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Plastic calculation of slender beam frames systematic method based on mechanism theory

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ABSTRACT

This work focuses on the limit analysis or plastic calculation of slender beam planar frames using a direct method based on the Virtual Works Principle (VWP). The method consists in looking for the structure's collapse mechanism from the equilibrium equations posed by means of virtual problems in displacements. This approach, in its classic formulation, is very unsystematic and requires a convenient choice of virtual problems.

In this work, it has been possible to systematize the plastic calculation of the structure by applying the theory of mechanisms to obtain, fully automatically, the equilibrium equations necessary to completely solve the structure. Also, the compatibility equations necessary to know the accumulated rotations in the plastic hinges at the instant of collapse are presented. Finally, the search for the collapse mechanism is not carried out by trial and error, but is posed by optimization based on energy criteria.

1. Introduction

Steel frames show a high nonlinear behavior due to the plasticity of the material and the slenderness of members. How to approach the behavior of steel frames has been a large subject in the research field of constructional computation. In general, the plastic-hinge approach is adopted to capture the inelasticity of material of a framed structure. The plastic-hinge approach demands only one beam-column element per physical member to assess approximately the nonlinear properties of the structures; so the computation time is considerably reduced. Wherefore, the improvement in the accurateness of the plastic-hinge approach has been an attractive topic.

In 1914 Kacinczy was the first to investigate the reserve of plastic resistance in a hyperstatic beam structure, introducing the concept of the plastic hinge and the collapse mechanism, independently Kist introduced same concepts in 1917.

During the past 60 years, the theories of plasticity, stability and computing technology have recorded great achievements that constitutes the basis allowing scientists to develop successfully plastic methods for structures. The framed structures are often regarded as benchmark to build up computation methods for other kinds of structure. Up to now, plastic methods for framed structures can be classed into two groups: step-by-step method and direct methods.

Step-by-step methods or elastic-plastic incremental methods are based on the standard methods of the elastic analysis. The loading process is divided into various steps. After reach loading step, the stiffness matrix is updated to take into account nonlinear effects. In comparison with the elastic solution, only the physical matrix is varied to consider the plastic behavior. The step-by-step methods take advantage of large experiences of the linear elastic analysis by the finite element method. One may find many useful computational algorithms and techniques in many text books [1].

The Direct methods consists in the rigid-plastic methods that the load multiplier can be directly identified without any intermediate states of structures. The direct methods are based on the static and kinematic theorems-two fundamental theorems of the limit analysis, which lead to static approach and kinematic approach, respectively.

The plastic behavior of structures in general and of framed structures in particular was well dealt with in many text books (e.g. Neal [2]). This work collects the basic theorems: static, kinematic and uniqueness; as well as the method for combining mechanisms. Such authors as M. Doblaré and L. Gracia [3]; H. R. Dalmau and J. Vilardell [4]; G. R.

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Calborg [5] applied to the study of the plastic analysis by bending of structures, mainly beams and basic frames.

The mechanism combination method is valid for solving very simple structures by hand. However, it has important drawbacks from the point of view of its practical application: first, it requires the appropriate virtual problems to be chosen for use both in displacement and in forces; and secondly, it requires possible collapse mechanisms to be tested, which even with few plastic hinges implies many possible collapse mechanisms that will have to be tested and verified. This makes the method very didactic and interesting from an academic point of view, but in practice hardly or not at all competitive with step by step calculation methods based on the rigidity method.

These drawbacks have been solved in this work by: one, the approach of the systematic equilibrium equations with the help of the Theory of Mechanisms; and two, the search for the structure's collapse mechanism using nonlinear optimization techniques based on energy criteria, which seeks to maximize the total energy of the system while making the dissipated energy minimal. A set of cases has been solved, with both point loads and uniform distributed load, in low calculation times.

This paper has been organized as follows: after this brief introduction, the methodology is presented, which is then validated by solving a simple case of which the exact solution is known; then, more cases are solved to see the potential of the calculation method presented and it is applied to a practical example (a gabled frame with point loads and distributed load), and finally the main conclusions and contributions of the work are summarized.

2. Methodology

In this section, we explain the methodology developed to systematize the direct method of limit analysis for slender beam frames using the Virtual Works Principle (VWP). We first establish the basic hypotheses, the candidate sections for the plastic hinge, the degree of the structure's hyperstaticity, the equilibrium equations (which are taken from the Virtual Displacement Principle (VDP)), and the structure's compatibility equations (taken from the Virtual Forces Principle (VFP)). All of this is calculated systematically using the theory of mechanisms.

2.1. Calculation hypothesis

The basic hypotheses on which the limit analysis or plastic calculation of this work is based are:

- The beams of the structure are assumed to be straight, of uniform section and sufficiently slender.
- The beams of the structure are free from residual or initial stresses.
- Plastic collapse involves an indefinite growth of displacement under constant load, and the load that produces it is the plastic collapse load.
- The section can only accumulate rotation when the bending moment reaches the value of the plastic moment, and this rotation can increase indefinitely.
- The plastic moment depends on the material and the section, and its possible reduction is neglected due to the effect of the rest of the forces transmitted by the section, for example axial and shear forces.
- The formation of the plastic hinge is supposed to take place suddenly and to be concentrated in the section that reaches the value of the plastic moment.
- The hypothesis of small displacements and rotations of the sections of the structure at the moment of collapse is assumed, therefore, the accumulated rotation in the plastic hinges must also be small values.

