Granular Computing
https://doi.org/10.1007/s41066-021-00306-x

ORIGINAL PAPER

Design of robust control for uncertain fuzzy quadruple-tank systems with time-varying delays

Ghali Naami[1] · Mohamed Ouahi[1] · Abdelhamid Rabhi[2] · Fernando Tadeo[3] · Viet Long Bui Tuan[2]

Received: 15 October 2021 / Accepted: 30 November 2021

Abstract

The robust H_∞ observer-based control design is addressed here for non-linear Takagi-Sugeno (T-S) fuzzy systems with time-varying delays, subject to uncertainties and external disturbances. This is motivated by the quadruple-tank with time delay control problem. The observer design methodology is based on constructing an appropriate Lyapunov–Krasovskii functional (LKF) for an augmented system formed from the original and the delayed states. The bilinear terms are transferred to the linear matrix inequalities, thanks to a change of variables which can be solved in one step. Furthermore, by employing the $\mathcal{L}_2$ performance index, the adverse effects of persistent bounded disturbances is largely avoided. The proposed method has the advantage of relating the controller and Lyapunov function to both the original and delayed states. Then, the controller and observer gains are obtained simultaneously by solving these inequalities with off-the-shelf software (Yalmip/MATLAB toolbox). Finally, an application to a simulated quadruple-tank system with time delay is carried out to demonstrate the benefits of the proposed technique, showing a compromise between controller simplicity and robustness that outperforms previous approaches.

Keywords Observer-based control · Quadruple-tank system · Takagi–Sugeno (T–S) fuzzy systems · Time-varying delays · H_∞ performance · Uncertainty · Linear matrix inequalities (LMIs)

1 Introduction

Takagi-Sugeno (T-S) systems are a kind of the fuzzy system introduced in Tanaka and Wang (2004) to facilitate the use of fuzzy system tools for some nonlinear systems. Because of the effective representation of a nonlinear system as a set of local linear models that are interpolated by nonlinear functions, T-S fuzzy methods have proved useful in a variety of problems (Ammar et al. 2018; Tuan VLB, and Hajjaji 2018; Chaibi et al. 2019; Naami et al. 2019; Ejegwa 2020; Yang et al. 2020; Saif et al. 2020; Dutta and Doley 2021; Zhang and Huang 2021; Kchaou and Jerbi 2021; Ech-charqy et al. 2020; D'Urso 2017; Najariyan et al. 2017). Stability analysis and controller design can then be handled with this technique (Takagi and Sugeno 1985; Tanaka et al. 1998) for nonlinear systems (Xie et al. (2020)), by an equivalent combination of linear systems. In this context, they have been shown to be useful as universal approximators in (Buckley (1992)) and (Castro (1995)), making possible to extend classical linear model techniques to a wide range of problems, including stabilization, observation, regulation and filtering.

Ghali Naami, Mohamed Ouahi, Abdelhamid Rabhi, Fernando Tadeo and Viet Long Bui Tuan contributed equally to this work.

✉ Fernando Tadeo
fernando.tadeo@uva.es

Ghali Naami
ghali.naami@usmba.ac.ma

Mohamed Ouahi
mohamed.ouahi@usmba.ac.ma

Abdelhamid Rabhi
abdelhamid.rabhi@u-picardie.fr

Viet Long Bui Tuan
btvlong@u-picardie.fr

1 Laboratory of Engineering, Systems and Applications, National School of Applied Sciences, BP 72 Imouzzer Road, Fez, Morocco

2 Modeling Information and Systems (MIS) Laboratory, University of Picardie Jules Verne, 33 Rue St Leu, Amiens, Morocco

3 Industrial Engineering School, University of Valladolid, Dr. Mergelina S/N, Valladolid, Spain

Published online: 02 February 2022

Over the last few decades, many research approaches have studied the observer-based control in the context of disturbances or noises that might create instabilities in nonlinear closed-loop systems (Wang et al. 2021). This control is often obtained under the assumption that the entire state vector can be accessed by output measurement. This is complex in practice when there are disturbances (Wang and Lam (2021)), or when some of the external disturbances to the system are unavailable (Wei and Ma (2021)). In several studies, the problem of constructing observers for systems with noise has been solved by using the H_∞ filtering approach (Tuan VLB, and Hajjaji 2018; Xie et al. 2019; Naami et al. 2021). Moreover, (He et al. (2021)) developed some important works on stabilisation for T-S fuzzy descriptor systems. Recently, a straightforward method for observer-based H_∞ control for discrete-time Takagi–Sugeno (T–S) fuzzy systems has been presented (Chang et al. (2015); Zhi (2021)). An observer-based controller as an alternative to direct static state feedback is discussed in (Chang et al. (2016); Mahmoud et al. (2021)). We also point out (Shahbazzadeh et al. (2021)) where dynamic output feedback controller was described for Lipschitz nonlinear systems under input saturation. On the other hand, the H_∞ observer design for uncertain one-sided Lipschitz systems with time-varying delay (albeit without taking the controller into account) is proposed in (Yan et al. (2020)).

In such context, the presence of parametric uncertainties, make the stabilization of the system more complicated. As a result, attention is being paid to uncertain processes. For example, the authors in (Salehifar et al. (2021)) have been investigated the problem of robust observer-based control for one-sided Lipschitz nonlinear systems subject to parametric uncertainties and external disturbances. Several other articles have addressed the stable stabilization problem of uncertain Takagi–Sugeno (T-S) fuzzy models (Ahammed and Azeem 2019; Dong et al. 2021; Zhu et al. 2021). In (Islam et al. (2020)), the robust controller design for an uncertain fuzzy system with time-varying time-delay was investigated using a fuzzy functional observer. Several results dealing with observer-based controller design method for Lipschitz nonlinear systems with uncertain parameters and disturbances have also been published (Zemouche et al. 2017; Rastegari et al. 2019; Yang et al. 2021; Dinh 2021). Among them we emphasize (Xu et al. (2019)), based on minimizing an index related to the state estimation performance, to optimize the actual value of the uncertainty. Furthermore, using a fuzzy description of the uncertainty bound, the optimal design of the controller is envisaged in (Yang et al. (2021)), using a comprehensive fuzzy performance index that involves the performance and the control cost.

The present work focuses on the design of H_∞ observer-based controllers for delayed continuous-time Takagi–Sugeno fuzzy systems, in the presence of parameter uncertainties and external disturbances. The research is motivated by a process composed of four interconnected tanks; the model of this process is also used to demonstrate the approach. There are some previous studies on observer-based controllers for fuzzy systems: for instance, in (Tuan VLB et al. (2019)), these controllers were applied to a similar quadruple-tank system; in (Naami et al. (2021)), the robust H_∞ control problem was studied with parameter uncertainties. However, these previous studies did not take into account the time delay, motivating this work. The main contribution of this paper is then the proposal of a direct approach for designing H_∞ observer-based controllers for T-S fuzzy systems with uncertainties, external disturbances and time-varying delays. The approach is based on proposing a Lyapunov–Krasovskii functional with time-delay information . Based on it, a controller is proposed based on using both the original state and the time-delay state. Using this approach allows to develop stability conditions expressed as LMIs, so the design conditions are also expressed as LMIs. The solution of these LMIs makes it possible to obtain the observer and controller gains.

The paper is organised as follows. Section 2 describes the mathematical modelling of the fuzzy quadruple-tank systems with time delay, and some previous results. Section 3 presents the main contributions , describing the observer-based controller and the design methodology. Section 4 provides the application to the quadruple-tank system with time delay. Finally, the paper draws some conclusions.

Notation The following standard notation are used in the paper. The superscript $(.)^T$ represent the matrix transpose. $P > 0$ means that P is a symmetric positive definite matrix, and $\mathcal{H}_e = \{P + P^T\}$. The symbol $*$ denotes a symmetric block. I denotes the identity matrix with appropriate dimensions.

