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Control of Discrete 2-D Takagi-Sugeno Systems Via a Sum-Of-Squares Approach

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Abstract The stabilization of Takagi-Sugeno (T-S) systems is solved here for the two-dimensional (2-D) polynomial discrete case, by using the sum of squares (SOS) approach. First, we provide a stabilization condition formulated in terms of Polynomial Multiple Lyapunov functions (PMLF). Then, a non-quadratic stabilization condition is developed by applying relaxed stabilization technique. Both conditions can be used for design, by solving them using numerical tools such as SOSTOOLS. A numerical example illustrates the effectiveness of the results.

Keywords discrete 2-D systems · Sum-of-Squares (SOS) · stabilization · Takagi-Sugeno systems.

1 Introduction

Two-dimensional (2-D) systems [1], [2], are drawning great attention in Systems Theory due to their extensive applications in practice, for instance, in multi-dimensional signal processing and transmission, and in thermal process. In fact, 2-D systems are very appropiate to model physical processes described by partial differential equations [3]. Moreover, 2-D techniques can also be applied as an analysis tool to solve complex control problems, for example, PI control of discrete linear repetitive processes [4], iterative learning control [5], and repetitive process control [2]. Several 2-D models have been proposed, depending on the application [6]. The approach here is inspired by [7], where sufficient conditions for robust H_∞ filtering were derived for 2-D discrete Roesser systems using homogenous polynomially parameter-dependent matrices of arbitrary degree, and in [8], that dealt with a parallel problem with time-varying delays.

These previous results focused on linear 2-D systems: we aim here to study the 2-D non-linear systems that can be represented using Takagi-Sugeno (T-S) models: these T-S models [9] are attracting a great deal of attention because they can effectively approximate a wide class of nonlinear systems, which can then be treated by adapting some linear systems techniques. For example, quadratic stabilization of T-S systems [10], [11] has been widely investigated based on a common quadratic Lyapunov function and the Parallel Distributed Compensation (PDC) [12], that unfortunately tends to give conservative conditions. Many other stability conditions for nonlinear systems have been investigated (see, for instance, [10], [13] and references therein). We emphasize here Multiple Lyapunov functions as they lead to good results in the sense that a common Lyapunov quadratic function may not exist but a multiple one exists (see [14], [15]), making possible to ensure stability for a wider class of nonlinear systems. Based on the T-S model, relaxed stabilization conditions are given in [16], by considering the information of some membership functions. Stability and stabilization conditions were derived for continuous-time T-S models in [17]. Various state-space models for 2-D systems have been suggested in [18] to solve complex problems in different fields, for example, robust H_∞ filtering of T-S systems, and stabilization of T-S systems with attenuation of stochastic perturbation. Relaxed stabilization conditions were developed in [19] by using non-quadratic stabilization conditions and a homogeneous polynomially

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parameter-dependent Lyapunov function. By using the basis-dependent Lyapunov function approach and adding slack matrices, the H_∞ filtering problem for 2-D T-S systems described by the Fornasini-Marchesini (FM) model was solved in [20]. Sufficient LMI conditions were also studied in [21] considering the effect of stochastic perturbations.

The techniques used here are instead based on a Sum-Of-Squares (SOS) approach [22], which makes possible to investigate the stability of polynomial systems for a range of problems that is wider than the more extensively used LMI approaches. SOS conditions can be numerically solved using off-the-shelf tools such as SOSTOOLS [22], [23]. More precisely, we use techniques based on those in [24], for polynomial T-S systems. SOS techniques have mainly been applied to 1-D continuous-time nonlinear systems: For example, the stability of polynomial systems with time-delay was presented in [25], using a novel polynomial Lyapunov-Krasovskii functional; Stability conditions for continuous-time polynomial systems were investigated in [26], by using polynomial Lyapunov functions; finally, we mention the effective technique for constructing Lyapunov functions for continuous-time multidimensional nonlinear systems in [27]. So far, few approaches have been proposed to deal with 2-D T-S system using SOS techniques: we can just cite [28], where a stabilization condition of the discrete 2-D T-S system based on polynomial Lyapunov functions has been provided. We must point out that the fact that in 2-D systems the information flows along two different directions makes controller synthesis for 2-D T-S systems more complex and challenging, especially when a polynomial model is considered, i.e., a T-S model whose consequent part is represented by a polynomial. Thus, this paper concentrates on designing the polynomial T-S control for nonlinear 2-D system that are characterized in the Roesser model; this is more challenging problem that parallel problems studied in the literature, such as those in [29].

Thus, this paper delivers an SOS-based methodology to solve the stabilization of Roesser-type polynomial discrete 2-D systems, whose consequent parts are depicted by polynomials. In this manner, the polynomial nonlinearities can be manipulated exactly and a large class of other nonlinearities can be treated by introducing auxiliary variables and constraints. We first derive a stabilization condition, based on polynomial multiple Lyapunov functions. Then, a non-quadratic stabilization condition is derived by using relaxation techniques. The proposed stabilization conditions are represented in terms of SOS and are numerically solved (partially symbolically) via SOSTOOLS [23]. We emphasize that the stabilization approach discussed in this paper is more general than existing LMI approaches to 2D T-S systems, thanks to the use of the Sum-Of-Squares approach.

The remainder of the paper is organized as follows: Section 2 provides the preliminaries and notation used throughout the paper; the main results are presented in Section 3; a numerical example is provided in Section 4 to demonstrate the effectiveness of the developed approach; finally, some conclusions are given in Section 5.

2 Problem Formulation

Notation:
- $\mathbb{R}$: the set of real numbers;
- $\mathbb{Z}^+$: the set of nonnegative integers
- $\mathbb{C}^1$: the set of complex numbers
- I : identity matrix (of size specified by the context);
- $\lambda_i(.)$: ith eigenvalue;
- $*$: represents a term that is induced by symmetry.

We consider here discrete nonlinear 2-D systems described as follows:

$$x^+(k,l) = \mathscr{X}(x(k,l)) + \mathscr{S}(x(k,l))u(k,l) \tag{1}$$

$$x^h(0,l) = f(l), \qquad x^v(k,0) = g(k) \tag{2}$$

with

$$x(k,l) = \begin{bmatrix} x^h(k,l) \\ x^v(k,l) \end{bmatrix}, x^+(k,l) = \begin{bmatrix} x^h(k+1,l) \\ x^v(k,l+1) \end{bmatrix},$$

where $x^h(.) \in \mathbb{R}^{n_1}$, $x^v(.) \in \mathbb{R}^{n_2}$ are the horizontal and the vertical state, respectively; $u(.) \in \mathbb{R}^m$ is the control vector; $\mathscr{X}(.)$ and $\mathscr{S}(.)$ are nonlinear functions satisfying $\mathscr{X}, \mathscr{S} \in \mathbb{C}^1$; k, l are two integers in $\mathbb{Z}^+$; finally, $f(l)$, $g(k)$ are the boundary conditions along the horizontal and vertical directions, respectively.