2.2. Candidate sections for the plastic hinge

The sections of the structure that are candidates for the formation of a possible plastic hinge, in principle, are:

- Knots of the structure, that is to say, zones of union of different beams, and beams with columns.
- Fixed support
- Sections of application of concentrated loads.
- Intermediate sections of beams or columns with distributed load.
- Zones with section change.

The total number of Possible Plastic Hinges is called (nPPH).

2.3. Balance and compatibility equations

Most plastic problems are solved using only the equilibrium equations with the Virtual Displacement Principle (VDP). This requires the number Static Structure (nSS) to be calculated, as it allows us to determine the number of Equilibrium Equations (nEE) necessary to pose and solve our problem: $nEE = nPPH - nSS$. These equilibrium equations are obtained, as has already been said, by applying the Virtual Displacements Principle (VDP) that requires compatible virtual or auxiliary problems in displacements.

The virtual problems that arise are possible mechanisms, that is, part or all of the structure experiences rigid solid movements that can be calculated systematically with the mechanisms theory. If we think of a fixed-ended frame with a panel mechanism, the final position of the mechanism clamp coincides with the fixed section on the right, its final position is known, and it is necessary to solve the inverse kinematic problem, that is, a value of small rotation for some of the torques based on the hypothesis of small displacements and rotations of the structure at the instant of plastic collapse.

Finally, by means of the Virtual Forces Principle (VFP), which requires virtual or auxiliary problems in balance, nSS compatibility equations are obtained, which will allow us to calculate the accumulated turns in the plastic ball joints.

2.4. Energy criteria

In the classic approach to the plastic problem, through the direct method based on the application of the Virtual Works Principle (VWP), we proceed to test possible mechanisms in order to search for collapse mechanisms. This procedure can be successful if the structure's collapse mechanism is tested at the beginning. However, in general, it is not known and the mechanisms that result from the combinations of the candidate sections of which the plastic hinges are formed have to be tested one by one, which can take a lot of calculations.

However, in this work, the method uses energy criteria and optimization techniques [8,9] so as to be able to avoid having to try possible mechanisms one by one. It is the constrained nonlinear optimization algorithm itself that, once well thought out, searches for the collapse mechanism corresponding to the structure with given loads and boundary conditions.

The optimization problem that arises consists in maximizing the total energy of the structure and minimizing the dissipated energy, a problem subject to the equality restrictions given by the equilibrium equations ($eq(\lambda, x_i, M_i, \theta_i) = 0$) together with the compatibility equations ($ce(\lambda, x_i, M_i, \theta_i) = 0$). In addition, there are, the restrictions of compatibility of the bending moments and accumulated rotations in the plastic hinges (see Fig. 1), which requires that they have the same sign:

$$M_i \cdot \theta_i \geqslant 0 \tag{1}$$

M_i bending moments in the sections of the structure and θ_i accumulated rotation in the plastic hinges. Regarding the signs of moments and rotations, are positive if they are counter-clockwise.

The case of the uniform distributed load introduces additional unknowns: the positions (x_i) of the possible plastic hinges are unknown a priori, and therefore, the optimization algorithm must ensure that the collapse load caused by the final structure (collapse mechanism) must be a minimum load value.