2 Model description and preliminaries

2.1 Fuzzy quadruple-tank systems

Figure 1 shows the schematic diagram of the quadruple-tank model with disturbances (Johansson and Nunes (1998)). The process is composed of four identical cylindrical tanks numbered 1–4, and a reservoir 5. The two pumps transfer the water from the reservoir to tank-3 and tank-4. The objective is to control the liquid levels in tank-3 and tank-4, by the observation of the dynamic states of

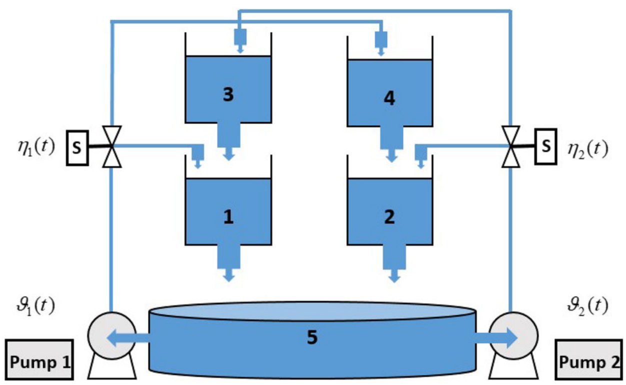

Fig. 1 Schematic Diagram of the Quadruple-Tank Systems

tank-1 and tank-2. A delay in the inputs of the system is introduced to make the control problem more challenging (Shah and Patel (2019)). The continuous-time nonlinear system is represented by the following equations:

$$\begin{aligned}
\dot{x}_1(t) &= -\frac{\tilde{a}_{p1}}{\tilde{A}_{p1}}\sqrt{2gx_1(t)} + \frac{\tilde{a}_{p3}}{\tilde{A}_{p1}}\sqrt{2gx_3(t-d(t))} + \frac{\eta_1(t)\tilde{k}_{\vartheta_1}}{\tilde{A}_{p1}}\vartheta_1(t-d(t)) \\
\dot{x}_2(t) &= -\frac{\tilde{a}_{p2}}{\tilde{A}_{s2}}\sqrt{2gx_2(t)} + \frac{\tilde{a}_{p4}}{\tilde{A}_{p2}}\sqrt{2gx_4(t-d(t))} + \frac{\eta_2(t)\tilde{k}_{\vartheta_2}}{\tilde{A}_{p2}}\vartheta_2(t-d(t)) \\
\dot{x}_3(t) &= -\frac{\tilde{a}_{p3}}{\tilde{A}_{p3}}\sqrt{2gx_3(t)} + \frac{(1-\eta_2(t))\tilde{k}_{\vartheta_2}}{\tilde{A}_{p3}}\vartheta_2(t-d(t)) \\
\dot{x}_4(t) &= -\frac{\tilde{a}_{s4}}{\tilde{A}_{p4}}\sqrt{2gx_3(t)} + \frac{(1-\eta_1(t))\tilde{k}_{\vartheta_1}}{\tilde{A}_{p4}}\vartheta_1(t-d(t))
\end{aligned} \tag{1}$$

where $x_k(t)$ is the liquid level in the tank number k, $\tilde{A}_{pk}$ is the cross sectional area of tank k, $\tilde{a}_{pk}$ is the cross-sectional area of the outlet of tank k, g is the gravity constant, $\eta_j(t) \in [0,\ 1]$ is the valve flow proportion, and $\vartheta_j(t)$ is the control signal of pump j, with the gain $\tilde{k}_{\vartheta j}$. The parameters of the process are condensed in Table 1. The measured output signals $y(t) \in \mathbb{R}^2$, correspond in this study to the levels of tanks 1 and 2. $d(t)$ is the time it takes for the liquid to move to tank-1 from tank-3, tank-2 from tank-4, tank-1 from pump-1, tank-4 from pump-1, tank-2 from pump-2, tank-3 from pump-2, and tank-3 and tank-4. This delay satisfies that:

$$0 \leqslant d(t) \leqslant \bar{d} \tag{2a}$$

$$\dot{d}(t) \leqslant \sigma < 1 \tag{2b}$$

The nonlinear terms $\tilde{F}_k(t) = \sqrt{x_k(t)}$ of the level in the tank-k $\forall k = 1, \ldots, 4$, the following sector rules apply:

If $x_k(t)$ is N_1 **Then** $\tilde{F}_k(t) = \tilde{C}_{Fz1}x_k(t)$

If $x_k(t)$ is N_2 **Then** $\tilde{F}_k(t) = \tilde{C}_{Fz2}x_k(t)$

with N_1 and N_2 the fuzzy sets.

As a result, the following is the T–S Fuzzy model:

$$\tilde{F}_k(t) = (\varphi_1(x_k)\tilde{C}_{Fz1} + \varphi_2(x_k)\tilde{C}_{Fz2})x_k(t) \tag{3}$$

It should be noted that the membership functions (MFs) are

$$\varphi_j(x_i) = \frac{\delta_j(x_k)}{\sum_{j=1}^{2}\delta_j(x_k)}, \quad j = 1,2,\ k = 1,\ldots,4 \tag{4}$$

with the following properties:

$$\sum_{j=1}^{2}\varphi_j(x_k) = 1, \quad \varphi_j(x_k) \in [0,\ 1], \quad j = 1,2,\ k = 1,\ldots,4 \tag{5}$$

where

$$\delta_j(x_k) = \frac{1}{\left[1 + \left\|\frac{x_k - \tilde{c}_j}{\tilde{a}_j}\right\|\right]^{2\tilde{b}_j}}, \quad j = 1,2,\ k = 1,\ldots,4 \tag{6}$$

Remark 1 The rules (3) are generalized to calculate only $\varphi_j(x_1)$, $j = 1,2$, by using the same fuzzy form and non-linear function characteristics of the model. Thanks to this, the number of membership functions is reduced from 8 to 2. Table 2 details the fuzzy parameters.

Time-varying parameters $\eta_1(t)$ and $\eta_2(t)$ are:

Table 1 Parameters of the process

Parameters	Concept (Unit)	Values
$\tilde{A}_{p1}$,$\tilde{A}_{p3}$	Areas of tanks 1.3 (m^2)	$2,8 \times 10^{-3}$
$\tilde{A}_{p2}$,$\tilde{A}_{p4}$	Areas of tanks 2, 4 (m^2)	3.2×10^{-3}
$\tilde{a}_{p1}, \tilde{a}_{p3}$	Areas of outlet in tanks 1, 3 (m^2)	7.1×10^{-6}
$\tilde{a}_{p2}, \tilde{a}_{p4}$	Areas of outlet in tanks 2, 4 (m^2)	5.7×10^{-6}
$\tilde{k}_{\vartheta_i}$	Coefficient of pump-i, $i = 1,2$ $(ml\ V^{-1}s^{-1})$	3.33, 3.35
x_i	The liquid level in tank-i (m)	
η_i	The gain of flow at valve i	
ϑ_i	The voltage control signal of pump i (V)	

Table 2 Parameters of membership functions

Parameters	Concept	Values
$\tilde{a}_1$	Width of MFs	0.0021
$\tilde{a}_2$		0.3078
$\tilde{b}_1$		0.7219
$\tilde{b}_2$		5.3137
$\tilde{c}_1$	Center of MFs	−0.1656
$\tilde{c}_2$		0.3155
$\tilde{C}_{Fz1}$	Coefficient of fuzzy set in region N_1	2.389×10^3
$\tilde{C}_{Fz2}$	Coefficient of fuzzy set in region N_2	0.8149

$$\tilde{\theta}_1(t)=\frac{\eta_1(t)}{2},\quad \tilde{\theta}_2(t)=\frac{(1-\eta_1(t))}{2},$$
$$\tilde{\theta}_3(t)=\frac{\eta_2(t)}{2},\quad \tilde{\theta}_4(t)=\frac{(1-\eta_2(t))}{2}. \tag{7}$$

then $\sum_{k=1}^{4}\tilde{\theta}_k(t)=1$

.

The quadruple-tank system can be expressed by the uncertain T–S Fuzzy model in (3) by:

$$\begin{aligned}\dot{x}(t)&=\sum_{i=1}^{m}\mu_i(\varphi(t))\{(A_i+\Delta A_i(t))x(t)\\&\quad+(A_{d_i}+\Delta A_{d_i}(t))x(t-d(t))+B_iu(t)\}+w(t)\\y(t)&=(C+\Delta C(t))x(t)\end{aligned} \tag{8}$$

where $\mu_i(\varphi(t))$ is the grade of membership of $\varphi(t)$, $x(t)\in\mathbb{R}^n$ is the state vector, $y(t)\in\mathbb{R}^p$ is the output vector, $u(t)\in\mathbb{R}^r$ is the input vector, $w(t)\in l_2^q$ is the unknown exogenous disturbance, $A_i\in\mathbb{R}^{n\times n}$, $A_{d_i}\in\mathbb{R}^{n\times n}$, $B_i\in\mathbb{R}^{n\times r}$ and $C_i\in\mathbb{R}^{p\times n}$ $(i=1,\ldots,m)$, are known constant matrices. Finally, $\Delta A_i(t)$, $\Delta A_{d_i}(t)$ $(i=1,\ldots,m)$ and $\Delta C(t)$ are unknown matrices that represent model uncertainty.

