional derivative is

$$\dot{V}_u(t_1,t_2) = \frac{\partial V^h(t_1,t_2)}{\partial t_1} + \frac{\partial V^v(t_1,t_2)}{\partial t_2}. \tag{8}$$

Lemma 2.3. *(Benzaouia et al., 2011a) The 2-D system* (6) *is asymptotically stable if its unidirectional derivative* (8) *is negative definite.*

Lemma 2.4. *(Seuret & Gouaisbaut, 2013) For a positive definite matrix $R > 0$, and a differentiable function $\{y(u), u \in [a,b]\}$ the following inequality holds:*

$$\int_a^b \dot{y}^T(\alpha)R\dot{y}(\alpha)d\alpha \geq \frac{1}{b-a}\Xi_1^T R \Xi_1 + \frac{3}{b-a}\Xi_2^T R \Xi_2, \tag{9}$$

where

$$\begin{aligned} \Xi_1 &= y(b) - y(a), \\ \Xi_2 &= y(b) + y(a) - \frac{2}{b-a}\int_a^b y(\alpha)d\alpha, \end{aligned}$$

Lemma 2.5. *(Sun et al., 2009) For a positive definite matrix $R > 0$, and a differentiable function $\{y(u), u \in [a,b]\}$ the following inequality holds:*

$$\int_a^b \int_\beta^b \dot{y}^T(\alpha)R\dot{y}(\alpha)d\alpha d\beta \geq 2\Xi_3^T R \Xi_3, \tag{10}$$

$$\int_a^b \int_a^\beta \dot{y}^T(\alpha)R\dot{y}(\alpha)d\alpha d\beta \geq 2\Xi_4^T R \Xi_4, \tag{11}$$

where

$$\begin{aligned} \Xi_3 &= y(b) - \frac{1}{b-a}\int_a^b y(\alpha)d\alpha, \\ \Xi_4 &= y(a) - \frac{1}{b-a}\int_a^b y(\alpha)d\alpha, \end{aligned}$$

Lemma 2.6. *(Reciprocal convexity lemma Park et al. (2011)) For any vector $\zeta \in \mathbb{R}^m$, positive definite matrices R_1, $R_2 \in \mathbb{R}^{n\times n}$, matrices X_1, $X_2 \in \mathbb{R}^{n\times m}$, $S \in \mathbb{R}^{n\times n}$, and* *real scalar $\sigma \in [0,1]$, the following inequality holds:*

$$-\frac{1}{\sigma}\zeta^T X_1^T R_1 X_1 \zeta - \frac{1}{1-\sigma}\zeta^T X_2^T R_2 X_2 \zeta \leq -\zeta^T \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}^T \begin{bmatrix} R_1 & S \\ * & R_2 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} \zeta,$$

subject to $\begin{bmatrix} R_1 & S \\ * & R_2 \end{bmatrix} > 0$.

Lemma 2.7. *(Xie, 1996) Given matrices $\Theta = \Theta^T$, Y and Z with appropriate dimensions, then for any $\mathcal{F}(t_1,t_2)$ satisfying $\mathcal{F}^T(t_1,t_2)\mathcal{F}(t_1,t_2) \leq I$,*

$$\Theta + Y\mathcal{F}(t_1,t_2)Z + Z^T\mathcal{F}^T(t_1,t_2)Y^T < 0,$$

if and only if there exists a scalar $\varepsilon > 0$, such that

$$\Theta + \varepsilon YY^T + \varepsilon^{-1}Z^TZ < 0.$$

Lemma 2.8. *(Schur complement Boyd et al. (1994)) For given symmetric matrices*

$$S = S^T = \begin{bmatrix} S_{11} & S_{12} \\ * & S_{22} \end{bmatrix},$$

where S_{11}, S_{22} are square matrices, the following conditions are equivalent

(1) $S < 0$;
(2) $S_{11} < 0, \quad S_{22} - S_{12}^T S_{11}^{-1} S_{12} < 0$;
(3) $S_{22} < 0, \quad S_{11} - S_{12} S_{22}^{-1} S_{12}^T < 0$.

In this paper, the robust H_∞ control problem is solved for the 2-D system (1) using the following state feedback controller:

$$u(t_1,t_2) = K \begin{bmatrix} x^h(t_1,t_2) \\ x^v(t_1,t_2) \end{bmatrix} \tag{12}$$

where $K = \begin{bmatrix} K_1 & K_2 \end{bmatrix}$ is the controller gain to be determined.

From (1) and (12), we obtain the following closed-loop system:

$$\begin{aligned} \begin{bmatrix} \frac{\partial x^h(t_1,t_2)}{\partial t_1} \\ \frac{\partial x^v(t_1,t_2)}{\partial t_2} \end{bmatrix} &= \hat{A}_c \begin{bmatrix} x^h(t_1,t_2) \\ x^v(t_1,t_2) \end{bmatrix} + \hat{A}_d \begin{bmatrix} x^h(t_1-h(t_1),t_2) \\ x^v(t_1,t_2-d(t_2)) \end{bmatrix} + \hat{B}w(t_1,t_2), \\ z(t_1,t_2) &= C_c \begin{bmatrix} x^h(t_1,t_2) \\ x^v(t_1,t_2) \end{bmatrix} + C_d \begin{bmatrix} x^h(t_1-h(t_1),t_2) \\ x^v(t_1,t_2-d(t_2)) \end{bmatrix} + Dw(t_1,t_2), \end{aligned} \tag{13}$$

with

$$\hat{A}_c = A + \Delta A + EK, \qquad C_c = C + FK.$$

Then, the robust H_∞ control problem to be addressed in this paper can be formulated as follows:

Given a 2-D system (1) and a prescribed level of noise attenuation $\gamma > 0$, determine the matrices K_1 and K_2 of the controller (12) such that the following requirements are satisfied:

(i) The closed-loop system (13) with $w(t_1, t_2) = 0$ is robustly asymptotically stable.
(ii) Under zero boundary condition, it holds that

$$||z||_2 < \gamma||w||_2, \tag{14}$$

for a prescribed $\gamma > 0$.

3. Main Results

3.1. Stability analysis

This subsection focuses on the problem of robust stability analysis for the uncertain 2-D continuous system with interval varying delays (6).

Theorem 3.1. *The 2-D continuous system* (6) *with parameter uncertainties* (2)-(3), *varying delays* (4), *and boundary conditions* (5) *is robustly asymptotically stable if there exist symmetric positive-definite matrices* P^h, P^v, Q_i^h, Q_i^v, R_j^h, R_j^v, Z_k^h, Z_k^v, *appropriately dimensioned matrices* M_i^h, M_i^v, S^h, S^v *and positive scalars* ε_k, $(i = 1,2,3)$, $(j = 1,2)$, $(k = 1,...,4)$, *such that the following LMIs hold.*

$$\Upsilon_{11} = \begin{bmatrix} \mathcal{W} + sym(\mathcal{U}_{11} + J_1^T \mathcal{M}\mathcal{A}J_1) & J_1^T\mathcal{M}G & \varepsilon_1 J_1^T H^T \\ * & -\varepsilon_1 I & 0 \\ * & * & -\varepsilon_1 I \end{bmatrix} < 0, \tag{15a}$$

$$\Upsilon_{12} = \begin{bmatrix} \mathcal{W} + sym(\mathcal{U}_{12} + J_1^T \mathcal{M}\mathcal{A}J_1) & J_1^T\mathcal{M}G & \varepsilon_2 J_1^T H^T \\ * & -\varepsilon_2 I & 0 \\ * & * & -\varepsilon_2 I \end{bmatrix} < 0, \tag{15b}$$

$$\Upsilon_{21} = \begin{bmatrix} \mathcal{W} + sym(\mathcal{U}_{21} + J_1^T \mathcal{M}\mathcal{A}J_1) & J_1^T\mathcal{M}G & \varepsilon_3 J_1^T H^T \\ * & -\varepsilon_3 I & 0 \\ * & * & -\varepsilon_3 I \end{bmatrix} < 0, \tag{15c}$$

$$\Upsilon_{22} = \begin{bmatrix} \mathcal{W} + sym(\mathcal{U}_{22} + J_1^T \mathcal{M}\mathcal{A}J_1) & J_1^T\mathcal{M}G & \varepsilon_4 J_1^T H^T \\ * & -\varepsilon_4 I & 0 \\ * & * & -\varepsilon_4 I \end{bmatrix} < 0, \tag{15d}$$

$$\Psi_1 = \begin{bmatrix} diag\{(R_2^h + Z_3^h), 3(R_2^h + Z_3^h)\} & S^h \\ * & diag\{(R_2^h + Z_4^h), 3(R_2^h + Z_4^h)\} \end{bmatrix} > 0, \tag{15e}$$

$$\Psi_2 = \begin{bmatrix} diag\{(R_2^v + Z_3^v), 3(R_2^v + Z_3^v)\} & S^v \\ * & diag\{(R_2^v + Z_4^v), 3(R_2^v + Z_4^v)\} \end{bmatrix} > 0, \tag{15f}$$

where

$$\begin{aligned} \mathcal{W} &= \Sigma + \Lambda^{hT}\Phi^h\Lambda^h + \Lambda^{vT}\Phi^v\Lambda^v, \\ \mathcal{U}_{11} &= \mathcal{G}^{hT}P^h\mathcal{D}_1^h + \mathcal{G}^{vT}P^v\mathcal{D}_1^v, \\ \mathcal{U}_{12} &= \mathcal{G}^{hT}P^h\mathcal{D}_1^h + \mathcal{G}^{vT}P^v\mathcal{D}_2^v, \\ \mathcal{U}_{21} &= \mathcal{G}^{hT}P^h\mathcal{D}_2^h + \mathcal{G}^{vT}P^v\mathcal{D}_1^v, \end{aligned}$$

$$\mathcal{U}_{22} \;=\; \mathcal{G}^{hT} P^h \mathcal{D}_2^h + \mathcal{G}^{vT} P^v \mathcal{D}_2^v,$$

