



Differential stability properties in convex scalar and vector optimization

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Abstract

This paper focuses on formulas for the ε -subdifferential of the optimal value function of scalar and vector convex optimization problems. These formulas can be applied when the set of solutions of the problem is empty. In the scalar case, both unconstrained problems and problems with an inclusion constraint are considered. For the last ones, limiting results are derived, in such a way that no qualification conditions are required. The main mathematical tool is a limiting calculus rule for the ε -subdifferential of the sum of convex and lower semicontinuous functions defined on a (non necessarily reflexive) Banach space. In the vector case, unconstrained problems are studied and exact formulas are derived by linear scalarizations. These results are based on a concept of infimal set, the notion of cone proper set and an ε -subdifferential for convex vector functions due to Taa.

Keywords Differential stability · ε -subdifferential · Parametric convex programming · Limiting calculus rule · Optimal value function · Approximate solution · Vector optimization · Infimal set · Cone proper set · Weak minimal solution

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Dedicated to Professor R. Tyrrell Rockafellar on the occasion of his 85th birthday.

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1 Introduction

Studying *differential stability* of optimization problems usually means to study differentiability properties of the optimal value function in parametric mathematical programming. We refer to [1, 2, 4, 7, 8, 25–27, 29, 31, 37] and the references therein for some old and new results in this direction.

Convex optimization is a subfield of mathematical optimization that studies the problem of minimizing convex functions over convex sets. In the early 1960's, Moreau and Rockafellar [31] introduced the concept of *subgradient* for convex functions, initiating the developments of theoretical and applied convex analysis. Ioffe and Tihomirov [21], Hiriart-Urruty and Lemaréchal [18, 19], Phelps [30], Zălinescu [37] and Borwein and Vanderwerff [9] presented a beautiful theory about convex sets and convex functions in finite and infinite dimensional spaces with many significant applications in mathematical programming, classical variational calculus, and optimal control theory.

In 1965, Brøndsted and Rockafellar [11] introduced the concept of approximate subdifferential (also called ε -subdifferential) of a convex function. It has become an important tool for the study of algorithms as well as for theoretical purposes in convex optimization. For more information, the reader is referred to [14, 17, 20, 28, 37] and the references therein.

This paper concerns with the study of differential stability properties to scalar and vector convex programming problems.

In the literature on differentiability properties of the optimal value function (also named efficient value mapping and marginal or perturbation function) of a parametric family of optimization problems including inclusion constraints, different qualification conditions are assumed (see the recent papers [1, 2, 4] and the references therein). However, it is well-known that these regularity conditions are usually difficult to check and they could not be satisfied. Then, an objective of this paper is to derive limiting versions of these differentiability properties where no qualification conditions are required. In addition, lots of differential stability results of the literature are based on the existence of solutions in the involved problem. Unfortunately, this assumption is not always satisfied (see [3, 28]). Thus, a second aim of this work is to derive such differential stability results for problems whose solution sets could be empty (see [7, 8] and the references therein for a different approach based on the regularization of the problem).

In vector optimization, additional technical difficulties happen. Namely, the optimal value mapping is set-valued and involves some concepts of infimal point (see [6, 12, 22, 23, 36]), which are the vector counterpart to the notion of infimum of a set of real numbers. As a result, differential stability properties in vector optimization problems are formulated in terms of graphical and epigraphical derivatives and most of the obtained results require the so-called domination property, which implies the existence of exact solutions of the problem (see [23, 24, 33, 34, 36]). Again, this paper concerns with differential stability of vector optimization problems that would not satisfy the domination property (in particular with empty solution set).

The contents of the paper are as follows. Section 2 collects some basic notations and concepts. In Section 3, a limiting calculus rule for the ε -subdifferential of the sum of m proper lower semicontinuous convex functions on a (non necessarily reflexive) Banach space is stated without requiring any qualification condition. It is derived by convex analysis tools and a generalization of the well-known Brøndsted-Rockafellar Theorem. Sections 4 and 5 are devoted to the differential stability of unconstrained and constrained convex optimization problems, respectively, whose solution sets can be empty. In Section 5, an inclusion

constraint is considered and a limiting formula for ε -subdifferential of the optimal value function is obtained. This result is a consequence of both the previous one in Section 4 dealing with unconstrained problems and the limiting sum rule stated in Section 2. It is also showed that it reduces to an exact formula provided that the so-called Robinson-Rockafellar condition is satisfied. Section 6 involves the differential stability of convex vector optimization problems. In deriving it, a linear scalarization approach is considered. The obtained differentiability properties are formulated by ε -subgradients of the scalarized function and also by ε -subgradients of the optimal value function of the scalarized problems. The main mathematical tools are an ε -subdifferential for vector functions due Taa (see [35]) and a concept of infimal point (see [6, 12]). As in the previous two sections, the set of solutions of the problem can be empty. Finally, in Section 7, the conclusions of this work are underlined.

2 Preliminaries and Mathematical Tools

Throughout $\overline{\mathbb{R}}$ stands for the set $\mathbb{R} \cup \{\pm\infty\}$, \mathbb{R}_+^p is the nonnegative orthant of \mathbb{R}^p and $\mathbb{R}_+ := \mathbb{R}_+^1$. Let X be a real locally convex Hausdorff topological linear space. The topological dual space of X is denoted by X^* .

For a set $C \subset X$, we denote by $\text{int } C$ and $\text{cl } C$ the topological interior and the closure of C , respectively. The core or algebraic interior of C is defined by

$$\text{core } C := \{\bar{x} \in C \mid \text{for each } x \in X, \exists \bar{\mu} > 0 \text{ such that } \bar{x} + \mu x \in C, \forall \mu \in [0, \bar{\mu}]\}.$$

Given a function $f : X \rightarrow \overline{\mathbb{R}}$, we denote the effective domain and the epigraph of f by $\text{dom } f$ and $\text{epi } f$ respectively, i.e.,

$$\begin{aligned} \text{dom } f &:= \{x \in X \mid f(x) < +\infty\}, \\ \text{epi } f &:= \{(x, \alpha) \in X \times \mathbb{R} \mid f(x) \leq \alpha\}. \end{aligned}$$

One says that f is proper if $f(x) > -\infty$ for all $x \in X$ and $\text{dom } f \neq \emptyset$. The function f is called lower-semicontinuous if $\text{epi } f$ is closed.

Definition 1 Let $f : X \rightarrow \overline{\mathbb{R}}$ be a proper convex function and $\varepsilon \geq 0$. The ε -subdifferential (or approximate subdifferential) of f at a point $x_0 \in \text{dom } f$ is the set

$$\partial_\varepsilon f(x_0) := \{x^* \in X^* \mid \langle x^*, x - x_0 \rangle \leq f(x) - f(x_0) + \varepsilon, \forall x \in X\}.$$

The function f is said to be ε -subdifferentiable at x_0 if $\partial_\varepsilon f(x_0) \neq \emptyset$.

Remark 1 Although a nonconvex function could be ε -subdifferentiable at some point in its effective domain, the proper class of functions where that notion has sense is the class of convex functions. More precisely, it is well-known that if f is a proper lower-semicontinuous convex function, then f is ε -subdifferentiable at every point $x_0 \in \text{dom } f$, for all $\varepsilon > 0$ (see [17]).

The set $\partial_\varepsilon f(x_0)$ reduces to the subdifferential $\partial f(x_0)$ when $\varepsilon = 0$, i.e., $\partial f(x_0) = \partial_0 f(x_0)$. Moreover, $\partial_\varepsilon f(x_0) = \bigcap_{\eta > \varepsilon} \partial_\eta f(x_0)$, for all $\varepsilon \geq 0$, and from the definition it follows that $\partial_\varepsilon f(x_0)$ is a weakly*-closed, convex set.

Given a nonempty subset $\Omega \subset X$, the indicator function δ_Ω of Ω is defined by setting $\delta_\Omega(x) = 0$ if $x \in \Omega$ and $\delta_\Omega(x) = +\infty$ if $x \notin \Omega$.

Definition 2 Let $\Omega \subset X$ be a nonempty convex set. The set $N_\varepsilon(x_0, \Omega)$ of ε -normal directions of Ω at $x_0 \in \Omega$ is defined by

$$N_\varepsilon(x_0, \Omega) := \{x^* \in X^* \mid \langle x^*, x - x_0 \rangle \leq \varepsilon, \forall x \in \Omega\}.$$

It is easy to see that $N_\varepsilon(x_0, \Omega) = \partial_\varepsilon \delta_\Omega(x_0)$ for every $\varepsilon \geq 0$. Moreover, when $\varepsilon = 0$, $N_\varepsilon(x_0, \Omega)$ reduces to the normal cone of Ω at x_0 , i.e., $N_0(x_0, \Omega) = N(x_0, \Omega)$. However, as a general rule, $N_\varepsilon(x_0, \Omega)$ is not a cone when $\varepsilon > 0$.

3 Sum Rules for the Approximate Subdifferential

In this section, X is assumed to be a Banach space with the norm $\|\cdot\|_X$. The dual space of X is denoted by X^* with the dual norm $\|\cdot\|_{X^*}$. Let f_1, f_2, \dots, f_m be proper lower-semicontinuous convex functions on X . The aim of this section is to derive a formula for the ε -subdifferential of the sum $\sum_{i=1}^m f_i$ at a given point $x_0 \in \bigcap_{i=1}^m \text{dom } f_i$ without requiring any constraint qualification.

The next result extends [20, Theorem 3.2], which is stated for the case $m = 2$.

Theorem 1 Let $f_i : X \rightarrow \overline{\mathbb{R}}$ be a proper lower-semicontinuous convex function, $i = 1, 2, \dots, m$, and let $x_0 \in \bigcap_{i=1}^m \text{dom } f_i$. Then, for every $\varepsilon > 0$, one has

$$\partial_\varepsilon \left(\sum_{i=1}^m f_i \right) (x_0) = \text{cl}_{w^*} \left(\bigcup_{\substack{(\varepsilon_1, \varepsilon_2, \dots, \varepsilon_m) \in \mathbb{R}_+^m \\ \sum_{i=1}^m \varepsilon_i = \varepsilon}} \sum_{i=1}^m \partial_{\varepsilon_i} f_i(x_0) \right), \tag{1}$$

where cl_{w^*} denotes the closure with respect to the weak* topology of X^* .

Proof Consider the Cartesian product $\tilde{X} := \underbrace{X \times X \times \dots \times X}_m$ and the continuous linear function $A : X \rightarrow \tilde{X}$, $A(x) = (x, x, \dots, x)$, for all $x \in X$. Define the function $f : \tilde{X} \rightarrow \overline{\mathbb{R}}$, with $f(x_1, x_2, \dots, x_m) = \sum_{i=1}^m f_i(x_i)$, for all $(x_1, x_2, \dots, x_m) \in \tilde{X}$. Clearly, f is a proper lower-semicontinuous convex function, $Ax_0 \in \text{dom } f$ and $f \circ A = \sum_{i=1}^m f_i$. By applying [20, Theorem 7.1] to the data A and f we obtain

$$\partial_\varepsilon \left(\sum_{i=1}^m f_i \right) (x_0) = \partial_\varepsilon (f \circ A) (x_0) = \text{cl}_{w^*} (A^* \partial_\varepsilon f (Ax_0)), \tag{2}$$

where $A^* : \tilde{X}^* \rightarrow X^*$ stands for the adjoint operator of A . It follows that

$$A^*(x_1^*, x_2^*, \dots, x_m^*) = \sum_{i=1}^m x_i^*, \quad \forall x_i^* \in X^*. \tag{3}$$

In addition, by [37, Corollary 2.4.5] we see that

$$\partial_\varepsilon f(Ax_0) = \bigcup_{\substack{(\varepsilon_1, \varepsilon_2, \dots, \varepsilon_m) \in \mathbb{R}_+^m \\ \sum_{i=1}^m \varepsilon_i = \varepsilon}} \prod_{i=1}^m \partial_{\varepsilon_i} f_i(x_0). \tag{4}$$

The result follows as a consequence of statements (2), (3) and (4). □

The next result extends the sequential sum rule in [14, Theorem 3] to more than two functions defined in a not necessarily reflexive Banach space. Although its proof is similar to the one in [14, Theorem 3], we include it for the sake of completeness.