Throughout the paper, the following assumptions are used:

Assumption 1 The pairs (A_i,B_i), (A_i,C), (A_{d_i},B_i) and (A_{d_i},C) $(i=1,\ldots,m)$, are stabilisable and detectable, respectively.

Assumption 2 The matrices $\Delta A_i(t)$, $\Delta A_{d_i}(t)$ $(i=1,\ldots,m)$ and $\Delta C(t)$ are unknown matrices that represent time-varying model uncertainties, as follows:

$$\begin{aligned}&\Delta A_i(t)=\tilde{M}_i\tilde{F}_i(t)\tilde{N}_i,\Delta A_{d_i}(t)=\tilde{M}_{i+1}\tilde{F}_i(t)\tilde{N}_{i+1}\\&\Delta C(t)=\tilde{M}_{m+1}\tilde{F}_{m+1}(t)\tilde{N}_{m+1}\end{aligned} \tag{9}$$

with

$$\tilde{F}_k^T(t)\tilde{F}_k(t)\leqslant I,\quad \forall k=1,2,\ldots,(m+1) \tag{10}$$

The control scheme can also minimize disturbance attenuation from the pump to tank levels, assuming that the disturbance can be described as follows:

$$w(t)=10^{-2}\times\begin{bmatrix}1.2sin(20\pi t) & 2.5cos(10\pi t) & -2.8sin(16\pi t) & 3.7sin(21\pi t)\end{bmatrix}^T$$

.

Using (1), the following are the nominal constant matrices:

$$A_i=\begin{bmatrix}\frac{-\tilde{a}_{p1}\sqrt{2g}\tilde{C}_{Fzj}}{\tilde{A}_{p1}} & 0 & 0 & 0\\ 0 & \frac{-a_{p2}\sqrt{2g}\tilde{C}_{Fzj}}{A_{p2}} & 0 & 0\\ 0 & 0 & \frac{-a_{p3}\sqrt{2g}\tilde{C}_{Fzj}}{A_{p3}} & 0\\ 0 & 0 & 0 & \frac{-a_{p4}\sqrt{2g}\tilde{C}_{Fzj}}{A_{p4}}\end{bmatrix}$$

$$A_{d_i}=\begin{bmatrix}0 & 0 & \frac{a_{p3}\sqrt{2g}\tilde{C}_{Fzj}}{\tilde{A}_{p1}} & 0\\ 0 & 0 & 0 & \frac{a_{p4}\sqrt{2g}\tilde{C}_{Fzj}}{A_{p2}}\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{bmatrix},$$

If $i=1,\ldots,4$ **Then** $j=1$, **And** $i=5,\ldots,m$ **Then** $j=2$, with $m=8$.

$$B_{1,5}=\begin{bmatrix}\frac{2\tilde{k}_{\vartheta_1}}{\tilde{A}_{p1}} & 0\\ 0 & 0\\ 0 & 0\\ 0 & 0\end{bmatrix},\qquad B_{2,6}=\begin{bmatrix}0 & 0\\ 0 & \frac{2\tilde{k}_{\vartheta_2}}{A_{p2}}\\ 0 & 0\\ 0 & 0\end{bmatrix},$$

$$B_{3,7}=\begin{bmatrix}0 & 0\\ 0 & 0\\ 0 & \frac{2\tilde{k}_{\vartheta_2}}{A_{p3}}\\ 0 & 0\end{bmatrix},\qquad B_{4,8}=\begin{bmatrix}0 & 0\\ 0 & 0\\ 0 & 0\\ \frac{2\tilde{k}_{\vartheta_1}}{A_{p4}} & 0\end{bmatrix}$$

$$\text{and } C=\begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\end{bmatrix}$$

The observer-based controller is as follows:

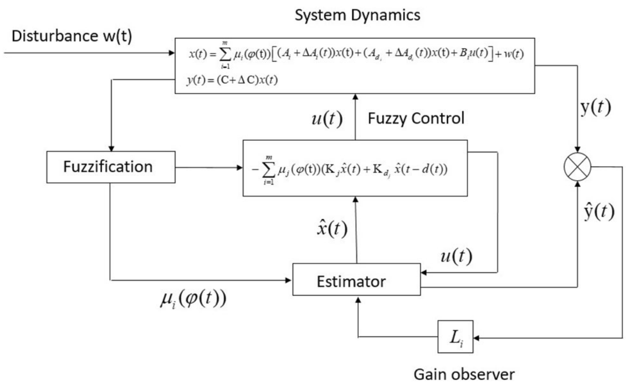

Fig. 2 The structural scheme of the fuzzy system

$$\dot{\hat{x}}(t) = \sum_{i=1}^{m} \mu_i(\varphi(t))\{A_i\hat{x}(t) + A_{d_i}\hat{x}(t-d(t)) + B_iu(t) + L_i(y(t) - C\hat{x}(t))\} \tag{11}$$

and the following controller is used for the system (8):

$$u(t) = -\sum_{j=1}^{m} \mu_j\varphi(t))\{(K_j\hat{x}(t) + K_{d_j}\hat{x}(t-d(t)\} \tag{12}$$

where $L_i \in \mathbb{R}^{n\times p}$, $K_j \in \mathbb{R}^{x\times n}$ and $K_{d_j} \in \mathbb{R}^{x\times n}$ $(i,j = 1,\ldots,m)$ are the observer and the controller gains, respectively, to be calculated, and $\hat{x}(t) \in \mathbb{R}^n$ is the estimation of $x(t)$ provided by the observer.

Let us define $e(t) = x(t) - \hat{x}(t)$, the dynamics of $e(t)$ which can be obtained as follows:

$$\begin{aligned} \dot{e}(t) \quad &= \sum_{i=1}^{m} \mu_i(\varphi(t))[(A_i - L_iC)e(t) + A_{d_i}e(t-d(t)) \\ &\quad +(\Delta A_i - L_i\Delta C)x(t). \\ &\quad +\Delta A_{d_i}x(t-d(t))] + w(t) \end{aligned} \tag{13}$$

By using the controller (12), then the system (8) in closed-loop is:

$$\begin{aligned} \dot{x}(t) \quad = \quad & \sum_{i=1}^{m} \mu_i(\varphi(t)) \sum_{j=1}^{m} \mu_j(\varphi(t))[(A_j - \Delta A_i(t) - B_iK_j)e(t) + B_iK_je(t) \\ & +B_iK_{d_j}e(t-d(t)) + (A_{d_i} - \Delta A_{d_i}(t) - B_iK_j)x(t-d(t)) + w(t)] \end{aligned} \tag{14}$$

Following this, the augmented system can be written as follows:

$$\begin{aligned} \underbrace{\begin{bmatrix} \dot{x}(t) \\ \dot{e}(t) \end{bmatrix}}_{\dot{\varsigma}(t)} &= \sum_{i=1}^{m}\sum_{j=1}^{m} \mu_i(\varphi(t))\mu_j(\varphi(t)) \underbrace{\begin{bmatrix} A_i - B_iK_j + \Delta A_i(t) & B_iK_j \\ \Delta A_i(t) - L_i\Delta C(t) & A_i - L_iC \end{bmatrix}}_{\tilde{A}_{ij}} \underbrace{\begin{bmatrix} x(t) \\ e(t) \end{bmatrix}}_{\varsigma(t)} \\ &+ \sum_{i=1}^{m}\sum_{j=1}^{m} \mu_i(\varphi(t))\mu_j(\varphi(t)) \underbrace{\begin{bmatrix} A_{d_i} - B_iK_{d_j} + \Delta A_{d_i}(t) & B_iK_{d_j} \\ \Delta A_{d_i}(t) & A_{d_i} \end{bmatrix}}_{\tilde{A}_{d_{ij}}} \\ &\times \underbrace{\begin{bmatrix} x(t-d(t)) \\ e(t-d(t)) \end{bmatrix}}_{\varsigma(t-d(t))} + \underbrace{\begin{bmatrix} I_n \\ I_n \end{bmatrix}}_{B_w} w(t) \end{aligned} \tag{15}$$

2.2 Problem statement

The main objective of the paper is to design a robust observer-based control design in the presence of process delays and disturbances, such that augmented system (15) is asymptotically stable and satisfies that

$$\frac{\|\zeta(t)\|_2}{\|w(t)\|_2} < \gamma \tag{16}$$

2.3 Preliminaries

The following lemmas are required to provide the main results in the following section.