with

$$\begin{aligned}
\mathcal{G}^h &= \begin{bmatrix} e_9 \\ e_1 - e_3 \\ e_3 - e_7 \end{bmatrix}, \mathcal{D}_1^h = \begin{bmatrix} e_1 \\ h_1 e_{11} \\ h_{12} e_{13} \end{bmatrix}, \mathcal{D}_2^h = \begin{bmatrix} e_1 \\ h_1 e_{11} \\ h_{12} e_{15} \end{bmatrix}, \mathcal{G}^v = \begin{bmatrix} e_{10} \\ e_2 - e_4 \\ e_4 - e_6 \end{bmatrix}, \\
\mathcal{D}_1^v &= \begin{bmatrix} e_2 \\ d_1 e_{12} \\ d_{12} e_{14} \end{bmatrix}, \mathcal{D}_2^v = \begin{bmatrix} e_2 \\ d_1 e_{12} \\ d_{12} e_{16} \end{bmatrix}, \\
J_1 &= \begin{bmatrix} e_1 \\ e_2 \\ e_5 \\ e_6 \\ e_9 \\ e_{10} \end{bmatrix}, \mathcal{M} = \begin{bmatrix} M_1^h & 0 \\ 0 & M_1^v \\ M_2^h & 0 \\ 0 & M_2^v \\ M_3^h & 0 \\ 0 & M_3^v \end{bmatrix}, \mathcal{A} = \begin{bmatrix} A_{11}^T & A_{21}^T \\ A_{12}^T & A_{22}^T \\ A_{d11}^T & A_{d21}^T \\ A_{d12}^T & A_{d22}^T \\ -I_{n_h} & 0 \\ 0 & -I_{n_v} \end{bmatrix}^T, H = \begin{bmatrix} H_{11}^T \\ H_{12}^T \\ H_{21}^T \\ H_{22}^T \\ 0 \\ 0 \end{bmatrix}^T, \\
\Lambda^h &= \begin{bmatrix} e_3 - e_5 \\ e_3 + e_5 - 2e_{13} \\ e_5 - e_7 \\ e_5 + e_7 - 2e_{15} \end{bmatrix}, \Lambda^v = \begin{bmatrix} e_4 - e_6 \\ e_4 + e_6 - 2e_{14} \\ e_6 - e_8 \\ e_6 + e_8 - 2e_{16} \end{bmatrix}, \\
\Phi^h &= \begin{bmatrix} diag\{R_2^h, 3R_2^h\} & S^h \\ * & diag\{R_2^h, 3R_2^h\} \end{bmatrix}, \Phi^h = \begin{bmatrix} diag\{R_2^v, 3R_2^v\} & S^v \\ * & dia\{R_2^v, 3R_2^v\} \end{bmatrix}, \\
\Sigma &= e_1^T(Q_1^h + Q_2^h + Q_3^h)e_1 - e_3^T Q_1^h e_3 - e_7^T Q_2^h e_7 - (1-\mu_1) e_5^T Q_3^h e_5 \\
&\quad + e_2^T(Q_1^v + Q_2^v + Q_3^v)e_2 - e_4^T Q_1^v e_4 - e_8^T Q_2^v e_8 - (1-\mu_2) e_6^T Q_3^v e_6 \\
&\quad + h_1^2 e_9^T R_1^h e_9 - (e_1 - e_3)^T R_1^h (e_1 - e_3) - 3(e_1 + e_3 - 2e_{11})^T R_1^h (e_1 + e_3 - 2e_{11}) \\
&\quad + d_1^2 e_{10}^T R_1^v e_{10} - (e_2 - e_4)^T R_1^v (e_2 - e_4) \\
&\quad - 3(e_2 + e_4 - 2e_{12})^T R_1^v (e_2 + e_4 - 2e_{12}) + h_{12}^2 e_9^T R_2^h e_9 \\
&\quad + d_{12}^2 e_{10}^T R_2^v e_{10} + \frac{h_1^2}{2} e_9^T Z_1^h e_9 - 2(e_1 - e_{11})^T Z_1^h (e_1 - e_{11}) \\
&\quad + \frac{d_1^2}{2} e_{10}^T Z_1^v e_{10} - 2(e_2 - e_{12})^T Z_1^v (e_2 - e_{12}) + \frac{h_1^2}{2} e_9^T Z_2^h e_9 \\
&\quad - 2(e_3 - e_{11})^T Z_1^h (e_3 - e_{11}) + \frac{d_1^2}{2} e_{10}^T Z_2^v e_{10} - 2(e_4 - e_{12})^T Z_2^v (e_4 - e_{12}) \\
&\quad + \frac{h_{12}^2}{2} e_9^T Z_3^h e_9 - 2(e_3 - e_{13})^T Z_3^h (e_3 - e_{13}) - 2(e_5 - e_{15})^T Z_3^h (e_5 - e_{15}) \\
&\quad + \frac{d_{12}^2}{2} e_{10}^T Z_3^v e_{10} - 2(e_4 - e_{14})^T Z_3^v (e_4 - e_{14}) - 2(e_6 - e_{16})^T Z_3^v (e_6 - e_{16}) \\
&\quad + \frac{h_{12}^2}{2} e_9^T Z_4^h e_9 - 2(e_5 - e_{13})^T Z_4^h (e_5 - e_{13}) - 2(e_7 - e_{15})^T Z_4^h (e_7 - e_{15}) \\
&\quad + \frac{d_{12}^2}{2} e_{10}^T Z_4^v e_{10} - 2(e_6 - e_{14})^T Z_4^v (e_6 - e_{14}) - 2(e_8 - e_{16})^T Z_4^v (e_8 - e_{16})
\end{aligned}$$

and the elementary matrices $e_m(m=1,2,...,16)$ *are defined by:*

$$e_m = \begin{cases} \begin{bmatrix} 0_{n_h,(p-1)n} & N_h & 0_{n_h,(8-p)n} \end{bmatrix}, & (p=\frac{m-1}{2}), \text{ if } m \text{ is odd;} \\ \begin{bmatrix} 0_{n_v,(p-1)n} & N_h & 0_{n_v,(8-p)n} \end{bmatrix}, & (p=\frac{m}{2}), \quad \text{if } m \text{ is even;} \end{cases}$$

with $N_h = \begin{bmatrix} I_{n_h} & 0_{n_h,n_v} \end{bmatrix}$, $N_v = \begin{bmatrix} 0_{n_v,n_h} & I_{n_v} \end{bmatrix}$.

Proof. In order to proof the stability for system (6), we select the following Lyapunov-Krasovskii functional candidate:

$$V(t_1,t_2) = V^h(t_1,t_2) + V^v(t_1,t_2), \tag{16}$$

with

$$\begin{aligned}
V^h(t_1,t_2) &= \sum_{i=0}^{7} V_i^h(t_1,t_2), \\
V_0^h(t_1,t_2) &= \zeta^{hT}(t_1,t_2)P^h\zeta^h(t_1,t_2), \\
V_1^h(t_1,t_2) &= \sum_{i=1}^{2}\int_{t_1-h_i}^{t_1} x^{hT}(\alpha,t_2)Q_i^h x^h(\alpha,t_2)d\alpha + \int_{t_1-h(t_1)}^{t_1} x^{hT}(\alpha,t_2)Q_3^h x^h(\alpha,t_2)d\alpha, \\
V_2^h(t_1,t_2) &= h_1\int_{-h_1}^{0}\int_{t_1+\beta}^{t_1} \dot{x}^{hT}(\alpha,t_2)R_1^h\dot{x}^h(\alpha,t_2)d\alpha d\beta, \\
V_3^h(t_1,t_2) &= h_{12}\int_{-h_2}^{-h_1}\int_{t_1+\beta}^{t_1} \dot{x}^{hT}(\alpha,t_2)R_2^h\dot{x}^h(\alpha,t_2)d\alpha d\beta, \\
V_4^h(t_1,t_2) &= \int_{-h_1}^{0}\int_{\lambda}^{0}\int_{t_1+\beta}^{t_1} \dot{x}^{hT}(\alpha,t_2)Z_1^h\dot{x}^h(\alpha,t_2)d\alpha d\beta d\lambda, \\
V_5^h(t_1,t_2) &= \int_{-h_1}^{0}\int_{-h_1}^{\lambda}\int_{t_1+\beta}^{t_1} \dot{x}^{hT}(\alpha,t_2)Z_2^h\dot{x}^h(\alpha,t_2)d\alpha d\beta d\lambda, \\
V_6^h(t_1,t_2) &= \int_{-h_2}^{-h_1}\int_{\lambda}^{-h_1}\int_{t_1+\beta}^{t_1} \dot{x}^{hT}(\alpha,t_2)Z_3^h\dot{x}^h(\alpha,t_2)d\alpha d\beta d\lambda, \\
V_7^h(t_1,t_2) &= \int_{-h_2}^{-h_1}\int_{-h_2}^{\lambda}\int_{t_1+\beta}^{t_1} \dot{x}^{hT}(\alpha,t_2)Z_4^h\dot{x}^h(\alpha,t_2)d\alpha d\beta d\lambda,
\end{aligned}$$

and

$$\begin{aligned}
V^v(t_1,t_2) &= \sum_{i=0}^{7} V_i^v(t_1,t_2), \\
V_0^v(t_1,t_2) &= \zeta^{vT}(t_1,t_2)P^v\zeta^v(t_1,t_2), \\
V_1^v(t_1,t_2) &= \sum_{i=1}^{2}\int_{t_2-d_i}^{t_2} x^{vT}(t_1,\alpha)Q_i^v x^v(t_2,\alpha)d\alpha + \int_{t_2-d(t_2)}^{t_2} x^{vT}(t_1,\alpha)Q_3^v x^v(t_1,\alpha)d\alpha, \\
V_2^v(t_1,t_2) &= d_1\int_{-d_1}^{0}\int_{t_2+\beta}^{t_2} \dot{x}^{vT}(t_1,\alpha)R_1^v\dot{x}^v(t_1,\alpha)d\alpha d\beta,
\end{aligned}$$

$$
\begin{aligned}
V_3^v(t_1,t_2) &= d_{12}\int_{-d_2}^{-d_1}\int_{t_2+\beta}^{t_2}\dot{x}^{vT}(t_1,\alpha)R_2^v\dot{x}^v(t_1,\alpha)d\alpha d\beta,\\
V_4^v(t_1,t_2) &= \int_{-d_1}^{0}\int_{\lambda}^{0}\int_{d_1+\beta}^{t_2}\dot{x}^{vT}(\alpha,t_2)Z_1^v\dot{x}^v(t_1,\alpha)d\alpha d\beta d\lambda,\\
V_5^v(t_1,t_2) &= \int_{-d_1}^{0}\int_{-d_1}^{\lambda}\int_{t_2+\beta}^{t_2}\dot{x}^{vT}(\alpha,t_2)Z_2^v\dot{x}^v(t_1,\alpha)d\alpha d\beta d\lambda,\\
V_6^v(t_1,t_2) &= \int_{-d_2}^{-d_1}\int_{\lambda}^{-d_1}\int_{t_2+\beta}^{t_2}\dot{x}^{vT}(t_1,\alpha)Z_3^v\dot{x}^v(t_1,\alpha)d\alpha d\beta d\lambda,\\
V_7^v(t_1,t_2) &= \int_{-d_2}^{-d_1}\int_{-d_2}^{\lambda}\int_{t_2+\beta}^{t_2}\dot{x}^{vT}(t_1,\alpha)Z_4^v\dot{x}^v(t_1,\alpha)d\alpha d\beta d\lambda,
\end{aligned}
$$

where

$$
\zeta^h(t_1,t_2)=\begin{bmatrix} x^h(t_1,t_2)\\ \int_{t_1-h_1}^{t_1}x^h(\alpha,t_2)d\alpha\\ \int_{t_1-h_2}^{t_1-h_1}x^h(\alpha,t_2)d\alpha\end{bmatrix},\qquad \zeta^v(t_1,t_2)=\begin{bmatrix} x^v(t_1,t_2)\\ \int_{t_2-d_1}^{t_2}x^v(t_1\alpha)d\alpha\\ \int_{t_2-d_2}^{t_2-d_1}x^v(t_1\alpha)d\alpha\end{bmatrix},
$$

and $\dot{x}^h(\alpha,t_2)=\frac{\partial x^h(t_1,t_2)}{\partial t_1}|_{t_1=\alpha}$, $\dot{x}^v(t_1,\alpha)=\frac{\partial x^v(t_1,t_2)}{\partial t_2}|_{t_2=\alpha}$,

Define

$$
\xi(t_1,t_2)=col\left\{\begin{pmatrix} x^h(t_1,t_2)\\ x^v(t_1,t_2)\\ x^h(t_1-h_1,t_2)\\ x^v(t_1,t_2-d_1)\\ x^h(t_1-h(t_1),t_2)\\ x^v(t_1,t_2-d(t_2))\\ x^h(t_1-h_2,t_2)\\ x^v(t_1,t_2-d_2)\\ \dot{x}^h(t_1,t_2)\\ \dot{x}^v(t_1,t_2)\end{pmatrix},\begin{pmatrix} \frac{1}{h_1}\int_{t_1-h_1}^{t_1}x^h(\alpha,t_2)d\alpha\\ \frac{1}{d_1}\int_{t_2-d_1}^{t_2}x^v(t_2,\alpha)d\alpha\\ \frac{1}{h(t_1)-h_1}\int_{t_1-h(t_1)}^{t_1-h_1}x^h(\alpha,t_2)d\alpha\\ \frac{1}{d(t_2)-d_1}\int_{t_2-d(t_2)}^{t_2-d_1}x^v(t_2,\alpha)d\alpha\\ \frac{1}{h_2-h(t_1)}\int_{t_1-h_2}^{t_1-h(t_1)}x^h(\alpha,t_2)d\alpha\\ \frac{1}{d_2-d(t_2)}\int_{t_2-d_2}^{t_2-d(t_2)}x^v(t_2,\alpha)d\alpha\end{pmatrix}\right\},
$$

$$
\begin{aligned}
\frac{\partial V^h(t_1,t_2)}{\partial t_1} &= 2\xi^{hT}(t_1,t_2)\mathcal{G}^{hT}P^h\mathcal{D}^h\xi(t_1,t_2)+x^{hT}(t_1,t_2)(Q_1^h+Q_2^h+Q_3^h)x^h(t_1,t_2)\\
&\quad -x^{hT}(t_1-h_1,t_2)Q_1^hx^h(t_1-h_1,t_2)-x^{hT}(t_1-h_2,t_2)Q_1^hx^h(t_1-h_2,t_2)\\
&\quad -(1-\dot{h}(t_1))x^{hT}(t_1-h(t_1),t_2)Q_1^hx^h(t_1-h(t_1),t_2)\\
&\quad +\frac{h_1^2}{2}\dot{x}^{hT}(t_1,t_2)(2R_1^h+Z_1^h+Z_2^h)\dot{x}^h(t_1,t_2)\\
&\quad +\frac{h_{12}^2}{2}\dot{x}^{hT}(t_1,t_2)(2R_2^h+Z_3^h+Z_4^h)\dot{x}^h(t_1,t_2)\\
&\quad -h_1\int_{t_1-h_1}^{t_1}\dot{x}^{hT}(\alpha,t_2)R_1^h\dot{x}^h(\alpha,t_2)d\alpha\\
&\quad -h_{12}\int_{t_1-h_2}^{t_1-h_1}\dot{x}^{hT}(\alpha,t_2)R_2^h\dot{x}^h(\alpha,t_2)d\alpha
\end{aligned}
$$