The following equivalent discrete 2-D T-S system will be used, to represent the nonlinear Roesser system (1):
Rule i: IF $z_1(k,l)$ IS M_{i1} AND,..., AND $z_s(k,l)$ IS M_{is} THEN

$$x^+(k,l) = A_i(x(k,l))\hat{x}(x(k,l)) + B_i(x(k,l))u(k,l) \tag{3}$$

$$x^h(0,l) = f(l), \qquad x^v(k,0) = g(k)$$

with

$$A_i(x(k,l)) = \begin{bmatrix} A^i_{11}(x(k,l)) & A^i_{12}(x(k,l)) \\ A^i_{21}(x(k,l)) & A^i_{22}(x(k,l)) \end{bmatrix},$$

$$B_i(x(k,l)) = \begin{bmatrix} B^i_1(x(k,l)) \\ B^i_2(x(k,l)) \end{bmatrix}$$

where $i = 1,\dots,r$, M_{is} are the fuzzy sets, $z_p(k)$, for $p = 1,\dots,s$, are the premise variables, r is the number of IF-THEN rules, $\hat{x}(x(k,l))$ is a column vector whose entries are all monomials in $x(k,l)$ and $A_i(x(k,l))$, $B_i(x(k,l))$ are polynomial matrices in $x(k,l)$, with appropriate dimensions.

These polynomial discrete 2-D T-S systems can also be expressed in a more compact form as follows

$$x^+(k,l) = \sum_{i=1}^{r} h_i(z(k,l))\{A_i(x(k,l))\hat{x}(x(k,l)) + B_i(x(k,l))u(k,l)\} \quad (4)$$

where

$$h_i(z(k,l)) = \frac{w_i(z(k,l))}{\sum_{i=1}^{r} w_i(z(k,l))},\ w_i(z(k,l)) = \prod_{j=1}^{s} M_{ij}(z(k,l))$$

where $M_{ij}(z(k,l))$ is the grade of membership of $z_j(k,l$ in M_{ij} and $w_i(z(k,l))$ represents the weight of the i^{th} rule. In this paper, we assume that $w_i(z(k,l)) \geq 0$, for $i = 1,2,\ldots,r$, and $\sum_{i=1}^{r} w_i(z(k,l)) > 0$ for all t. Therefore, we get $h_i(z(k,l)) \geq 0$, for $i = 1,2,\ldots,r$ and $\sum_{i=1}^{r} h_i(z(k,l)) = 1$ for all t.

Remark 1 *The region of validity of this system is defined by* $\mathscr{V}_0$:

$$\mathscr{V}_0 = \{x(k) \in \mathbb{R}^n; |L_{(N)}x(k)| \leq \eta_{(N)}\} \quad (5)$$

where $\eta_{(N)} > 0$ *and* $L_{(N)} \in \mathbb{R}^{1\times n}$ *for* $N = 1,\ldots \mathfrak{n}$ *with* $\mathfrak{n}$ *representing the number of constraints that characterize the allowed region for the closed-loop system in the state space. For detailed discussions about the use of T-S models to represent exactly nonlinear systems inside a region* $\mathscr{V}_0$ *we can cite [35, 36]: although those results are for one dimensional systems, they can be directly extended to multidimensional systems, like the 2D T-S in this paper.*

Some stabilization conditions for the 2-D polynomial discrete systems in (4) will be later derived using the SOS approach, which is now introduced.

Lemma 1 *[30] For two symmetric matrices* $P > 0$ *and* $Q > 0$, *the inequality* $A^TQA - P < 0$ *holds, if there exist a matrix* G *such that*

$$\begin{bmatrix} P & * \\ GA & G+G^T-Q \end{bmatrix} > 0$$

Definition 1 *[22] A multivariate polynomial* $f(x)$, *for* $x \in \mathbb{R}^N$, *is a SOS if there exist polynomials* $f_i(x)$, $i = 1,\ldots,n$ *such that*

$$f(x) = \sum_{i=1}^{n} f_i^2(x) \quad (6)$$

This implies $f(x) \geq 0$ *for any* $x \in \mathbb{R}^n$.

Lemma 2 *[31] Let* $f(x)$ *be a polynomial in* $x \in \mathbb{R}^n$ *of degree* $2d$. *Let* $Z(x)$ *be a column vector whose entries are all monomial in x, with a degree no greater than* d. *Then,* $f(x)$ *is said to be SOS if and only if there exists a positive semi-definite matrix* Q *such that*

$$f(x) = Z(x)^TQZ(x) \quad (7)$$

3 Main Results

To simplify the calculations, the following notations will be adopted:

$$h_i = h_i(z(k,l)),\quad M_z(\tilde{x}) = \sum_{i=1}^{r} h_iM_i(x(k,l)),$$

$$M_z^{-1}(\tilde{x}) = (\sum_{i=1}^{r} h_iM_i(x(k,l)))^{-1}$$

$K = \{k_1,k_2,\ldots,k_n\}$ denotes the set of row indices of $B_i(\tilde{x})$ whose corresponding row is equal to zeros; we then define $\tilde{x} = (x_{k_1},x_{k_2},\ldots,x_{k_n})$. We will consider $T(x(k,l))$ that is a polynomial matrix defined by $\hat{x}(x(k,l)) = T(x(k,l))x(k,l)$.

3.1 Stabilization Conditions Via Polynomial Multiple Lyapunov function

In [33], a PDC control scheme and a polynomial Lyapunov function are proposed for 1-D to obtain less conservative stabilization conditions. In this paper we extend the PDC scheme used in the literature for 1-D T-S polynomial fuzzy systems to the 2-D systems studied in this paper. This 2-D controller can be expressed as follows

Rule i: IF $z_1(k,l)$ IS M_{i1} AND,..., AND $z_s(k,l)$ IS M_{is} THEN

$$u(k,l) = F_i(\tilde{x})\hat{x}(x(k,l))$$

with $i = 1,\ldots,r$, and $F_i(\tilde{x})$ polynomial matrices of appropriate dimensions to be determined. The overall 2-D controller can be represented by

$$u(k,l) = \sum_{i=1}^{r} h_iF_i(\tilde{x})\hat{x}(x(k,l)) = F_z(\tilde{x})\hat{x}(x(k,l)) \quad (8)$$

the closed-loop system is given by

$$\begin{aligned} x^+(k,l) &= \sum_{i=1}^{r}\sum_{j=1}^{r} h_ih_j\{A_i(\tilde{x}) + B_i(\tilde{x})F_j(\tilde{x})\}\hat{x}(x(k,l)) \\ &= (A_z(\tilde{x}) + B_z(\tilde{x})F_z(\tilde{x}))\hat{x}(x(k,l)) \end{aligned} \quad (9)$$