Lemma 1 (*Lien* (2004)) $\forall\, \sigma$, *positive constant, and real matrices* M, N, *and* $F \in \mathbb{R}$ *of appropriate dimensions, such that* $F^T(t)F(t) \leqslant I$, *the following holds*:

$$MFN + N^TF^TM^T \leqslant \frac{1}{\varrho}MM^T + \varrho N^TN \tag{17}$$

Lemma 2 (*Chang et al.* (2015)) *For matrices* T, Q, U, *and* W, *with appropriate dimensions, and scalar* $\tilde{\zeta}$ *the inequality*,

$$T + W^TQ^T + QW < 0 \tag{18}$$

is satisfied if the following condition holds:

$$\begin{bmatrix} T & \tilde{\zeta}Q + W^TU^T \\ * & -\tilde{\zeta}U - \tilde{\zeta}U^T \end{bmatrix} < 0 \tag{19}$$

Lemma 3 (*Dong et al.* (2017)) *For any constant matrix* $Z = Z^T > 0$ *and scalar* $h \geq 0$ *such that the following integrations are well defined, then*

$$\int_{t-h}^{t}\xi^T(s)Z\xi(s)ds \leqslant -\frac{1}{h}\left(\int_{t-h}^{t}\xi(s)ds\right)^T Z(\int_{t-h}^{t}\xi(s)ds)$$

$$\int_{-h}^{0}\int_{t+\theta}^{t}\xi^T(s)Z\xi(s)dsd\theta \leqslant$$
$$-\frac{2}{h^2}\left(\int_{-h}^{0}\int_{t+\theta}^{t}\xi(s)dsd\theta\right)^T Z(\int_{-h}^{0}\int_{t+\theta}^{t}\xi(s)dsd\theta).$$

3 Main results

3.1 Robust observer-based control design

This section aims to investigate the stability problem of system (8) in the presence of uncertainties and disturbances.

Theorem 1 *The robust observer-based system (15) is asymptotically stable with H_∞ performance, for given scalars $\gamma, \tilde{\zeta}_1, \tilde{\zeta}_2$ and positive scalars $\sigma_k > 0\ (k = 1, \ldots, m)$, if there exist symmetric positive-definite matrices $P_1, P_2, Q_1, Q_2, R_1, R_2 \in \mathbb{R}^{n\times n}$ and unknown matrices $S_1, S_2 \in \mathbb{R}^{r\times r}$, $Z_{1j}, Z_{2j} \in \mathbb{R}^{r\times n}$, $W_i \in \mathbb{R}^{n\times p}$, $(i, j = 1, \ldots, m)$ satisfying the following LMIs*:

$$\begin{bmatrix} \Upsilon_1 & \Xi_{12} & \Xi_{13} & a_1 & b_1^T & a_2 & b_2^T \\ * & \Xi_{22} & 0 & 0 & 0 & 0 & 0 \\ * & * & \Xi_{33} & 0 & 0 & 0 & 0 \\ * & * & * & -\sigma_1 I & 0 & 0 & 0 \\ * & * & * & * & -\frac{1}{\sigma_1}I & 0 & 0 \\ * & * & * & * & * & -\sigma_2 I & 0 \\ * & * & * & * & * & * & -\frac{1}{\sigma_2}I \end{bmatrix} < 0, \tag{20}$$

where

$$\Upsilon_1 = \begin{bmatrix} \Upsilon_{11}+T_{11} & \Lambda_{12}+T_{12} & 0 & 0 & P \\ * & \Upsilon_{22} & 0 & 0 & 0 \\ * & * & -\frac{1}{\bar{d}}R_1 & 0 & 0 \\ * & * & * & -\frac{1}{\bar{d}^2}R_2 & 0 \\ * & * & * & * & -\gamma^2 I \end{bmatrix},$$

$$\Upsilon_{11} = \Lambda_{11} + \Lambda_{11}^T + Q_1 + Q_2 + \bar{d}R_1 + \frac{\bar{d}^2}{2}R_2 + I,$$

$$\Xi_{22} = -(Q_1 + Q_2) + \sigma Q_2,$$

$$\Lambda_{11} = \begin{bmatrix} P_1 A_i & 0 \\ 0 & P_2 A_i - W_i C \end{bmatrix},$$

$$\Lambda_{12} = \begin{bmatrix} P_1 A_{d_i} & 0 \\ 0 & P_2 A_{d_i} \end{bmatrix}, \quad T_{11} = H_e\{E_{11}\},$$
$$T_{12} = H_e\{E_{12}\},$$

$$E_{11} = \begin{bmatrix} -B_i Z_1 & B_i Z_1 \\ 0 & 0 \end{bmatrix},$$

$$E_{12} = \begin{bmatrix} -B_i Z_2 & B_i Z_2 \\ 0 & 0 \end{bmatrix},$$

$$\Xi_{12} = \begin{bmatrix} \begin{pmatrix} \tilde{\zeta}_1(P_1 B_i - B_i S_2) \\ 0 \end{pmatrix} \\ \begin{pmatrix} -Z_2^T \\ Z_2 \end{pmatrix} \\ 0 \\ 0 \\ 0 \end{bmatrix},$$

$$P = \begin{bmatrix} P_1 & 0 \\ 0 & P_2 \end{bmatrix},$$

$$\Xi_{13} = \begin{bmatrix} \begin{pmatrix} \tilde{\zeta}_2(P_1 B_i - B_i S_1) - Z_1^T \\ Z_1^T \end{pmatrix} \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix},$$

$$\Xi_{13} = \begin{bmatrix} \begin{pmatrix} \tilde{\zeta}_2(P_1 B_i - B_i S_1) - Z_1^T \\ Z_1^T \end{pmatrix} \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix},$$

$$\Upsilon_{22} = -\tilde{\zeta}_1 S_2 - \tilde{\zeta}_1 S_2^T,$$

$$\Upsilon_{33} = -\tilde{\zeta}_2 S_1 - \tilde{\zeta}_2 S_1^T,$$

and the gains of the controller and observer can be obtained as $K_j = S_1^{-1}Z_1$, $K_{d_i} = S_2^{-1}Z_2$ and $L_i = P_2^{-1}W_i$.

Proof The following L-K functional is used to demonstrate the result:

$$V(t) = \varsigma^T(t)P\varsigma(t) + \int_{t-\bar{d}}^{t}\varsigma^T(s)Q_1\varsigma(s)ds + \int_{t-d(t)}^{t}\varsigma^T(s)Q_2\varsigma(s)ds + \int_{-\bar{d}}^{0}\int_{t+\theta}^{t}\dot{\varsigma}^T(s)R_1\dot{\varsigma}(s)dsd\theta + \int_{-\bar{d}}^{0}\int_{\theta}^{0}\int_{t+\theta}^{t}\dot{\varsigma}^T(s)R_2\dot{\varsigma}(s)dsd\theta \tag{21}$$