$$
\begin{aligned}
&-\int_{-h_1}^{0}\int_{t_1+\beta}^{t_1}\dot{x}^{hT}(\alpha,t_2)Z_1^h\dot{x}^h(\alpha,t_2)d\alpha d\beta\\
&-\int_{-h_1}^{0}\int_{t_1-h_1}^{t_1+\beta}\dot{x}^{hT}(\alpha,t_2)Z_2^h\dot{x}^h(\alpha,t_2)d\alpha d\beta\\
&-\int_{-h_2}^{-h_1}\int_{t_1+\beta}^{t_1-h_1}\dot{x}^{hT}(\alpha,t_2)Z_3^h\dot{x}^h(\alpha,t_2)d\alpha d\beta\\
&-\int_{-h_2}^{-h_1}\int_{t_1-h_2}^{t_1+\beta}\dot{x}^{hT}(\alpha,t_2)Z_4^h\dot{x}^h(\alpha,t_2)d\alpha d\beta,
\end{aligned}
$$

where

$$
\mathcal{D}^h = \left[\begin{array}{ccc} e_1^T & h_1e_{11}^T & (h_2-h(t_1))e_{13}^T+(h(t_1)-h_1)e_{15}^T \end{array}\right]^T,
$$

and

$$
\begin{aligned}
\frac{\partial V^v(t_1,t_2)}{\partial t_2} &= 2\xi^{vT}(t_1,t_2)\mathcal{G}^{vT}P^v\mathcal{D}^v\xi^v(t_1,t_2)+x^{vT}(t_1,t_2)(Q_1^v+Q_2^v+Q_3^v)x^v(t_1,t_2)\\
&-x^{vT}(t_1,t_2-d_1)Q_1^vx^v(t_1,t_2-d_1)-x^{vT}(t_1,t_2-d_2)Q_2^vx^v(t_1,t_2-d_2)\\
&-(1-\dot{d}(t_2))x^{vT}(t_1,t_2-d(t_2))Q_1^vx^v(t_1,t_2-d(t_2))\\
&+\frac{d_1^2}{2}\dot{x}^{vT}(t_1,t_2)(2R_1^v+Z_1^v+Z_2^v)\dot{x}^v(t_1,t_2)\\
&+\frac{d_{12}^2}{2}\dot{x}^{vT}(t_1,t_2)(2R_2^v+Z_3^v+Z_4^v)\dot{x}^v(t_1,t_2)\\
&-d_1\int_{t_2-d_1}^{t_2}\dot{x}^{vT}(t_1,\alpha)R_1^v\dot{x}^v(t_1\alpha)d\alpha\\
&-d_{12}\int_{t_2-d_2}^{t_2-d_1}\dot{x}^{vT}(t_1,\alpha)R_2^v\dot{x}^v(t_1,\alpha)d\alpha\\
&-\int_{-d_1}^{0}\int_{t_2+\beta}^{t_2}\dot{x}^{vT}(t_2,\alpha)Z_1^v\dot{x}^v(t_1,\alpha)d\alpha d\beta\\
&-\int_{-d_1}^{0}\int_{t_2-d_1}^{t_2+\beta}\dot{x}^{vT}(t_1,\alpha)Z_2^v\dot{x}^v(t_1,\alpha)d\alpha d\beta\\
&-\int_{-d_2}^{-d_1}\int_{t_2+\beta}^{t_2-d_1}\dot{x}^{vT}(t_1,\alpha)Z_3^v\dot{x}^v(t_1,\alpha)d\alpha d\beta\\
&-\int_{-d_2}^{-d_1}\int_{t_2-d_2}^{t_2+\beta}\dot{x}^{vT}(t_1,\alpha)Z_4^v\dot{x}^v(t_1,\alpha)d\alpha d\beta
\end{aligned}
$$

where

$$
\mathcal{D}^v = \left[\begin{array}{ccc} e_2^T & d_1e_{12}^T & (d_2-d(t_2))e_{14}^T+(d(t_2)-d_1)e_{16}^T \end{array}\right]^T,
$$

By defining $\lambda^h=\frac{h(t_1)-h_1}{h_{12}}$ and $\lambda^v=\frac{d(t_2)-d_1}{d_{12}}$ we can write

$$
\mathcal{D}^h = \lambda^h\mathcal{D}_1^h+(1-\lambda^h)\mathcal{D}_2^h,
$$

$$
\begin{aligned}
&= \lambda^v\lambda^h\mathcal{D}_1^h + (1-\lambda^v)\lambda^h\mathcal{D}_1^h + \lambda^v(1-\lambda^h)\mathcal{D}_2^h + (1-\lambda^v)(1-\lambda^h)\mathcal{D}_2^h,\\
\mathcal{D}^v &= \lambda^v\mathcal{D}_1^v + (1-\lambda^v)\mathcal{D}_2^v,\\
&= \lambda^h\lambda^v\mathcal{D}_1^v + (1-\lambda^h)\lambda^v\mathcal{D}_1^v + \lambda^h(1-\lambda^v)\mathcal{D}_2^v + (1-\lambda^h)(1-\lambda^v)\mathcal{D}_2^v.
\end{aligned}
$$

It follows from the integral inequalities in Lemma 2.4 and Lemma 2.5 that:

$$
\begin{aligned}
&\star h_1\int_{t_1-h_1}^{t_1}\dot{x}^{hT}(\alpha,t_2)R_1^h\dot{x}^h(\alpha,t_2)d\alpha \geq \xi^T(t_1,t_2)\{(e_1-e_3)^TR_1^h(e_1-e_3)\\
&\quad + 3(e_1+e_3-2e_{11})^TR_1^h(e_1+e_3-2e_{11})\}\xi(t_1,t_2), \qquad (17a)\\
&\star d_1\int_{t_2-d_1}^{t_2}\dot{x}^{vT}(t_1,\alpha)R_1^v\dot{x}^v(t_1\alpha)d\alpha \geq \xi^T(t_1,t_2)\{(e_2-e_4)^TR_1^v(e_2-e_4)\\
&\quad + 3(e_2+e_4-2e_{12})^TR_1^v(e_2+e_4-2e_{12})\}\xi(t_1,t_2), \qquad (17b)\\
&\star h_{12}\int_{t_1-h_2}^{t_1-h_1}\dot{x}^{hT}(\alpha,t_2)R_2^h\dot{x}^h(\alpha,t_2)d\alpha = h_{12}\int_{t_1-h(t_1)}^{t_1-h_1}\dot{x}^{hT}(\alpha,t_2)R_2^h\dot{x}^h(\alpha,t_2)d\alpha\\
&\quad + h_{12}\int_{t_1-h_2}^{t_1-h(t_1)}\dot{x}^{hT}(\alpha,t_2)R_2^h\dot{x}^h(\alpha,t_2)d\alpha \geq\\
&\quad \frac{h_{12}}{h_2-h(t_1)}\xi^T(t_1,t_2)\{(e_5-e_7)^TR_2^h(e_5-e_7)\\
&\quad + 3(e_5+e_7-2e_{15})^TR_2^h(e_5+e_7-2e_{15})\}\xi(t_1,t_2)\\
&\quad + \frac{h_{12}}{h(t_1)-h_1}\xi^T(t_1,t_2)\{(e_3-e_5)^TR_2^h(e_3-e_5)\\
&\quad + 3(e_3+e_5-2e_{13})^TR_2^h(e_3+e_5-2e_{13})\}\xi(t_1,t_2), \qquad (17c)\\
&\star d_{12}\int_{t_2-d_2}^{t_2-d_1}\dot{x}^{vT}(t_1,\alpha)R_2^v\dot{x}^v(t_1,\alpha)d\alpha = d_{12}\int_{t_2-d(t_2)}^{t_2-d_1}\dot{x}^{vT}(t_2,\alpha)R_2^v\dot{x}^v(t_2,\alpha)d\alpha\\
&\quad + d_{12}\int_{t_2-d_2}^{t_2-d(t_2)}\dot{x}^{vT}(t_2,\alpha)R_2^v\dot{x}^v(t_2,\alpha)d\alpha \geq\\
&\quad \frac{d_{12}}{d_2-d(t_2)}\xi^T(t_1,t_2)\{(e_6-e_8)^TR_2^v(e_6-e_8)\\
&\quad + 3(e_6+e_8-2e_{16})^TR_2^v(e_6+e_8-2e_{16})\}\xi(t_1,t_2)\\
&\quad + \frac{d_{12}}{d(t_2)-d_1}\xi^T(t_1,t_2)\{(e_4-e_6)^TR_2^v(e_4-e_6)\\
&\quad + 3(e_4+e_6-2e_{14})^TR_2^v(e_4+e_6-2e_{14})\}\xi(t_1,t_2), \qquad (17d)\\
&\star\int_{-h_1}^{0}\int_{t_1+\beta}^{t_1}\dot{x}^{hT}(\alpha,t_2)Z_1^h\dot{x}^h(\alpha,t_2)d\alpha d\beta \geq\\
&\quad \xi^T(t_1,t_2)\{2(e_1-e_{11})^TZ_1^h(e_1-e_{11})\}\xi(t_1,t_2), \qquad (17e)\\
&\star\int_{-d_1}^{0}\int_{t_2+\beta}^{t_2}\dot{x}^{vT}(t_2,\alpha)Z_1^v\dot{x}^v(t_1,\alpha)d\alpha d\beta \geq\\
&\quad \xi^T(t_1,t_2)\{2(e_2-e_{12})^TZ_1^v(e_2-e_{12})\}\xi(t_1,t_2), \qquad (17f)\\
&\star\int_{-h_1}^{0}\int_{t_1-h_1}^{t_1+\beta}\dot{x}^{hT}(\alpha,t_2)Z_2^h\dot{x}^h(\alpha,t_2)d\alpha d\beta\\
&\quad \xi^T(t_1,t_2)\{2(e_3-e_{11})^TZ_2^h(e_3-e_{11})\}\xi(t_1,t_2), \qquad (17g)
\end{aligned}
$$

$$\star \int_{-d_1}^{0} \int_{t_2-d_1}^{t_2+\beta} \dot{x}^{vT}(t_1,\alpha) Z_2^v \dot{x}^v(t_1,\alpha) d\alpha d\beta$$

$$\xi^T(t_1,t_2)\{2(e_4-e_{12})^T Z_2^v (e_4-e_{12})\}\xi(t_1,t_2), \tag{17h}$$

$$\star \int_{-h_2}^{-h_1} \int_{t_1+\beta}^{t_1-h_1} \dot{x}^{hT}(\alpha,t_2) Z_3^h \dot{x}^h(\alpha,t_2) d\alpha d\beta =$$

$$\int_{-h(t_1)}^{-h_1} \int_{t_1+\beta}^{t_1-h_1} \dot{x}^{hT}(\alpha,t_2) Z_3^h \dot{x}^h(\alpha,t_2) d\alpha d\beta$$

$$+ \int_{-h_2}^{-h(t_1)} \int_{t_1+\beta}^{t_1-h(t_1)} \dot{x}^{hT}(\alpha,t_2) Z_3^h \dot{x}^h(\alpha,t_2) d\alpha d\beta$$

$$+ (h_2-h(t_1)) \int_{t_1-h(t_1)}^{t_1-h_1} \dot{x}^{hT}(\alpha,t_2) Z_3^h \dot{x}^h(\alpha,t_2) d\alpha d\beta \geq$$

$$\xi^T(t_1,t_2)\{2(e_3-e_{13})^T Z_3^h (e_3-e_{13}) + 2(e_5-e_{15})^T Z_3^h (e_5-e_{15})$$