Theorem 2 *Let $f_i : X \rightarrow \overline{\mathbb{R}}$ be a proper lower-semicontinuous convex function, $i = 1, 2, \dots, m$, and let $x_0 \in \bigcap_{i=1}^m \text{dom} f_i$. Then, $x^* \in \partial_\varepsilon \left(\sum_{i=1}^m f_i \right) (x_0)$ if and only if there exist $\varepsilon_i \geq 0$, $\sum_{i=1}^m \varepsilon_i = \varepsilon$ and nets $(x_{i,\alpha})_\alpha \subset \text{dom} f_i$, $x_{i,\alpha} \xrightarrow{\| \cdot \|_X} x_0$, $(x_{i,\alpha}^*)_\alpha \subset X^*$, $\sum_{i=1}^m x_{i,\alpha}^* \xrightarrow{w^*} x^*$ such that*

$$x_{i,\alpha}^* \in \partial_{\varepsilon_i} f_i(x_{i,\alpha}), \quad \forall \alpha, \tag{5}$$

$$f_i(x_{i,\alpha}) - \langle x_{i,\alpha}^*, x_{i,\alpha} - x_0 \rangle \rightarrow f_i(x_0). \tag{6}$$

Proof Consider $x^* \in \partial_\varepsilon \left(\sum_{i=1}^m f_i \right) (x_0)$. By formula (1) there exist nets $(\varepsilon_{i,\alpha})_\alpha \subset \mathbb{R}_+$ and $(u_{i,\alpha}^*)_\alpha \subset X^*$ such that $\sum_{i=1}^m \varepsilon_{i,\alpha} = \varepsilon$, $u_{i,\alpha}^* \in \partial_{\varepsilon_{i,\alpha}} f_i(x_0)$ and $\sum_{i=1}^m u_{i,\alpha}^* \xrightarrow{w^*} x^*$, for every α . For each i , we have that $\varepsilon_{i,\alpha} \in [0, \varepsilon]$, for all α . Without restriction of generality, we can assume that $\varepsilon_{i,\alpha} \rightarrow \varepsilon_i$, where $\varepsilon_i \geq 0$, $\sum_{i=1}^m \varepsilon_i = \varepsilon$.

In addition, by applying [14, Proposition 2] to each i and α it follows that there exist nets $(x_{i,\alpha})_\alpha \subset \text{dom} f_i$ and $(x_{i,\alpha}^*)_\alpha \subset X^*$ satisfying

$$\begin{cases} x_{i,\alpha}^* \in \partial_{\varepsilon_i} f_i(x_{i,\alpha}), \\ \|x_{i,\alpha} - x_0\|_X \leq \sqrt{a_{i,\alpha}}, \\ \|x_{i,\alpha}^* - u_{i,\alpha}^*\|_{X^*} \leq \sqrt{a_{i,\alpha}}, \\ |f_i(x_{i,\alpha}) - \langle x_{i,\alpha}^*, x_{i,\alpha} - x_0 \rangle - f_i(x_0)| \leq 2a_{i,\alpha}, \end{cases} \tag{7}$$

where $a_{i,\alpha} = |\varepsilon_{i,\alpha} - \varepsilon_i|$. Clearly, assertions (5), (6) and $x_{i,\alpha} \xrightarrow{\| \cdot \|_X} x_0$ are satisfied. In addition, as $\sum_{i=1}^m u_{i,\alpha}^* \xrightarrow{w^*} x^*$, from the third statement of (7) we deduce

$$\sum_{i=1}^m x_{i,\alpha}^* = \sum_{i=1}^m (x_{i,\alpha}^* - u_{i,\alpha}^*) + \sum_{i=1}^m u_{i,\alpha}^* \xrightarrow{w^*} x^*$$

and the proof of the necessary condition is complete.

Conversely, assume that there exist $\varepsilon_i \geq 0$, $\sum_{i=1}^m \varepsilon_i = \varepsilon$, nets $(x_{i,\alpha})_\alpha \subset \text{dom} f_i$ and $(x_{i,\alpha}^*)_\alpha \subset X^*$ satisfying $x_{i,\alpha} \xrightarrow{\| \cdot \|_X} x_0$, $\sum_{i=1}^m x_{i,\alpha}^* \xrightarrow{w^*} x^*$, conditions (5) and (6). From (5), we

obtain for each i :

$$f_i(x) \geq f_i(x_{i,\alpha}) - \varepsilon_i + \langle x_{i,\alpha}^*, x - x_{i,\alpha} \rangle, \quad \forall x \in X,$$

and so

$$\begin{aligned} \sum_{i=1}^m f_i(x) &\geq \sum_{i=1}^m f_i(x_{i,\alpha}) - \varepsilon + \sum_{i=1}^m \langle x_{i,\alpha}^*, x - x_{i,\alpha} \rangle \\ &= \sum_{i=1}^m (f_i(x_{i,\alpha}) - \langle x_{i,\alpha}^*, x_{i,\alpha} - x_0 \rangle) - \varepsilon + \langle \sum_{i=1}^m x_{i,\alpha}^*, x - x_0 \rangle, \quad \forall x \in X. \end{aligned}$$

By (6) and condition $\sum_{i=1}^m x_{i,\alpha}^* \xrightarrow{w^*} x^*$ it follows that, for each $x \in X$,

$$\sum_{i=1}^m (f_i(x_{i,\alpha}) - \langle x_{i,\alpha}^*, x_{i,\alpha} - x_0 \rangle) - \varepsilon + \langle \sum_{i=1}^m x_{i,\alpha}^*, x - x_0 \rangle \rightarrow \sum_{i=1}^m f_i(x_0) - \varepsilon + \langle x^*, x - x_0 \rangle.$$

Therefore,

$$\sum_{i=1}^m f_i(x) \geq \sum_{i=1}^m f_i(x_0) - \varepsilon + \langle x^*, x - x_0 \rangle, \quad \forall x \in X.$$

In other words, $x^* \in \partial_\varepsilon \left(\sum_{i=1}^m f_i \right) (x_0)$, and the proof finishes. □

Remark 2 It is not hard to check that the next set is convex:

$$\bigcup_{\substack{(\varepsilon_1, \varepsilon_2, \dots, \varepsilon_m) \in \mathbb{R}_+^m \\ \sum_{i=1}^m \varepsilon_i = \varepsilon}} \sum_{i=1}^m \partial_{\varepsilon_i} f_i(x_0).$$

Therefore, if X is a reflexive Banach space, formula (1) can be formulated as follows:

$$\partial_\varepsilon \left(\sum_{i=1}^m f_i \right) (x_0) = \text{cl}_{\| \cdot \|_{X^*}} \left(\bigcup_{\substack{(\varepsilon_1, \varepsilon_2, \dots, \varepsilon_m) \in \mathbb{R}_+^m \\ \sum_{i=1}^m \varepsilon_i = \varepsilon}} \sum_{i=1}^m \partial_{\varepsilon_i} f_i(x_0) \right)$$

and then nets in Theorem 2 can be replaced by sequences and the condition $\sum_{i=1}^m x_{i,\alpha}^* \xrightarrow{w^*} x^*$

by the statement $\sum_{i=1}^m x_{i,\alpha}^* \xrightarrow{\| \cdot \|_{X^*}} x^*$, where $\text{cl}_{\| \cdot \|_{X^*}}$ denotes the closure for the strong topology on X^* . In particular, Theorem 2 reduces to [14, Theorem 3] by considering $m = 2$ and a reflexive Banach space X .

4 Differential Stability of Unconstrained Optimization Problems

Let X and Y be real locally convex Hausdorff topological linear spaces. Let $\varphi : X \times Y \rightarrow \overline{\mathbb{R}}$ be an extended real-valued function. Consider the parametric unconstrained optimization problem

$$\min\{\varphi(x, y) \mid y \in Y\} \tag{P_x}$$

depending on the parameter $x \in X$. Function φ is called the objective function of problem (P_x) . The optimal value function $\mu : X \rightarrow \overline{\mathbb{R}}$ of (P_x) is

$$\mu(x) := \inf \{ \varphi(x, y) \mid y \in Y \}.$$

For each $x \in X$, the set of approximate solutions of (P_x) with error $\eta \geq 0$ is denoted by $M_\eta(x)$, i.e.,

$$M_\eta(x) := \{ y \in Y \mid \varphi(x, y) \leq \mu(x) + \eta \}.$$

Then, $M(x) := M_0(x)$ is the set of (exact) solutions of problem (P_x) .

The following result was derived in [28, Corollary 5], [20, Corollary 5.1], [37, Theorem 2.6.2, p. 109] and [3, Theorem 4.1] for the case where φ is a convex function. Next we state it by a proof that makes clear that such a convex assumption is only required to avoid the nonemptiness of the involved ε -subdifferentials (see Remark 1). Particularly, it is worthy to stress that the proof does not involve the conjugate function of the optimal value function μ . In [37] the reader can find other properties of ε -subdifferentials that are stated without assuming convexity assumptions.