The time derivative of $V(t)$ along the trajectory of (15) is the following:

$$\begin{aligned}\dot{V}(t) = {} & \sum_{i=1}^{m}\sum_{j=1}^{m}\mu_i(\varphi(t))\mu_j(\varphi(t)) \\ & \Big\{\varsigma^T(t)(P\tilde{A}_{ij} + \tilde{A}_{ij}^T P)\dot{\varsigma}(t) + \varsigma^T(t)(P\tilde{A}_{d_{ij}} \\ & + \tilde{A}_{d_{ij}}^T P)\dot{\varsigma}(t-d(t))\Big\} + \varsigma^T(t)Q_1\varsigma(t) - \varsigma^T(t-\bar{d})Q_1\varsigma(t-\bar{d}) \\ & + \varsigma^T(t)Q_2\varsigma(t) - (1-\sigma)\varsigma^T(t-d(t)) \\ & Q_2\varsigma(t-d(t)) + \bar{d}\varsigma^T(s)R_1\varsigma(s) \\ & - \int_{t-\bar{d}}^{t}\varsigma^T(s)R_1\varsigma(s)ds + \frac{d^2}{2}\varsigma^T(s)R_2\varsigma(s) \\ & - \int_{-\bar{d}}^{0}\int_{t+\Theta}^{t}\varsigma^T(s)R_2\varsigma(s)dsd\Theta\end{aligned} \tag{22}$$

By Lemma 3, it is obtained that

$$\begin{aligned}\dot{V}(t) \le {} & \sum_{i=1}^{m}\sum_{j=1}^{m}\mu_i(\varphi(t))\mu_j(\varphi(t)) \\ & \Big\{\varsigma^T(t)(P\tilde{A}_{ij} + \tilde{A}_{ij}^T P)\dot{\varsigma}(t) + \varsigma^T(t)(P\tilde{A}_{d_{ij}} \\ & + \tilde{A}_{d_{ij}}^T P)\dot{\varsigma}(t-d(t))\Big\} + \varsigma^T(t)[Q_1 + Q_2]\varsigma(t) \\ & - \varsigma^T(t-\bar{d})Q_1\varsigma(t-\bar{d}) \\ & - (1-\sigma)\varsigma^T(t-d(t))Q_2\varsigma(t-d(t)) + \varsigma^T(s) \\ & \left[\bar{d}R_1 + \frac{\bar{d}^2}{2}R_2\right]\varsigma(s) \\ & - \frac{1}{\bar{d}}\left(\int_{t-\bar{d}}^{t}\varsigma^T(s)ds\right)^T R_1\left(\int_{t-\bar{d}}^{t}\varsigma^T(s)ds\right) \\ & - \frac{2}{\bar{d}^2}\left(\int_{-\bar{d}}^{0}\int_{t+\Theta}^{t}\varsigma^T(s)dsd\Theta\right)^T R_2 \\ & \left(\int_{-\bar{d}}^{0}\int_{t+\Theta}^{t}\varsigma^T(s)dsd\Theta\right)\end{aligned} \tag{23}$$

Following that, it is now shown that for any $w(t) \in L_2[0,\infty)$ of the T-S system with parametric uncertainty (8), the following condition must be fulfilled:

$$J(\varsigma(t), w(t)) = \int_0^\infty \left[\varsigma(t)^T\varsigma(t) - \gamma^2 w(t)^T w(t)\right]dt < 0 \tag{24}$$

Under null initial conditions, the Lyapunov function fulfill $V(0) = 0$ and $V(\infty) \geqslant 0$, which leads to:

$$J(\varsigma(t), w(t)) = \int_0^\infty \left[\dot{V}(\varsigma(t)) + \varsigma(t)^T\varsigma(t) - \gamma^2 w(t)^T w(t)\right]dt - V(\infty)$$

$$J(\varsigma(t), w(t)) \leqslant \int_0^\infty \left[\dot{V}(\varsigma(t)) + \varsigma(t)^T\varsigma(t) - \gamma^2 w(t)^T w(t)\right]dt \tag{25}$$

To achieve the attenuation level in (16), we must fulfill inequality in the following:

$$\dot{V}(\varsigma(t)) + \varsigma(t)^T\varsigma(t) - \gamma^2 w(t)^T w(t) < 0 \tag{26}$$

Combining (23) and (26), we have:

$$\dot{V}(t) \le \psi^T\Omega\psi \tag{27}$$

where $\psi^T = \left[\varsigma^T(t) \quad \varsigma^T(t-d(t)) \quad \int_{t-\bar{d}}^{t}\varsigma^T(s)ds \quad \int_{-\bar{d}}^{0}\int_{t+\Theta}^{t}\varsigma^T(s)dsd\Theta\right]$

$$\Upsilon = \sum_{i=1}^{m}\sum_{j=1}^{m}\mu_i(\varphi(t))\mu_j(\varphi(t)) \begin{bmatrix}\hat{\Upsilon}_{11} & \hat{\Upsilon}_{12} & 0 & 0 & P \\ * & \hat{\Upsilon}_{22} & 0 & 0 & 0 \\ * & * & -\frac{1}{\bar{d}}R_1 & 0 & 0 \\ * & * & * & -\frac{1}{\bar{d}^2}R_2 & 0 \\ * & * & * & * & -\gamma^2 I\end{bmatrix} < 0, \tag{28}$$

$$\hat{\Upsilon}_{11} = P\tilde{A}_{ij} + \tilde{A}_{ij}^T P + Q_1 + Q_2 + \bar{d}R_1 + \frac{\bar{d}^2}{2}R_2, nbsp;$$

$$\hat{\Upsilon}_{12} = P\tilde{A}_{d_{ij}}, \quad \hat{\Upsilon}_{22} = -(Q_1 + Q_2) + \sigma Q_2.$$

After that, Eq. (28) can be rewritten as follows:

$$\Upsilon = \tilde{\Upsilon}_1 + \tilde{\Upsilon}_2 + \tilde{\Upsilon}_\Delta < 0, \tag{29}$$

$$where\ \tilde{\Upsilon}_1 = \begin{bmatrix}\Upsilon_{11} & \Lambda_{12} & 0 & 0 & P \\ * & \Upsilon_{22} & 0 & 0 & 0 \\ * & * & -\frac{1}{\bar{d}}R_1 & 0 & 0 \\ * & * & * & -\frac{1}{\bar{d}^2}R_2 & 0 \\ * & * & * & * & -\gamma^2 I\end{bmatrix}$$

$$\Upsilon_{11} = \Lambda_{11} + \Lambda_{11}^T + Q_1 + Q_2 + \bar{d}R_1 + \frac{\bar{d}^2}{2}R_2 + I,$$

$$\Lambda_{11} = \begin{bmatrix} P_1A_i & 0 \\ 0 & P_2A_i - W_iC \end{bmatrix},$$

$$\Lambda_{12} = \begin{bmatrix} P_1A_{d_i} & 0 \\ 0 & P_2A_{d_i} \end{bmatrix},$$

$$P = \begin{bmatrix} P_1 & 0 \\ 0 & P_2 \end{bmatrix}, P_2L_i = W_i,$$

$$\tilde{\Upsilon}_2 = \begin{bmatrix} \bar{\Upsilon}_{11} & \bar{\Lambda}_{12} & 0 & 0 & 0 \\ * & 0 & 0 & 0 & 0 \\ * & * & 0 & 0 & 0 \\ * & * & * & 0 & 0 \\ * & * & * & * & 0 \end{bmatrix},$$

$$\tilde{\Upsilon}_\Delta = \begin{bmatrix} \breve{\Upsilon}_{11} & \breve{\Lambda}_{12} & 0 & 0 & 0 \\ * & 0 & 0 & 0 & 0 \\ * & * & 0 & 0 & 0 \\ * & * & * & 0 & 0 \\ * & * & * & * & 0 \end{bmatrix},$$

$$\bar{\Upsilon}_{11} = \bar{\Lambda}_{11} + \bar{\Lambda}_{11}^T, \quad \breve{\Upsilon}_{11} = \breve{\Lambda}_{11} + \breve{\Lambda}_{11}^T,$$

$$\bar{\Lambda}_{11} = \begin{bmatrix} -P_1B_iK_j & P_1B_iK_j \\ 0 & 0 \end{bmatrix}$$

$$\bar{\Lambda}_{12} = \begin{bmatrix} -P_1B_iK_{d_j} & P_1B_iK_{d_j} \\ 0 & 0 \end{bmatrix},$$

$$\breve{\Lambda}_{11} = \begin{bmatrix} P_1\Delta A_i(t) & 0 \\ P_2\Delta A_i(t) - R\Delta C(t) & 0 \end{bmatrix}$$

$$\breve{\Lambda}_{12} = \begin{bmatrix} P_1\Delta A_{d_i}(t) & 0 \\ P_2\Delta A_{d_i}(t) - R\Delta C(t) & 0 \end{bmatrix}.$$

