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Numerical integration of an age-structured population model with infinite life span

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A B S T R A C T

The choice of age as a physiological parameter to structure a population and to describe its dynamics involves the election of the life-span. The analysis of an unbounded life-span age-structured population model is motivated because, not only new models continue to appear in this framework, but also it is required by the study of the asymptotic behaviour of its dynamics. The numerical integration of the corresponding model is usually performed in bounded domains through the truncation of the age life-span. Here, we propose a new numerical method that avoids the truncation of the unbounded age domain. It is completely analyzed and second order of convergence is established. We report some experiments to exhibit numerically the theoretical results and the behaviour of the problem in the simulation of the evolution of the Nicholson's blowflies model.

1. Introduction

Population dynamics modelization entails making different choices. Once it is decided that the individuals in the population are non homogeneous, the variables in which these are organized in the population have to be chosen. This is crucial because the variables that determine the structure of the population are largely responsible for its dynamics. Age is one of the most natural and important parameters structuring a population. It plays a main role in demography but, nowadays, it is not restricted to it and also appears in other fields such as ecology, epidemiology and cell growth. Other structuring variables (usually known as size), in general, are intimately dependent on age.

A second concern relates to the description of the phenomena that influence the dynamics of the population (mortality, fertility, migration, etc.). It goes back and forth between a discrete age-time setting and a continuous one. We consider the model within the continuous framework.

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In a continuous age-structured population model with finite age-span $[0, a_\dagger)$, the natural interpretation for the mortality rate function ($\mu = \mu(a)$) is through the survival probability to age a,

$$\Pi(a) = \exp\left(-\int_0^a \mu(\sigma)d\sigma\right).$$

Then, it is assumed that $\Pi(a_\dagger) = 0$, in consistency with the model. As a consequence, the mortality rate (μ) must be unbounded at $a_\dagger$ [1,2]. Although the maximum finite age condition has become a benchmark of the study on age-structured problems [2–4], the analysis of an unbounded age-span under the conditions in Iannelli and Milner [2] is useful.

The model we study is described by the following first order hyperbolic partial differential equation,

$$u_t + u_a = -\mu(a, I_\mu(t), t)\, u, \quad a > 0,\ t > 0, \tag{1.1}$$

which represents how the dynamics of the individuals in the population evolve. Here independent variables a and t represent age and time, respectively. The dependent function $u(a,t)$ denotes the population density with respect to the age a at time t, and the function $\mu(a, I_\mu(t), t) \geq 0$ is the nonnegative age-specific mortality rate function. The partial differential Eq. (1.1) is supplemented with a nonlinear and nonlocal boundary condition which represents the individuals birth law, given by

$$u(0,t) = \int_0^\infty \beta(a, I_\beta(t), t)\, u(a,t)\, da\,, \quad t > 0, \tag{1.2}$$

where $\beta(a, I_\beta(t), t) \geq 0$ is the nonegative age-specific fertility rate function. An initial condition

$$u(a,0) = u_0(a), \quad a \geq 0, \tag{1.3}$$

where $u_0(a)$ is a nonnegative function, is also given. In this general nonlinear model, the nonnegative vital functions μ and β also depend on the functionals

$$I_\mu(t) = \int_0^\infty \gamma_\mu(a)\, u(a,t)\, da, \quad I_\beta(t) = \int_0^\infty \gamma_\beta(a)\, u(a,t)\, da\,,\ t > 0, \tag{1.4}$$

to take into account the influence of the age distribution of the population on the life history of the individuals. In these functionals (1.4), the nonnegative function parameters $\gamma_\mu(a)$ and $\gamma_\beta(a)$ give an account of the different influence of the individuals in the population, depending on their age, on the mortality and birth rates.