Theorem 3 *Suppose that φ is convex and μ is finite at $\bar{x} \in X$. For each $\eta > 0$, consider a point $\bar{y}_\eta \in M_\eta(\bar{x})$. Then, for every $\varepsilon \geq 0$, one has*

$$\begin{aligned} \partial_\varepsilon \mu(\bar{x}) &= \bigcap_{\eta > 0} \bigcap_{y_\eta \in M_\eta(\bar{x})} \{ x^* \in X^* \mid (x^*, 0) \in \partial_{\varepsilon+\eta} \varphi(\bar{x}, y_\eta) \} \\ &= \bigcap_{\eta > 0} \{ x^* \in X^* \mid (x^*, 0) \in \partial_{\varepsilon+\eta} \varphi(\bar{x}, \bar{y}_\eta) \} \\ &= \bigcap_{\eta > 0} \bigcup_{y \in Y} \{ x^* \in X^* \mid (x^*, 0) \in \partial_{\varepsilon+\eta} \varphi(\bar{x}, y) \}. \end{aligned} \tag{8}$$

Proof Let $x^* \in X^*$, $\eta > 0$ and $y_\eta \in M_\eta(\bar{x})$. From the definition, we have that

$$\begin{aligned} x^* \in \partial_\varepsilon \mu(\bar{x}) &\iff \mu(x) \geq \mu(\bar{x}) - \varepsilon + \langle x^*, x - \bar{x} \rangle, \quad \forall x \in X \\ &\iff \varphi(x, y) \geq \mu(\bar{x}) - \varepsilon + \langle x^*, x - \bar{x} \rangle, \quad \forall x \in X, y \in Y \\ &\implies \varphi(x, y) \geq \varphi(\bar{x}, y_\eta) - (\varepsilon + \eta) + \langle x^*, x - \bar{x} \rangle, \quad \forall x \in X, y \in Y \\ &\iff \varphi(x, y) \geq \varphi(\bar{x}, y_\eta) - (\varepsilon + \eta) + \langle x^*, x - \bar{x} \rangle + \langle 0, y - y_\eta \rangle, \quad \forall x \in X, y \in Y \\ &\iff (x^*, 0) \in \partial_{\varepsilon+\eta} \varphi(\bar{x}, y_\eta). \end{aligned}$$

Therefore,

$$\begin{aligned} \partial_\varepsilon \mu(\bar{x}) &\subseteq \bigcap_{\eta > 0} \bigcap_{y_\eta \in M_\eta(\bar{x})} \{ x^* \in X^* \mid (x^*, 0) \in \partial_{\varepsilon+\eta} \varphi(\bar{x}, y_\eta) \} \\ &\subseteq \bigcap_{\eta > 0} \{ x^* \in X^* \mid (x^*, 0) \in \partial_{\varepsilon+\eta} \varphi(\bar{x}, \bar{y}_\eta) \} \\ &\subseteq \bigcap_{\eta > 0} \bigcup_{y \in Y} \{ x^* \in X^* \mid (x^*, 0) \in \partial_{\varepsilon+\eta} \varphi(\bar{x}, y) \}. \end{aligned}$$

Reciprocally, consider $x^* \in X^*, \delta \geq 0$ and $y \in Y$. Then,

$$\begin{aligned} (x^*, 0) \in \partial_\delta \varphi(\bar{x}, y) &\iff \varphi(x, v) \geq \varphi(\bar{x}, y) - \delta + \langle x^*, x - \bar{x} \rangle, \quad \forall x \in X, v \in Y \quad (9) \\ &\implies \varphi(x, v) \geq \mu(\bar{x}) - \delta + \langle x^*, x - \bar{x} \rangle, \quad \forall x \in X, v \in Y. \end{aligned}$$

Taking infimum on $v \in Y$, we obtain $\mu(x) \geq \mu(\bar{x}) - \delta + \langle x^*, x - \bar{x} \rangle, \quad \forall x \in X$. This means that $x^* \in \partial_\delta \mu(\bar{x})$. Thus,

$$\bigcap_{\eta > 0} \bigcup_{y \in Y} \{x^* \in X^* | (x^*, 0) \in \partial_{\varepsilon+\eta} \varphi(\bar{x}, y)\} \subseteq \bigcap_{\eta > 0} \partial_{\varepsilon+\eta} \mu(\bar{x}) = \partial_\varepsilon \mu(\bar{x})$$

and the proof finishes. □

Remark 3 Notice from (9) that if φ is proper, then $(x^*, 0) \in \partial \varphi(\bar{x}, y)$ implies $\mu(x) > -\infty$, for all $x \in X$. Thus, the inclusion of Theorem 3

$$\bigcap_{\eta > 0} \bigcup_{y \in Y} \{x^* \in X^* | (x^*, 0) \in \partial_{\varepsilon+\eta} \varphi(\bar{x}, y)\} \subset \partial_\varepsilon \mu(\bar{x})$$

holds true without requiring the assumption $\mu(\bar{x}) \in \mathbb{R}$.

Corollary 1 *Suppose that φ is convex and μ is finite at $\bar{x} \in X$ and $M(\bar{x}) \neq \emptyset$. Then, for every $\varepsilon \geq 0$ and $y \in M(\bar{x})$, one has*

$$\partial_\varepsilon \mu(\bar{x}) = \{x^* \in X^* | (x^*, 0) \in \partial_\varepsilon \varphi(\bar{x}, y)\}.$$

Proof Let $\varepsilon \geq 0$ and $y \in M(\bar{x})$. Since $M(\bar{x}) \subset M_\eta(\bar{x})$, for all $\eta > 0$, we can apply equality (8) to $\bar{y}_\eta := y$ and we obtain

$$\begin{aligned} \partial_\varepsilon \mu(\bar{x}) &= \bigcap_{\eta > 0} \{x^* \in X^* | (x^*, 0) \in \partial_{\varepsilon+\eta} \varphi(\bar{x}, y)\} \\ &= \left\{ x^* \in X^* | (x^*, 0) \in \bigcap_{\eta > 0} \partial_{\varepsilon+\eta} \varphi(\bar{x}, y) \right\} \\ &= \{x^* \in X^* | (x^*, 0) \in \partial_\varepsilon \varphi(\bar{x}, y)\} \end{aligned}$$

and the result is proved. □

Let us illustrate with a simple example that Theorem 3 and Corollary 1 also work for a nonconvex function φ (see Remark 1 and the paragraph just before Theorem 3).

Example 1 Consider the function $\varphi : \mathbb{R}^2 \rightarrow \mathbb{R}, \varphi(x, y) = e^{x|y|}$, for all $(x, y) \in \mathbb{R}^2$. Clearly, $\mu(x) = 1$, for all $x \geq 0$ and $\mu(x) = 0$ otherwise. Therefore,

$$\partial_\varepsilon \mu(x) = \begin{cases} \{0\} & \text{if } x < 0, \varepsilon \geq 0, \\ \emptyset & \text{if } x \geq 0, 0 \leq \varepsilon < 1, \\ \{0\} & \text{if } x \geq 0, \varepsilon \geq 1. \end{cases}$$

In addition, it is not hard to check that

$$\partial_\varepsilon \varphi(x, y) = \begin{cases} \emptyset & \text{if } 0 \leq \varepsilon < e^{x|y|}, \\ \{(0, 0)\} & \text{if } \varepsilon \geq e^{x|y|}, \end{cases}$$

and

$$M_\eta(x) = \begin{cases} \mathbb{R} & \text{if } x = 0, \eta > 0, \\ (-\infty, -\ln \eta/x] \cup [\ln \eta/x, +\infty) & \text{if } x < 0, 0 < \eta < 1, \\ \mathbb{R} & \text{if } x < 0, \eta \geq 1, \\ [-\ln(1 + \eta)/x, \ln(1 + \eta)/x] & \text{if } x > 0, \eta > 0. \end{cases}$$

Next we derive $\partial_\varepsilon \mu(x)$ via formula (8). If $x < 0$ and $\varepsilon \geq 0$, or $x \geq 0$ and $\varepsilon \geq 1$, it follows that

$$\begin{aligned} \partial_\varepsilon \mu(x) &= \bigcap_{\eta > 0} \{x^* \in \mathbb{R} \mid (x^*, 0) \in \partial_{\varepsilon+\eta} \varphi(x, y_\eta)\} \\ &= \bigcap_{\eta > 0} \{x^* \in \mathbb{R} \mid (x^*, 0) \in \{(0, 0)\}\} \\ &= \{0\}. \end{aligned}$$

For $x \geq 0$ and $\varepsilon \in [0, 1)$ notice that $\partial_{\varepsilon+\eta} \varphi(x, y_\eta) = \emptyset$ for $\eta = \frac{1-\varepsilon}{2}$. Then, we have that

$$\begin{aligned} \partial_\varepsilon \mu(x) &= \bigcap_{\eta > 0} \{x^* \in \mathbb{R} \mid (x^*, 0) \in \partial_{\varepsilon+\eta} \varphi(x, y_\eta)\} \\ &\subset \{x^* \in \mathbb{R} \mid (x^*, 0) \in \partial_{\varepsilon+\frac{1-\varepsilon}{2}} \varphi(x, y_\eta)\} \\ &= \emptyset. \end{aligned}$$

To state $\partial_\varepsilon \mu(x)$ via Corollary 1 observe that

$$M(x) = \begin{cases} \emptyset & \text{if } x < 0, \\ \mathbb{R} & \text{if } x = 0, \\ \{0\} & \text{if } x > 0. \end{cases}$$

Then, for all $x \geq 0, y \in M(x)$ and $\varepsilon \geq 0$,

$$\partial_\varepsilon \mu(x) = \{x^* \in \mathbb{R} \mid (x^*, 0) \in \partial_\varepsilon \varphi(x, y)\} = \begin{cases} \emptyset & \text{if } 0 \leq \varepsilon < 1, \\ \{0\} & \text{if } \varepsilon \geq 1. \end{cases}$$

5 Differential Stability of Constrained Convex Optimization Problems

Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be two Banach spaces and $\varphi : X \times Y \rightarrow \overline{\mathbb{R}}$ be an extended real-valued function. Let $G : X \rightrightarrows Y$ be a set-valued map. The graph and the domain of G are given, respectively, by

$$\begin{aligned} \text{gph } G &:= \{(x, y) \in X \times Y \mid y \in G(x)\}, \\ \text{dom } G &:= \{x \in X \mid G(x) \neq \emptyset\}. \end{aligned}$$

If $\text{gph } G$ is a closed (respectively convex) subset of $X \times Y$, G is said to be closed (respectively convex). In addition, if $\text{dom } G \neq \emptyset$, G is called proper. Equipping $X \times Y$ with the norm $\|(x, y)\|_{X \times Y} = \|x\|_X + \|y\|_Y$ for any $(x, y) \in X \times Y$, the product space $X \times Y$ is a Banach space.

Consider the parametric optimization problem under an inclusion constraint

$$\min\{\varphi(x, y) \mid y \in G(x)\} \tag{P_x^c}$$

depending on the parameter $x \in X$. The multifunction G is called constraint multifunction of (P_x^c) . The optimal value function $\mu^c : X \rightarrow \overline{\mathbb{R}}$ of (P_x^c) is

$$\mu^c(x) := \inf \{ \varphi(x, y) \mid y \in G(x) \}.$$

The usual convention $\inf \emptyset = +\infty$ forces $\mu^c(x) = +\infty$ for every $x \notin \text{dom}G$. The solution map $M^c : X \rightrightarrows Y$ of (P_x^c) is defined by

$$M^c(x) = \{ y \in G(x) \mid \mu^c(x) = \varphi(x, y) \}.$$

For each $\eta > 0$, the approximate solution map $M_\eta^c : X \rightrightarrows Y$ of (P_x^c) is given by

$$M_\eta^c(x) = \{ y \in G(x) \mid \varphi(x, y) \leq \mu^c(x) + \eta \}.$$

Lemma 1 *Let $\varphi : X \times Y \rightarrow \overline{\mathbb{R}}$ be a proper lower-semicontinuous convex function and $G : X \rightrightarrows Y$ a proper closed, convex multifunction. For each $(\bar{x}, \bar{y}) \in \text{dom}\varphi \cap \text{gph}G$ and $\gamma > 0$ it follows that $(x^*, y^*) \in \partial_\gamma(\varphi + \delta_{\text{gph}G})(\bar{x}, \bar{y})$ if and only if there exist $\gamma_1, \gamma_2 \geq 0$, with $\gamma_1 + \gamma_2 = \gamma$ and nets $((x_{1,\alpha}, y_{1,\alpha})_\alpha \subset \text{dom}\varphi, (x_{1,\alpha}, y_{1,\alpha}) \xrightarrow{\|\cdot\|_{X \times Y}} (\bar{x}, \bar{y}), ((x_{2,\alpha}, y_{2,\alpha})_\alpha \subset \text{gph}G, (x_{2,\alpha}, y_{2,\alpha}) \xrightarrow{\|\cdot\|_{X \times Y}} (\bar{x}, \bar{y}), ((x_{1,\alpha}^*, y_{1,\alpha}^*)_\alpha, ((x_{2,\alpha}^*, y_{2,\alpha}^*)_\alpha \subset X^* \times Y^*, (x_{1,\alpha}^* + x_{2,\alpha}^*, y_{1,\alpha}^* + y_{2,\alpha}^*) \xrightarrow{w^*} (x^*, y^*)$ such that*

$$\begin{aligned} (x_{1,\alpha}^*, y_{1,\alpha}^*) &\in \partial_{\gamma_1}\varphi(x_{1,\alpha}, y_{1,\alpha}), \quad \forall \alpha, \\ (x_{2,\alpha}^*, y_{2,\alpha}^*) &\in N_{\gamma_2}((x_{2,\alpha}, y_{2,\alpha}), \text{gph}G), \quad \forall \alpha, \\ \varphi(x_{1,\alpha}, y_{1,\alpha}) - \langle (x_{1,\alpha}^*, y_{1,\alpha}^*), (x_{1,\alpha}, y_{1,\alpha}) - (\bar{x}, \bar{y}) \rangle &\rightarrow \varphi(\bar{x}, \bar{y}), \\ \langle (x_{2,\alpha}^*, y_{2,\alpha}^*), (x_{2,\alpha}, y_{2,\alpha}) - (\bar{x}, \bar{y}) \rangle &\rightarrow 0. \end{aligned}$$