Theorem 1 *The polynomial T-S system (9) is asymptotically stable if there exists symmetric polynomial matrices* $X_1^j(\tilde{x}) \in \mathbb{R}^{n_1\times n_1}$, $X_2^j(\tilde{x}) \in \mathbb{R}^{n_2\times n_2}$, *and a polynomial matrices* $K_1^j(\tilde{x}) \in \mathbb{R}^{m_1\times n_1}$, $K_2^j(\tilde{x}) \in \mathbb{R}^{m_2\times n_2}$, *where* $\varepsilon_1^i(\tilde{x}) > 0$, $\varepsilon_2^i(\tilde{x}) > 0$ *for* $(\tilde{x} \neq 0)$ *and* $\varepsilon_1^{ij}(\tilde{x}) \geq 0$, $\varepsilon_2^{ij}(\tilde{x}) \geq 0$ *for all* $\tilde{x}$ *such that the following SOS holds:*

$$\nu_1^T(X_1^i(\tilde{x}) - \varepsilon_1^i(\tilde{x})I)\nu_1 \quad \text{is SOS}\ \ i = 1,\ldots,r \quad (10)$$

$$\nu_1^T(X_2^i(\tilde{x}) - \varepsilon_2^i(\tilde{x})I)\nu_1 \quad \text{is SOS}\ \ i = 1,\ldots,r \quad (11)$$

$$\nu_2^T\Omega_{ii}^{mn}(\tilde{x})\nu_2 \quad \text{is SOS}\ \ i,m,n = 1,\ldots,r \quad (12)$$

$$\nu_2^T(\Omega_{ij}^{mn}(\tilde{x}) + \Omega_{ji}^{mn}(\tilde{x}))\nu_2 \ \ \text{is SOS} \quad (13)$$

$$i < j \quad i,j,m,n = 1,\dots,r$$

with v_1, v_2 vectors of appropriate dimensions, that are independent of $x(k,l)$, and

$$\Omega_{ij}^{mn}(\tilde{x}) = \begin{bmatrix} \Omega_{ij11}(\tilde{x}) & * & * & * \\ 0 & \Omega_{ij22}(\tilde{x}) & * & * \\ \Omega_{ij31}(\tilde{x}) & \Omega_{ij32}(\tilde{x}) & \Omega_{ij33}^{m}(\tilde{x}) & * \\ \Omega_{ij41}(\tilde{x}) & \Omega_{ij42}(\tilde{x}) & 0 & \Omega_{ij44}^{n}(\tilde{x}) \end{bmatrix} > 0 \quad (14)$$

$$\Omega_{ij11}(\tilde{x}) = X_1^j(\tilde{x}) - \varepsilon_1^{ij}(\tilde{x})I$$
$$\Omega_{ij22}(\tilde{x}) = X_2^j(\tilde{x}) - \varepsilon_2^{ij}(\tilde{x})I$$
$$\Omega_{ij31}(\tilde{x}) = T_1(\tilde{x}^+)A_{11}^i(\tilde{x})X_1^j(\tilde{x}) + T_1(\tilde{x}^+)B_1^i(\tilde{x})K_1^j(\tilde{x})$$
$$\Omega_{ij32}(\tilde{x}) = T_1(\tilde{x}^+)A_{12}^i(\tilde{x})X_2^j(\tilde{x}) + T_1(\tilde{x}^+)B_1^i(\tilde{x})K_2^j(\tilde{x})$$
$$\Omega_{ij33}^m(\tilde{x}) = X_1^m(x(k+1,l)) - \varepsilon_1^{ij}(\tilde{x})I$$
$$\Omega_{ij41}(\tilde{x}) = T_2(\tilde{x}^+)A_{21}^i(\tilde{x})X_1^j(\tilde{x}) + T_2(\tilde{x}^+)B_2^i(\tilde{x})K_1^j(\tilde{x})$$
$$\Omega_{ij42}(\tilde{x}) = T_2(\tilde{x}^+)A_{22}^i(\tilde{x})X_2^j(\tilde{x}) + T_2(\tilde{x}^+)B_2^i(\tilde{x})K_2^j(\tilde{x})$$
$$\Omega_{ij44}^n(\tilde{x}) = X_2^n(x(k,l+1)) - \varepsilon_2^{ij}(\tilde{x})I$$

Proof Considering the following polynomial multiple Lyapunov function:

$$V(x(k,l)) = \hat{x}^T(x(k,l))P_z(\tilde{x})\hat{x}(x(k,l)) \quad (15)$$

where $P_z(\tilde{x})$ is a polynomial matrix in $x(k,l)$ such that

$$P_z(\tilde{x}) = \begin{bmatrix} P_1^z(\tilde{x}) & 0 \\ 0 & P_2^z(\tilde{x}) \end{bmatrix} > 0$$

$$\lambda_{min}(P_z(\tilde{x}))\|\hat{x}(x(k,l))\|_2^2 \le V(x(k,l)) \le \lambda_{max}(P_z(\tilde{x}))\|\hat{x}(x(k,l))\|_2^2$$

where $\lambda_{min}(.)$ and $\lambda_{max}(.)$ denote the minimum and maximum eigenvalueS of a matrix, respectively.

Note that if $P_z(\tilde{x})$ is a constant matrix and $\hat{x}(x(k,l)) = x(k,l)$, then (15) reduces to the multiple Lyapunov function $x^T(k,l)P_z x(k,l)$ used in the literature: therefore (15) is a more general representation, and will reduce conservatism in the developed conditions.

The variation of (15) is given by

$$\begin{aligned} \Delta V(x(k,l)) &= \hat{x}^{+T}(x(k,l))P_z(\tilde{x}^+)\hat{x}^+(x(k,l)) \\ &- \hat{x}^T(x(k,l))P_z(\tilde{x})\hat{x}(x(k,l)) \\ &= \hat{x}^T(x(k,l))[(\tilde{A}_z(\tilde{x}) + \tilde{B}_z(\tilde{x})F_z(\tilde{x}))^T P_z(\tilde{x}^+) \\ &\times (\tilde{A}_z(\tilde{x}) + \tilde{B}_z(\tilde{x})F_z(\tilde{x})) - P_z(\tilde{x})]\hat{x}(x(k,l)) \\ &= -\hat{x}^T(x(k,l))Q_{1z}(\tilde{x})\hat{x}(x(k,l)) \end{aligned} \quad (16)$$

where

$$\begin{aligned} -Q_{1z}(\tilde{x}) &= (\tilde{A}_z(\tilde{x}) + \tilde{B}_z(\tilde{x})F_z(\tilde{x}))^T P_z(\tilde{x}^+) \\ &\times (\tilde{A}_z(\tilde{x}) + \tilde{B}_z(\tilde{x})F_z(\tilde{x})) - P_z(\tilde{x}) \end{aligned}$$

and

$$\tilde{A}_z(\tilde{x}) = T(\tilde{x}^+)A_z(\tilde{x}), \;\; \tilde{B}_z(\tilde{x}) = T(\tilde{x}^+)B_z(\tilde{x})$$

$$T(\tilde{x}^+) = \begin{bmatrix} T_1(\tilde{x}^+) & 0 \\ 0 & T_2(\tilde{x}^+) \end{bmatrix} = \begin{bmatrix} T_1(x(k+1,l) & 0 \\ 0 & T_2(x(k,l+1)) \end{bmatrix}$$

Note that $Q_{1z}(\tilde{x})$ are positive definite symmetric polynomial matrices, and $\lambda_{min}(Q_{1z}(\tilde{x})) > 0$, so

$$\hat{x}^T(x(k,l))Q_{1z}(\tilde{x})\hat{x}(x(k,l)) \ge \lambda_{min}(Q_{1z}(\tilde{x}))\|\hat{x}(x(k,l))\|_2^2$$

Thus, $\Delta V(x(k,l))$ is bounded as follows:

$$\Delta V(x(k,l)) \le -\lambda_{min}(Q_{1z}(\tilde{x}))\|\hat{x}(x(k,l))\|_2^2, \; \forall \|\hat{x}(x(k,l))\|_2 < \eta_{(N)}$$