$$+ \frac{h_2-h(t_1)}{h(t_1)-h_1}(e_3-e_5)^T Z_3^h (e_3-e_5)$$

$$+ 3\frac{h_2-h(t_1)}{h(t_1)-h_1}(e_3+e_5-2e_{13})^T Z_3^h (e_3+e_5-2e_{13})\}\xi(t_1,t_2), \tag{17i}$$

$$\star \int_{-d_2}^{-d_1} \int_{t_2+\beta}^{t_2-d_1} \dot{x}^{vT}(t_1,\alpha) Z_3^v \dot{x}^v(t_1,\alpha) d\alpha d\beta =$$

$$\int_{-d(t_2)}^{-d_1} \int_{t_2+\beta}^{t_2-d_1} \dot{x}^{vT}(t_1,\alpha) Z_3^v \dot{x}^v(t_1,\alpha) d\alpha d\beta$$

$$+ \int_{-d_2}^{-d(t_2)} \int_{t_2+\beta}^{t_2-d(t_2)} \dot{x}^{vT}(t_1,\alpha) Z_3^v \dot{x}^v(t_1,\alpha) d\alpha d\beta$$

$$+ (d_2-d(t_2)) \int_{t_2-d(t_2)}^{t_2-d_2} \dot{x}^{vT}(t_2,\alpha) Z_3^v \dot{x}^v(t_1,\alpha) d\alpha d\beta \geq$$

$$\xi^T(t_1,t_2)\{2(e_4-e_{14})^T Z_3^v (e_4-e_{14}) + 2(e_6-e_{16})^T Z_3^v (e_6-e_{16})$$

$$+ \frac{d_2-d(t_2)}{d(t_2)-d_1}(e_4-e_6)^T Z_3^v (e_4-e_6)$$

$$+ 3\frac{d_2-d(t_2)}{d(t_2)-d_1}(e_4+e_6-2e_{14})^T Z_3^v (e_4+e_6-2e_{14})\}\xi(t_1,t_2), \tag{17j}$$

$$\star \int_{-h_2}^{-h_1} \int_{t_1-h_2}^{t_1+\beta} \dot{x}^{hT}(\alpha,t_2) Z_4^h \dot{x}^h(\alpha,t_2) d\alpha d\beta =$$

$$\int_{-h(t_1)}^{-h_1} \int_{t_1-h(t_1)}^{t_1+\beta} \dot{x}^{hT}(\alpha,t_2) Z_4^h \dot{x}^h(\alpha,t_2) d\alpha d\beta$$

$$+ \int_{-h_2}^{-h(t_1)} \int_{t_1-h_2}^{t_1+\beta} \dot{x}^{hT}(\alpha,t_2) Z_4^h \dot{x}^h(\alpha,t_2) d\alpha d\beta$$

$$+ (h(t_1)-h_1) \int_{t_1-h_2}^{t_1-h(t_1)} \dot{x}^{hT}(\alpha,t_2) Z_4^h \dot{x}^h(\alpha,t_2) d\alpha d\beta \geq$$

$$\xi^T(t_1,t_2)\{2(e_5-e_{13})^T Z_4^h (e_5-e_{13})\}\xi(t_1,t_2) + 2(e_7-e_{15})^T Z_4^h (e_7-e_{15})$$

$$+ \frac{h(t_1)-h_1}{h_2-h(t_1)}(e_5-e_7)^T Z_4^h (e_5-e_7)$$

$$+ 3\frac{h(t_1) - h_1}{h_2 - h(t_1)}(e_5 + e_7 - 2e_{15})^T Z_4^h(e_5 + e_7 - 2e_{15})\}\xi(t_1, t_2), \tag{17k}$$

$$\star \int_{-d_2}^{-d_1} \int_{t_2-d_2}^{t_2+\beta} \dot{x}^{vT}(t_1, \alpha) Z_4^v \dot{x}^v(t_1, \alpha) d\alpha d\beta =$$
$$\int_{-d(t_2)}^{-d_1} \int_{t_2-d(t_2)}^{t_2+\beta} \dot{x}^{vT}(t_2, \alpha) Z_4^v \dot{x}^h(t_2, \alpha) d\alpha d\beta$$
$$+ \int_{-d_2}^{-d(t_2)} \int_{t_2-d_2}^{t_2+\beta} \dot{x}^{vT}(t_2, \alpha)) Z_4^v \dot{x}^v(t_2, \alpha) d\alpha d\beta$$
$$+ (d(t_2) - d_1) \int_{t_2-d_2}^{t_2-d(t_2)} \dot{x}^{vT}(t_2, \alpha) Z_4^v \dot{x}^v(t_2, \alpha) d\alpha d\beta \geq$$
$$\xi^T(t_1, t_2)\{2(e_6 - e_{14})^T Z_4^v(e_6 - e_{14}) + 2(e_8 - e_{16})^T Z_4^v(e_8 - e_{16})$$
$$+ \frac{d(t_2) - d_1}{d_2 - d(t_2)}(e_6 - e_8)^T Z_4^v(e_6 - e_8)$$
$$+ 3\frac{d(t_2) - d_1}{d_2 - d(t_2)}(e_6 + e_8 - 2e_{16})^T Z_4^v(e_6 + e_8 - 2e_{16})\}\xi(t_1, t_2). \tag{17l}$$

According to Lemma 2.6, we have

$$\frac{1}{\lambda^h}\xi(t_1, t_2)^T\{(e_3 - e_5)^T(R_2^h + Z_3^h)(e_3 - e_5)$$
$$+ 3(e_3 + e_5 - 2e_{13})^T(R_2^h + Z_3^h)(e_3 + e_5 - 2e_{13})\}\xi(t_1, t_2)$$
$$+ \frac{1}{1 - \lambda^h}\xi(t_1, t_2)^T\{(e_5 - e_7)^T(R_2^h + Z_4^h)(e_5 - e_7)$$
$$+ 3(e_5 + e_7 - 2e_{15})^T(R_2^h + Z_4^h)(e_5 + e_7 - 2e_{15})\}\xi(t_1, t_2)$$
$$- \xi(t_1, t_2)^T\{(e_3 - e_5)^T Z_3^h(e_3 - e_5) + (e_3 + e_5 - 2e_{13})^T Z_3^h(e_3 + e_5 - 2e_{13})$$
$$+ (e_5 - e_7)^T Z_4^h(e_5 - e_7) + 3(e_5 + e_7 - 2e_{15})^T Z_4^h(e_5 + e_7 - 2e_{15})\}\xi(t_1, t_2)$$
$$\geq \xi^T(t_1, t_2)\Lambda_h^T \Phi_h \Lambda^h \xi(t_1, t_2),$$

and

$$\frac{1}{\lambda^v}\xi(t_1, t_2)^T\{(e_4 - e_6)^T(R_2^v + Z_3^v)(e_4 - e_6)$$
$$+ 3(e_4 + e_6 - 2e_{14})^T(R_2^v + Z_3^v)(e_4 + e_6 - 2e_{14})\}\xi(t_1, t_2)$$
$$+ \frac{1}{1 - \lambda^h}\xi(t_1, t_2)^T\{(e_6 - e_8)^T(R_2^v + Z_4^v)(e_6 - e_8)$$
$$+ 3(e_6 + e_8 - 2e_{16})^T(R_2^v + Z_4^v)(e_6 + e_8 - 2e_{16})\}\xi(t_1, t_2)$$
$$- \xi(t_1, t_2)^T\{(e_4 - e_6)^T Z_3^v(e_4 - e_6) + (e_4 + e_6 - 2e_{14})^T Z_3^v(e_4 + e_6 - 2e_{14})$$
$$+ (e_6 - e_8)^T Z_4^v(e_6 - e_8) + 3(e_6 + e_8 - 2e_{16})^T Z_4^v(e_6 + e_8 - 2e_{16})\}\xi(t_1, t_2)$$
$$\geq \xi^T(t_1, t_2)\Lambda_v^T \Phi_v \Lambda^v \xi(t_1, t_2),$$

Following (6), for any free matrices M_1^h, M_2^h, M_3^h, M_1^v, M_2^v and M_3^v, with appropriate dimensions, we have:

$$
\begin{aligned}
0 &= 2\left\{ \begin{bmatrix} x^h(t_1,t_2) \\ x^v(t_1,t_2) \end{bmatrix}^T \begin{bmatrix} M_1^h & 0 \\ 0 & M_1^v \end{bmatrix} + \begin{bmatrix} x^h(t_1-h(t_1),t_2) \\ x^v(t_1,t_2-d(t_2)) \end{bmatrix}^T \begin{bmatrix} M_2^h & 0 \\ 0 & M_2^v \end{bmatrix} \right. \\
&\quad \left. + \begin{bmatrix} \frac{\partial x^h(t_1,t_2)}{\partial t_1} \\ \frac{\partial x^v(t_1,t_2)}{\partial t_2} \end{bmatrix}^T \begin{bmatrix} M_3^h & 0 \\ 0 & M_3^v \end{bmatrix} \right\} \\
&\quad \times \left\{ \hat{A} \begin{bmatrix} x^h(t_1,t_2) \\ x^v(t_1,t_2) \end{bmatrix} + \hat{A}_d \begin{bmatrix} x^h(t_1-h(t_1),t_2) \\ x^v(t_1,t_2-d(t_2)) \end{bmatrix} - \begin{bmatrix} \frac{\partial x^h(t_1,t_2)}{\partial t_1} \\ \frac{\partial x^v(t_1,t_2)}{\partial t_2} \end{bmatrix} \right\} \\
&= \xi^T(t_1,t_2)\{sym(J_1^T \mathcal{M}(\mathcal{A}+G\mathcal{F}(t_1,t_2)H)J_1)\}\xi(t_1,t_2), \qquad (20)
\end{aligned}
$$

Combining together with (17a)-(20) yields

$$
\begin{aligned}
\dot{V}_u(t_1,t_2) &\le \xi^T(t_1,t_2)\{\mathcal{W} + sym(\mathcal{U} + J_1^T\mathcal{M}\mathcal{A}J_1 + J_1^T G\mathcal{F}(t_1,t_2)HJ_1)\}\xi(t_1,t_2) \\
&= \xi^T(t_1,t_2)\{\lambda^h\lambda^v\Pi_{11} + (1-\lambda^h)\lambda^v\Pi_{12} + \lambda^h(1-\lambda^v)\Pi_{21} \\
&\qquad +(1-\lambda^h)(1-\lambda^v)\Pi_{22}\}\xi(t_1,t_2),
\end{aligned}
$$

where

$$
\begin{aligned}
\Pi_{11} &= \mathcal{W} + sym(\mathcal{U}_{11} + J_1^T\mathcal{M}\mathcal{A}J_1 + J_1^T\mathcal{M}G\mathcal{F}(t_1,t_2)HJ_1), \\
\Pi_{12} &= \mathcal{W} + sym(\mathcal{U}_{12} + J_1^T\mathcal{M}\mathcal{A}J_1 + J_1^T\mathcal{M}G\mathcal{F}(t_1,t_2)HJ_1), \\
\Pi_{21} &= \mathcal{W} + sym(\mathcal{U}_{21} + J_1^T\mathcal{M}\mathcal{A}J_1 + J_1^T\mathcal{M}G\mathcal{F}(t_1,t_2)HJ_1), \\
\Pi_{22} &= \mathcal{W} + sym(\mathcal{U}_{22} + J_1^T\mathcal{M}\mathcal{A}J_1 + J_1^T\mathcal{M}G\mathcal{F}(t_1,t_2)HJ_1).
\end{aligned}
$$

Hence, if $\Pi_{11} < 0$, $\Pi_{12} < 0$, $\Pi_{21} < 0$, and $\Pi_{22} < 0$, are satisfied, then $\dot{V}_u(t_1,t_2) < 0$, which ensures the robust asymptotical stability of system (6). Then, applying Lemma 2.7, if there exists positive scalars ε_1, ε_2, ε_3 and ε_4, the following inequalities hold:

$$
\begin{aligned}
&\mathcal{W} + sym(\mathcal{U}_{11} + J_1^T\mathcal{M}\mathcal{A}J_1) + \varepsilon_1^{-1}J_1^T\mathcal{M}GG^T\mathcal{M}^TJ_1 + \varepsilon_1 J_1^T H^T H J_1 < 0, \qquad (21a) \\
&\mathcal{W} + sym(\mathcal{U}_{12} + J_1^T\mathcal{M}\mathcal{A}J_1) + \varepsilon_2^{-1}J_1^T\mathcal{M}GG^T\mathcal{M}^TJ_1 + \varepsilon_2 J_1^T H^T H J_1 < 0, \qquad (21b) \\
&\mathcal{W} + sym(\mathcal{U}_{21} + J_1^T\mathcal{M}\mathcal{A}J_1) + \varepsilon_3^{-1}J_1^T\mathcal{M}GG^T\mathcal{M}^TJ_1 + \varepsilon_3 J_1^T H^T H J_1 < 0, \qquad (21c) \\
&\mathcal{W} + sym(\mathcal{U}_{22} + J_1^T\mathcal{M}\mathcal{A}J_1) + \varepsilon_4^{-1}J_1^T\mathcal{M}GG^T\mathcal{M}^TJ_1 + \varepsilon_4 J_1^T H^T H J_1 < 0. \qquad (21d)
\end{aligned}
$$

By using lemma 2.8, the inequalities in (15a),(15b),(15c) and (15d) are equivalent to conditions (21a),(21b),(21c) and (21d), respectively. This completes the proof. □

In the absence of uncertainties, Theorem 3.1 reduces to the following corollary.