Proof We apply Theorem 2 to the case f_1, f_2 play the role of the functions φ and $\delta_{\text{gph}G}$, respectively. Hence $(x^*, y^*) \in \partial_\gamma(\varphi + \delta_{\text{gph}G})(\bar{x}, \bar{y})$ if and only if there exist $\gamma_1, \gamma_2 \geq 0$, $\gamma_1 + \gamma_2 = \gamma$ and nets $((x_{1,\alpha}, y_{1,\alpha})_\alpha \subset \text{dom}\varphi, (x_{1,\alpha}, y_{1,\alpha}) \xrightarrow{\|\cdot\|_{X \times Y}} (\bar{x}, \bar{y}), ((x_{2,\alpha}, y_{2,\alpha})_\alpha \subset \text{gph}G, (x_{2,\alpha}, y_{2,\alpha}) \xrightarrow{\|\cdot\|_{X \times Y}} (\bar{x}, \bar{y}), ((x_{1,\alpha}^*, y_{1,\alpha}^*)_\alpha, ((x_{2,\alpha}^*, y_{2,\alpha}^*)_\alpha \subset X^* \times Y^*, (x_{1,\alpha}^*, y_{1,\alpha}^*) + (x_{2,\alpha}^*, y_{2,\alpha}^*) \xrightarrow{w^*} (x^*, y^*)$ such that

$$\begin{aligned} (x_{1,\alpha}^*, y_{1,\alpha}^*) &\in \partial_{\gamma_1}\varphi(x_{1,\alpha}, y_{1,\alpha}), \quad \forall \alpha, \\ (x_{2,\alpha}^*, y_{2,\alpha}^*) &\in \partial_{\gamma_2}\delta_{\text{gph}G}(x_{2,\alpha}, y_{2,\alpha}), \quad \forall \alpha, \\ \varphi(x_{1,\alpha}, y_{1,\alpha}) - \langle (x_{1,\alpha}^*, y_{1,\alpha}^*), (x_{1,\alpha}, y_{1,\alpha}) - (\bar{x}, \bar{y}) \rangle &\rightarrow \varphi(\bar{x}, \bar{y}), \\ \delta_{\text{gph}G}(x_{2,\alpha}, y_{2,\alpha}) - \langle (x_{2,\alpha}^*, y_{2,\alpha}^*), (x_{2,\alpha}, y_{2,\alpha}) - (\bar{x}, \bar{y}) \rangle &\rightarrow \delta_{\text{gph}G}(\bar{x}, \bar{y}), \end{aligned}$$

and the result is proved since $\partial_{\gamma_2}\delta_{\text{gph}G}(x_{2,\alpha}, y_{2,\alpha}) = N_{\gamma_2}((x_{2,\alpha}, y_{2,\alpha}), \text{gph}G)$ and $\delta_{\text{gph}G}(x_{2,\alpha}, y_{2,\alpha}) = \delta_{\text{gph}G}(\bar{x}, \bar{y}) = 0$. □

We are now in a position to formulate the main result of this section.

Theorem 4 *Assume that $\varphi : X \times Y \rightarrow \overline{\mathbb{R}}$ is proper, lower-semicontinuous convex and $G : X \rightrightarrows Y$ is proper, closed and convex. Suppose that μ^c is finite at $\bar{x} \in X$. Consider $\varepsilon \geq 0$ and a point $\bar{y}_\eta \in M_\eta^c(\bar{x})$ for all $\eta > 0$. Then, $x^* \in \partial_\varepsilon\mu^c(\bar{x})$ if and only if for each $\eta > 0$ there exist $\gamma_1, \gamma_2 \geq 0$, $\gamma_1 + \gamma_2 = \varepsilon + \eta$ and nets $((x_{1,\alpha}, y_{1,\alpha})_\alpha \subset \text{dom}\varphi, (x_{1,\alpha}, y_{1,\alpha}) \xrightarrow{\|\cdot\|_{X \times Y}} (\bar{x}, \bar{y}_\eta), ((x_{2,\alpha}, y_{2,\alpha})_\alpha \subset \text{gph}G,$*

$(x_{2,\alpha}, y_{2,\alpha}) \xrightarrow{\|\cdot\|_{X \times Y}} (\bar{x}, \bar{y}_\eta), ((x_{1,\alpha}^*, y_{1,\alpha}^*))_\alpha, ((x_{2,\alpha}^*, y_{2,\alpha}^*))_\alpha \subset X^* \times Y^*, (x_{1,\alpha}^* + x_{2,\alpha}^*, y_{1,\alpha}^* + y_{2,\alpha}^*) \xrightarrow{w^*} (x^*, 0)$ such that

$$(x_{1,\alpha}^*, y_{1,\alpha}^*) \in \partial_{\gamma_1} \varphi(x_{1,\alpha}, y_{1,\alpha}), \quad \forall \alpha, \tag{10}$$

$$(x_{2,\alpha}^*, y_{2,\alpha}^*) \in N_{\gamma_2}((x_{2,\alpha}, y_{2,\alpha}), \text{gph}G), \quad \forall \alpha, \tag{11}$$

$$\varphi(x_{1,\alpha}, y_{1,\alpha}) - \langle (x_{1,\alpha}^*, y_{1,\alpha}^*), (x_{1,\alpha}, y_{1,\alpha}) - (\bar{x}, \bar{y}_\eta) \rangle \rightarrow \varphi(\bar{x}, \bar{y}_\eta), \tag{12}$$

$$\langle (x_{2,\alpha}^*, y_{2,\alpha}^*), (x_{2,\alpha}, y_{2,\alpha}) - (\bar{x}, \bar{y}_\eta) \rangle \rightarrow 0. \tag{13}$$

Proof Notice that μ^c coincides with the optimal value function μ given by $\varphi + \delta_{\text{gph}G}$ and M_η^c coincides in $\text{dom}G$ with the approximate solution map M_η defined by $\varphi + \delta_{\text{gph}G}$. Then, we apply Theorem 3 to the case where φ plays the role of the function $\varphi + \delta_{\text{gph}G}$. Hence

$$\partial_\varepsilon \mu^c(\bar{x}) = \bigcap_{\eta > 0} \{x^* \in X^* \mid (x^*, 0) \in \partial_{\varepsilon+\eta}(\varphi + \delta_{\text{gph}G})(\bar{x}, \bar{y}_\eta)\}.$$

Moreover, by Lemma 1, $(x^*, 0) \in \partial_{\varepsilon+\eta}(\varphi + \delta_{\text{gph}G})(\bar{x}, \bar{y}_\eta)$ if and only if there exist $\gamma_1, \gamma_2 \geq 0, \gamma_1 + \gamma_2 = \varepsilon + \eta$ and $((x_{1,\alpha}, y_{1,\alpha}))_\alpha \subset \text{dom}\varphi, (x_{1,\alpha}, y_{1,\alpha}) \xrightarrow{\|\cdot\|_{X \times Y}} (\bar{x}, \bar{y}_\eta), ((x_{2,\alpha}, y_{2,\alpha}))_\alpha \subset \text{gph}G, (x_{2,\alpha}, y_{2,\alpha}) \xrightarrow{\|\cdot\|_{X \times Y}} (\bar{x}, \bar{y}_\eta), ((x_{1,\alpha}^*, y_{1,\alpha}^*))_\alpha, ((x_{2,\alpha}^*, y_{2,\alpha}^*))_\alpha \subset X^* \times Y^*, (x_{1,\alpha}^* + x_{2,\alpha}^*, y_{1,\alpha}^* + y_{2,\alpha}^*) \xrightarrow{w^*} (x^*, 0)$ such that (10)–(13) are satisfied. So we obtain the statement of the theorem. \square

Theorem 4 improves [3, Theorems 4.2 and 4.3] as it does not require any qualification condition. In fact, it reduces to them when certain qualification conditions are satisfied. This claim is shown in the next result. For each $\delta \geq 0$ we denote

$$\Gamma(\delta) := \{(\gamma_1, \gamma_2) \in \mathbb{R}_+^2 \mid \gamma_1 + \gamma_2 = \delta\}.$$

Corollary 2 *Let $\varphi : X \times Y \rightarrow \bar{\mathbb{R}}$ be a proper lower-semicontinuous convex function and $G : X \rightrightarrows Y$ a proper closed convex multifunction. Suppose that μ^c is finite at $\bar{x} \in X$. Consider for each $\eta > 0$ a point $\bar{y}_\eta \in M_\eta^c(\bar{x})$. Assume that the following Robinson-Rockafellar condition*

$$(0, 0) \in \text{core}(\text{dom}\varphi - \text{gph}G) \tag{14}$$

is satisfied. Then, for all $\varepsilon \geq 0$ one has

$$\partial_\varepsilon \mu^c(\bar{x}) = \left\{ x^* \in X^* \mid (x^*, 0) \in \bigcap_{\eta > 0} \bigcup_{(\gamma_1, \gamma_2) \in \Gamma(\varepsilon+\eta)} \partial_{\gamma_1} \varphi(\bar{x}, \bar{y}_\eta) + N_{\gamma_2}((\bar{x}, \bar{y}_\eta), \text{gph}G) \right\}.$$

Proof Suppose that $x^* \in \partial_\varepsilon \mu^c(\bar{x})$ and $\eta > 0$. According to Theorem 4 there exist $(\gamma_1, \gamma_2) \in \Gamma(\varepsilon + \eta)$ and nets $((x_{1,\alpha}, y_{1,\alpha}))_\alpha \subset \text{dom}\varphi, (x_{1,\alpha}, y_{1,\alpha}) \xrightarrow{\|\cdot\|_{X \times Y}} (\bar{x}, \bar{y}_\eta), ((x_{2,\alpha}, y_{2,\alpha}))_\alpha \subset \text{gph}G, (x_{2,\alpha}, y_{2,\alpha}) \xrightarrow{\|\cdot\|_{X \times Y}} (\bar{x}, \bar{y}_\eta), ((x_{1,\alpha}^*, y_{1,\alpha}^*))_\alpha, ((x_{2,\alpha}^*, y_{2,\alpha}^*))_\alpha \subset X^* \times Y^*, (x_{1,\alpha}^* + x_{2,\alpha}^*, y_{1,\alpha}^* + y_{2,\alpha}^*) \xrightarrow{w^*} (x^*, 0)$ such that assertions (10)–(13) hold true.