Using non-singular matrices S_1, S_2 and defining $K_j = S_1^{-1}Z_{1j}, K_{d_j} = S_2^{-1}Z_{2j}$ it is possible to show that,

$$\begin{aligned} P1B_iK_j &= (P_1B_i - B_iS_1)S_1^{-1}Z_1 + B_iZ_1, \\ P1B_iK_{d_j} &= (P_1B_i - B_iS_2)S_2^{-1}Z_2 + B_iZ_2. \end{aligned} \tag{30}$$

Then, using Eq. (30), inequality (29) is equivalent to:

$$\Upsilon = \overbrace{\Upsilon_1 + \tilde{\Upsilon}_\Delta}^{R} + \mathcal{H}_e \left\{ \underbrace{\underbrace{\begin{bmatrix} \begin{pmatrix} P_1B_i - B_iS_1 \\ 0 \end{pmatrix} \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}}_{X_3} S_1^{-1} \underbrace{[(-Z_1 \quad Z_1) \quad 0 \quad 0 \quad 0 \quad 0]}_{X_4}}_{\Gamma} \right\}$$
$$+\mathcal{H}_e \left\{ \underbrace{\begin{bmatrix} \begin{pmatrix} P_1B_i - B_iS_2 \\ 0 \end{pmatrix} \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}}_{X_1} S_2^{-1} \underbrace{[0 \quad (-Z_2 \quad Z_2) \quad 0 \quad 0 \quad 0]}_{X_2} \right\} < 0 \tag{31}$$

where

$$\Upsilon_1 = \begin{bmatrix} \Upsilon_{11} + T_{11} & \Lambda_{12} + T_{12} & 0 & 0 & P \\ * & \Upsilon_{22} & 0 & 0 & 0 \\ * & * & -\frac{1}{d}R_1 & 0 & 0 \\ * & * & * & -\frac{1}{d^2}R_2 & 0 \\ * & * & * & * & -\gamma^2 I \end{bmatrix}$$

,

$$T_{11} = H_e\{E_{11}\}, \qquad T_{12} = H_e\{E_{12}\},$$
$$E_{11} = \begin{bmatrix} -B_iZ_1 & B_iZ_1 \\ 0 & 0 \end{bmatrix}, \qquad E_{12} = \begin{bmatrix} -B_iZ_2 & B_iZ_2 \\ 0 & 0 \end{bmatrix},$$

Using Lemma 2, inequality (31) can be guaranteed by the following condition:

$$\Upsilon = \begin{bmatrix} \Upsilon_1 + \tilde{\Upsilon}_\Delta + \Gamma & \tilde{\zeta}_1X_1 + X_2^TS_2^T \\ * & -\tilde{\zeta}_1S_2 - \tilde{\zeta}_1S_2^T \end{bmatrix} < 0 \tag{32}$$

$$\text{where } \tilde{\zeta}_1X_1 + X_2^TS_2^T = \begin{bmatrix} \begin{pmatrix} \tilde{\zeta}_1(P_1B_i - B_iS_2) \\ 0 \end{pmatrix} \\ \begin{pmatrix} -Z_2^T \\ Z_2^T \end{pmatrix} \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

then

$$\Upsilon = \begin{bmatrix} \Upsilon_1 + \tilde{\Upsilon}_\Delta + \Gamma & \tilde{\zeta}_1 X_1 + X_2^T S_2^T \\ * & -\tilde{\zeta}_1 S_2 - \tilde{\zeta}_1 S_2^T \end{bmatrix} + \mathcal{H}_e \left\{ \underbrace{\begin{bmatrix} \begin{pmatrix} P_1 B_i - B_i S_1 \\ 0 \end{pmatrix} \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}}_{X_3} S_1^{-1} \underbrace{\begin{bmatrix} 0 & \begin{pmatrix} -Z_1 & Z_1 \end{pmatrix} & 0 & 0 & 0 \end{bmatrix}}_{X_4} \right\} < 0 \tag{33}$$

Lemma 2 makes possible to verify (33), using the matrix condition of inequality:

$$\Upsilon = \begin{bmatrix} \Upsilon_1 + \tilde{\Upsilon}_\Delta & \tilde{\zeta}_1 X_1 + X_2^T S_2^T & \tilde{\zeta}_2 X_3 + X_4^T S_1^T \\ * & -\tilde{\zeta}_1 S_2 - \tilde{\zeta}_1 S_2^T & 0 \\ * & * & -\tilde{\zeta}_2 S_1 - \tilde{\zeta}_2 S_1^T \end{bmatrix} < 0 \tag{34}$$

where

$$\tilde{\zeta}_2 X_3 + X_4^T S_1^T = \begin{bmatrix} \begin{pmatrix} \tilde{\zeta}_2 (P_1 B_i - B_i S_1) - Z_1^T \\ Z_1^T \end{pmatrix} \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Taking into account the special structure of $\Delta A_i(t), \Delta A_{d_i}(t), \Delta C(t)$ and integrating Lemma 3, we obtain

$$\begin{aligned} \tilde{\Upsilon}_\Delta = 2 &\begin{bmatrix} \begin{pmatrix} P_1 \tilde{M}_i \\ P_2 \tilde{M}_i \end{pmatrix} \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \tilde{F}_i(t) \begin{bmatrix} \begin{pmatrix} \tilde{N}_i & 0 \end{pmatrix} & \begin{pmatrix} \tilde{N}_{i+1} & 0 \end{pmatrix} & 0 & 0 & 0 \end{bmatrix} \\ &+ 2 \begin{bmatrix} \begin{pmatrix} P_1 \tilde{M}_{m+1} \\ P_2 \tilde{M}_{m+1} \end{pmatrix} \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \tilde{F}_{m+1}(t) \begin{bmatrix} \begin{pmatrix} \tilde{N}_{m+1} & 0 \end{pmatrix} & \begin{pmatrix} \tilde{N}_{m+i+1} & 0 \end{pmatrix} & 0 & 0 & 0 \end{bmatrix} \\ &+ 2 \begin{bmatrix} \begin{pmatrix} 0 \\ -W_i \tilde{M}_i \end{pmatrix} \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \tilde{F}_i(t) \begin{bmatrix} \begin{pmatrix} \tilde{N}_i & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 \end{pmatrix} & 0 & 0 & 0 \end{bmatrix} \end{aligned} \tag{35}$$

We have the inequality:

$$\tilde{\Upsilon}_\Delta \leqslant \sigma_1 Y_{1i} Y_{1i}^T + \sigma_1^{-1} Y_{2i} Y_{2i}^T + \sigma_2 Y_{3i} Y_{3i}^T + \sigma_2^{-1} Y_{4i} Y_{4i}^T \tag{36}$$

for any $\sigma_1 > 0$ and $\sigma_2 > 0$, such that

$$Y_{1i}^T = \begin{bmatrix} \begin{pmatrix} P_1 \tilde{M}_i \\ P_2 \tilde{M}_i \end{pmatrix} \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix},$$

$$Y_{2i} = \begin{bmatrix} \begin{pmatrix} \tilde{N}_i & 0 \end{pmatrix} & \begin{pmatrix} \tilde{N}_{i+1} & 0 \end{pmatrix} & 0 & 0 & 0 \end{bmatrix},$$

$$Y_{3i}^T = \begin{bmatrix} \begin{pmatrix} 0 \\ -W_i \tilde{M}_i \end{pmatrix} \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix},$$

$$Y_{4i}^T = \begin{bmatrix} \begin{pmatrix} \tilde{N}_i & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 \end{pmatrix} & 0 & 0 & 0 \end{bmatrix}.$$

Adding (36) to (34) and applying the Schur complement, inequality (20) can be achieved immediately, completing the proof. □

Remark 2 The approach introduced in Theorem 1 requires only one step, making it much easier to implement than the two-step LMI method in (Zemouche et al. (2017)), and than the methods based on cone-complementary algorithm linearisation, such as (Tuan VLB, and Hajjaji (2018); Tuan VLB et al. (2019)).

Remark 3 Comparing with earlier works in (Tuan VLB, and Hajjaji (2018); Tuan VLB et al. (2019); Naami et al. (2021)), an observer-based control design methodology for T-S fuzzy systems is proposed: these previous studies use Lyapunov functions related only to the original state, and unrelated to the delayed state. However, in many applications the process delay is a major concern, as it is responsible for the deterioration in the performance. From the above analysis, we can infer that the result presented in this study is more general and practical for real applications.