$\Delta V(x(k,l))$ is negative if $-Q_{1z}(\tilde{x}) < 0$

$$\begin{aligned} -Q_{1z}(\tilde{x}) = [&(\tilde{A}_z(\tilde{x}) + \tilde{B}_z(\tilde{x})F_z(\tilde{x}))^T P_z(\tilde{x}^+)(\tilde{A}_z(\tilde{x}) \\ &+ \tilde{B}_z(\tilde{x})F_z(\tilde{x})) - P_z(\tilde{x})] < 0 \end{aligned} \quad (17)$$

Pre- and post-multiplying both sides of (17) by $P_z^{-1}(\tilde{x})$, we obtain (18) where $P_z^{-1}(\tilde{x}) = X_z(\tilde{x})$ and $K_z(\tilde{x}) = F_z(\tilde{x})X_z(\tilde{x})$, $i = 1,\dots,r$:

$$\begin{aligned} &[X_z(\tilde{x})(\tilde{A}_z(\tilde{x}) + \tilde{B}_z(\tilde{x})F_z(\tilde{x}))^T X_z^{-1}(\tilde{x}^+)(\tilde{A}_z(\tilde{x}) \\ &+ \tilde{B}_z(\tilde{x})F_z(\tilde{x}))X_z(\tilde{x}) - X_z(\tilde{x})] < 0 \end{aligned} \quad (18)$$

applying the Schur complement, we then have

$$\Omega_z^{mn}(\tilde{x}) = \begin{bmatrix} X_z(\tilde{x}) & (\tilde{A}_z(\tilde{x})X_z(\tilde{x}) + \tilde{B}_z(\tilde{x})K_z(\tilde{x}))^T \\ * & X_z(\tilde{x}^+) \end{bmatrix} > 0 \quad (19)$$

where

$$\Omega_z^{mn}(\tilde{x}) = \sum_{m=1}^{r}\sum_{n=1}^{r}\sum_{i=1}^{r}\sum_{j=1}^{r} h_m(z(k+1,l))h_n(z(k,l+1))h_i h_j \Omega_{ij}^{mn}(\tilde{x})$$

$$X_z(\tilde{x}) = \begin{bmatrix} X_1^z(\tilde{x}) & 0 \\ 0 & X_2^z(\tilde{x}) \end{bmatrix}, X_z(\tilde{x}^+) = \begin{bmatrix} X_1^z(\tilde{x}^+) & 0 \\ 0 & X_2^z(\tilde{x}^+) \end{bmatrix}$$

$$X_1^z(\tilde{x}^+) = \sum_{m=1}^{r} h_m(z(k+1,l))X_1^m(x(k+1,l))$$

$$X_2^z(\tilde{x}^+) = \sum_{n=1}^{r} h_n(z(k,l+1))X_2^n(x(k,l+1))$$

and $\Omega_{ij}^{mn}(\tilde{x})$ are defined in (14).

3.2 Stabilization Conditions for Polynomial System using Non-PDC Control

In order to obtain more relaxed stabilization conditions for polynomial discrete 2-D T-S systems, a Non-PDC control law is used, of the following form [28]:

$$u(k,l) = K_z(\tilde{x})Y_z^{-1}(\tilde{x})\hat{x}(x(k,l)) \tag{20}$$

where $K_z(\tilde{x})$ and $Y_z(\tilde{x})$ are polynomial matrices of appropriate dimensions to be determined

$$K_z(\tilde{x}) = \begin{bmatrix} K_1^z(\tilde{x}) \; K_2^z(\tilde{x}) \end{bmatrix},\; Y_z(\tilde{x}) = \begin{bmatrix} Y_1^z(\tilde{x}) & 0 \\ 0 & Y_2^z(\tilde{x}) \end{bmatrix}$$

the closed-loop system of (4) and (20) is then as follows:

$$x^+(k,l) = (A_z(\tilde{x}) + B_z(\tilde{x})K_z(\tilde{x})Y_z^{-1}(\tilde{x}))\hat{x}(x(k,l)) \tag{21}$$

Theorem 2 *The 2-D system (21) is asymptotically stable if there exist symmetric polynomial matrices $P_1^i(\tilde{x}) \in \mathbb{R}^{n_1\times n_1}$, $P_2^i(\tilde{x}) \in \mathbb{R}^{n_2\times n_2}$, and polynomial matrices $K_1^i(\tilde{x}) \in \mathbb{R}^{m_1\times n_1}$, $K_2^i(\tilde{x}) \in \mathbb{R}^{m_2\times n_2}$, $Y_1^i(\tilde{x}) \in \mathbb{R}^{n_1\times n_1}$, $Y_2^i(\tilde{x}) \in \mathbb{R}^{n_2\times n_2}$, $Z_{ii}^{mn}(\tilde{x})$, $Z_{ij}^{mn}(\tilde{x}) = (Z_{ji}^{mn})^T(\tilde{x})$ such that (22), (23), (24), (25) and (26) are satisfied, where $\varepsilon_1^i(\tilde{x})$, $\varepsilon_2^i(\tilde{x})$, $\varepsilon_1^{ij}(\tilde{x})$ and $\varepsilon_2^{ij}(\tilde{x})$ are non negative polynomials such that $\varepsilon_1^i(\tilde{x}) > 0$, $\varepsilon_2^i(\tilde{x}) > 0$ for $(\tilde{x} \neq 0)$ and $\varepsilon_1^{ij}(\tilde{x}) \geq 0$, $\varepsilon_2^{ij}(\tilde{x}) \geq 0$ for all $\tilde{x}$.*

$$\nu_1^T(P_1^i(\tilde{x}) - \varepsilon_1^i(\tilde{x})I)\nu_1 \quad \text{is SOS} \tag{22}$$

$$\nu_1^T(P_2^i(\tilde{x}) - \varepsilon_2^i(\tilde{x})I)\nu_1 \quad \text{is SOS} \tag{23}$$

$$\nu_2^T(T_{ii}^{mn}(\tilde{x}) - Z_{ii}^{mn}(\tilde{x}))\nu_2 \;\; \text{is SOS} \;\; i,m,n = 1,\dots,r \tag{24}$$

$$\nu_2^T(T_{ij}^{mn}(\tilde{x}) + T_{ji}^{mn}(\tilde{x}) - Z_{ij}^{mn}(\tilde{x}) - Z_{ji}^{mn}(\tilde{x}))\nu_2 \quad \text{is SOS} \tag{25}$$

$$i<j, \quad i,j,m,n = 1,\dots,r.$$

$$\begin{bmatrix} Z_{11}^{mn}(\tilde{x}) & Z_{12}^{mn}(\tilde{x}) & \dots & Z_{1r}^{mn}(\tilde{x}) \\ Z_{21}^{mn}(\tilde{x}) & Z_{22}^{mn}(\tilde{x}) & \dots & Z_{2r}^{mn}(\tilde{x}) \\ \vdots & \vdots & \ddots & \vdots \\ Z_{r1}^{mn}(\tilde{x}) & Z_{r2}^{mn}(\tilde{x}) & \dots & Z_{rr}^{mn}(\tilde{x}) \end{bmatrix} > 0 \tag{26}$$

$m,n = 1,\dots,r.$

with ν_1 and ν_2 vectors that are independent of $x(k,l)$.