We claim that net $((x_{1,\alpha}^*, y_{1,\alpha}^*))_\alpha$ is weak* bounded. Indeed, consider an arbitrary point $(w_1, w_2) \in X \times Y$ and denote

$$r_{1,\alpha} := \varphi(x_{1,\alpha}, y_{1,\alpha}) - \langle (x_{1,\alpha}^*, y_{1,\alpha}^*), (x_{1,\alpha}, y_{1,\alpha}) - (\bar{x}, \bar{y}_\eta) \rangle, \tag{15}$$

$$r_{2,\alpha} := \langle (x_{2,\alpha}^*, y_{2,\alpha}^*), (x_{2,\alpha}, y_{2,\alpha}) - (\bar{x}, \bar{y}_\eta) \rangle. \tag{16}$$

By applying the Robinson-Rockafellar condition there exist $t > 0, (u_1, u_2) \in \text{dom}\varphi$ and $(v_1, v_2) \in \text{gph}G$ such that $t(w_1, w_2) = (u_1, u_2) - (v_1, v_2)$. Thus,

$$\begin{aligned} \langle (x_{1,\alpha}^*, y_{1,\alpha}^*), t(w_1, w_2) \rangle &= \langle (x_{1,\alpha}^*, y_{1,\alpha}^*), (u_1, u_2) - (x_{1,\alpha}, y_{1,\alpha}) \rangle \\ &\quad + \langle (x_{1,\alpha}^*, y_{1,\alpha}^*), (x_{1,\alpha}, y_{1,\alpha}) - (\bar{x}, \bar{y}_\eta) \rangle \\ &\quad - \langle (x_{1,\alpha}^*, y_{1,\alpha}^*), (v_1, v_2) - (\bar{x}, \bar{y}_\eta) \rangle. \end{aligned} \tag{17}$$

Using the definition of approximate subdifferential and (10), (11) (15) and (16) one has

$$\begin{aligned} RHS_{(17)} &\leq \varphi(u_1, u_2) + \gamma_1 - r_{1,\alpha} - \langle (x_{1,\alpha}^*, y_{1,\alpha}^*), (v_1, v_2) - (\bar{x}, \bar{y}_\eta) \rangle \\ &= \varphi(u_1, u_2) + \gamma_1 - r_{1,\alpha} - \langle (x_{1,\alpha}^* + x_{2,\alpha}^*, y_{1,\alpha}^* + y_{2,\alpha}^*), (v_1, v_2) - (\bar{x}, \bar{y}_\eta) \rangle \\ &\quad + \langle (x_{2,\alpha}^*, y_{2,\alpha}^*), (v_1, v_2) - (x_{2,\alpha}, y_{2,\alpha}) \rangle + r_{2,\alpha} \\ &\leq \varphi(u_1, u_2) + \gamma_1 - r_{1,\alpha} \\ &\quad - \langle (x_{1,\alpha}^* + x_{2,\alpha}^*, y_{1,\alpha}^* + y_{2,\alpha}^*), (v_1, v_2) - (\bar{x}, \bar{y}_\eta) \rangle + \gamma_2 + r_{2,\alpha}. \end{aligned}$$

By (12) and (13) it follows that $r_{1,\alpha} \rightarrow \varphi(\bar{x}, \bar{y}_\eta)$, and $r_{2,\alpha} \rightarrow 0$. In addition, $(u_1, u_2) \in \text{dom}\varphi$ and

$$\langle (x_{1,\alpha}^* + x_{2,\alpha}^*, y_{1,\alpha}^* + y_{2,\alpha}^*), (v_1, v_2) - (\bar{x}, \bar{y}_\eta) \rangle \rightarrow \langle x^*, v_1 - \bar{x} \rangle$$

since $(x_{1,\alpha}^* + x_{2,\alpha}^*, y_{1,\alpha}^* + y_{2,\alpha}^*) \xrightarrow{w^*} (x^*, 0)$. Consequently, $(\langle (x_{1,\alpha}^*, y_{1,\alpha}^*), t(w_1, w_2) \rangle)_\alpha$ is bounded. As $(w_1, w_2) \in X \times Y$ is arbitrary we deduce that $((x_{1,\alpha}^*, y_{1,\alpha}^*))_\alpha$ is weak* bounded. Thanks to Banach-Steinhaus Theorem it follows that it is bounded and also the sequence $((x_{2,\alpha}^*, y_{2,\alpha}^*))_\alpha$. By Banach-Alaoglu Theorem we can suppose without loss of generality that $(x_{i,\alpha}^*, y_{i,\alpha}^*) \xrightarrow{w^*} (x_i^*, y_i^*), i = 1, 2$. It is clear that, $(x_1^*, y_1^*) + (x_2^*, y_2^*) = (x^*, 0)$. In addition, by [37, Theorem 2.4.2(ix)] one has $(x_1^*, y_1^*) \in \partial_{\gamma_1}\varphi(\bar{x}, \bar{y}_\eta)$ and $(x_2^*, y_2^*) \in N_{\gamma_2}((\bar{x}, \bar{y}_\eta), \text{gph}G)$. Therefore, $(x^*, 0) \in \partial_{\gamma_1}\varphi(\bar{x}, \bar{y}_\eta) + N_{\gamma_2}((\bar{x}, \bar{y}_\eta), \text{gph}G)$.

Conversely, let $(x^*, 0) \in X^* \times Y^*$ be such that for each $\eta > 0$ there exists $(\gamma_1, \gamma_2) \in \Gamma(\varepsilon + \eta)$ satisfying $(x^*, 0) \in \partial_{\gamma_1}\varphi(\bar{x}, \bar{y}_\eta) + N_{\gamma_2}((\bar{x}, \bar{y}_\eta), \text{gph}G)$. Thus, there exists $(x_i^*, y_i^*) \in X^* \times Y^*, i = 1, 2, (x_1^*, y_1^*) + (x_2^*, y_2^*) = (x^*, 0)$ satisfying

$$\begin{aligned} \varphi(x, y) &\geq \varphi(\bar{x}, \bar{y}_\eta) - \gamma_1 + \langle (x_1^*, y_1^*), (x, y) - (\bar{x}, \bar{y}_\eta) \rangle, \quad \forall (x, y) \in X \times Y, \\ \gamma_2 &\geq \langle (x_2^*, y_2^*), (x, y) - (\bar{x}, \bar{y}_\eta) \rangle, \quad \forall (x, y) \in \text{gph}G. \end{aligned}$$

By summing up both inequalities and taking into account that $\gamma_1 + \gamma_2 = \varepsilon + \eta$ and $(x_1^*, y_1^*) + (x_2^*, y_2^*) = (x^*, 0)$ we see that

$$\varphi(x, y) + \eta \geq \varphi(\bar{x}, \bar{y}_\eta) - \varepsilon + \langle x^*, x - \bar{x} \rangle, \quad \forall (x, y) \in \text{gph}G.$$

Therefore, as $\bar{y}_\eta \in G(\bar{x})$,

$$\begin{aligned} \mu^c(x) + \eta &\geq \varphi(\bar{x}, \bar{y}_\eta) - \varepsilon + \langle x^*, x - \bar{x} \rangle \\ &\geq \mu^c(\bar{x}) - \varepsilon + \langle x^*, x - \bar{x} \rangle, \quad \forall x \in X. \end{aligned}$$

Since η is an arbitrary positive number it follows that

$$\mu^c(x) \geq \mu^c(\bar{x}) - \varepsilon + \langle x^*, x - \bar{x} \rangle, \quad \forall x \in X,$$

i.e., $x^* \in \partial_\varepsilon \mu^c(\bar{x})$, and the proof finishes. □

We now give the relationship between qualification condition (14) in Corollary 2 and qualification conditions in [3, Theorem 4.3]. The first assertion is obvious and the second one follows from [29, Lemma 1.58].

Proposition 1 *Consider a proper function $\varphi : X \times Y \rightarrow \overline{\mathbb{R}}$ and a proper multifunction $G : X \rightrightarrows Y$. Then the qualification condition $(0, 0) \in \text{int}(\text{dom}\varphi - \text{gph}G)$ implies (14). If, in addition, φ and G are convex, then qualification condition (14) is equivalent to say that $\mathbb{R}_+(\text{dom}\varphi - \text{gph}G) = X \times Y$.*

6 Vector Optimization Problems

In this section, differential stability properties of convex vector optimization problems are obtained. Namely, in unconstrained problems, the ε -subdifferential of the infimal value mapping is characterized in terms of ε -subdifferentials of linear scalarizations of the problem.

Consider the parametric unconstrained vector optimization problem

$$\text{Min}\{f(x, y) \mid y \in Y\}, \tag{VP_x}$$

where X, Y , and Z are real locally convex Hausdorff topological linear spaces, and $f : X \times Y \rightarrow Z$ is a vector-valued function. The final space Z is ordered by a convex cone D , which is assumed to be proper ($D \neq Z$) and solid ($\text{int}D \neq \emptyset$). We denote

$$\begin{aligned} z_1, z_2 \in Z, \quad z_1 \leq_D z_2 &\iff z_2 - z_1 \in D, \\ z_1 \ll_D z_2 &\iff z_2 - z_1 \in \text{int}D. \end{aligned}$$

In addition, D^+ stands for the (positive) polar cone of D :

$$D^+ := \{\lambda \in Z^* \mid \lambda(d) \geq 0, \forall d \in D\}.$$

Consider a nonempty set $A \subset Z$ and $\mathcal{E} \in D$. A point $\bar{a} \in A$ is said to be a weak (respectively \mathcal{E} -weak) minimal point of A if there is not a point $a \in A$ satisfying $a \ll_D \bar{a}$ (respectively $a \ll_D \bar{a} - \mathcal{E}$). The set of all weak (respectively \mathcal{E} -weak) minimal points of A is denoted by $\text{WMin}(A, D)$ (respectively $\text{WMin}(A, D, \mathcal{E})$). Clearly, $\text{WMin}(A, D, 0) = \text{WMin}(A, D)$. Notice that a point $z \in Z$ belongs to $\text{WMin}(A, D, \mathcal{E})$ if and only if $z \in A$ and $(A - z + \mathcal{E}) \cap (-\text{int}D) = \emptyset$.

We say that a nonempty set $A \subset Z$ is D -proper if $A + D \neq Z$ (see [16, Definition 2.15]). This concept defines a really general kind of lower boundedness with respect to the ordering \ll_D (see [15, Section 3]). Notice that A is D -proper if and only if $\text{cl}A$ is D -proper.