Remark 4 The following optimisation can be used to find the optimal performance index γ:

$$\begin{cases} min & \gamma, \\ s.t. & P_1 > 0, P_2 > 0, Q_1 > 0, Q_2 > 0, R_1 > 0, R_2 > 0, LMI(20) \end{cases} \quad (37)$$

4 Simulation results and discussions

To compensate for the inherent differences between system states and measurement outputs, an uncertain model is used. The uncertainty matrices for the fuzzy quadruple-tank system (1) are then the following:

$$\tilde{M}_i = [0.1 \quad 0.1 \quad 0.1 \quad 0.1]^T,$$

$$\tilde{N}_i = \lambda_i [0.1 \quad 0.1 \quad 0.1 \quad 0.1],$$

$$\tilde{M}_{m+1} = [0.1 \quad 0.1]^T,$$

$$\tilde{M}_{m+1} = \lambda_{m+1} [0.1 \quad 0.1],$$

for $i = 1, \ldots, 8$, and $m = 8$.

Choosing time-varying delay $d(t) = 0.66 + 0.5sin(t)$, $0 \leqslant d(t) \leqslant 2$ and the parameter constant values are $\lambda_i \in [1, 120]$, $i = 1, \ldots, 8$, $m = 8$, $\tilde{\zeta}_1 = 0.001$, $\tilde{\zeta}_2 = 0.002$. The simulation starting from the initial conditions $x(0) = [0.34 \quad 0.14\, 0.41\, 0.31]^T$, $\hat{x}(0) = [0.21 \quad 0.01 \quad 0.22 \quad 0.10]^T$,

respectively.

Some simulation results are given in Figs. 3 and 4. Figure 3 shows the evolution of the measured levels (x_1, x_2), although Fig. 4 illustrates the evolution of the unmeasured states (x_3, x_4). Despite the existence of model uncertainties and disturbances, the observer estimates adequately the level of the four tanks, as seen in these figures.

Now, by using the YALMIP toolbox (Löfberg (2004)) in Matlab (Higham and Higham (2005)), the LMI (20) in Theorem 1 can be solved, obtaining the following controller gains:

$$K_1 = S_1^{-1} Z_{11} = \begin{bmatrix} -2.34 & -3.42 \\ -7.59 & -2.15 \\ 1.35 & 4.38 \\ 3.57 & 1.56 \end{bmatrix}^T,$$

$$K_2 = S_1^{-1} Z_{12} = \begin{bmatrix} -1.47 & -1.52 \\ -2.16 & -2.24 \\ 2.15 & 4.65 \\ 7.25 & 1.25 \end{bmatrix}^T,$$

$$K_3 = S_1^{-1} Z_{13} = \begin{bmatrix} -2.91 & -3.03 \\ -1.44 & -1.49 \\ 8.85 & 8.11 \\ 2.41 & 2.52 \end{bmatrix}^T,$$

$$K_4 = S_1^{-1} Z_{14} = \begin{bmatrix} -1.98 & -2.83 \\ -1.51 & -1.48 \\ 7.25 & 9.09 \\ 2.34 & 2.51 \end{bmatrix}^T,$$

$$K_5 = S_1^{-1} Z_{15} = \begin{bmatrix} 3.98 & 4.95 \\ 2.50 & 2.72 \\ -6.35 & -1.25 \\ -3.99 & -4.23 \end{bmatrix}^T,$$

$$K_6 = S_1^{-1} Z_{16} = \begin{bmatrix} 4.08 & 9.90 \\ 7.66 & 4.46 \\ -2.46 & -2.97 \\ -8.96 & -6.86 \end{bmatrix}^T,$$

$$K_7 = S_1^{-1} Z_{17} = \begin{bmatrix} 4.79 & 3.25 \\ 3.02 & 1.22 \\ -5.47 & -1.24 \\ -4.86 & -1.74 \end{bmatrix}^T,$$

$$K_8 = S_1^{-1} Z_{18} = \begin{bmatrix} 3.15 & 7.33 \\ 4.28 & 2.42 \\ -3.59 & -1.14 \\ -1.19 & -2.10 \end{bmatrix}^T,$$

$$K_{d_1} = S_2^{-1} Z_{21} = \begin{bmatrix} -5.53 & -5.94 \\ -2.53 & -1.95 \\ 1.39 & 1.39 \\ 4.23 & 3.30 \end{bmatrix}^T,$$

$$K_{d_2} = S_2^{-1} Z_{22} = \begin{bmatrix} -5.20 & -3.65 \\ -2.51 & -2.13 \\ 1.17 & 9.98 \\ 4.14 & 3.31 \end{bmatrix}^T,$$

$$K_{d_3} = S_2^{-1} Z_{23} = \begin{bmatrix} -5.45 & -4.87 \\ -2.54 & -1.91 \\ 1.44 & 9.45 \\ 4.22 & 3.27 \end{bmatrix}^T,$$

$$K_{d_4} = S_2^{-1} Z_{24} = \begin{bmatrix} -6.01 & -9.16 \\ -2.59 & -1.91 \\ 1.10 & 1.63 \\ 4.24 & 3.28 \end{bmatrix}^T,$$

$$K_{d_5} = S_2^{-1} Z_{25} = \begin{bmatrix} 5.77 & 1.05 \\ 3.20 & 3.61 \\ -8.06 & -2.17 \\ -5.13 & -5.66 \end{bmatrix}^T,$$

$$K_{d_6} = S_2^{-1} Z_{26} = \begin{bmatrix} 2.44 & 2.02 \\ 7.16 & 6.33 \\ -6.45 & -4.75 \\ -1.11 & -9.89 \end{bmatrix}^T,$$

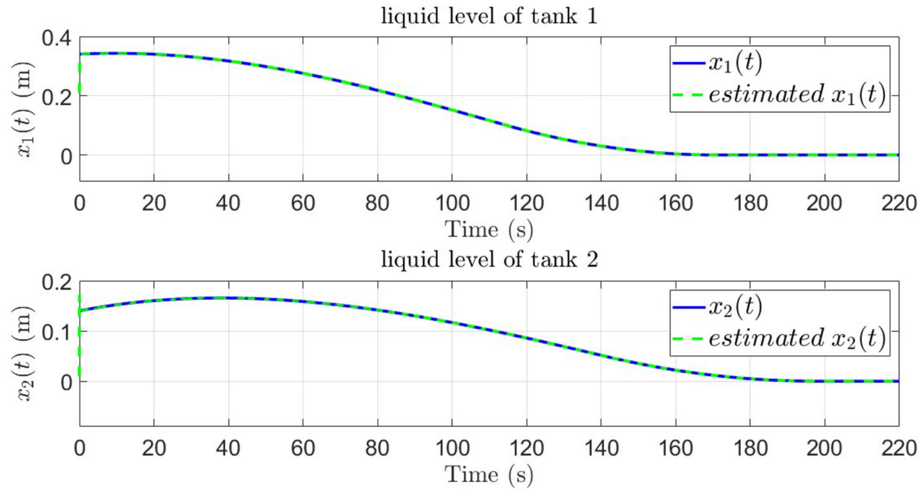

Fig. 3 Evolution of measurable states and its estimation

$$K_{d_7} = S_2^{-1} Z_{27} = \begin{bmatrix} 6.86 & 1.07 \\ 3.79 & 2.87 \\ -7.41 & -2.62 \\ -6.14 & -4.39 \end{bmatrix}^T,$$

$$K_{d_8} = S_2^{-1} Z_{28} = \begin{bmatrix} 2.13 & 1.29 \\ 1.01 & 6.15 \\ -4.14 & -2.39 \\ -1.61 & -9.82 \end{bmatrix}^T,$$

and the observer gains

$$L_1 = P_2^{-1} W_1 = \begin{bmatrix} -1.24 & 2.25 \\ -5.14 & 2.14 \\ -1.45 & 4.35 \\ -2.23 & 1.23 \end{bmatrix},$$

$$L_2 = P_2^{-1} W_2 = \begin{bmatrix} -1.36 & 2.24 \\ -9.34 & 1.25 \\ -3.25 & 2.97 \\ -2.82 & 1.74 \end{bmatrix},$$

$$L_3 = P_2^{-1} W_3 = \begin{bmatrix} -2.02 & 3.50 \\ -8.86 & 1.52 \\ -2.94 & 5.23 \\ -1.41 & 2.44 \end{bmatrix},$$

$$L_4 = P_2^{-1} W_4 = \begin{bmatrix} -2.02 & 3.50 \\ -8.85 & 1.52 \\ -2.94 & 5.23 \\ -1.41 & 2.44 \end{bmatrix},$$

$$L_5 = P_2^{-1} W_5 = \begin{bmatrix} 53.41 & 66.149 \\ 2.32 & 2.88 \\ 80.08 & 99.19 \\ 3.71 & 4.61 \end{bmatrix},$$

$$L_6 = P_2^{-1} W_6 = \begin{bmatrix} 19.29 & 1.53 \\ 83.62 & 6.66 \\ 28.93 & 2.29 \\ 1.33 & 1.06 \end{bmatrix},$$