$$T_{ij}^{mn}(\tilde{x}) = \begin{bmatrix} T_{ij11}(\tilde{x}) & * & * & * \\ 0 & T_{ij22}(\tilde{x}) & * & * \\ T_{ij31}(\tilde{x}) & T_{ij32}(\tilde{x}) & T_{ij33}^m(\tilde{x}) & * \\ T_{ij41}(\tilde{x}) & T_{ij42}(\tilde{x}) & 0 & T_{ij44}^n(\tilde{x}) \end{bmatrix} > 0 \tag{27}$$

$$\begin{aligned}
T_{ij11}(\tilde{x}) &= P_1^j(\tilde{x}) - \varepsilon_1^i(\tilde{x})I \\
T_{ij22}(\tilde{x}) &= P_2^j(\tilde{x}) - \varepsilon_2^i(\tilde{x})I \\
T_{ij31}(\tilde{x}) &= T_1(\tilde{x}^+)A_{11}^i(\tilde{x})Y_1^j(\tilde{x}) + T_1(\tilde{x}^+)B_1^i(\tilde{x})K_1^j(\tilde{x}) \\
T_{ij32}(\tilde{x}) &= T_1(\tilde{x}^+)A_{12}^i(\tilde{x})Y_2^j(\tilde{x}) + T_1(\tilde{x}^+)B_1^i(\tilde{x})K_2^j(\tilde{x}) \\
T_{ij33}^m(\tilde{x}) &= Y_1^m(x(k+1,l)) + Y_1^{mT}(x(k+1,l)) \\
&\quad - P_1^m(x(k+1,l)) - \varepsilon_1^{ij}(\tilde{x})I \\
T_{ij41}(\tilde{x}) &= T_2(\tilde{x}^+)A_{21}^i(\tilde{x})Y_1^j(\tilde{x}) + T_2(\tilde{x}^+)B_2^i(\tilde{x})K_1^j(\tilde{x}) \\
T_{ij42}(\tilde{x}) &= T_2(\tilde{x}^+)A_{22}^i(\tilde{x})Y_2^j(\tilde{x}) + T_2(\tilde{x}^+)B_2^i(\tilde{x})K_2^j(\tilde{x}) \\
T_{ij44}^n(\tilde{x}) &= Y_2^n(x(k,l+1)) + Y_2^{nT}(x(k,l+1)) \\
&\quad - P_2^n(x(k,l+1) - \varepsilon_2^{ij}(\tilde{x})I
\end{aligned}$$

Proof Consider the following polynomial Lyapunov function [28]:

$$V(x(k,l)) = \hat{x}^T(x(k,l))Y_z^{-T}(\tilde{x})P_z(\tilde{x})Y_z^{-1}(\tilde{x})\hat{x}(x(k,l)) \tag{28}$$

Then,

$$\begin{aligned}
\hat{x}^T(x(k,l))\nu_{1min}Y_z^{-T}(\tilde{x})Y_z^{-1}(\tilde{x})\hat{x}(x(k,l)) &\leq V(x(k,l)) \\
&\leq \hat{x}^T(x(k,l))\nu_{1max}Y_z^{-T}(\tilde{x})Y_z^{-1}(\tilde{x})\hat{x}(x(k,l))
\end{aligned} \tag{29}$$

where

$$\nu_{1min} = \lambda_{min_z}(P_z(\tilde{x})), \;\; \nu_{1max} = \lambda_{max_z}(P_z(\tilde{x}))$$

As $(Y_z^{-T}(\tilde{x})Y_z^{-1}(\tilde{x})) = Y_z(\tilde{x})Y_z^T(\tilde{x})$ with

$$\nu_{2min} = \lambda_{min_z}(Y_z(\tilde{x})Y_z^T(\tilde{x})), \;\; \nu_{2max} = \lambda_{max_z}(Y_z(\tilde{x})Y_z^T(\tilde{x}))$$

(29) becomes

$$\nu_{1min}\nu_{2max}^{-1}\|\hat{x}(x(k,l))\|^2 \leq V(x(k,l)) \leq \nu_{1max}\nu_{2min}^{-1}\|\hat{x}(x(k,l))\|^2$$

which ensures that $V(x(k,l))$ is a polynomial Lyapunov function.

Then, the variation of (28) is given by

$$\begin{aligned}
\Delta V(x(k,l)) &= \hat{x}^T(x(k,l))[(\tilde{A}_z(\tilde{x}) + \tilde{B}_z(\tilde{x})K_z(\tilde{x}) \\
&\quad \times Y_z^{-1}(\tilde{x}))^T Y_z^{-T}(\tilde{x}^+)P_z(\tilde{x}^+)Y_z^{-1}(\tilde{x}^+)(\tilde{A}_z(\tilde{x}) \\
&\quad + \tilde{B}_z(\tilde{x})K_z(\tilde{x})Y_z^{-1}(\tilde{x})) \\
&\quad - Y_z^{-T}(\tilde{x})P_z(\tilde{x})Y_z^{-1}(\tilde{x})]\hat{x}(x(k,l)) \\
&= -\hat{x}^T(x(k,l))Q_{2z}(\tilde{x})\hat{x}(x(k,l))
\end{aligned} \tag{30}$$

where

$$\begin{aligned}
-Q_{2z}(\tilde{x}) &= [(\tilde{A}_z(\tilde{x}) + \tilde{B}_z(\tilde{x})K_z(\tilde{x}) \\
&\quad \times Y_z^{-1}(\tilde{x}))^T Y_z^{-T}(\tilde{x}^+)P_z(\tilde{x}^+)Y_z^{-1}(\tilde{x}^+)(\tilde{A}_z(\tilde{x}) \\
&\quad + \tilde{B}_z(\tilde{x})K_z(\tilde{x})Y_z^{-1}(\tilde{x})) \\
&\quad - Y_z^{-T}(\tilde{x})P_z(\tilde{x})Y_z^{-1}(\tilde{x})]
\end{aligned} \tag{31}$$

$$Y^z(\tilde{x}^+) = \begin{bmatrix} Y_1^z(\tilde{x}^+) & 0 \\ 0 & Y_2^z(\tilde{x}^+) \end{bmatrix}, P^z(\tilde{x}^+) = \begin{bmatrix} P_1^z(\tilde{x}^+) & 0 \\ 0 & P_2^z(\tilde{x}^+) \end{bmatrix}$$

$$Y_1^z(\tilde{x}^+) = \sum_{m=1}^{r} h_m(z(k+1,l))Y_1^m(x(k+1,l))$$
$$Y_2^z(\tilde{x}^+) = \sum_{n=1}^{r} h_n(z(k,l+1))Y_2^n(x(k,l+1))$$
$$P_1^z(\tilde{x}^+) = \sum_{m=1}^{r} h_m(z(k+1,l))P_1^m(x(k+1,l))$$

$$P_2^z(\tilde{x}^+) = \sum_{n=1}^{r} h_n(z(k,l+1))P_2^n(x(k,l+1))$$

and $Q_{2z}(\tilde{x})$ are symmetric polynomial matrices, positive definite.

$$\hat{x}^T(x(k,l))Q_{2z}(\tilde{x})\hat{x}(x(k,l)) \geq \lambda_{min}(Q_{2z}(\tilde{x}))\|\hat{x}(x(k,l))\|_2^2$$

Thus, $\Delta V(x(k,l))$ is bounded as follows:

$$\Delta V(x(k,l)) \leq -\lambda_{min}(Q_{2z}(\tilde{x}))\|\hat{x}(x(k,l))\|_2^2, \quad \forall \|\hat{x}(x(k,l))\|_2 < \eta_{(N)}$$

Note that in the 2-D system the information is propagated along two independent directions; hence, $h_i(z(k+1,l))$ and $h_j(z(k,l+1))$ are two different membership functions.