For convenience, the final space Z is extended to $\overline{Z} := Z \cup \{-\infty_D\} \cup \{+\infty_D\}$, where $+\infty_D$ (respectively $-\infty_D$) stands for the greatest (respectively least) element of Z with respect to the ordering \ll_D . In addition, we assume $-\infty_D \ll_D z \ll_D +\infty_D$ and $z \pm \infty_D = \pm \infty_D + z = \pm \infty_D$, for all $z \in Z$. As a result, for each nonempty set $A \subset \overline{Z}$ and $\mathcal{E} \in D$ we have $\text{WMin}(A, D, \mathcal{E}) = \{-\infty_D\}$ whenever $-\infty_D \in A$. Clearly, if $\text{WMin}(A, D, \mathcal{E}) \cap Z \neq \emptyset$, then $-\infty_D \notin A$.

In order to deal with an optimal value function corresponding to problem (VP_x) a notion of infimal point is required. Here we consider a concept linked with D -proper sets, in the sense that these sets have always infimal points under a cone closedness assumption. Namely, a nonempty set $A \subset Z$ is said to be D -closed if $A + D$ is a closed set. For all

$z^* \in Z^*$ we denote

$$\operatorname{argmin}_A z^* := \{\bar{a} \in A \mid z^*(\bar{a}) \leq z^*(a), \forall a \in A\}.$$

Lemma 2 Consider a nonempty set $A \subset Z$. The next properties are true.

- (i) If $\operatorname{WMin}(\operatorname{cl}A, D) \neq \emptyset$, then A is D -proper;
- (ii) $\operatorname{WMin}(\operatorname{cl}A, D) + \operatorname{int}D \subset A + \operatorname{int}D$;
- (iii) Assume that A is D -proper and $\operatorname{cl}A$ is D -closed. Then, $\operatorname{WMin}(\operatorname{cl}A, D) \neq \emptyset$ and

$$A \subset \operatorname{WMin}(\operatorname{cl}A, D) + D, \tag{18}$$

$$A + \operatorname{int}D = \operatorname{WMin}(\operatorname{cl}A, D) + \operatorname{int}D. \tag{19}$$

- (iv) Assume that $\operatorname{cl}A$ is D -closed. Then, $\operatorname{WMin}(\operatorname{cl}A, D) \neq \emptyset$ if and only if A is D -proper.
- (v) For each $a \in \operatorname{WMin}(\operatorname{cl}A, D)$, $q \in \operatorname{int}D$ and $\eta > 0$,

$$(A - a) \cap (-\eta q + D) \cap (\eta q - D) \neq \emptyset.$$

- (vi) We have that $\bigcup_{\lambda \in D^+ \setminus \{0\}} \operatorname{argmin}_{\operatorname{cl}A} \lambda \subset \operatorname{WMin}(\operatorname{cl}A, D)$.

Proof

- (i) Suppose that A is not D -proper. Then, $\operatorname{cl}A$ is not D -proper. As $D + \operatorname{int}D = \operatorname{int}D$, for each point $a \in \operatorname{cl}A$ we have

$$\begin{aligned} (\operatorname{cl}A - a) \cap (-\operatorname{int}D) = \emptyset &\iff (\operatorname{cl}A + D - a) \cap (-\operatorname{int}D) = \emptyset \\ &\iff Z \cap (-\operatorname{int}D) = \emptyset \\ &\iff \operatorname{int}D = \emptyset. \end{aligned}$$

Since D is solid, it follows that $(\operatorname{cl}A - a) \cap (-\operatorname{int}D) \neq \emptyset$, for all $a \in \operatorname{cl}A$. Thus, $\operatorname{WMin}(\operatorname{cl}A, D) = \emptyset$ and statement (i) is proved.

- (ii) It is not hard to check that $\operatorname{cl}A + \operatorname{int}D = A + \operatorname{int}D$. Then, the result follows since $\operatorname{WMin}(\operatorname{cl}A, D) \subset \operatorname{cl}A$.
- (iii) Fix $q \in \operatorname{int}D$ and an arbitrary point $a \in A$. Define $\bar{t} := \sup \Lambda$, where

$$\Lambda := \{t \geq 0 \mid a - tq \in \operatorname{cl}A + D\}.$$

We claim that $\bar{t} \in \mathbb{R}$. Indeed, $\Lambda \neq \emptyset$ as $0 \in \Lambda$. Moreover, $[0, t] \subset \Lambda$ whenever $t \in \Lambda$, since if $t \in \Lambda$ and $s \in [0, t]$ one has

$$a - sq = (a - tq) + (t - s)q \in \operatorname{cl}A + D + D = \operatorname{cl}A + D.$$

Thus, the boundedness from above of Λ is obtained if there exists $M > 0$ such that $M \notin \Lambda$. Reasoning by contradiction, suppose that $n \in \Lambda$, for all $n \in \mathbb{N}$. Then,

$$\bigcup_{n \in \mathbb{N}} (a - nq + D) \subset \operatorname{cl}A + D + D = \operatorname{cl}A + D. \tag{20}$$

As $q \in \operatorname{int}D$, it is easy to check that $Z = \bigcup_{n \in \mathbb{N}} (-nq + D)$. Therefore,

$$\bigcup_{n \in \mathbb{N}} (a - nq + D) = a + Z = Z,$$

that is a contradiction, since $\operatorname{cl}A$ is D -proper.

By the nonemptiness and boundedness from above of Λ it follows that $\bar{t} \in \mathbb{R}$. Define $\bar{z} := a - \bar{t}q$. Clearly, $\bar{z} \in \text{cl}A + D$, since $\text{cl}A$ is D -closed. Thus, $\bar{z} = \bar{a} + \bar{d}$, for some $\bar{a} \in \text{cl}A$ and $\bar{d} \in D$.

We claim $\bar{a} \in \text{WMin}(\text{cl}A, D)$. Indeed, reasoning by contradiction, suppose that there exists $a' \in \text{cl}A$ such that $a' \ll_D \bar{a}$. Then, there exists $q' \in \text{int}D$ such that $a' = \bar{a} - q'$. As $\bar{q} := q' + \bar{d} \in \text{int}D$, there exists $\alpha > 0$ such that $\bar{q} - \alpha q \in D$. Thus,

$$a - (\bar{t} + \alpha)q = \bar{z} - \alpha q = \bar{a} + \bar{d} - \alpha q = a' + q' + \bar{d} - \alpha q = a' + \bar{q} - \alpha q \in \text{cl}A + D,$$

i.e., $\bar{t} + \alpha \in \Lambda$, that is a contradiction, since $\bar{t} = \sup \Lambda$ and $\alpha > 0$.

Thus, $\bar{a} \in \text{WMin}(\text{cl}A, D)$ and $\text{WMin}(\text{cl}A, D) \neq \emptyset$. In addition,

$$a = \bar{z} + \bar{t}q = \bar{a} + \bar{d} + \bar{t}q \in \text{WMin}(\text{cl}A, D) + D.$$

As point $a \in A$ is arbitrary, we deduce $A \subset \text{WMin}(\text{cl}A, D) + D$, and assertion (18) is proved. Finally, (19) follows by (18) and statement (ii), since $D + \text{int}D = \text{int}D$.

- (iv) It is a direct consequence of statements (i) and (iii).
- (v) This statement is obvious, since $\text{WMin}(\text{cl}A, D) \subset \text{cl}A$ and $(-\eta q + D) \cap (\eta q - D)$ is a neighborhood of zero, for all $\eta > 0$ and $q \in \text{int}D$.
- (vi) It is a well-known result (see, for instance, [13, Theorem 3.5(ii)]).

□

Remark 4 The cone closedness assumption of Lemma 2(iii) cannot be dropped. Consider, for instance, the set $A = \{(x, e^x) \in \mathbb{R}^2 : x \in \mathbb{R}\}$. We have that $\text{WMin}(A, \mathbb{R}_+^2) = \emptyset$ and A is closed, \mathbb{R}_+^2 -proper, but it is not \mathbb{R}_+^2 -closed.

The next notion of infimal set is a weak version of the one introduced in [6, 12].

Definition 3 The weak infimal set of a nonempty set $A \subset Z$ is defined to be

$$\text{WInf}(A, D) := \begin{cases} \text{WMin}(\text{cl}A, D) & \text{if } A \text{ is } D\text{-proper,} \\ \{-\infty_D\} & \text{otherwise.} \end{cases}$$

Remark 5 (i) According to Lemma 2(iv), it follows that D -properness is the lower boundedness condition that guaranties the existence of infimal points in the class of sets whose closure is D -closed. Notice that closed sets are D -closed in different settings. For instance, if Z is finite dimensional, D is closed and $A_\infty \cap (-D) = \{0\}$, where A_∞ denotes the asymptotic cone of A (see [5, Definition 2.1.2]), then $\text{cl}A$ is D -closed (see [5, Corollary 2.3.4]).

(ii) In the literature on differentiability properties of the optimal value function of a parametric family of vector optimization problems (see [23, 24, 33, 34, 36] and the references therein), it is always assumed the domination property: the range of each problem in the family is included in the conical extension of the set of optimal values. For instance, in the setting of problem (VP_x) , where the concept of weak minimality is considered to define its solutions and optimal values, that assumption is formulated as follows:

$$f(x, Y) \subset \text{WMin}(f(x, Y), D) + D, \quad \forall x \in X. \tag{21}$$

Some examples are [24, Theorem 10] and its required domination property, [23, Theorem 3.4], [33, Theorem 5.3] and their \bar{K} -domination requirement and [34, Theorems 5.1 and 5.2], [36, Theorem 5.1] and their P -minicompleteness assumption.

Clearly, that assumption implies the existence of the considered solutions, which is a too strong condition in the study of the optimal value function (see [3, 28]). For instance, in Example 2 such an assumption is not satisfied as each problem in the parametric family does not have weak efficient solutions.

The differentiability properties of this paper overcome such a handicap since infimal solutions are considered. Notice that requirement (21) is replaced with assertion (18), which involves $\text{cl}A$ instead of the set A .

By the concept of weak infimal set, we define the weak optimal value mapping $M : X \rightrightarrows \bar{Z}$ of problem (VP_x) as follows:

$$M(x) = \text{WInf}(f(x, Y), D),$$

where $f(x, Y)$ is the range of $f(x, \cdot) : Y \rightarrow Z$. By Lemma 2(iv), if $\text{cl}f(x, Y)$ is D -closed, then $\emptyset \neq M(x) \subset Z$ whenever $f(x, Y)$ is D -proper, and $M(x) = \{-\infty_D\}$ otherwise.

The notion of \mathcal{E} -weak subdifferential of a set-valued function is required (see [35]) in order to state differential stability properties of mapping M . Denote by $L(X, Z)$ the set of all continuous linear functions from X to Z , and consider a set-valued mapping $F : X \rightrightarrows Z$ and points $\mathcal{E} \in D$ and $(\bar{x}, \bar{z}) \in \text{gph}F, \bar{z} \in Z$. We say that $T \in L(X, Z)$ is a \mathcal{E} -weak subgradient for \bar{z} of F at \bar{x} , and it is denoted by $T \in \partial_{\mathcal{E}}^W F(\bar{x}, \bar{z})$, if

$$\bar{z} - T(\bar{x}) \in \text{WMin}((F - T)(X), D, \mathcal{E}),$$

where $(F - T)(X) = \bigcup_{x \in X} (F(x) - T(x))$. The set $\partial_{\mathcal{E}}^W F(\bar{x}, \bar{z})$ is called \mathcal{E} -weak subdifferential for \bar{z} of F at \bar{x} . When $\mathcal{E} = 0$, it reduces to the subdifferential for \bar{z} of F at \bar{x} (see [32, Definition 6.2.8], [34] and [10, Definition 7.4.2(c)]):

$$\partial^W F(\bar{x}, \bar{z}) := \partial_0^W F(\bar{x}, \bar{z}) = \{T \in L(X, Z) | \bar{z} - T(\bar{x}) \in \text{WMin}((F - T)(X), D)\}.$$

Remark 6 It is not difficult to see that $-\infty_D \notin F(X)$ whenever F is \mathcal{E} -weak subdifferentiable at $\bar{x} \in X$ for a point $\bar{z} \in Z$. Indeed, if $-\infty_D \in F(X)$, then $\text{WMin}((F - T)(X), D) = \{-\infty_D\}$ and so $\partial_{\mathcal{E}}^W F(\bar{x}, \bar{z}) = \emptyset$, for all $(\bar{x}, \bar{z}) \in \text{gph}F, \bar{z} \in Z$.