Corollary 3.2. *The 2-D continuous system* (6) *without parameter uncertainties* (2)-(3)*, time varying delays* (4)*, and boundary conditions* (5) *is asymptotically stable if there exist symmetric positive-definite matrices* P^h, P^v, Q_i^h, Q_i^v, R_j^h, R_j^v, Z_k^h, Z_k^v *and appropriately dimensioned matrices* M_i^h, M_i^v, S^h, S^v, $(i = 1,2,3)$, $(j = 1,2)$,

$(k = 1, ..., 4)$, *such that the LMIs* (15e), (15f), (22a), (22b),(22c) *and* (22d) *hold.*

$$\mathcal{W} + sym(\mathcal{U}_{11} + J_1^T \mathcal{M}\mathcal{A}J_1) < 0, \tag{22a}$$

$$\mathcal{W} + sym(\mathcal{U}_{12} + J_1^T \mathcal{M}\mathcal{A}J_1) < 0, \tag{22b}$$

$$\mathcal{W} + sym(\mathcal{U}_{21} + J_1^T \mathcal{M}\mathcal{A}J_1) < 0, \tag{22c}$$

$$\mathcal{W} + sym(\mathcal{U}_{22} + J_1^T \mathcal{M}\mathcal{A}J_1) < 0, \tag{22d}$$

Remark 3. The number of decision variables involved in Corollary 3.2 in this paper and Theorem 1 in Ghous & Xiang (2016a) are $12n_h^2 + 12n_v^2 + 6n_h + 6n_v$ and $11.5n_h^2 + 11.5n_v^2 + 2.5n_h + 2.5n_v$, respectively.

Remark 4. It is well-known that the choice of the Lyapunov-Krasovskii functional play an important role in reducing the conservativeness of the stability criteria. In the present paper an augmented Lyapunov functional including some integral terms has been used, which leads to exploit more information on the sizes of delays, in order to develop a stability condition that does not create significant conservativeness in the results. In addition, compared with the existing results, the Lyapunov-Krasovskii functional used in this paper contains some additional triple-integral terms, which plays an important role in the reduction of conservativeness. To the best of the authors knowledge, it is the first time this Lyapunov functional is employed to solve the problem of robust delay dependent stability for 2-D continuous systems with interval delays.

Remark 5. It is well known that the conservatism of the delay-dependent stability criteria depends on not only the choice of Lyapunov-Krasovskii functional but also the estimation of the integral terms appearing in the derivative of some Lyapunov-Krasovskii functional. Different from the free-weighting matrices technique employed in Ghous & Xiang (2016a), this paper uses the Wirtinger inequality to estimate the derivative of Lyapunov-Krasovskii functional. As a result, extra cross terms such as:

$$3\frac{h_{12}}{h_2 - h(t_1)}\xi^T(t_1,t_2)\{(e_5 + e_7 - 2e_{15})^T R_2^h (e_5 + e_7 - 2e_{15})\}\xi(t_1,t_2)$$

were used in the delay-dependent stability condition, which are effective in the reduction of conservatism.

4. H_∞ Performance Analysis

Theorem 4.1. *For given scalars* $0 \le h_1 \le h_2$, $0 \le d_1 \le d_2$, μ_1, μ_2 *and* γ, *system*(1) *with* $u(t_1, t_2) = 0$ *is robustly asymptotically stable with a prescribed* H_∞ *performance* γ *if there exist symmetric positive-definite matrices* P^h, P^v, Q_i^h, Q_i^v, R_j^h, R_j^v, Z_k^h, Z_k^v, *appropriately dimensioned matrices* M_i^h, M_i^v, S^h, S^v *and positive scalars* ε_k, $(i = 1, 2, 3)$, $(j = 1, 2)$, $(k = 1, ..., 4)$, *such that the LMIs* (15e), (15f), (23a), (23b),(23c) *and* (23d)*, hold.*

$$\Upsilon_{11w} = \begin{bmatrix} \mathcal{W}_w + sym(\mathcal{U}_{11w} + J_w^T \mathcal{M}_w \mathcal{A}_w J_w) - \gamma^2 f_w^T f_w & f_z^T & J_w^T \mathcal{M}_w G & \varepsilon_1 J_w^T H_w^T \\ * & -I & 0 & 0 \\ * & * & -\varepsilon_1 I & 0 \\ * & * & * & -\varepsilon_1 I \end{bmatrix} < 0, \tag{23a}$$

$$\Upsilon_{12w} = \begin{bmatrix} \mathcal{W}_w + sym(\mathcal{U}_{12w} + J_w^T \mathcal{M}_w \mathcal{A}_w J_w) - \gamma^2 f_w^T f_w & f_z^T & J_w^T \mathcal{M}_w G & \varepsilon_2 J_w^T H_w^T \\ * & -I & 0 & 0 \\ * & * & -\varepsilon_2 I & 0 \\ * & * & * & -\varepsilon_2 I \end{bmatrix} < 0, \tag{23b}$$

$$\Upsilon_{21w} = \begin{bmatrix} \mathcal{W}_w + sym(\mathcal{U}_{21w} + J_w^T \mathcal{M}_w \mathcal{A}_w J_w) - \gamma^2 f_w^T f_w & f_z^T & J_w^T \mathcal{M}_w G & \varepsilon_3 J_w^T H_w^T \\ * & -I & 0 & 0 \\ * & * & -\varepsilon_3 I & 0 \\ * & * & * & -\varepsilon_3 I \end{bmatrix} < 0, \tag{23c}$$

$$\Upsilon_{22w} = \begin{bmatrix} \mathcal{W}_w + sym(\mathcal{U}_{22w} + J_w^T \mathcal{M}_w \mathcal{A}_w J_w) - \gamma^2 f_w^T f_w & f_z^T & J_w^T \mathcal{M}_w G & \varepsilon_4 J_w^T H_w^T \\ * & -I & 0 & 0 \\ * & * & -\varepsilon_4 I & 0 \\ * & * & * & -\varepsilon_4 I \end{bmatrix} < 0, \tag{23d}$$

where

$$\begin{aligned} \mathcal{W}_w &= \Sigma_w + \Lambda_w^{hT} \Phi^h \Lambda_w^h + \Lambda_w^{vT} \Phi^v \Lambda_w^v, \\ \mathcal{U}_{11w} &= \mathcal{G}_w^{hT} P^h \mathcal{D}_{1w}^h + \mathcal{G}_w^{vT} P^v \mathcal{D}_{1w}^v, \\ \mathcal{U}_{12w} &= \mathcal{G}_w^{hT} P^h \mathcal{D}_{1w}^h + \mathcal{G}_w^{vT} P^v \mathcal{D}_{2w}^v, \\ \mathcal{U}_{21w} &= \mathcal{G}_w^{hT} P^h \mathcal{D}_{2w}^h + \mathcal{G}_w^{vT} P^v \mathcal{D}_{1w}^v, \\ \mathcal{U}_{22w} &= \mathcal{G}_w^{hT} P^h \mathcal{D}_{2w}^h + \mathcal{G}_w^{vT} P^v \mathcal{D}_{2w}^v, \end{aligned}$$

with

$$\begin{aligned} \mathcal{G}_w^h &= \begin{bmatrix} f_9 \\ f_1 - f_3 \\ f_3 - f_7 \end{bmatrix}, \mathcal{D}_{1w}^h = \begin{bmatrix} f_1 \\ h_1 f_{11} \\ h_{12} f_{13} \end{bmatrix}, \mathcal{D}_{2w}^h = \begin{bmatrix} f_1 \\ h_1 f_{11} \\ h_{12} f_{15} \end{bmatrix}, \mathcal{G}_w^v = \begin{bmatrix} f_{10} \\ f_2 - f_4 \\ f_4 - f_6 \end{bmatrix}, \\ \mathcal{D}_{1w}^v &= \begin{bmatrix} f_2 \\ d_1 f_{12} \\ d_{12} f_{14} \end{bmatrix}, \mathcal{D}_2^v = \begin{bmatrix} f_2 \\ d_1 f_{12} \\ d_{12} f_{16} \end{bmatrix}, \\ J_w &= \begin{bmatrix} f_1 \\ f_2 \\ f_5 \\ f_6 \\ f_9 \\ f_{10} \\ f_w \end{bmatrix}, \mathcal{M}_w = \begin{bmatrix} M_1^h & 0 \\ 0 & M_1^v \\ M_2^h & 0 \\ 0 & M_2^v \\ M_3^h & 0 \\ 0 & M_3^v \\ 0 & 0 \end{bmatrix}, \mathcal{A} = \begin{bmatrix} A_{11}^T & A_{21}^T \\ A_{12}^T & A_{22}^T \\ A_{d11}^T & A_{d21}^T \\ A_{d12}^T & A_{d22}^T \\ -I_{n_h} & 0 \\ 0 & -I_{n_v} \\ B_1^T & B_2^T \end{bmatrix}^T, H_w = \begin{bmatrix} H_{11}^T \\ H_{12}^T \\ H_{21}^T \\ H_{22}^T \\ 0 \\ 0 \\ H_3^T \end{bmatrix}^T, \end{aligned}$$

$$
\begin{aligned}
\Lambda_w^h &= \begin{bmatrix} f_3-f_5 \\ f_3+f_5-2f_{13} \\ f_5-f_7 \\ f_5+f_7-2f_{15} \end{bmatrix}, \Lambda_w^v = \begin{bmatrix} f_4-f_6 \\ f_4+f_6-2f_{14} \\ f_6-f_8 \\ f_6+f_8-2f_{16} \end{bmatrix}, f_w = \begin{bmatrix} 0_{n_w,8n} & I_{n_w} \end{bmatrix}, \\
f_z &= C_1f_1+C_2f_2+C_{d1}f_5+C_{d2}f_6+Df_w, \\
\Sigma_w &= f_1^T(Q_1^h+Q_2^h+Q_3^h)f_1 - f_3^TQ_1^hf_3 - f_7^TQ_2^hf_7 - (1-\mu_1)f_5^TQ_3^hf_5 \\
&\quad + f_2^T(Q_1^v+Q_2^v+Q_3^v)f_2 - f_4^TQ_1^vf_4 - f_8^TQ_2^vf_8 - (1-\mu_2)f_6^TQ_3^vf_6 \\
&\quad + h_1^2f_9^TR_1^hf_9 - (f_1-f_3)^TR_1^h(f_1-f_3) - 3(f_1+f_3-2f_{11})^TR_1^h(f_1+f_3-2f_{11}) \\
&\quad + d_1^2f_{10}^TR_1^vf_{10} - (f_2-f_4)^TR_1^v(f_2-f_4) \\
&\quad - 3(f_2+f_4-2f_{12})^TR_1^v(f_2+f_4-2f_{12}) + h_{12}^2f_9^TR_2^hf_9 \\
&\quad + d_{12}^2f_{10}^TR_2^vf_{10} + \frac{h_1^2}{2}f_9^TZ_1^hf_9 - 2(f_1-f_{11})^TZ_1^h(f_1-f_{11}) \\
&\quad + \frac{d_1^2}{2}f_{10}^TZ_1^vf_{10} - 2(f_2-f_{12})^TZ_1^v(f_2-f_{12}) + \frac{h_1^2}{2}f_9^TZ_2^hf_9 \\
&\quad - 2(f_3-f_{11})^TZ_1^h(f_3-f_{11}) + \frac{d_1^2}{2}f_{10}^TZ_2^vf_{10} - 2(f_4-f_{12})^TZ_2^v(f_4-f_{12}) \\
&\quad + \frac{h_{12}^2}{2}f_9^TZ_3^hf_9 - 2(f_3-f_{13})^TZ_3^h(f_3-f_{13}) - 2(f_5-f_{15})^TZ_3^h(f_5-f_{15}) \\
&\quad + \frac{d_{12}^2}{2}f_{10}^TZ_3^vf_{10} - 2(f_4-f_{14})^TZ_3^v(f_4-f_{14}) - 2(f_6-f_{16})^TZ_3^v(f_6-f_{16}) \\
&\quad + \frac{h_{12}^2}{2}f_9^TZ_4^hf_9 - 2(f_5-f_{13})^TZ_4^h(f_5-f_{13}) - 2(f_7-f_{15})^TZ_4^h(f_7-f_{15}) \\
&\quad + \frac{d_{12}^2}{2}f_{10}^TZ_4^vf_{10} - 2(f_6-f_{14})^TZ_4^v(f_6-f_{14}) - 2(f_8-f_{16})^TZ_4^v(f_8-f_{16}),
\end{aligned}
$$

and the matrices $f_m (m = 1, 2, ..., 16)$ are defined by

$$
f_m = \begin{cases} \begin{bmatrix} e_m & 0_{n_h,n_w} \end{bmatrix}, & \textit{if } m \textit{ is odd;} \\ \begin{bmatrix} e_m & 0_{n_v,n_w} \end{bmatrix}, & \textit{if } m \textit{ is even;} \end{cases}
$$