Multiplying the right side of (31) by $Y_z(\tilde{x})$, and the left side by $Y_z^T(\tilde{x})$, we have:

$$(Y_z^T(\tilde{x})\tilde{A}_z^T(\tilde{x}) + K_z^T(\tilde{x})\tilde{B}_z^T(\tilde{x}))Y_z^{-T}(\tilde{x}^+)P_z(\tilde{x}^+)Y_z^{-1}(\tilde{x}^+) \times (\tilde{A}_z(\tilde{x})Y_z(\tilde{x}) + \tilde{B}_z(\tilde{x})K_z(\tilde{x})) - P_z(\tilde{x}) < 0. \tag{32}$$

By using Lemma 1, where $A = Y_z^{-1}(\tilde{x}^+)(\tilde{A}_z(\tilde{x})Y_z(\tilde{x}) + \tilde{B}_z(\tilde{x})K_z(\tilde{x}))$, (32) can be expressed as

$$T_z(\tilde{x}) = \begin{bmatrix} P_z(\tilde{x}) & * \\ \Upsilon_{21} & \Upsilon_{22} \end{bmatrix} > 0 \tag{33}$$

where

$$\Upsilon_{21} = \tilde{A}_z(\tilde{x})Y_z(\tilde{x}) + \tilde{B}_z(\tilde{x})K_z(\tilde{x})$$
$$\Upsilon_{22} = Y_z(\tilde{x}^+) + Y_z^T(\tilde{x}^+) - P_z(\tilde{x}^+)$$

$$T_z(\tilde{x}) = \sum_{m=1}^{r}\sum_{n=1}^{r} h_m(z(k+1,l))h_n(z(k,l+1)) \times (\sum_{i=1}^{r} h_i^2 T_{ii}^{mn}(\tilde{x}) + \sum_{i=1}^{r-1}\sum_{j>i} h_i h_j (T_{ij}^{mn}(\tilde{x}) + T_{ji}^{mn}(\tilde{x})))$$

where $T_{ij}^{mn}(\tilde{x})$ are defined in (27)

If the conditions (25) and (26) are satisfied, then the following are also satisfied:

$$\begin{aligned} T_z(\tilde{x}) \geq & \sum_{m=1}^{r}\sum_{n=1}^{r} h_m(z(k+1,l))h_n(z(k,l+1)) \\ & \times (\sum_{i=1}^{r} h_i^2 Z_{ii}^{mn}(\tilde{x}) + \sum_{i=1}^{r-1}\sum_{j>i} h_i h_j (Z_{ij}^{mn}(\tilde{x}) + Z_{ji}^{mn}(\tilde{x}))) \\ & = \sum_{m=1}^{r}\sum_{n=1}^{r} h_m(z(k+1,l))h_n(z(k,l+1))\theta^T Z^{mn}(\tilde{x})\theta \end{aligned} \tag{34}$$

where $\theta^T = [h_1 I \;\; h_2 I \;\; \ldots \;\; h_r I]$ and

$$Z^{mn}(\tilde{x}) = \begin{bmatrix} Z_{11}^{mn}(\tilde{x}) & Z_{12}^{mn}(\tilde{x}) & \ldots & Z_{1r}^{mn}(\tilde{x}) \\ Z_{21}^{mn}(\tilde{x}) & Z_{22}^{mn}(\tilde{x}) & \ldots & Z_{2r}^{mn}(\tilde{x}) \\ \vdots & \vdots & \ddots & \vdots \\ Z_{r1}^{mn}(\tilde{x}) & Z_{r2}^{mn}(\tilde{x}) & \ldots & Z_{rr}^{mn}(\tilde{x}) \end{bmatrix}$$

Remark 2 *The feasibility of the developed SOS condition is influenced by the polynomials $\varepsilon_i^1(x)$, $\varepsilon_i^2(x)$, $\varepsilon_{ij}^1(x)$ and $\varepsilon_{ij}^2(x)$; therefore, in practice the polynomial structure of $\varepsilon_i^1(x)$, $\varepsilon_i^2(x)$, $\varepsilon_{ij}^1(x)$ and $\varepsilon_{ij}^2(x)$ has to be carefully selected.*

Remark 3 *Following the previous study on polynomial fuzzy models, [24, 33], the major drawback of the results is the numerical computational cost. In fact, when the degree 2d increases, the computational complexity increases, and the computational time required increases. Nonetheless, the SOS approach has a clear advantage, as a generalization of the existing approaches to T-S fuzzy system, being more effective in representing nonlinear control systems. In fact, the SOS approach used here is more relaxed than the LMI approach previously used in the literature [32].*

4 Computer simulations

Example 1 *Let us consider the following nonlinear differential equation borrowed from [32]*

$$\frac{\partial^2 q(x,t)}{\partial x \partial t} = a_1 \frac{\partial q(x,t)}{\partial t} + a_2 \frac{\partial q(x,t)}{\partial x} + a_0 \sin^2(q(x,t)) + bf(x,t))$$

with the initial and boundary conditions $q(x,0) = q_1(x)$ and $q(t,0) = q_2(t)$. $q(x,t)$ is the state function, a_0, a_1, a_2, b are real coefficients, and $f(x,t)$ is the input function. If we define

$$x_c^h(x,t) = \frac{\partial q(x,t)}{\partial t} - a_2 q(x,t)$$
$$x_c^v(x,t) = q(x,t)$$

then we obtain the following 2-D model:

$$\begin{bmatrix} \frac{\partial x_c^h(x,t)}{\partial x} \\ \frac{\partial x_c^v(x,t)}{\partial t} \end{bmatrix} = \begin{bmatrix} a_1 & a_1 a_2 + a_0 \sin^2(x_c^v(x,t)) \\ 1 & a_2 \end{bmatrix} \times \begin{bmatrix} x_c^h(x,t) \\ x_c^v(x,t) \end{bmatrix} + \begin{bmatrix} b \\ 0 \end{bmatrix} u_c(x,t)$$

with boundary conditions

$$x_c^h(0,t) = \dot{q}_2(t) - a_2 q_2(t)$$
$$x_c^v(x,t) = q_1(x)$$

Consider the two following rules obtained for $\sin^2(x_c^v(x,t))$*:*

IF $\sin^2(x_c^v(x,t))$ *is about 0, THEN*

$$\begin{bmatrix} \frac{\partial x_c^h(x,t)}{\partial t} \\ \frac{\partial x_c^v(x,t)}{\partial t} \end{bmatrix} = A_1^c \begin{bmatrix} x_c^h(x,t) \\ x_c^v(x,t) \end{bmatrix} + B_1^c u_c(x,t)$$

IF $\sin^2(x_c^v(x,t))$ *is about* ∓ 1*, THEN*

$$\begin{bmatrix} \frac{\partial x_c^h(x,t)}{\partial t} \\ \frac{\partial x_c^v(x,t)}{\partial t} \end{bmatrix} = A_2^c \begin{bmatrix} x_c^h(x,t) \\ x_c^v(x,t) \end{bmatrix} + B_2^c u_c(x,t)$$

with

$$A_1^c = \begin{bmatrix} a_1 & a_1 a_2 \\ 1 & a_2 \end{bmatrix}, B_1^c = \begin{bmatrix} b \\ 0 \end{bmatrix}$$

$$A_2^c = \begin{bmatrix} a_1 & a_1 a_2 + a_0 \\ 1 & a_2 \end{bmatrix}, B_2^c = B_1^c$$