Consider a point $\bar{x} \in X$ satisfying $M(\bar{x}) \neq \{-\infty_D\}$. For each $\bar{z} \in M(\bar{x}), q \in \text{int}D$ and $\eta > 0$, we denote

$$M_{\eta}(\bar{x}, \bar{z}, q) := \{y \in Y | f(\bar{x}, y) \in \bar{z} + (-\eta q + D) \cap (\eta q - D)\}.$$

By Lemma 2(v), we have that $M_{\eta}(\bar{x}, \bar{z}, q) \neq \emptyset$.

For each $\lambda \in Z^*, \mu_{\lambda \circ f}$ stands for the optimal value function corresponding to the function $\varphi = \lambda \circ f$.

Recall that a vector function $h : X \rightarrow Z$ is said to be D -convex if for all $\alpha \in (0, 1)$ and $x_1, x_2 \in X$ it follows that

$$h(\alpha x_1 + (1 - \alpha)x_2) \leq_D \alpha h(x_1) + (1 - \alpha)h(x_2).$$

Theorem 5 *Suppose that f is D -convex and $\text{cl}f(x, Y)$ is D -closed, for all $x \in X$. Let $\bar{x} \in X, M(\bar{x}) \neq \{-\infty_D\}$. Consider $(\bar{x}, \bar{z}) \in \text{gph}M, \mathcal{E} \in D, q \in \text{int}D$ and $y_{\eta} \in M_{\eta}(\bar{x}, \bar{z}, q)$, for all $\eta > 0$. If $T \in \partial_{\mathcal{E}}^W M(\bar{x}, \bar{z})$, then there exists $\lambda \in D^+ \setminus \{0\}, \lambda(q) = 1$, such that*

$$\lambda \circ T \in \bigcap_{\eta > 0} \{x^* \in X^* | (x^*, 0) \in \partial_{\eta + \lambda(\mathcal{E})}(\lambda \circ f)(\bar{x}, y_{\eta})\}.$$

Proof First, notice that $\bar{z} \in Z$, since $M(\bar{x}) \neq \{-\infty_D\}$. Let $T \in L(X, Z)$ be a \mathcal{E} -weak subgradient for \bar{z} of M at \bar{x} . By Remark 6 we see that $-\infty_D \notin M(X)$ and so $f(x, Y)$ is D -proper, for all $x \in X$. Thus,

$$(M - T)(X) = \bigcup_{x \in X} (\text{WInf}(f(x, Y), D) - T(x)) \subset Z.$$

By the definition we have that $\bar{z} - T(\bar{x}) \in \text{WMin}((M - T)(X), D, \mathcal{E})$, i.e.,

$$\left(\bigcup_{x \in X} (\text{WInf}(f(x, Y), D) - T(x)) - (\bar{z} - T(\bar{x})) + \mathcal{E} \right) \cap (-\text{int}D) = \emptyset.$$

By (19), for each $x \in X$ one has $\text{WInf}(f(x, Y), D) + \text{int}D = f(x, Y) + \text{int}D$. Therefore, as $\text{int}D + \text{int}D = \text{int}D$ it follows that $\mathcal{F}_T(\bar{x}, \bar{z}) \cap (-\text{int}D) = \emptyset$, where

$$\mathcal{F}_T(\bar{x}, \bar{z}) := \bigcup_{x \in X} (f(x, Y) - T(x) + \text{int}D) - (\bar{z} - T(\bar{x})) + \mathcal{E}.$$

As f is D -convex, we see that $\mathcal{F}_T(\bar{x}, \bar{z})$ is convex. Then, by Eidelheit Theorem [37, Theorem 1.1.3], there exists $\lambda \in Z^* \setminus \{0\}$ such that for all $x \in X, y \in Y, d_1, d_2 \in D$,

$$(\lambda \circ f)(x, y) - (\lambda \circ T)(x) + \lambda(d_1) - \lambda(\bar{z}) + (\lambda \circ T)(\bar{x}) + \lambda(\mathcal{E}) \geq -\lambda(d_2).$$

As D is a cone, we have $-\lambda(d_2) \leq 0$, for all $d_2 \in D$. Therefore, $\lambda \in D^+ \setminus \{0\}$. Clearly $\lambda(q) > 0$ as $q \in \text{int}D$, and we can suppose without loss of generality that $\lambda(q) = 1$. Then, by taking $d_1 = d_2 = 0$ we obtain

$$(\lambda \circ f)(x, y) \geq \lambda(\bar{z}) - \lambda(\mathcal{E}) + (\lambda \circ T)(x - \bar{x}), \quad \forall x \in X, y \in Y. \tag{22}$$

Consider $\eta > 0$. As $y_\eta \in M_\eta(\bar{x}, \bar{z}, q)$, $\lambda \in D^+$ and $\lambda(q) = 1$ we have that

$$\lambda(\bar{z}) \geq (\lambda \circ f)(\bar{x}, y_\eta) - \eta. \tag{23}$$

Therefore, by statements (22) and (23) it follows that

$$(\lambda \circ f)(x, y) \geq (\lambda \circ f)(\bar{x}, y_\eta) - (\eta + \lambda(\mathcal{E})) + (\lambda \circ T)(x - \bar{x}), \quad \forall x \in X, y \in Y,$$

i.e., $(\lambda \circ T, 0) \in \partial_{\eta + \lambda(\mathcal{E})}(\lambda \circ f)(\bar{x}, y_\eta)$, and the proof finishes. □

Theorem 6 *Let $\bar{x} \in X, M(\bar{x}) \neq \{-\infty_D\}$. Consider $(\bar{x}, \bar{z}) \in \text{gph}M, T \in L(X, Z), \mathcal{E} \in D, q \in \text{int}D$ and $y_\eta \in M_\eta(\bar{x}, \bar{z}, q)$, for all $\eta > 0$. If there exists $\lambda \in D^+ \setminus \{0\}, \lambda(q) = 1$, such that*

$$\lambda \circ T \in \bigcap_{\eta > 0} \{x^* \in X^* \mid (x^*, 0) \in \partial_{\eta + \lambda(\mathcal{E})}(\lambda \circ f)(\bar{x}, y_\eta)\},$$

then, $T \in \partial_{\mathcal{E}}^W M(\bar{x}, \bar{z})$.

Proof Assume that there exists $\lambda \in D^+ \setminus \{0\}, \lambda(q) = 1$, such that $(\lambda \circ T, 0) \in \partial_{\eta + \lambda(\mathcal{E})}(\lambda \circ f)(\bar{x}, y_\eta)$, for all $\eta > 0$. Then,

$$(\lambda \circ f)(x, y) \geq (\lambda \circ f)(\bar{x}, y_\eta) - (\eta + \lambda(\mathcal{E})) + (\lambda \circ T)(x - \bar{x}), \quad \forall x \in X, y \in Y. \tag{24}$$

Notice that $-\infty_D \notin M(X)$. Indeed, if $M(x) = \{-\infty_D\}$ for some $x \in X$, then $f(x, Y)$ is not D -proper. Thus, for all $n \in \mathbb{N}$ there exist $y_n \in Y$ and $d_n \in D$ such that $-nq = f(x, y_n) + d_n$, which implies

$$(\lambda \circ f)(x, y_n) = -n - \lambda(d_n) \xrightarrow{n \rightarrow \infty} -\infty,$$

that is contrary to assertion (24).

Reasoning by contradiction, suppose that $T \notin \partial_{\mathcal{E}}^W M(\bar{x}, \bar{z})$. Thus, $\bar{z} - T(\bar{x}) \notin \text{WMin}((M - T)(X), D, \mathcal{E})$. As $-\infty_D \notin M(X)$ it follows that there exist $x_0 \in X$, $z \in M(x_0)$, $z \in Z$ and $d \in \text{int}D$ such that

$$\bar{z} - T(\bar{x}) = z - T(x_0) + d + \mathcal{E}. \tag{25}$$

As $y_\eta \in M_\eta(\bar{x}, \bar{z}, q)$, there exist $d_\eta, e_\eta \in D$ such that

$$\bar{z} = f(\bar{x}, y_\eta) + \eta q - d_\eta = f(\bar{x}, y_\eta) - \eta q + e_\eta.$$

Clearly, $2\eta = \lambda(d_\eta) + \lambda(e_\eta)$ and then $\lambda(e_\eta)$ tends to zero when $\eta \rightarrow 0$.

From (25) we deduce

$$f(\bar{x}, y_\eta) - \eta q + e_\eta - T(\bar{x}) = z - T(x_0) + d + \mathcal{E}. \tag{26}$$

In addition, by Lemma 2(ii) we see that

$$z + d \in \text{WInf}(f(x_0, Y), D) + \text{int}D = f(x_0, Y) + \text{int}D$$

and so there exists $y_0 \in Y$ and $d' \in \text{int}D$ such that $z + d = f(x_0, y_0) + d'$. By (26) we deduce that

$$f(\bar{x}, y_\eta) - \eta q + e_\eta - T(\bar{x}) = f(x_0, y_0) + d' - T(x_0) + \mathcal{E}$$

and then

$$\begin{aligned} (\lambda \circ f)(x_0, y_0) &= (\lambda \circ f)(\bar{x}, y_\eta) - (\eta + \lambda(\mathcal{E})) + (\lambda \circ T)(x_0 - \bar{x}) + \lambda(e_\eta) - \lambda(d') \\ &< (\lambda \circ f)(\bar{x}, y_\eta) - (\eta + \lambda(\mathcal{E})) + (\lambda \circ T)(x_0 - \bar{x}) \end{aligned}$$

for η small enough since $-\lambda(d') < 0$. This assertion contradicts (24). Therefore $T \in \partial_{\mathcal{E}}^W M(\bar{x}, \bar{z})$ and the proof is completed. □

In problems where the set $\text{cl}f(x, Y)$ is D -closed, for all $x \in X$ Theorems 5 and 6 allow us to characterize the \mathcal{E} -weak subdifferential of the optimal value function. Next we show one of this characterizations, which is a direct consequence of the cone closedness criterion introduced in Remark 5.