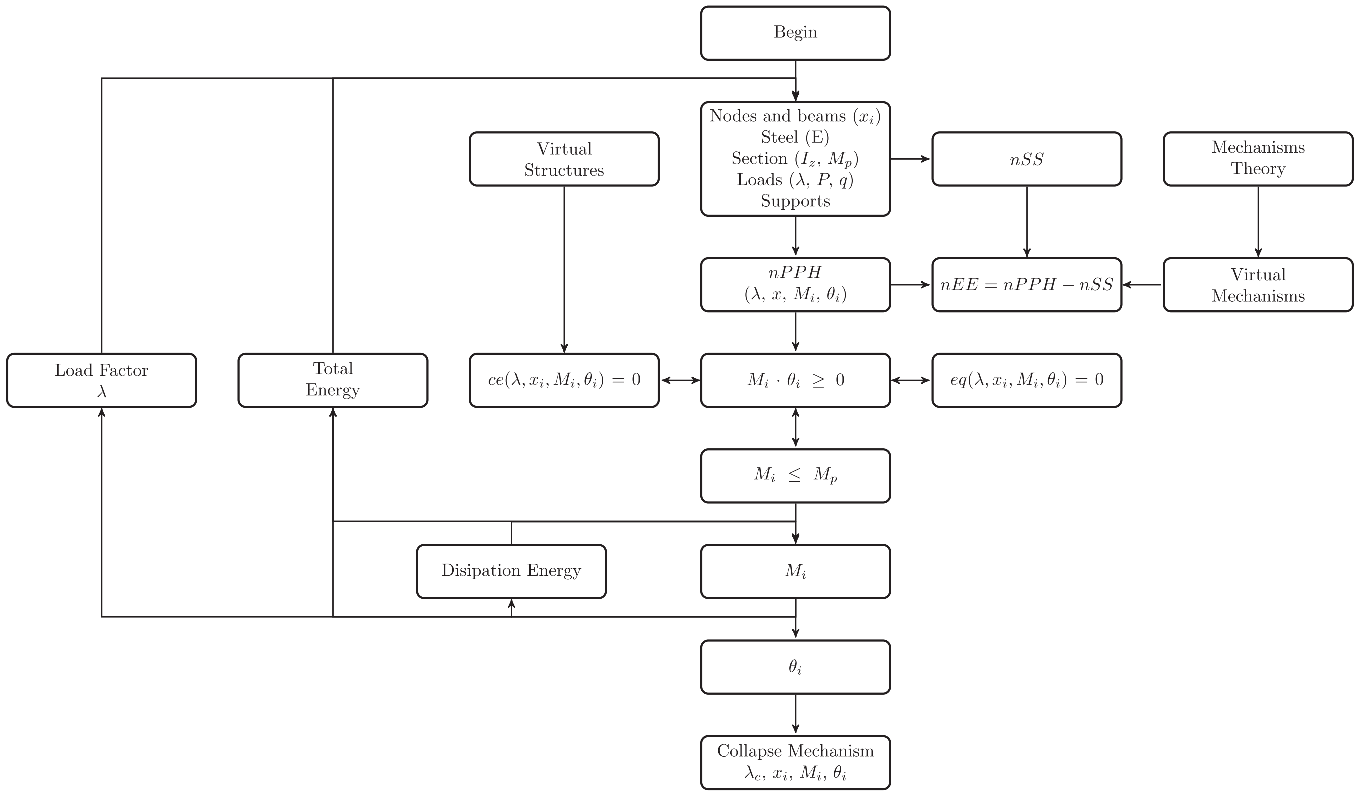

Fig. 1. Methodology flow chart.

The methodology has been tested with small structures, with few beams and/or columns that implies a low number of parameters, therefore Gradient Based techniques have been used. It is clear that, if the number of parameters is increased, it may be necessary to switch to Population Based techniques..

Fig. 1 includes a flow chart that summarizes the methodology followed in this work.

3. Validation problem: fixed-fixed frame

In this section, a first case is included as a validation problem, consisting of a basic frame fixed ended in the base of both columns (both columns of length L), beam length ($2L$) and whose loads are as indicated in Fig. 2, both of value P. All columns and beam have the same mechanical and geometric properties, so the maximum moment for all the bars is the plastic moment, M_p. The numerical data of the problem are: $L = 4\,m$; $P = 1000\,N$; $E = 2.1{\cdot}10^{11}\,Pa$; $I_z = 8360.0{\cdot}10^{-8}\,m^4$; $S =$

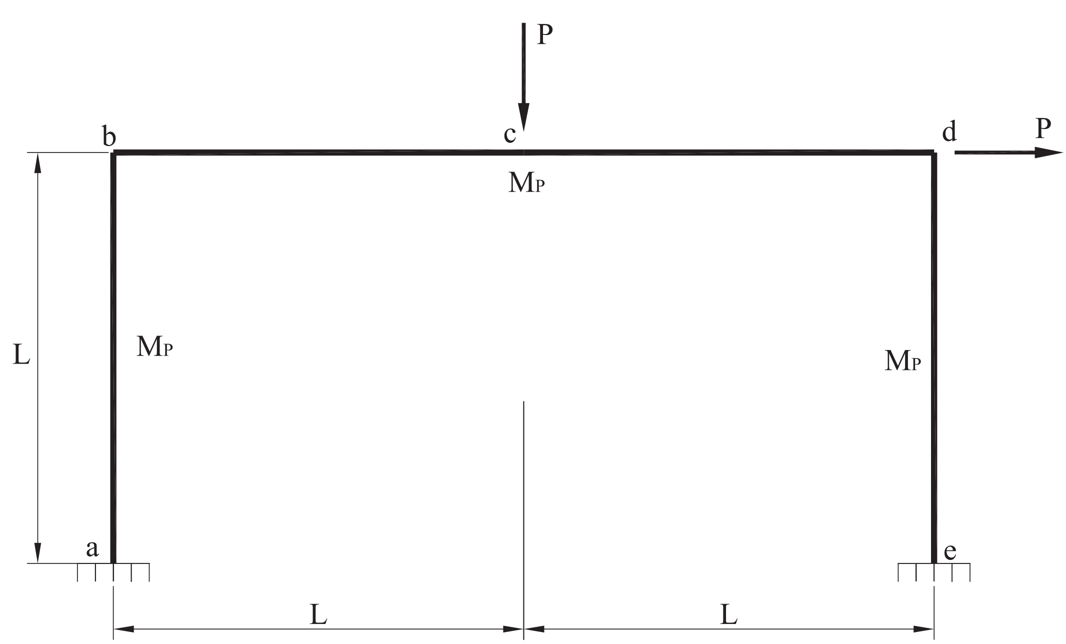

Fig. 2. Fixed-fixed frame.