The membership functions are given by:

$$h_1(x,t) = 1 - \sin^2(x_c^v(x,t)), \quad h_2(x,t) = \sin^2(x_c^v(x,t))$$

For control purpose the 2-D T-S system is discretized with sampling intervals T_1 *and* T_2 *corresponding to variables x and t, respectively, with the system parameters given in [32, 29] and [34], obtaining the following system:*

IF $\sin^2(x^v(k,l))$ *is about 0, THEN*

$$\begin{bmatrix} x^h(k+1,l) \\ x^v(k,l+1) \end{bmatrix} = A_1 \begin{bmatrix} x^h(k,l) \\ x^v(k,l) \end{bmatrix} + B_1 u(k,l)$$

IF $\sin^2(x^v(k,l))$ *is about* ∓ 1*, THEN*

$$\begin{bmatrix} x^h(k+1,l) \\ x^v(k,l+1) \end{bmatrix} = A_2 \begin{bmatrix} x^h(k,l) \\ x^v(k,l) \end{bmatrix} + B_2 u(k,l)$$

$$A_1 = \begin{bmatrix} 1 + a_1 T_1 & a_1 a_2 T_1 \\ T_2 & 1 + a_2 T_2 \end{bmatrix}, B_1 = \begin{bmatrix} b T_1 \\ 0 \end{bmatrix}$$

$$A_2 = \begin{bmatrix} 1 + a_1 T_1 & (a_1 a_2 + a_0) T_1 \\ T_2 & 1 + a_2 T_2 \end{bmatrix}, B_2 = B_1$$

The membership functions then become: $h_1(k,l) = 1 - \sin^2(x^v(k,l))$*,* $h_2(k,l) = \sin^2(x^v(k,l))$*. We can refer to [40] for more discussions about the discretization of the continuous-time T-S fuzzy model, see also the papers dealing with the delta operator, like [41]. For the numerical simulation we consider the following parameters:* $a_1 = -3$*,* $a_0 = -2$*,* $b = -1$*,* $T_1 = 0.5$*,* $T_2 = 0.8$*.*

Table 1 Feasibility intervals for a_2

Methods	Feasibility intervals
Usual PDC [32]	$[-1.990, -0.512]$
Theorem 1 of [32]	$[-2.012, -0.494]$
Corollary 2 [32]	$[-2.292, -0.212]$
Theorem 2 non-PDC of [32]	$[-2.492, -0.014]$
Theorem 1 PDC of [28]	$[-2.499, -0.018]$
Theorem 2 non-PDC of [28]	$[-2.65, -0.01]$
Theorem 1 of [29] with $g = 2$	$[-2.502, -0.013]$
Theorem 2 of [29] with $g = d = 2$	$[-2.561, -0.012]$
Theorem 3 of [29] with $g = d = 2$	$[-2.566, -0.011]$
Theorem 1 PDC with $2d = 2$	$[-2.525, -0.001]$
Theorem 2 non-PDC with $2d = 2$	$[-2.743, -0.001]$

The feasible intervals for a_2 *are shown in Table 1 for the proposed technique, and compared with those in the literature. From Table 1, we can observe that the intervals obtained by our proposed Theorems 1 and 2 are larger than the results given by [32], [29] and [28]. Fixing* $a_2 = -2.4$*, and then solving (22) to (26) with the SOSTOOLS solver gives:*

$$K_1(\tilde{x}) = \begin{bmatrix} K_1^1(\tilde{x}) & K_2^1(\tilde{x}) \end{bmatrix}, K_2(\tilde{x}) = \begin{bmatrix} K_1^2(\tilde{x}) & K_2^2(\tilde{x}) \end{bmatrix},$$

$$Y_1(\tilde{x}) = \begin{bmatrix} Y_1^1(\tilde{x}) & 0 \\ 0 & Y_2^1(\tilde{x}) \end{bmatrix}, Y_2(\tilde{x}) = \begin{bmatrix} Y_1^2(\tilde{x}) & 0 \\ 0 & Y_2^2(\tilde{x}) \end{bmatrix}$$

$$K_1^1(\tilde{x}) = -7.5365.10^{-6} x^h(k,l)^2 - 435.36 x^v(k,l)^2$$
$$K_2^1(\tilde{x}) = 0.000070999 x^h(k,l)^2 + 14012.0 x^v(k,l)^2$$
$$K_1^2(\tilde{x}) = -7.2643.10^{-6} x^h(k,l)^2 - 433.25 x^v(k,l)^2$$
$$K_2^2(\tilde{x}) = 0.000051495 x^h(k,l)^2 + 10150.0 x^v(k,l)^2$$
$$Y_1^1(\tilde{x}) = 7.0585.10^{-6} x^h(k,l)^2 + 427.83 x^v(k,l)^2$$
$$Y_2^1(\tilde{x}) = 0.000010406 x^h(k,l)^2 + 1945.4 x^v(k,l)^2$$
$$Y_1^2(\tilde{x}) = 7.1543.10^{-6} x^h(k,l)^2 + 428.39 x^v(k,l)^2$$
$$Y_2^2(\tilde{x}) = 0.000010334 x^h(k,l)^2 + 1951.3 x^v(k,l)^2$$

with the boundary conditions

$$x^h(0,l) = 0.2, \quad 0 \le l \le 30$$
$$x^v(k,0) = 0.3, \quad 0 \le k \le 30$$
$$x^h(0,l) = 0.05, \quad x^v(k,0) = 0.1, \quad i,j > 30$$

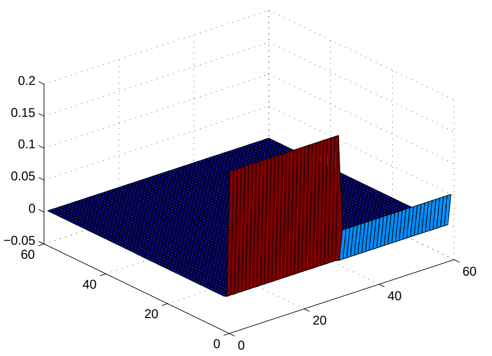

Fig. 1 Horizontal states $x^h(k,l)$ for $a_2 = -2.4$ using the proposed controller.

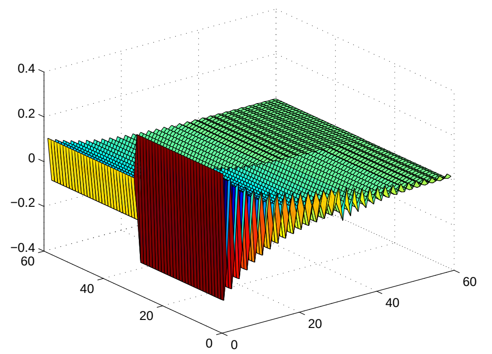

Fig. 2 Vertical states $x^v(k,l)$ for $a_2 = -2.4$ using the proposed controller.