Proof. According to (1) with $u(t_1,t_2) = 0$, $w(t_1,t_2) \in \mathcal{L}_2\{[0,\infty),[0,\infty)\}$, and similar to equality (20) we obtain:

$$
\begin{aligned}
0 &= 2\left\{ \begin{bmatrix} x^h(t_1,t_2) \\ x^v(t_1,t_2) \end{bmatrix}^T \begin{bmatrix} M_1^h & 0 \\ 0 & M_1^v \end{bmatrix} + \begin{bmatrix} x^h(t_1-h(t_1),t_2) \\ x^v(t_1,t_2-d(t_2)) \end{bmatrix}^T \begin{bmatrix} M_2^h & 0 \\ 0 & M_2^v \end{bmatrix} \right. \\
&\quad \left. + \begin{bmatrix} \frac{\partial x^h(t_1,t_2)}{\partial t_1} \\ \frac{\partial x^v(t_1,t_2)}{\partial t_2} \end{bmatrix}^T \begin{bmatrix} M_3^h & 0 \\ 0 & M_3^v \end{bmatrix} \right\} \\
&\quad \times \left\{ \hat{A} \begin{bmatrix} x^h(t_1,t_2) \\ x^v(t_1,t_2) \end{bmatrix} + \hat{A}_d \begin{bmatrix} x^h(t_1-h(t_1),t_2) \\ x^v(t_1,t_2-d(t_2)) \end{bmatrix} + \hat{B}w(t_1,t_2) - \begin{bmatrix} \frac{\partial x^h(t_1,t_2)}{\partial t_1} \\ \frac{\partial x^v(t_1,t_2)}{\partial t_2} \end{bmatrix} \right\} \\
&= \xi_w^T(t_1,t_2)\{sym(J_w^T\mathcal{M}_w(\mathcal{A}+G\mathcal{F}(t_1,t_2)H_w)J_w)\}\xi_w(t_1,t_2), \qquad (24)
\end{aligned}
$$

where $\xi_w(t_1,t_2) = \begin{bmatrix} \xi^T(t_1,t_2) & w^T(t_1,t_2) \end{bmatrix}^T$.

In addition, defining

$$\mathcal{J} = \int_0^\infty \int_0^\infty \{z^T(t_1,t_2)z(t_1,t_2) - w^T(t_1,t_2)w(t_1,t_2)\}dt_1dt_2. \quad (25)$$

By considering the Lyapunov-Krasovskii functionals in (16), and assuming the zero boundary condition,

$$\begin{aligned}
\mathcal{J} &\leq \int_0^\infty \int_0^\infty \{\dot{V}_u(t_1,t_2) + z^T(t_1,t_2)z(t_1,t_2) - w^T(t_1,t_2)w(t_1,t_2)\}dt_1dt_2, \\
&= \int_0^\infty \int_0^\infty \xi_w^T(t_1,t_2)\{\mathcal{W}_w + sym(\mathcal{U}_w + J_w^T\mathcal{M}_w\mathcal{A}_wJ_w + J_w^TGF(t_1,t_2)H_wJ_w) \\
&\quad + f_z^Tf_z - \gamma^2 f_w^Tf_w\}\xi_w(t_1,t_2)dt_1dt_2, \\
&= \int_0^\infty \int_0^\infty \xi_w^T(t_1,t_2)\{\lambda^h\lambda^v\Pi_{11w} + (1-\lambda^h)\lambda^v\Pi_{12w} + \lambda^h(1-\lambda^v)\Pi_{21w} \\
&\quad + (1-\lambda^h)(1-\lambda^v)\Pi_{22w}\}\xi_w(t_1,t_2)dt_1dt_2,
\end{aligned}$$

where

$$\begin{aligned}
\Pi_{11w} &= \mathcal{W}_w + sym(\mathcal{U}_{11w} + J_w^T\mathcal{M}_w\mathcal{A}_wJ_w + J_w^T\mathcal{M}_wG\mathcal{F}(t_1,t_2)H_wJ_w) \\
&\quad + f_z^Tf_z - \gamma^2 f_w^Tf_w, \\
\Pi_{12w} &= \mathcal{W}_w + sym(\mathcal{U}_{12w} + J_w^T\mathcal{M}_w\mathcal{A}_wJ_w + J_w^T\mathcal{M}_wG\mathcal{F}(t_1,t_2)H_wJ_w) \\
&\quad + f_z^Tf_z - \gamma^2 f_w^Tf_w, \\
\Pi_{21w} &= \mathcal{W}_w + sym(\mathcal{U}_{21w} + J_w^T\mathcal{M}_w\mathcal{A}_wJ_w + J_w^T\mathcal{M}_wG\mathcal{F}(t_1,t_2)H_wJ_w) \\
&\quad + f_z^Tf_z - \gamma^2 f_w^Tf_w, \\
\Pi_{22w} &= \mathcal{W}_w + sym(\mathcal{U}_{22w} + J_w^T\mathcal{M}_w\mathcal{A}_wJ_w + J_w^T\mathcal{M}_wG\mathcal{F}(t_1,t_2)H_wJ_w) \\
&\quad + f_z^Tf_z - \gamma^2 f_w^Tf_w.
\end{aligned}$$

if $\Pi_{11w} < 0$, $\Pi_{12w} < 0$, $\Pi_{21w} < 0$, and $\Pi_{22w} < 0$, we obtain $\mathcal{J} < 0$, which implies:

$$||z||_2^2 < \gamma^2 ||w||_2^2.$$

Then, applying Lemma 2.7, if there exists positive scalars ε_1, ε_2, ε_3 and ε_4, the following inequalities hold

$$\mathcal{W}_w + sym(\mathcal{U}_{11w} + J_w^T\mathcal{M}_w\mathcal{A}_wJ_w) + \varepsilon_1^{-1}J_w^T\mathcal{M}_wGG^T\mathcal{M}_w^TJ_w + \varepsilon_1J_w^TH_w^TH_wJ_w + f_z^Tf_z - \gamma^2 f_w^Tf_w < 0, \quad (26a)$$

$$\mathcal{W}_w + sym(\mathcal{U}_{12w} + J_w^T\mathcal{M}_w\mathcal{A}_wJ_w) + \varepsilon_2^{-1}J_w^T\mathcal{M}_wGG^T\mathcal{M}_w^TJ_w + \varepsilon_2J_w^TH_w^TH_wJ_w + f_z^Tf_z - \gamma^2 f_w^Tf_w < 0, \quad (26b)$$

$$\mathcal{W}_w + sym(\mathcal{U}_{21w} + J_w^T\mathcal{M}_w\mathcal{A}_wJ_w) + \varepsilon_3^{-1}J_w^T\mathcal{M}_wGG^T\mathcal{M}_w^TJ_w + \varepsilon_3J_w^TH_w^TH_wJ_w + f_z^Tf_z - \gamma^2 f_w^Tf_w < 0, \quad (26c)$$

$$\mathcal{W}_w + sym(\mathcal{U}_{22w} + J_w^T\mathcal{M}_w\mathcal{A}_wJ_w) + \varepsilon_4^{-1}J_w^T\mathcal{M}_wGG^T\mathcal{M}_w^TJ_w + \varepsilon_4J_w^TH_w^TH_wJ_w + f_z^Tf_z - \gamma^2 f_w^Tf_w < 0, \quad (26d)$$

By using Lemma 2.8, LMIs (23a), (23b), (23c) and (23d), are equivalent to those in (26a), (26b), (26c) and (26d) respectively. This completes the proof. □

5. Robust H_∞ Controller Design

Theorem 5.1. *For some given scalars $h_1 \leq h_2 \leq 0$, $d_1 \leq d_2 \leq 0$ and μ_1, μ_2, the closed-loop system* (13) *is robustly asymptotically stable with a prescribed H_∞ performance γ if there exist symmetric positive-definite matrices $\bar{P}^h$, $\bar{P}^v$, $\bar{Q}^h_i$, $\bar{Q}^v_i$, $\bar{R}^h_j$, $\bar{R}^v_j$, $\bar{Z}^h_k$, $\bar{Z}^v_k$, appropriately dimensioned matrices W^h_i, W^v_i, $\bar{S}^h$, $\bar{S}^v$ Y_1 Y_2 and positive scalars η_k, $(i = 1, 2, 3)$, $(j = 1, 2)$, $(k = 1, ..., 4)$, such that the following LMIs hold:*

$$\begin{bmatrix} \bar{\mathcal{W}}_w + sym(\bar{\mathcal{U}}_{11w} + J_w^T \bar{\mathcal{M}}_w \bar{\mathcal{A}}_w J_w) - \gamma^2 f_w^T f_w & \bar{f}_z^T & J_w^T \bar{\mathcal{M}}_w G & \varepsilon_1 J_w^T \bar{H}_w^T \\ * & -I & 0 & 0 \\ * & * & -\eta_1 I & 0 \\ * & * & * & -\eta_1 I \end{bmatrix} < 0, \quad (27a)$$

$$\begin{bmatrix} \bar{\mathcal{W}}_w + sym(\bar{\mathcal{U}}_{12w} + J_w^T \bar{\mathcal{M}}_w \bar{\mathcal{A}}_w J_w) - \gamma^2 f_w^T f_w & \bar{f}_z^T & J_w^T \bar{\mathcal{M}}_w G & \varepsilon_2 J_w^T \bar{H}_w^T \\ * & -I & 0 & 0 \\ * & * & -\eta_2 I & 0 \\ * & * & * & -\eta_2 I \end{bmatrix} < 0, \quad (27b)$$

$$\begin{bmatrix} \bar{\mathcal{W}}_w + sym(\bar{\mathcal{U}}_{21w} + J_w^T \bar{\mathcal{M}}_w \bar{\mathcal{A}}_w J_w) - \gamma^2 f_w^T f_w & \bar{f}_z^T & J_w^T \bar{\mathcal{M}}_w G & \varepsilon_3 J_w^T H_w^T \\ * & -I & 0 & 0 \\ * & * & -\eta_3 I & 0 \\ * & * & * & -\eta_3 I \end{bmatrix} < 0, \quad (27c)$$

$$\begin{bmatrix} \bar{\mathcal{W}}_w + sym(\bar{\mathcal{U}}_{22w} + J_w^T \bar{\mathcal{M}}_w \bar{\mathcal{A}}_w J_w) - \gamma^2 f_w^T f_w & \bar{f}_z^T & J_w^T \bar{\mathcal{M}}_w G & \varepsilon_4 J_w^T \bar{H}_w^T \\ * & -I & 0 & 0 \\ * & * & -\eta_4 I & 0 \\ * & * & * & -\eta_4 I \end{bmatrix} < 0, \quad (27d)$$