$628.0{\cdot}10^{-6}\,m^3$; $f_y = 275.0{\cdot}10^6\,Pa$; $M_p = S{\cdot}f_y = 172700.0\,N{\cdot}m$, where L is the length of the columns and half of the beam, P is the concentrated loads, E is Young's module, I_z the moment of inertia, S the section plastic module, f_y the yield strength of the steel and M_P the plastic moment.

The methodology outlined in Section 2 is applied using a MATLAB script that systematically solves the plastic problem without user intervention [6,7]. To do so, it first calculates the equilibrium equations ($nEE = 2$) and the compatibility equations ($nSS = 3$) for this problem, which it then introduces as equality constraints in the optimization algorithm. It is the optimization algorithm that searches for the collapse mechanism, maximizing the total energy of the system; in this, case the collapse mechanism of the structure involves the formation of plastic hinges in sections **a, c, d** and **e**.

The collapse load factor results in the value: $\lambda_c = 129.5247$; therefore, the collapse load of the structure is $P_c = \lambda_c{\cdot}P$, which corresponds to the exact theoretical value,

$$P_c = \frac{3M_p}{L} \tag{2}$$

where P_c is the collapse load.

To obtain the accumulated rotations in the plastic hinges at the instant of collapse, the condition that the dissipated energy is minimal is imposed, thus obtaining the following results: $\theta_a = 0.0\,rad; \theta_c = 0.0065\,rad; \theta_d = -0.01312\,rad; \theta_e = 0.00656\,rad$; which again exactly matches the theoretical solution:

$$\theta_c = \frac{M_p{\cdot}L}{6EI_z}$$
$$\theta_d = -\frac{M_p{\cdot}L}{3EI_z} \tag{3}$$
$$\theta_e = \frac{M_p{\cdot}L}{6EI_z}$$

The bending moments in the sections where plastic ball joints are formed are:

$$\begin{aligned} M_a &= -M_p \\ M_c &= +M_p \\ M_d &= -M_p \\ M_e &= +M_p \end{aligned} \tag{4}$$

Therefore, the solution obtained meets (1), is compatible and is the solution of the plastic calculation problem.

4. Examples

Then, in the following three sections, a series of cases is studied: in the first (Example 1), the support conditions of the frame are changed; in the second (Example 3), the values of the loads are changed; and in the third (Example 3), the type of loads, which are now uniformly distributed, is changed.

4.1. Example 1: fixed-pinned frame

This first case study is very similar to the one seen in the previous section, in fact it only differs in that the support of the base of the righthand column has been changed from fixed to hinge support, see Fig. 3.

The data in this example is again: $L = 4\,m$; $P = 1000\,N$; $E = 2.1{\cdot}10^{11}\,Pa$; $I_z = 8360.0{\cdot}10^{-8}\,m^4$; $S = 628.0{\cdot}10^{-6}\,m^3$; $f_y = 275.0{\cdot}10^6\,Pa$; $M_p = S{\cdot}f_y = 172700.0\,N{\cdot}m$.

For this methodology, the equilibrium equations ($nEE = 2$) are first applied, followed by the compatibility equations ($nSS = 2$). Now, in the section (e), where the hinge support is located, no plastic join can be formed. The optimization algorithm searches for the collapse mechanism, which in this case involves the formation of plastic hinges in sections **b, c** and **d**. A collapse mechanism is formed, which is termed complete because just $nSS + 1$ plastic hinges are formed in the structure.

The collapse load factor has the value: $\lambda_c = 172.7$; therefore, the collapse load of the structure is $P_c = \lambda_c{\cdot}P$, which corresponds to the exact theoretical value,

$$P_c = \frac{4M_p}{L} \tag{5}$$

where P_c is the collapse load.

As in the first case, to obtain the accumulated rotations in the plastic hinges at the instant of collapse, the condition that the dissipated energy is minimal is imposed, thus obtaining the following results: $\theta_b = 0.0\,rad; \theta_c = 0.06558\,rad; \theta_d = -0.03498\,rad$; which again exactly match the theoretical solution:

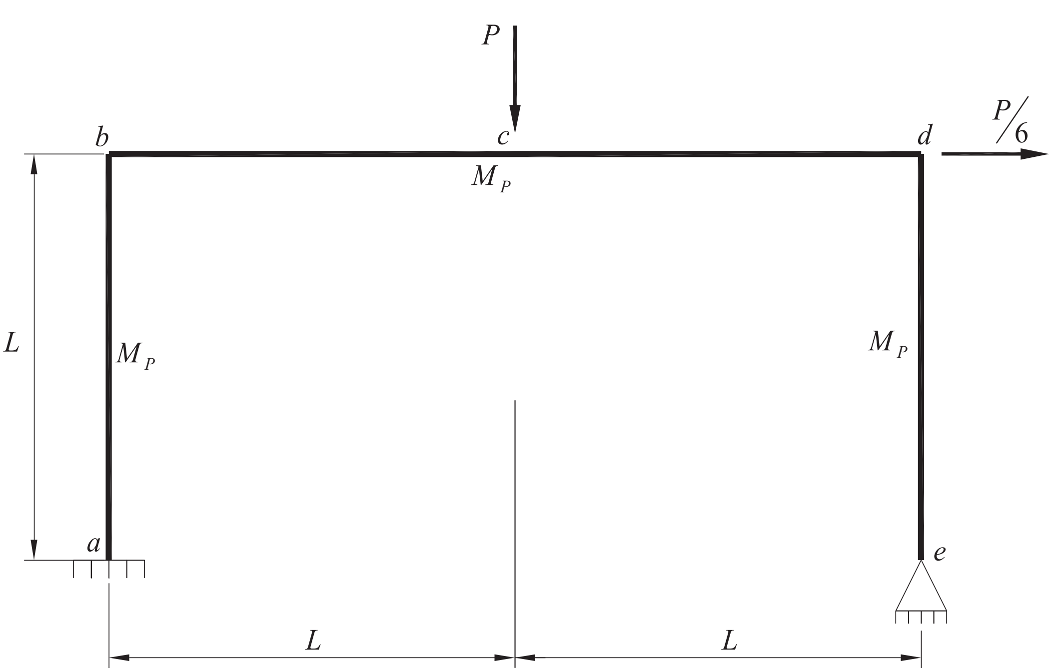

Fig. 3. Fixed-hinge frame.

$$\theta_c = \frac{5M_p \cdot L}{3EI_z}$$
$$\theta_d = -\frac{8M_p \cdot L}{9EI_z} \quad (6)$$

The bending moments in the sections where plastic hinges are formed are:

$$M_a = -0.667 \cdot M_p$$
$$M_b = -M_p$$
$$M_c = +M_p$$
$$M_d = -M_p \quad (7)$$

Therefore, the solution obtained meets (1), is compatible and it is the solution of the plastic calculation case raised.

4.2. Example 2: frame with point loads

This second example is in every way similar to the structure of the validation problem except that the concentrated loads have different values, see Fig. 4.

The equilibrium equations ($nEE = 2$) and the compatibility equations are posed ($nSS = 3$), looking for the collapse mechanism: sections **b, c** and **d**. The collapse mechanism formed is an incomplete mechanism because less than $nSS+1$ plastic hinges are formed in the structure.

The collapse load factor is: $\lambda_c = 172.7$; therefore, the collapse load of the structure is $P_c = \lambda_c \cdot P$, which corresponds to the exact value,

$$P_c = \frac{4M_p}{L} \quad (8)$$

where P_c is collapse load.

The rotations accumulated in the plastic hinges at the instant of collapse are: $\theta_b = 0.0\,rad$; $\theta_c = 0.03279\,rad$; $\theta_d = -0.01312\,rad$; which again exactly match the theoretical solution:

$$\theta_c = \frac{5M_p \cdot L}{6EI_z}$$
$$\theta_d = -\frac{M_p \cdot L}{3EI_z} \quad (9)$$

The bending moments in the sections where plastic hinges are formed are:

$$M_a = 0.167 \cdot M_p$$
$$M_b = -M_p$$
$$M_c = +M_p$$
$$M_d = -M_p$$
$$M_e = 0.833 \cdot M_p \quad (10)$$

The solution obtained is the plastic problem solution because it meets (1), and is, therefore, compatible.

4.3. Example 3: frame with uniform distributed load

This case is even more interesting because the load type is changed and includes a uniform distributed load along the total length of the lefthand column, which can, for example, simulate the action of the wind, see Fig. 5. The data in this case are: $L = 1\,m$; $q = 1000\,N/m$; $E = 2.1 \cdot 10^{11}\,Pa$; $I_z = 8360.0 \cdot 10^{-8}\,m^4$; $S = 628.0 \cdot 10^{-6}\,m^3$; $f_y = 275.0 \cdot 10^6\,Pa$; $M_p = S \cdot f_y = 172700.0\,N \cdot m$, where q is the intensity of the distributed load.

As pointed out above, this example is very interesting because, when introducing a distributed type load, an intermediate plastic hinge may originate in the column, which does not a priori know where it is going to occur, in section (b) for example. One more unknown (x, position of the plastic hinge) appears for each additional distributed load. To solve the plastic analysis in such a case, it is necessary for the solution to ensure that the collapse load that originates the final situation of the structure, the plastic collapse load, is a minimum value, which requires another optimization loop.

Once the problem is well posed in a script of the Matlab program, the optimization algorithm itself is in charge of looking for the minimum collapse load that causes the plastic collapse of the structure of interest; in this case, the mechanism that involves plastic hinges in the following sections: **a, b, d** and **e**, again the final mechanism formed is a complete mechanism. The collapse load factor is: $\lambda_c = 143.228$; therefore, the collapse load of the structure is $q_c = \lambda_c \cdot q$, which corresponds to the exact theoretical value,

$$q_c = \frac{2(2+\sqrt{3})M_p}{9L^2} \quad (11)$$

where q_c is the collapse load.