The model described by (1.1)–(1.3) was introduced first in Gurtin and MacCamy by Gurtin and McCamy, with the use of vital functions (μ and β) only depending on the age and the total population $P(t) = \int_0^\infty u(a,t)\, da$. In that work, they proved the existence and uniqueness of a nonnegative solution $u \in \mathcal{C}^1(\mathbb{R}^+ \times [0,T])$ under appropriate compatibility properties between initial and boundary conditions, continuity of the derivatives of mortality and fertility rates, continuity of the initial condition and $u_0' \in \mathcal{L}_1(\mathbb{R}^+)$. They also demonstrated the global existence of the solution u for all time when the fertility was uniformly bounded and the existence of a unique equilibrium age-distribution. The study of stability of such equilibrium was only made by considering particular expresions of the vital functions due to the high difficulty in dealing with the general problem. In [6], the analysis of the problem (1.1)–(1.3), assuming that the vital functions depend on time and on a weighted population, was addressed in a similar approach as in Gurtin and MacCamy [5]. An extensive summary about the theoretical results for problem (1.1)–(1.3) can be found in Iannelli and Milner [2]. Moreover, we will require additional regularity hypotheses on the vital rates functions and the initial condition to prove convergence of our proposal. We should mention that, for age-structured population models with infinite life span, Webb [7] and Iannelli and Milner [2] considered situations with a birth function of the form $\beta(a) = e^{-pa}\sum_{j=1}^m \beta_j\, a^j$, that corresponds to a reproductive process in which individuals of the species continue to reproduce as long as they survive.

The model (1.1)–(1.3) includes enough information to describe the evolution of a population structured by age. However, the explicit analytical solution of such a general model is unaffordable and can only be obtained in very special cases. Moreover, its asymptotic behaviour requires to be careful with the shape of the age-specific birth and death rates at infinity (see [2,5,6], where specific form for the rates β and μ were considered). This means that numerical integration is a valuable and feasible alternative.

From a numerical point of view, the model (1.1)–(1.3) has been studied in different situations (linear and nonlinear models and applications, for example, in epidemiology) for the last 40 years. We can find an outline of the most significant numerical methods developed so far in Abia et al. [8]. These numerical studies were based on the hypothesis of an initial condition with compact support $[0, A]$, and a finite integration time T, which allowed us to integrate in a finite age interval $[0, A+T]$. Under these assumptions, the numerical schemes provided are accurate. Nevertheless, close attention needs to be paid to the unbounded life span model. On the one hand, new models still arise in this framework as, for example, [9]. On the other hand, the asymptotic study of these models makes the use of an unbonded life span essential: the finite setting does not allowed a far enough numerical integration, this is why numerical methods that approximate the solution in an unbounded life-span should be addressed.

The paper is organized as follows: in Section 2, we reformulate the age-structured population problem (1.1)–(1.3) by introducing a change in both the structured dependent and independent variables that transforms the infinitum life span age-structured problem into a mathematical equivalent structured population model with an artificial structure. In Section 3, we propose a second order numerical method which is described in detail, and we carry out its convergence analysis in Section 4: it is the highest order method analyzed for this setting. Finally, Section 5 is devoted to the numerical experimentation where we introduce some numerical tests to reveal the experimental second order of convergence, and some features of the long-time integration of the Nicholson's blowflies model.

2. An artificial size-structured population model

The framework we propose to analyze the infinite life-span model involves an artificially structured population model, defined over a bounded state interval, which is equivalent to the original unbounded age-specific one. Then, we integrate numerically the transformed model to obtain the solution to the initial problem.

First of all, we will consider a change in both the structured dependent and independent variables to transform the age-structured problem into an equivalent population model structured by an artificial size variable. Thus, the problem is reformulated over a bounded domain of the new artificial structuring variable. For this kind of problem, different numerical methods have been considered in the recent literature [10]. Then, we can apply a numerical method for this well-known structured population model. Finally, from this approximation we will recover a numerical solution to the original unbounded life-span problem. The new independent variable can be seen as a computational variable in terms of which the numerical procedure, that we will describe in the following section, makes a truncation of the unbounded age interval that changes with the discretization parameter.