$$\begin{bmatrix} diag\{(\bar{R}^h_2 + \bar{Z}^h_3), 3(\bar{R}^h_2 + \bar{Z}^h_3)\} & \bar{S}^h \\ * & diag\{(\bar{R}^h_2 + \bar{Z}^h_4), 3(\bar{R}^h_2 + \bar{Z}^h_4)\} \end{bmatrix} > 0, \quad (27e)$$

$$\begin{bmatrix} diag\{(\bar{R}^v_2 + \bar{Z}^v_3), 3(\bar{R}^v_2 + \bar{Z}^v_3)\} & \bar{S}^v \\ * & diag\{(\bar{R}^v_2 + \bar{Z}^v_4), 3(\bar{R}^v_2 + \bar{Z}^v_4)\} \end{bmatrix} > 0, \quad (27f)$$

where

$$\begin{aligned} \bar{\mathcal{W}}_w &= \bar{\Sigma}_w + \Lambda_w^{hT} \bar{\Phi}^h \Lambda_w^h + \Lambda_w^{vT} \bar{\Phi}^v \Lambda_w^v, \\ \bar{\mathcal{U}}_{11w} &= \mathcal{G}_w^{hT} \bar{P}^h \mathcal{D}_{1w}^h + \mathcal{G}_w^{vT} \bar{P}^v \mathcal{D}_{1w}^v, \\ \bar{\mathcal{U}}_{12w} &= \mathcal{G}_w^{hT} \bar{P}^h \mathcal{D}_{1w}^h + \mathcal{G}_w^{vT} \bar{P}^v \mathcal{D}_{2w}^v, \\ \bar{\mathcal{U}}_{21w} &= \mathcal{G}_w^{hT} \bar{P}^h \mathcal{D}_{2w}^h + \mathcal{G}_w^{vT} \bar{P}^v \mathcal{D}_{1w}^v, \\ \bar{\mathcal{U}}_{22w} &= \mathcal{G}_w^{hT} \bar{P}^h \mathcal{D}_{2w}^h + \mathcal{G}_w^{vT} \bar{P}^v \mathcal{D}_{2w}^v, \end{aligned}$$

with

$$
\begin{aligned}
\bar{\mathcal{M}}_w &= \begin{bmatrix} I_{n_h} & 0 \\ 0 & I_{n_v} \\ I_{n_h} & 0 \\ 0 & I_{n_v} \\ I_{n_h} & 0 \\ 0 & I_{n_v} \\ 0 & 0 \end{bmatrix}, \bar{\mathcal{A}}_w = \begin{bmatrix} W^h A_{11}^T + Y_1^T E_1^T & W^h A_{21}^T + Y_1^T E_2^T \\ W^v A_{12}^T + Y_2^T E_1^T & W^v A_{22}^T + Y_2^T E_2^T \\ W^h A_{d11}^T & W^h A_{d21}^T \\ W^v A_{d12}^T & W^v A_{d22}^T \\ -W^h & 0 \\ 0 & -W^v \\ B_1^T & B_2^T \end{bmatrix}^T, \\
\bar{H}_w &= \begin{bmatrix} H_{11}W^h & H_{12}W^v & H_{21}W^h & H_{22}W^v & 0 & 0 & H_3 \end{bmatrix}, \\
\bar{\Phi}^h &= \begin{bmatrix} diag\{\bar{R}_2^h, 3\bar{R}_2^h\} & \bar{S}^h \\ * & diag\{\bar{R}_2^h, 3\bar{R}_2^h\} \end{bmatrix}, \bar{\Phi}^h = \begin{bmatrix} diag\{\bar{R}_2^v, 3\bar{R}_2^v\} & \bar{S}^v \\ * & diag\{\bar{R}_2^v, 3\bar{R}_2^v\} \end{bmatrix}, \\
\bar{f}_z &= (C_1 W^{hT} + FY_1)f_1 + (C_2 W^{vT} + FY_2)f_2 + C_{d1}W^{hT}f_5 + C_{d2}W^{vT}f_6 + Df_w, \\
\bar{\Sigma}_w &= f_1^T(\bar{Q}_1^h + \bar{Q}_2^h + \bar{Q}_3^h)f_1 - f_3^T\bar{Q}_1^h f_3 - f_7^T\bar{Q}_2^h f_7 - (1-\mu_1)f_5^T\bar{Q}_3^h f_5 \\
&+ f_2^T(\bar{Q}_1^v + \bar{Q}_2^v + \bar{Q}_3^v)f_2 - f_4^T\bar{Q}_1^v f_4 - f_8^T\bar{Q}_2^v f_8 - (1-\mu_2)f_6^T\bar{Q}_3^v f_6 \\
&+ h_1^2 f_9^T\bar{R}_1^h f_9 - (f_1 - f_3)^T\bar{R}_1^h(f_1 - f_3) - 3(f_1 + f_3 - 2f_{11})^T\bar{R}_1^h(f_1 + f_3 - 2f_{11}) \\
&+ d_1^2 f_{10}^T\bar{R}_1^v f_{10} - (f_2 - f_4)^T\bar{R}_1^v(f_2 - f_4) \\
&- 3(f_2 + f_4 - 2f_{12})^T\bar{R}_1^v(f_2 + f_4 - 2f_{12}) + h_{12}^2 f_9^T\bar{R}_2^h f_9 \\
&+ d_{12}^2 f_{10}^T\bar{R}_2^v f_{10} + \frac{h_1^2}{2}f_9^T Z_1^h f_9 - 2(f_1 - f_{11})^T\bar{Z}_1^h(f_1 - f_{11}) \\
&+ \frac{d_1^2}{2}f_{10}^T\bar{Z}_1^v f_{10} - 2(f_2 - f_{12})^T\bar{Z}_1^v(f_2 - f_{12}) + \frac{h_1^2}{2}f_9^T\bar{Z}_2^h f_9 \\
&- 2(f_3 - f_{11})^T\bar{Z}_1^h(f_3 - f_{11}) + \frac{d_1^2}{2}f_{10}^T\bar{Z}_2^v f_{10} - 2(f_4 - f_{12})^T\bar{Z}_2^v(f_4 - f_{12}) \\
&+ \frac{h_{12}^2}{2}f_9^T\bar{Z}_3^h f_9 - 2(f_3 - f_{13})^T Z_3^h(f_3 - f_{13}) - 2(f_5 - f_{15})^T\bar{Z}_3^h(f_5 - f_{15}) \\
&+ \frac{d_{12}^2}{2}f_{10}^T\bar{Z}_3^v f_{10} - 2(f_4 - f_{14})^T\bar{Z}_3^v(f_4 - f_{14}) - 2(f_6 - f_{16})^T\bar{Z}_3^v(f_6 - f_{16}) \\
&+ \frac{h_{12}^2}{2}f_9^T\bar{Z}_4^h f_9 - 2(f_5 - f_{13})^T\bar{Z}_4^h(f_5 - f_{13}) - 2(f_7 - f_{15})^T\bar{Z}_4^h(f_7 - f_{15}) \\
&+ \frac{d_{12}^2}{2}f_{10}^T\bar{Z}_4^v f_{10} - 2(f_6 - f_{14})^T\bar{Z}_4^v(f_6 - f_{14}) - 2(f_8 - f_{16})^T\bar{Z}_4^v(f_8 - f_{16}),
\end{aligned}
$$

Moreover, the stabilizing feedback controller gains are given: $K_1 = Y_1(W^h)^{-T}$*, and* $K_2 = Y_2(W^v)^{-T}$*.*

Proof. Replace A_{11}, A_{12}, A_{21}, A_{22}, C_1 and C_2 in (23a), (23b), (23c) and (23d) with $A_{11}+E_1K_1$, $A_{12}+E_1K_2$, $A_{21}+E_2K_1$, $A_{22}+E_2K_2$, C_1+FK_1 and C_2+FK_2 respectively, and setting $M_1^h = M_2^h = M_3^h = M^h$ and $M_1^v = M_2^v = M_v^h = M^v$.

In addition, define

$$
\begin{aligned}
L &= diag = \{M^{h-1}, M^{v-1}, M^{h-1}, M^{v-1}, ..., M^{h-1}, M^{v-1}\} \in \mathbb{R}^{8n\times 8n}, \\
L_q &= diag\{L, I_{n_w}, I_{n_z}, \varepsilon_k^{-1}I_n, \varepsilon_k^{-1}I_n\}, \qquad k = \{1,2,3,4\}, \\
L_5 &= diag = \{M^{h-1}, M^{h-1}, M^{h-1}, M^{h-1}\}, \\
L_6 &= diag = \{M^{v-1}, M^{v-1}, M^{h-1}, M^{v-1}\},
\end{aligned}
$$

And set

$$
\begin{array}{l}
\bar{P}^h = L_5 P^h L_5^T; \quad \bar{P}^v = L_6 P^v L_6^T; \quad \bar{Q}_i^h = M^{h-1} Q_i^h M^{h-T}; \quad \bar{Q}_i^v = M^{v-1} Q_i^v M^{v-T};\\
\bar{R}_j^h = M^{h-1} R_j^h M^{h-T}; \quad \bar{R}_j^v = M^{v-1} R_j^v M^{v-T}; \quad \bar{Z}_k^h = M^{h-1} Z_k^h M^{h-T}; \quad \eta_k = \varepsilon_k^{-1};\\
\bar{Z}_k^v = M^{v-1} Z_k^v M^{v-T}; \quad \bar{S}^h = diag\{M^{h-1}, M^{h-1}\} S^h diag\{M^{h-T}, M^{h-T}\};\\
\bar{S}^v = diag\{M^{v-1}, M^{v-1}\} S^v diag\{M^{v-T}, M^{v-T}\}; \quad W_h = M^{h-1}; \quad W_v = M^{v-1};\\
Y_1 = K_1 M^{h-T}; \quad Y_2 = K_2 M^{v-T}; \quad (i = 1, 2, 3), \quad (j = 1, 2), \quad (k = 1, ..., 4).
\end{array}
$$

Then inequalities (28a)-(28f) are equivalent to LMIs (27a)-(27f) respectively.

$$
L_1^T \Upsilon_{11w} L_1 < 0; \tag{28a}
$$

$$
L_2^T \Upsilon_{12w} L_2 < 0; \tag{28b}
$$

$$
L_3^T \Upsilon_{21w} L_3 < 0; \tag{28c}
$$

$$
L_4^T \Upsilon_{22w} L_4 < 0; \tag{28d}
$$

$$
L_5^T \Psi_1 L_5 < 0; \tag{28e}
$$

$$
L_6^T \Psi_2 L_6 < 0; \tag{28f}
$$

This completes the proof. □

Remark 6. Recently, the Wirtinger inequality has been applied to develop less conservative delay-dependent stability conditions for one-dimensional systems (Park et al., 2015; Seuret & Gouaisbaut, 2013); however, most of existing results have focused only on stability analysis, not considering the controller design problem. The main reason for this, is that the Wirtinger inequality involves the introduction of an augmented Lyapunov-Krasovskii functional, which makes the controller design task complex. In this paper, we have solved the problem of robust H_∞ controller design for uncertain 2-D continuous systems with interval time-varying delays, by the use of some free matrices in (24), which facilitated the design.

6. Numerical examples

Example 6.1. Consider the well-known dynamical system (involved in gas absorption water stream heating and air drying) described by the following Darboux equation with time delays, which is used in Ghous & Xiang (2016a):

$$
\frac{\partial^2 q(x,t)}{\partial x \partial t} = a_1 \frac{\partial q(x,t)}{\partial t} + a_2 \frac{\partial q(x,t)}{\partial t} + a_0 q(x,t) + a_3 q(x, t - d(t)) + bu(x,t), \tag{29}
$$

where $q(x,t)$ is unknown function at $x(space) \in [0, x_f]$ and $t(time) \in [0, \infty)$, a_0, a_1, a_2, a_3 and b are real coefficients, h_2 i is a varying delay and $u(x,t)$ is the input function. Let us define

$$
x^h(x,t) = \frac{\partial q(x,t)}{\partial t} - a_2 q(x,t), \qquad x^v(x,t) = q(x,t).
$$

Table 1. Calculated upper delay bound d_2 for different d_1 and $\mu_2 = 0.3$.