The rotations accumulated in the plastic ball joints at the instant of

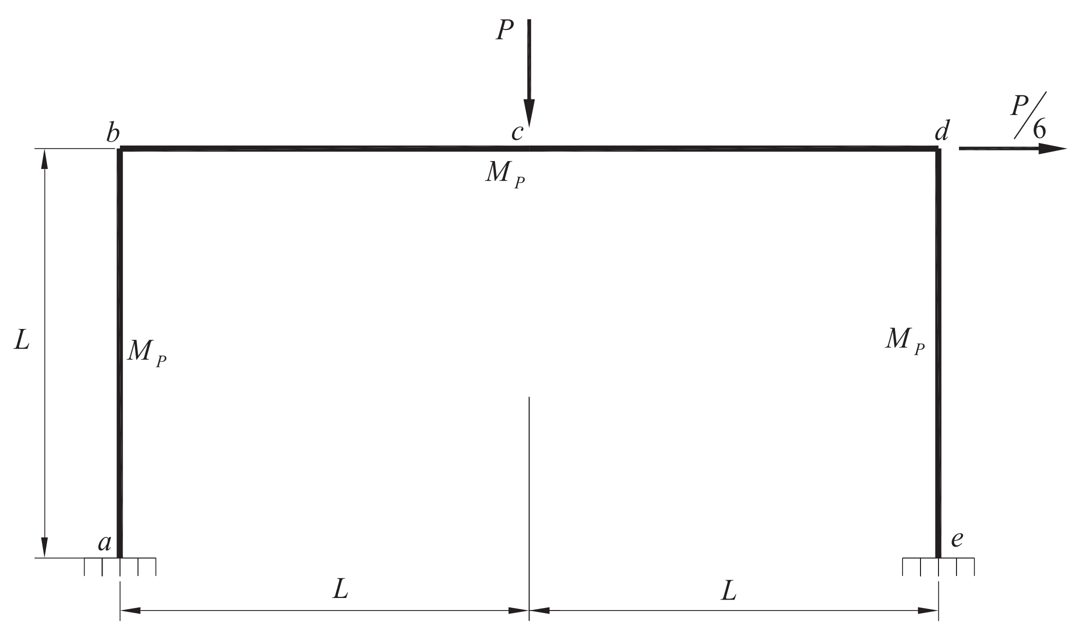

Fig. 4. Frame with point loads.

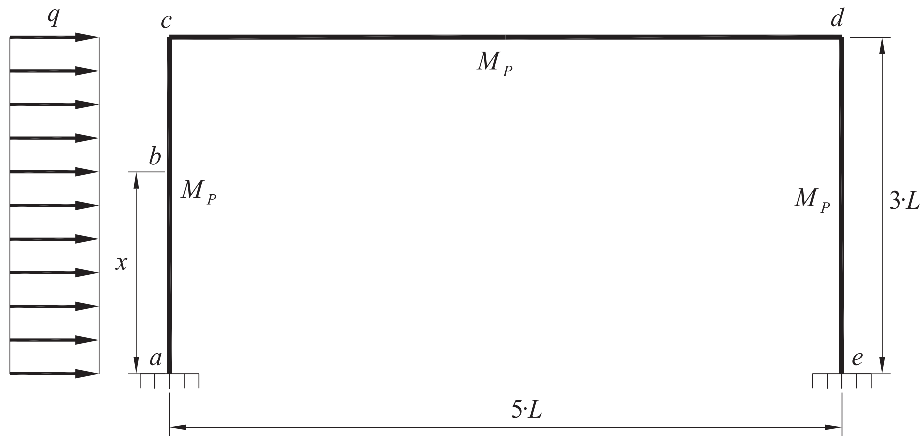

Fig. 5. Frame with uniform distributed load.

collapse are: $\theta_a = -0.0182716\,rad$; $\theta_b = 0.0000646283\,rad$; $\theta_d = 0.0\,rad$; $\theta_e = 0.0103941\,rad$; which again exactly match the theoretical solution:

$$\begin{aligned}\theta_a &= \frac{11(2\sqrt{3}-9)M_p{\cdot}L}{12(1+\sqrt{3})EI_z}\\ \theta_b &= \frac{(7-4\sqrt{3})M_p{\cdot}L}{4(1+\sqrt{3})EI_z}\\ \theta_d &= 0\\ \theta_e &= \frac{5(3-\sqrt{3})M_p{\cdot}L}{6EI_z}\end{aligned} \tag{12}$$

The bending moments are:

$$\begin{aligned}M_a &= -M_p\\ M_b &= +M_p\\ M_c &= 0.732{\cdot}M_p\\ M_d &= -M_p\\ M_e &= +M_p\end{aligned} \tag{13}$$

And the intermediate section in the sector requested by the distributed load (section b) is:

$$x = 2.1941\,m \tag{14}$$

It is important to highlight that a methodology based on energy criteria allows problems to be solved both with loads concentrated in certain sections and with uniform distributed load on some beams and/or columns.