Corollary 3 *Suppose that $Z = \mathbb{R}^p$, D is closed, f is D -convex and $f(x, Y)_\infty \cap (-D) = \{0\}$, for all $x \in X$. Let $\bar{x} \in X$, $M(\bar{x}) \neq \{-\infty_D\}$. Consider $(\bar{x}, \bar{z}) \in \text{gph}M$, $T \in L(X, \mathbb{R}^p)$, $\mathcal{E} \in D$, $q \in \text{int}D$ and $y_\eta \in M_\eta(\bar{x}, \bar{z}, q)$, for all $\eta > 0$. It follows that $T \in \partial_{\mathcal{E}}^W M(\bar{x}, \bar{z})$ if and only if there exists $\lambda \in D^+ \setminus \{0\}$, $\lambda(q) = 1$, such that*

$$\lambda \circ T \in \bigcap_{\eta > 0} \{x^* \in X^* \mid (x^*, 0) \in \partial_{\eta + \lambda(\mathcal{E})}(\lambda \circ f)(\bar{x}, y_\eta)\}.$$

Example 2 Let us illustrate Corollary 3 with the problem (VP_x) defined by the data $X = Y = \mathbb{R}$, $Z = \mathbb{R}^2$, $D = \mathbb{R}_+^2$ and $f(x, y) = (e^y, x + e^y)$, for all $x, y \in \mathbb{R}$.

Consider $x \in \mathbb{R}$. Clearly, $f(x, \mathbb{R})$ is the ray $(0, x) + (0, +\infty)\{(1, 1)\}$. Therefore, $f(x, \mathbb{R})$ is \mathbb{R}_+^2 -proper, $f(x, \mathbb{R})_\infty = \mathbb{R}_+\{(1, 1)\}$ and $f(x, \mathbb{R})_\infty \cap (-\mathbb{R}_+^2) = \{(0, 0)\}$. In addition, direct calculations from the definitions show that $\text{WMin}(f(x, \mathbb{R}), \mathbb{R}_+^2) = \emptyset$ and $M(x) = \text{WMin}(\text{cl}f(x, \mathbb{R}), \mathbb{R}_+^2) = \{(0, x)\}$.

Notice that f is a \mathbb{R}_+^2 -convex function, since its components $f_1(x, y) := e^y$ and $f_2(x, y) := x + e^y$ are convex functions. In addition, consider $\bar{x} \in \mathbb{R}$, $\mathcal{E} := (d_1, d_2) \in \mathbb{R}_+^2$ and $q := (1, 1) \in \text{int}\mathbb{R}_+^2$. We have $M_\eta(\bar{x}, (0, \bar{x}), (1, 1)) = (-\infty, \ln \eta]$.

In order to apply Corollary 3 to obtain $\partial_{\mathcal{E}}^W M(\bar{x}, (0, \bar{x}))$, take $\lambda = (\lambda_1, \lambda_2) \in \mathbb{R}_+^2$ such that $\lambda_1 + \lambda_2 = 1, T = (a_1, a_2) \in L(\mathbb{R}, \mathbb{R}^2)$ and $y_\eta := \ln \eta$, for all $\eta > 0$. Direct calculations show that

$$\begin{aligned} (\lambda \circ T, 0) \in \partial_{\eta+\lambda(\mathcal{E})}(\lambda \circ f)(\bar{x}, y_\eta) &\iff \\ e^y + \lambda_2 x \geq \lambda_2 \bar{x} - (\lambda_1 d_1 + \lambda_2 d_2) + (\lambda_1 a_1 + \lambda_2 a_2)(x - \bar{x}), \quad \forall x, y \in \mathbb{R} &\iff \\ (\lambda_2 - (\lambda_1 a_1 + \lambda_2 a_2))x \geq \lambda_2 \bar{x} - (\lambda_1 d_1 + \lambda_2 d_2) - (\lambda_1 a_1 + \lambda_2 a_2)\bar{x}, \quad \forall x \in \mathbb{R} &\iff \\ \lambda_1 a_1 + \lambda_2 a_2 = \lambda_2. & \end{aligned}$$

Therefore, there exists $\lambda \in \mathbb{R}_+^2$ satisfying $\lambda(q) = 1$ such that $(\lambda \circ T, 0) \in \partial_{\eta+\lambda(\mathcal{E})}(\lambda \circ f)(\bar{x}, y_\eta)$ for all $\eta > 0$ if and only if $\lambda_1 a_1 + \lambda_2 a_2 = \lambda_2$ and so it follows that

$$\partial_{\mathcal{E}}^W M(\bar{x}, (0, \bar{x})) = \{(a_1, a_2) \in \mathbb{R}^2 \mid a_1 \leq 0, a_2 \geq 1\} \cup \{(a_1, a_2) \in \mathbb{R}^2 \mid a_1 \geq 0, a_2 \leq 1\}. \tag{27}$$

Notice that most of the differential stability properties stated in [23, 24, 33, 34, 36] cannot be applied to this example as $\text{WMin}(f(x, \mathbb{R}), \mathbb{R}_+^2) = \emptyset$, for all $x \in \mathbb{R}$ (see Remark 5(ii)).

Next we relate the differential stability of problem (VP_x) with the one of the family of optimization problems defined from (VP_x) by linear scalarizations:

$$\min\{(\lambda \circ f)(x, y) \mid y \in Y\}, \tag{P_{\lambda,x}}$$

where $\lambda \in D^+ \setminus \{0\}$. We stand the optimal value function of this problem for $\mu_{\lambda \circ f}$.

Proposition 2 *Suppose that f is D -convex and $\text{cl}f(x, Y)$ is D -closed, for all $x \in X$. Let $\bar{x} \in X, M(\bar{x}) \neq \{-\infty_D\}$. Consider $(\bar{x}, \bar{z}) \in \text{gph}M, \mathcal{E} \in D$ and $q \in \text{int}D$. If $T \in \partial_{\mathcal{E}}^W M(\bar{x}, \bar{z})$, then there exists $\lambda \in D^+ \setminus \{0\}, \lambda(q) = 1$, such that $\lambda \circ T \in \partial_{\lambda(\mathcal{E})} \mu_{\lambda \circ f}(\bar{x})$.*

Proof In order to apply Theorem 5, notice from Lemma 2(v) that the set $M_\eta(\bar{x}, \bar{z}, q)$ is nonempty, for all $\eta > 0$. Then, by considering for each $\eta > 0$ a point $y_\eta \in M_\eta(\bar{x}, \bar{z}, q)$ we deduce from Theorem 5 that for each $T \in \partial_{\mathcal{E}}^W M(\bar{x}, \bar{z})$ there exists $\lambda \in D^+ \setminus \{0\}, \lambda(q) = 1$, such that

$$\lambda \circ T \in \bigcap_{\eta>0} \{x^* \in X^* \mid (x^*, 0) \in \partial_{\eta+\lambda(\mathcal{E})}(\lambda \circ f)(\bar{x}, y_\eta)\}.$$

Moreover, by Remark 3, we have

$$\bigcap_{\eta>0} \{x^* \in X^* \mid (x^*, 0) \in \partial_{\eta+\lambda(\mathcal{E})}(\lambda \circ f)(\bar{x}, y_\eta)\} \subset \partial_{\lambda(\mathcal{E})} \mu_{\lambda \circ f}(\bar{x})$$

and the result follows. □

Proposition 3 *Suppose that $\bar{x} \in X, q \in \text{int}D$ and $\lambda \in D^+ \setminus \{0\}$ satisfying $\lambda(q) = 1$ and $\mu_{\lambda \circ f}(\bar{x})$ is finite. Consider $\bar{z} \in \text{argmin}_{\text{cl}f(\bar{x}, Y)} \lambda, \mathcal{E} \in D$ and $T \in L(X, Z)$. If $\lambda \circ T \in \partial_{\lambda(\mathcal{E})} \mu_{\lambda \circ f}(\bar{x})$,*

then $T \in \partial_{\mathcal{E}}^W M(\bar{x}, \bar{z})$.

Proof By Lemma 2(vi) we deduce that $\bar{z} \in \text{WMin}(\text{cl}f(\bar{x}, Y), D)$. Then, Lemma 2(i) implies that $\text{cl}f(\bar{x}, Y)$ is a D -proper set, and so $\text{WMin}(\text{cl}f(\bar{x}, Y), D) = M(\bar{x})$. Thus, $\bar{z} \in M(\bar{x})$. Clearly, $\mu_{\lambda \circ f}(\bar{x}) = \lambda(\bar{z})$.

By Lemma 2(v), the set $M_\eta(\bar{x}, \bar{z}, q)$ is nonempty. In addition,

$$M_\eta(\bar{x}, \bar{z}, q) \subset \{y \in Y \mid (\lambda \circ f)(\bar{x}, y) \leq \mu_{\lambda \circ f}(\bar{x}) + \eta\}.$$

Indeed, consider a point $y \in M_\eta(\bar{x}, \bar{z}, q)$. Then, $f(\bar{x}, y) \in \bar{z} + (-\eta q + D) \cap (\eta q - D)$ and there exists $d \in D$ such that $f(\bar{x}, y) = \bar{z} + \eta q - d$. Thus,

$$(\lambda \circ f)(\bar{x}, y) = \lambda(\bar{z}) + \eta - \lambda(d) \leq \mu_{\lambda \circ f}(\bar{x}) + \eta.$$

Then, for each $\eta > 0$ take a point $y_\eta \in M_\eta(\bar{x}, \bar{z}, q)$. By Theorem 3 we have that

$$\partial_{\lambda(\varepsilon)} \mu_{\lambda \circ f}(\bar{x}) = \bigcap_{\eta > 0} \{x^* \in X^* \mid (x^*, 0) \in \partial_{\eta + \lambda(\varepsilon)} (\lambda \circ f)(\bar{x}, y_\eta)\}$$

and the result follows by applying Theorem 6. □

Example 2 (continuation) Let us obtain $\partial_\varepsilon^W M(\bar{x}, (0, \bar{x}))$ as a result of Propositions 2 and 3. Let $q = (1, 1)$ and $\lambda = (\lambda_1, \lambda_2) \in \mathbb{R}^2$ such that $\lambda_1 + \lambda_2 = 1$. Notice that

$$\mu_{\lambda \circ f}(x) = \inf\{\lambda((0, x) + e^y(1, 1)) \mid y \in \mathbb{R}\} = \lambda_2 x + \inf\{e^y \mid y \in \mathbb{R}\} = \lambda_2 x.$$

Therefore, $\partial_{\lambda(\varepsilon)} \mu_{\lambda \circ f}(x) = \lambda_2$, for all $x \in \mathbb{R}$ and so, by Propositions 2 and 3 we have that $T = (a_1, a_2) \in \partial_\varepsilon^W M(\bar{x}, (0, \bar{x}))$ if and only if there exists $\lambda = (\lambda_1, \lambda_2) \in \mathbb{R}_+^2$ satisfying $\lambda_1 + \lambda_2 = 1$ such that $\lambda_1 a_1 + \lambda_2 a_2 = \lambda_2$, i.e., $(\lambda_1, \lambda_2) \perp (a_1, a_2 - 1)$. Thus, assertion (27) is obtained again.

7 Conclusions

In this paper several formulas are stated for computing the approximate subdifferential of the optimal value function of scalar and vector convex optimization problems whose solution sets could be empty. Notice that the scalar constrained convex optimization problems studied here are the same as those analyzed in [3]. However, in this paper we use a limiting approach to differential stability of those problems. As a result, no regularity conditions are required.

In vector optimization problems, the optimal value function is set-valued and involve infimal points. Its differentiability properties involve an ε -subdifferential for vector functions introduced by Taa. They are formulated by ε -subgradients of linear scalarizations of the problem and ε -subgradients of the optimal value function corresponding to such linear scalarizations.

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