Figures 1 and 2 show the evolutions of the two states $x^h(k,l)$ and $x^v(k,l)$, respectively: it is clear that the polynomial discrete 2-D T-S system (4) with the polynomial controller (20) is asymptotically stable.

Remark 4 *The same system can be studied with the saturation. In fact, 2-D systems with saturation has been already studied in [39] (FM T-S system) where the sinusoid was used in the example. The 2-D saturated systems for Roesser model was also solved in [38], where the saturation function used is the standard symmetric one. Note that we have already dealt with the problem of stabilization of nonlinear discrete-time T-S fuzzy systems with actuator saturation in [37] for (1-D), where the sinusoid was used in the example, and we will consider the discrete 2-D T-S problem as future work.*

For a polynomial $X(\tilde{x}) \in \mathbb{R}^{n_d \times n_d}$, the complexity of computing the SOS decomposition, depends on two factors: the number of variables and the degree of the polynomial $(2d+1)n_d^2$.
The number of variable (N.V) and the number of (LMIs / SOS) are shown in Table 2 for the example. Notice that with this technique lies when the number of variables or the degree of the polynomial are increased, the conservatism of the result decreases, but the computational complexity increases.

Example 2 *Consider the following discrete-time nonlinear plant represented by the polynomial 2-D T-S fuzzy Roesser model with the following two rules:*

$$A_1 = \begin{bmatrix} 0.25 & 0.675 \\ 0.75x^h(k,l) & 0.325 \end{bmatrix}, B_1 = \begin{bmatrix} -0.36x^v(k,l) \\ 0 \end{bmatrix}$$

$$A_2 = \begin{bmatrix} 0.25 & 0.105 \\ 0.75x^h(k,l) & 0.325 \end{bmatrix}, B_2 = B_1$$

The membership functions are given as: $h_1(k,l) = 1 - \sin^2(x^v(k,l))$, $h_2(k,l) = \sin^2(x^v(k,l))$.

By solving the SOS design condition in Theorems 2, the asymptotical stability for the above 2-D polynomial fuzzy system is ensured. The corresponding controller gain matrices are given by:

$$K_1(\tilde{x}) = \begin{bmatrix} K_1^1(\tilde{x}) & K_2^1(\tilde{x}) \end{bmatrix}, K_2(\tilde{x}) = \begin{bmatrix} K_1^2(\tilde{x}) & K_2^2(\tilde{x}) \end{bmatrix},$$

$$Y_1(\tilde{x}) = \begin{bmatrix} Y_1^1(\tilde{x}) & 0 \\ 0 & Y_2^1(\tilde{x}) \end{bmatrix}, Y_2(\tilde{x}) = \begin{bmatrix} Y_1^2(\tilde{x}) & 0 \\ 0 & Y_2^2(\tilde{x}) \end{bmatrix}$$

$$K_1^1(\tilde{x}) = 0.45443x^v(k,l) + 3.4487.10^{-18}$$
$$K_2^1(\tilde{x}) = 1627.9x^v(k,l) - 1.0845.10^{-15}$$
$$K_1^2(\tilde{x}) = 0.45512x^v(k,l) + 1.4207.10^{-17}$$
$$K_2^2(\tilde{x}) = 253.6x^v(k,l) - 1.4194.10^{-17}$$
$$Y_1^1(\tilde{x}) = 6.0479.10^{-6}x^h(k,l)^2 + 6.6041x^v(k,l)^2$$
$$Y_2^1(\tilde{x}) = 0.0040157x^h(k,l)^2 + 8681.2x^v(k,l)^2$$
$$Y_1^2(\tilde{x}) = 3.9496.10^{-6}x^h(k,l)^2 + 6.609x^v(k,l)^2$$
$$Y_2^2(\tilde{x}) = 0.0037871x^h(k,l)^2 + 8693.1x^v(k,l)^2$$

with the boundary conditions

$$x^h(0,l) = 0.2, \quad 0 \le l \le 30$$
$$x^v(k,0) = 0.3, \quad 0 \le k \le 30$$
$$x^h(0,l) = 0.05, \quad x^v(k,0) = 0.1, \quad i,j > 30$$

Figs. 3-4 show the state trajectory of the system state variables $x^h(k,l)$ and $x^v(k,l)$, respectively. The simulation results in Figs. 3-4, show that the polynomial discrete 2-D T-S system is asymptotically stable. Hence, the effectiveness of the proposed approach has been illustrated in the numerical example.

Table 2 Number of variables and number of (LMIs/ SOS constraints), where $n_1 = n_2 = n_d$ and $m_1 = m_2 = m_d$

Methods	N.V	Number of (LMIs/ SOS constraints)
Theorem 1 PDC in [32]	$n_d(n_d+1)+2m_dn_dr+16n_d^2r+n_d(4n_d+1)r(r+1)$	$r+\frac{r}{2}(r-1)+1$
Theorem 1 PDC for degre 2d	$(2d+1)[n_d(n_d+1)r+2n_dm_dr]$	$2r+r^3+\frac{r^3}{2}(r-1)$
Theorem 2 NON-PDC in [32]	$n_d(n_d+1)r+2m_dn_dr+2n_d^2r+16n_d^2r^3+n_d(4n_d+1)r^3(r+1)$	$r^3+\frac{r^3}{2}(r-1)+r^2$
Theorem 2 NON-PDC for degre 2d	$(2d+1)[n_d(n_d+1)r+2m_dn_dr+2n_d^2r+16n_d^2r^3+n_d(4n_d+1)r^3(r+1)]$	$2r+r^3+\frac{r^3}{2}(r-1)+r^2$

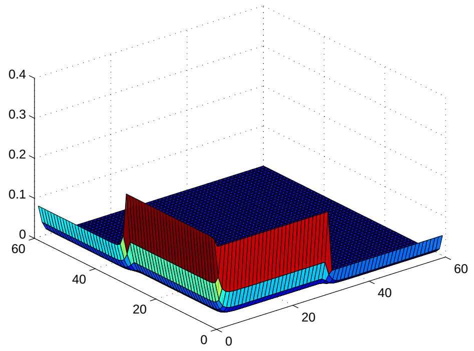

Fig. 3 Trajectory of the 2-D polynomial system state $x^h(k,l)$.

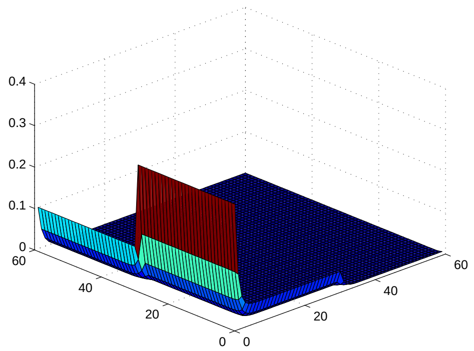

Fig. 4 Trajectory of the 2-D polynomial system state $x^v(k,l)$.

5 Conclusion

Stabilization of a class of 2-D nonlinear systems has been solved in this paper using a Sum-of-Square approach. More precisely, stabilization of the class of 2-D discrete systems that can be described in terms of Takagi-Sugeno Roesser-type polynomial models has been studied using PDC controllers. Firstly, an stabilization condition based on polynomial multiple Lyapunov functions has been derived. Secondly, a non-quadratic stabilization condition was developed using a relaxation technique. These design conditions are formulated in terms of SOS, to make them numerically tractable. A numerical example has been provided to show the effectiveness of the proposed results.

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