5. Problem application: gable frame

The resolution of the practical problem of a gable type frame with distributed loads and point loads is considered in this section, see Fig. 6. The data of the problem are: $L_p = 4m$; $L_d = 6m$; $\beta = 10^\circ$; $q = 1000\,N/m$; $P = 1000\,N$; $E = 2.1{\cdot}10^{11}\,Pa$; $I_z = 8360.0{\cdot}10^{-8}\,m^4$; $S = 628.0{\cdot}10^{-6}\,m^3$; $f_y = 275.0{\cdot}10^6\,Pa$; $M_p = S{\cdot}f_y = 172700.0\,N{\cdot}m$, where L_p is the height of the columns, L_d is half the width of the frame and β is the angle of inclination of the beam.

After applying the same methodology as in the examples in the previous section, a collapse load factor of value $\lambda_c = 26.65$ is obtained. A complete plastic collapse mechanism is formed that involves the formation of hinges in sections **a, d, f** and **g.** The values of the accumulated rotation in the plastic hinges at the instant of collapse are: $\theta_a = -0.00615\,rad$; $\theta_d = 0.0\,rad$; $\theta_f = -0.0231\,rad$; $\theta_g = 0.0200\,rad$.

The values of the bending moments of the frame in the candidate sections are:

$$\begin{aligned}M_a &= -M_p\\ M_b &= -0.4393{\cdot}M_p;\ (x_b = 2.0\,m)\\ M_c &= -0.4956{\cdot}M_p\\ M_d &= +M_p;\ (x_d = 4.2978\,m)\\ M_e &= 0.7811{\cdot}M_p\\ M_f &= -M_p\\ M_g &= +M_p\end{aligned} \tag{15}$$

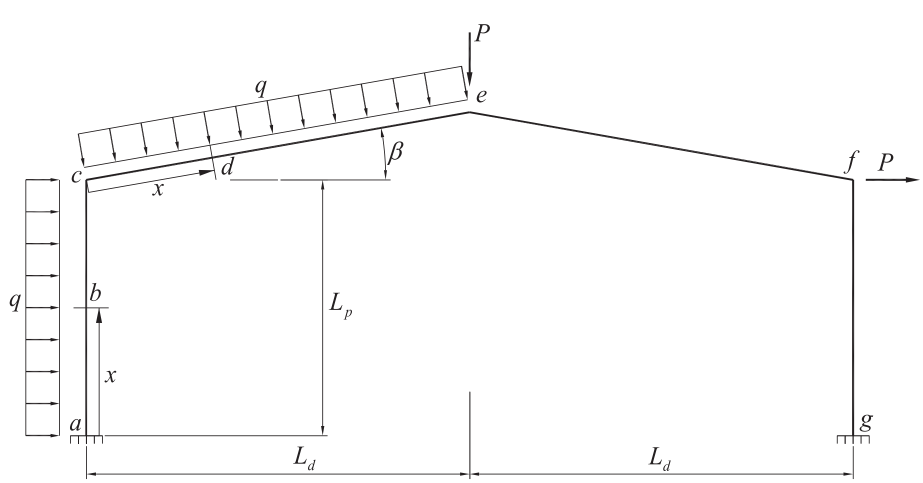

Fig. 6. Gable frame.

It is necessary to clarify that this case of geometry, loads and supports does not originate a plastic hinge in section b, although does in section d requested by distributed load.

6. Conclusions

The plastic methods for planar beam frames have been investigated for decades by numerous researchers. There are two analysis methods: "Direct methods" or "Step-by-step methods". Direct methods in their classic formulation are very unsystematic. They are based on the Virtual Works Principle (VWP) and use equilibrium equations to find the structure's collapse mechanism using virtual problems in displacements (virtual mechanisms).

This work uses a Direct method, but the solution is not presented in a traditional way, combining mechanisms and testing until the collapse mechanism is found.

However, in this work it has been possible to systematize the equations by applying the Theory of Mechanisms. Likewise, the compatibility equations necessary to know the accumulated rotations in the plastic hinges at the moment of collapse are presented.

One of the contributions of this work is that optimization techniques are used to look for the last state of the structure, the collapse mechanism and the associated load factor.

This work is a very useful two-dimensional method for the analysis of real industrial buildings, since they are large structures with a continuous section that can be studied using flat frames. The methodology of this work is useful to analyze the structure regardless of the type of loads: concentrated loads or distributed load.

Declaration of Competing Interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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Appendix A. Virtual Works Principle (VWP)

The methodology of this work is based on the Virtual Works Principle (VWP), which consists of an energy balance: the work carried out by external loads is equal to the deformation energy of the structure plus the energy dissipated in the plastic hinges, which is formulated as:

$$\sum_{j=1}^{nP} P_j v_j(\theta_i) + \sum_{l=1}^{nq} \int_0^{L_l} q_l(x) v_l(\theta_i) dx = \sum_{k=1}^{nb} \int_0^{L_k} M(x) \frac{m(x)}{EI_z} dx + \sum_{i=1}^{nPPH} M_i \theta_i \tag{16}$$

where nP is the number of sections with point loads, P_j is the point load, $v_j(\theta_i)$ is the transverse displacement that depends on the accumulated rotations in the plastic hinges, nq is the number of beams and/or columns with uniform distributed load, q_l is the value of the distributed load, nb is the number of beams and columns in the structure, $M(x)$ is the bending moment in the beams and columns of the structure, $m(x)$ is the bending moment of the auxiliary or virtual problem, and θ_i is the accumulated rotation in the plastic hinges.