Method	$d_1 = 0$	$d_1 = 0.5$	$d_1 = 1$
(El-Kasri et al., 2013)	2.1843	—	—
(Ghous & Xiang, 2016a)	3.9829	—	—
Corollary 3.2	4.1685	4.2949	4.3945

It is easy to verify that equation (29) can be converted into the model (6) with

$$A = \begin{bmatrix} a_1 & a_0 + a_1 a_2 \\ 1 & a_2 \end{bmatrix}, \qquad A_d = \begin{bmatrix} 0 & a_3 \\ 0 & 0 \end{bmatrix}$$

To carry out a numerical study the following parameters are also fixed: $a_0 = 0.2$, $a_1 = -3$, $a_2 = -1$, $a_3 = -0.4$, $b = 0$.

The stability of this system cannot be solved by the delay-independent methods in Benzaouia et al. (2011a); Hmamed et al. (2013). However, solving the LMIs developed in El-Kasri et al. (2013); Ghous & Xiang (2016a) and those in Corollary 3.2 yields the upper bounds on d_2 that ensure stability of system (29) for $\mu = 0.3$ and various d_1 in Table 1. It can be seen clearly that our results provides larger delay bound than the previous results of other studies when $d_1 = 0$. In addition, the stability conditions provided by (El-Kasri et al., 2013; Ghous & Xiang, 2016a) cannot deal with the case when $d_1 \neq 0$.

Remark 7. One of essential concerns of delay-dependent stability conditions, is to obtain a maximum allowable upper bound of delay as large as possible such that the system can remain stable. Thus, the obtained maximum allowable upper bound can be considered as a significant index to evaluate the conservatism of the delay dependent stability criterion. According to Table 1, we can conclude that the stability criterion presented in this paper is less conservative for this example than that in (El-Kasri et al., 2013; Ghous & Xiang, 2016a).

Example 6.2. Consider a 2-D system (1) with the parameters that follows:

$$\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} = \begin{bmatrix} 0.1 & 0.1 \\ 0.2 & 0.1 \end{bmatrix}, \quad \begin{bmatrix} A_{d11} & A_{d12} \\ A_{d21} & A_{d22} \end{bmatrix} = \begin{bmatrix} -0.1 & -1 \\ 0 & -0.9 \end{bmatrix},$$
$$\begin{bmatrix} B_1 \\ B_2 \end{bmatrix} = \begin{bmatrix} 0.1 \\ 0.3 \end{bmatrix}, \quad \begin{bmatrix} E_1 \\ E_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad \begin{bmatrix} G_1 \\ G_2 \end{bmatrix} = \begin{bmatrix} 0.3 \\ 0.4 \end{bmatrix},$$
$$\begin{bmatrix} C_1 & C_2 \end{bmatrix} = \begin{bmatrix} 0.8 & 2 \end{bmatrix}, \quad \begin{bmatrix} C_{d1} & C_{d2} \end{bmatrix} = \begin{bmatrix} 0 & 0 \end{bmatrix}, \quad D = 1, \quad F = 0.1,$$
$$\begin{bmatrix} H_{11} & H_{12} \end{bmatrix} = \begin{bmatrix} 0.1 & 0.3 \end{bmatrix}, \quad \begin{bmatrix} H_{21} & H_{22} \end{bmatrix} = \begin{bmatrix} 0.1 & 0.2 \end{bmatrix}, \quad H_3 = 0.1,$$

The purpose is to design a robust controller in the form of (12) such that the closed-loop system is robustly asymptotically stable and satisfies the H_∞ performance constraint (14).

Table 2. Minimum values of H_∞ performance γ_{min} for given delays d_2 and h_2 with $\mu_1 = \mu_2 = 0.9$.

Method	$d_2 = h_2 = 0.4$	$d_2 = h_2 = 0.8$	$d_2 = h_2 = 1.2$	NoDv
(Ghous & Xiang, 2016a)	1.1025	1.5198	Infeasible	32
Theorem 5.1	1.0922	1.1776	1.4958	47

Table 3. Comparison of minimum values of H_∞ performance γ_{min} for $\mu_1 = \mu_2 = 0.6$.

$d_1 = h_1$	$d_2 = h_2$	γ_{min}	Controller gain K
0.2	0.6	1.1227	$[-1.4518 \quad -5.6650]$
	0.8	1.1749	$[-1.1434 \quad -4.7879]$
	1	1.2486	$[-0.9303 \quad -3.8634]$
0.4	0.8	1.1719	$[-1.1523 \quad -4.8618]$
	1	1.2455	$[-0.9266 \quad -3.9396]$
	1.2	1.4105	$[-0.8034 \quad -2.9520]$
0.6	1	1.2411	$[-1.1523 \quad -4.8618]$
	1.2	1.3800	$[-0.8059 \quad -2.9895]$
	1.4	Infeasible	—

To compare our results with those in Ghous & Xiang (2016a), we use Theorem 5.1 with $d_1 = h_1 = 0$. Table 2 shows a comparison results on minimum disturbance attenuation γ_{min} for different d_2 and h_2 and $\mu = 0.9$, and shows also the number of decision variables (NoDv) involved in each method.

In the case of $h_1 > 0$ and $d_1 > 0$, Table 3 shows the minimum H_∞ performance γ_{min} and the corresponding controller gains based on Theorem 5.1. It is obvious that the achieved minimum γ_{min} and the corresponding controller gain K are related to lower and upper bounds of delays.

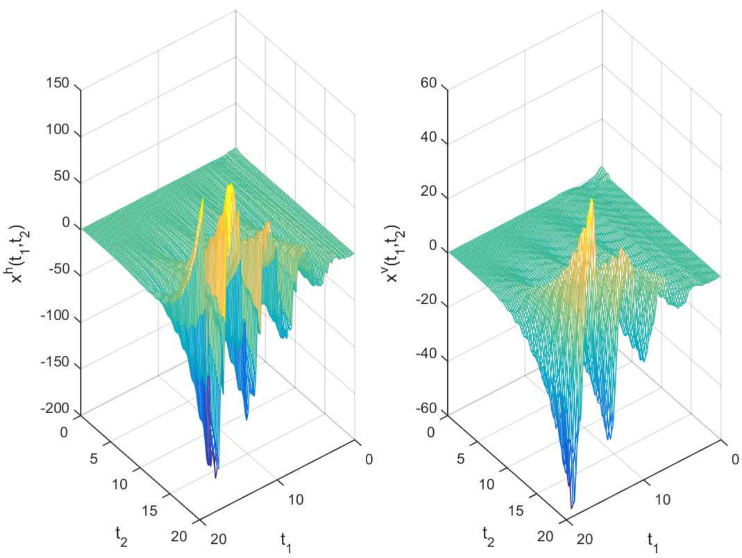

Figure 1. State responses of the open-loop system.

For the simulations we define:

$$
\begin{aligned}
&\mathcal{F}(t_1, t_2) = sin(0.3(t_1 + t_2)),\\
&w(t_1, t_2) = 0.1e^{-0.5(t_1+t_2)}cos(0.1(t_1 + t_2)),\\
&h(t_1) = 0.9 + 0.3cos(0.6\pi t_1),\\
&d(t_2) = 0.9 + 0.3cos(0.6\pi t_2).
\end{aligned}
$$

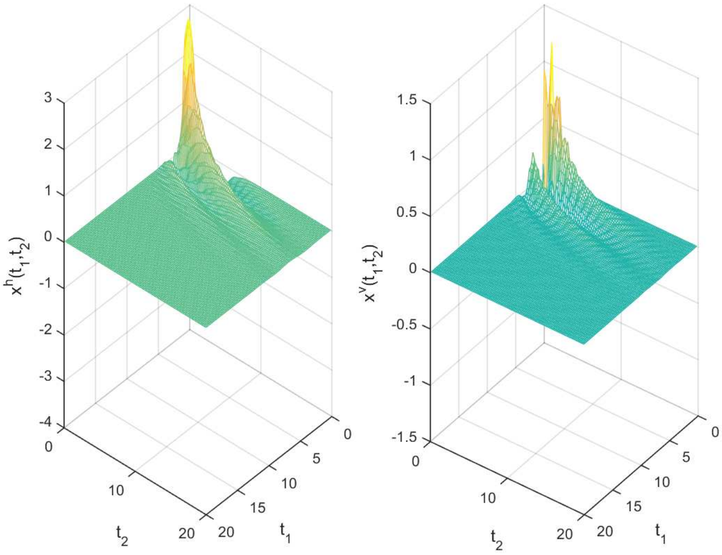

Figure 2. State responses of the closed-loop system.

The varying delays $h(t_1)$ and $d(t_2)$ satisfying:

$$\begin{aligned} 0.6 \le h(t_1) \le 1.2, \qquad & \dot{h}(t_1) \le 0.6, \\ 0.6 \le d(t_2) \le 1.2, \qquad & \dot{d}(t_2) \le 0.6. \end{aligned}$$

The boundary conditions are assumed to be:

$$\begin{cases} x^h(\theta, t_2) = 2, & -h_2 \le \theta \le 0, \quad 0 \le t_2 \le 2.4, \\ x^h(\theta, t_2) = 0, & -h_2 \le \theta \le 0, \quad t_2 \ge 1.2, \\ x^v(t_1, \delta) = 2, & -d_2 \le \delta \le 0, \quad 0 \le t_1 \le 2.4, \\ x^v(t_1, \delta) = 0, & -d_2 \le \delta \le 0, \quad t_1 \ge 1.2, \end{cases},$$

It should be emphasized that the open-loop system is unstable (see Figure 1). This problem cannot be solved by the approach in Ghous & Xiang (2016a), due to the fact that $h_1 \neq 0$ and $d_1 \neq 0$. On the contrary by applying Theorem 5.1 in this paper we obtain a feasible solution for the minimum H_∞ performance γ_{min}, and the optimal controller gain matrix K and they are $\gamma_{min} = 1.3800$ and $K = [-0.8059 \quad -2.9895]$. After applying the controller $u(t_1, t_2) = K \left[\, x^{hT}(t_1, t_2) \;\; x^{vT}(t_1, t_2) \, \right]^T$, the closed-loop system is stabilized as depicted in the state responses and the measured output of the closed-loop given in Figures 2 and 3, respectively, which confirm that the designed state feedback controller is efficient.

Remark 8. It should be pointed out that, the delay dependent stability and H_∞ control conditions proposed in this paper, can address the situation that the lower bounds of delays are not restricted to be zero, while the conditions in Ghous & Xiang (2016a) fail to be applied in this case. On the other hand, according to Remark 3, Table 1 and 2, it can seen that our method developed in this paper gives less conservative results than the method in Ghous & Xiang (2016a) by sacrificing more number of

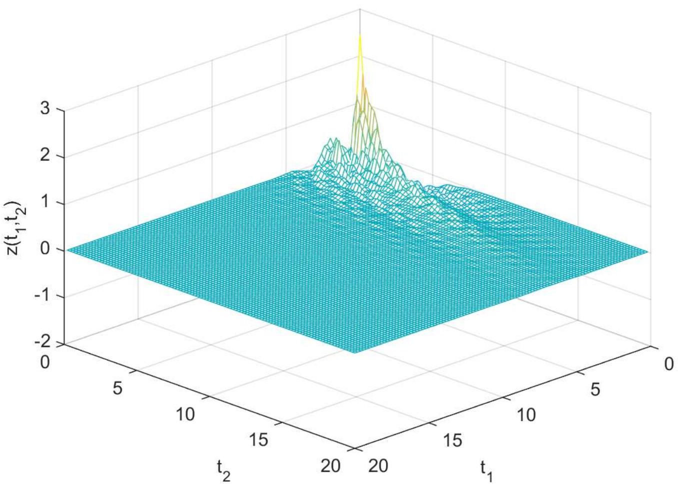

Figure 3. Measured output of the closed loop system.

decision variables. The main reason for obtaining such larger number is that our results are derived based on the augmented Lyapunov-Krasovskii functionals (16), which takes into account more information on the sizes of delays and especially the lower bounds. In the future research, we will focus on reducing the number of decision variables in stability and H_∞ control for uncertain 2-D with interval time-varying delays.

7. Conclusions

In the present paper, the Wirtinger inequality has been exploited to solve the stability analysis and robust H_∞ controller design problems for uncertain 2-D continuous systems, with delays varying within a given interval, and affected by norm-bounded parameter uncertainties. More precisely, a new delay-dependent stability condition is proposed that thanks to the augmented structure of the proposed Lyapunov functional and the use of Wirtinger inequality, is less conservative than previous ones from the 2-D systems literature. Based on this condition, a state feedback controller has been designed to solve the associated H_∞ control problem. Numerical examples demonstrate the effectiveness of the proposed method.

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