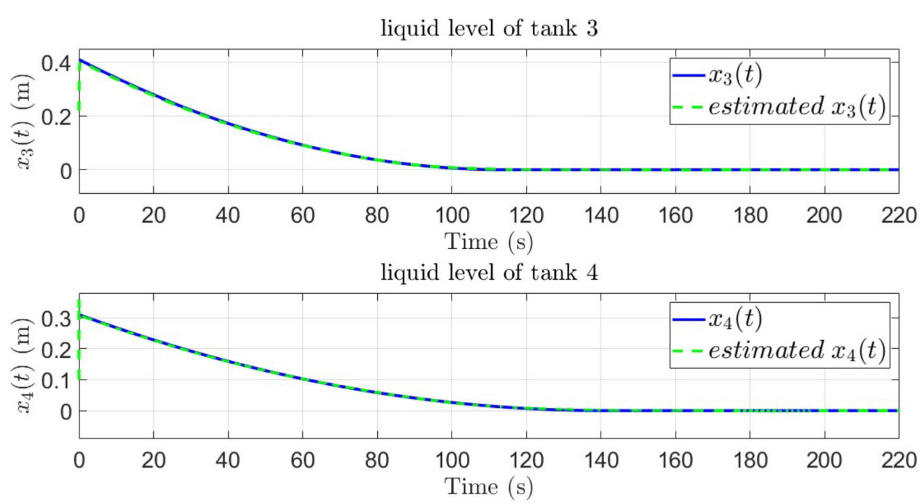

Fig. 4 Evolution of unmeasurable states and its estimation

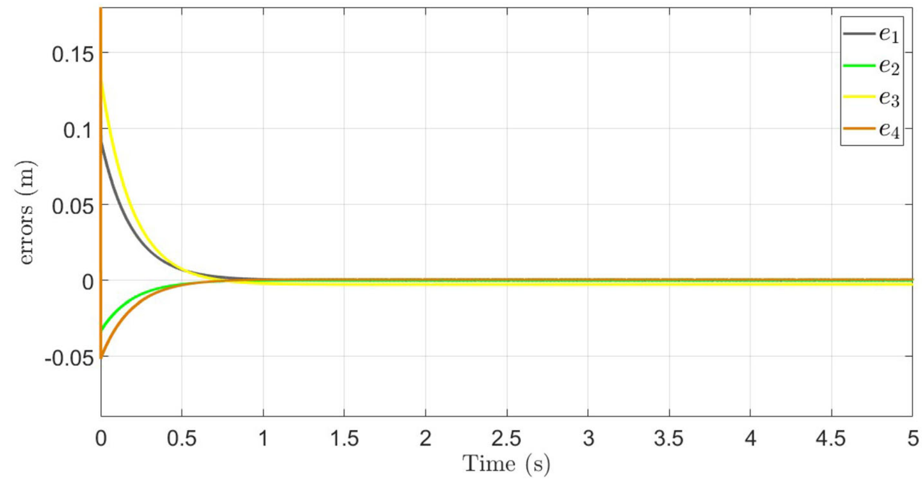

Fig. 5 Evolution of the estimation errors of tank levels

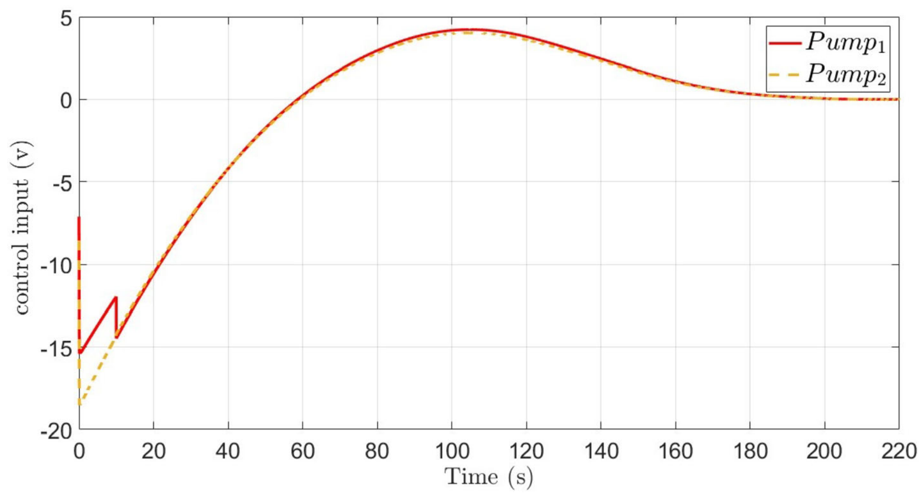

Fig. 6 Evolution of the pump control signals

$$L_7 = P_2^{-1} W_7 = \begin{bmatrix} 53.64 & 65.72 \\ 2.33 & 2.86 \\ 80.43 & 98.56 \\ 3.73 & 4.58 \end{bmatrix},$$

$$L_8 = P_2^{-1} W_8 = \begin{bmatrix} 30.42 & 2.25 \\ 2.13 & 3.14 \\ 32.98 & 1.14 \\ 2.63 & 3.14 \end{bmatrix}.$$

The flow distribution rate in the tanks is considered, based on the position of the control valves $\eta_1(t)$ and $\eta_2(t)$, which will be modified according to the following rules:

If $t_s(p) = 0, \ldots, 20$ **Then** $\eta_1(t) + \eta_2(t) \in [1, 2]$, **MP**

If $t_s(p) = 20, \ldots, 120$ **Then** $\eta_1(t) + \eta_2(t) \in [0, 1]$, **NMP**

If $t_s(p) = 120, \ldots, 220$ **Then** $\eta_1(t) + \eta_2(t) \in [1, 2]$, **MP**

If $t_s(p) = 220, \ldots, 300$ **Then** $\eta_1(t) + \eta_2(t) \in [0, 1]$, **NMP**

where **MP** is Minimum Phase setting (Johansson (1997)), and **NMP** is No-Minimum Phase setting. Using the setting rules, the estimation error of each tank level is presented in Fig. 5. Figure 6 shows the control signals of the observer-based controller designed using the approach in this paper. The results demonstrate the closed-loop stability and the reduction of tracking errors in various situations, including in the presence of parameter uncertainties and perturbations.

5 Conclusions

This paper has proposed a direct approach to design robust observer-based controllers for T-S fuzzy systems with time-varying delays in the presence of parameter uncertainties and admissible external disturbances. Lyapunov-

Krasovskii functional have been used to ensure the robust asymptotic stability, making possible to then select both the controller and observer gains in a single step, by solving a set of LMIs. This methodology ensures that the closed-loop system is robust asymptotically stable with a certain performance γ. The results reported in this paper are also relevant from the perspective of discrete-time state estimation. The approach was demonstrated using a simulated quadruple tank laboratory process: an observer-based controller was developed, and tested in simulation, to show the effectiveness of the proposed approach. It must be pointed out that the functional used in this article does not include the integral of the derivative, but this issue appears in the application, contributing to non-linear terms that are more complex. This merits further research.

Funding Open Access funding provided thanks to the CRUE-CSIC agreement with Springer Nature.

Data availability The datasets generated during and/or analysed during the current study are available from the corresponding author on reasonable request.

Declarations

Conflict of interest The authors declare that they have no conflict of interest.

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