The previous expression allows us to calculate: the total energy, the energy of bending deformation and the dissipated energy of the structure of interest.

If the integral expression makes use of a virtual or auxiliary problem in displacements (see Annex 2), it is called the Virtual Displacements Principle (VDP) and provides equilibrium equations for the analyzed structure.

On the contrary, if the VWP is posed using a virtual or auxiliary problem in equilibrium (see Annex 3), it is called the Virtual Forces Principle (VFP) and provides compatibility equations for the analyzed structure.

Appendix B. Virtual Mechanism

Applying the Virtual Displacements Principle (VPD) requires posing auxiliary problems in compatible displacements that can be mechanisms, which involve null deformations and stresses. For this, the inverse kinematic problem of the theory of mechanisms is used.

To better illustrate the technique, it is applied to the problem in the Fig. 2. First, the coordinate transformation matrix of each beam element is formulated:

$$T_0^1 = \begin{pmatrix} \cos(\theta_a) & -\sin(\theta_a) & 0 & 0 \\ \sin(\theta_a) & \cos(\theta_a) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \simeq \begin{pmatrix} 1 & -\theta_a & 0 & 0 \\ \theta_a & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{17}$$

$$T_1^2 = \begin{pmatrix} \cos(\theta_b) & -\sin(\theta_b) & 0 & 0 \\ \sin(\theta_b) & \cos(\theta_b) & 0 & L \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \simeq \begin{pmatrix} 1 & -\theta_b & 0 & 0 \\ \theta_b & 1 & 0 & L \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{18}$$

$$T_2^3 = \begin{pmatrix} \cos(\theta_c) & -\sin(\theta_c) & 0 & L \\ \sin(\theta_c) & \cos(\theta_c) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \simeq \begin{pmatrix} 1 & -\theta_c & 0 & L \\ \theta_c & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{19}$$

$$T_3^4 = \begin{pmatrix} \cos(\theta_d) & -\sin(\theta_d) & 0 & L \\ \sin(\theta_d) & \cos(\theta_d) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \simeq \begin{pmatrix} 1 & -\theta_d & 0 & L \\ \theta_d & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{20}$$

$$T_4^5 = \begin{pmatrix} \cos(\theta_e) & -\sin(\theta_e) & 0 & 0 \\ \sin(\theta_e) & \cos(\theta_e) & 0 & -L \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \simeq \begin{pmatrix} 1 & -\theta_e & 0 & 0 \\ \theta_e & 1 & 0 & -L \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{21}$$

Therefore, the system coordinate transformation matrix results:

$$T_0^5 = T_0^1 \cdot T_1^2 \cdot T_2^3 \cdot T_3^4 \cdot T_4^5 = \begin{pmatrix} 1 & 0 & 0 & 2\cdot L \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{22}$$

where, in the transformation matrices of each beam of the system, the hypothesis of small displacements and rotations has been assumed. The origin of the coordinate system has been located in section a, the x axis oriented from section a to section e, with the y axis being vertical.

The final position and orientation vector is known and corresponds to section e:

$$P = \begin{pmatrix} 2\cdot L \\ 0 \\ 0 \end{pmatrix} \tag{23}$$

Giving initial values to the turns of the pairs, for example, the values $\theta_a = -0.0001\,rad; \theta_c = 0.0\,rad$, and solving the inverse kinematic problem; the following torque values are obtained:

$$\begin{aligned} \theta_b &= 0.0001\,rad \\ \theta_d &= -0.0001\,rad \\ \theta_e &= 0.0001\,rad \end{aligned} \tag{24}$$

which corresponds to the mechanism known as mechanism board.

Appendix C. Virtual structure

Applying the Virtual Forces Principle (VFP) requires posing auxiliary problems in balance. This then is simple, at least in principle, as it only includes concentrated forces and/or moments.

By posing virtual problems with loads and/or specific moments, the calculation of the work of the external loads is simple and the calculation of the deformation energy can be systematized.

If the structure of the problem of interest only has concentrated loads, the following expression results for each beam:

$$\int_0^L M(x)\frac{m(x)}{EI_z}dx = \frac{L}{6EI_z}(m_a(2M_a + M_b) + m_b(M_a + 2M_b)) \tag{25}$$

where $M(x)$ are the moments of the problem of interest and $m(x)$ are the moments of the virtual problem.

If the structure beam is requested by uniform distributed load, the following expression is as follows:

$$\int_0^L M(x)\frac{m(x)}{EI_z}dx = \frac{L}{6EI_z}(m_a(2M_a + M_b) + m_b(M_a + 2M_b) + \frac{qL^2}{4}(m_a + m_b)) \tag{26}$$

where q is the value of the uniform distributed load requested at the beam.

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