Therefore, we propose the following change of variables: we relate the age with a new artificial structuring variable x by means of $a = \alpha(x)$, $x \in [0, 1)$, with α an increasing function such that $\alpha(0) = 0$ and $\lim_{x\to 1^-} \alpha(x) = +\infty$. Next, the new unknown density function $v(x, t)$ is defined by the identity

$$g(x)\, v(x, t) = u(\alpha(x), t),\ x \in [0, 1),\ t \in \mathbb{R},$$

where $g(x) = \dfrac{1}{\alpha'(x)}$, $x \in [0, 1)$. Thus, the new population density, $v(x, t)$, satisfies the following structured model

$$v_t + (g(x)\, v)_x = -\mu^*(x, I^*_\mu(t), t)\, v, \quad 0 < x < 1,\ t > 0, \tag{2.1}$$

$$g(0)\, v(0, t) = \int_0^1 \beta^*(x, I^*_\beta(t), t)\, v(x, t)\, dx\,, \quad t > 0, \tag{2.2}$$

$$v(x, 0) = \frac{1}{g(x)}\, u_0(\alpha(x)), \quad 0 \le x < 1, \quad v(1, 0) = 0, \tag{2.3}$$

where, we could represent $I^*_\mu(t)$ and $I^*_\beta(t)$ as

$$I^*_\mu(t) = \int_0^1 \gamma^*_\mu(x)\, v(x, t)\, dx, \quad I^*_\beta(t) = \int_0^1 \gamma^*_\beta(x)\, v(x, t)\, dx\,,\ t > 0, \tag{2.4}$$

with $\mu^*(x, I^*_\mu(t), t) = \mu(\alpha(x), I_\mu(t), t)$; $\beta^*(x, I^*_\beta(t), t) = \beta(\alpha(x), I_\beta(t), t)$; $\gamma^*_\mu(x) = \gamma_\mu(\alpha(x))$ and $\gamma^*_\beta(x) = \gamma_\beta(\alpha(x))$.

In this size-structured population model, the variable that structures the population, x, is artificial without biological meaning. Thus, function α is linked with the (size-specific) growth rate function in the associated size-structured population model. In this setting, the selection of function α has to follow some properties that allow us to define a nonnegative continuous size-specific growth rate function on [0,1], with $g(1) = 0$. Then, $\alpha \in \mathcal{C}^1([0, 1))$, $\alpha'(x) > 0$, $x \in [0, 1)$ (that is, strictly increasing) and $\lim_{x\to 1^-} \alpha'(x) = +\infty$. Note also that $\lim_{x\to 1^-} g(x)\, v(x, t) = 0 = \lim_{x\to 1^-} u(\alpha(x), t)$. In order to consider the new problem for the new unknown function, we assume $\lim_{x\to 1^-} \dfrac{u_0(\alpha(x))}{g(x)} = 0$. Note that this condition is not very restrictive: an initial data with compact support on $[0, +\infty)$ verifies this condition

In this paper, our main objective is to introduce a second order numerical method to solve (1.1)–(1.3) through the discretization of the associated size-structured population problem (2.1)–(2.3).

In the following, we specify the minimum requirements upon the data functions and the solution of the problem for the later convergence analysis. Thus, let be $T, \sigma \in \mathbb{R}^+$, we define

$$\mathcal{A}_\sigma = \{f \in \mathcal{C}^2([0, \infty));\ f^{(n)}(a) = \mathcal{O}(e^{-\sigma a}), \text{as } a \to \infty, n = 0, 1, 2\}$$

and

$$\mathcal{A}^{\mathrm{b}}_\sigma = \{f \in \mathcal{C}^2([0, \infty));\ f \text{ bounded and } f^{(n)}(a) = \mathcal{O}(e^{-\sigma a}), \text{as } a \to \infty, n = 1, 2\}.$$

Throughout the paper, we assume the following hypotheses,

(H1) $u \in \mathcal{C}^2([0, \infty) \times [0, T])$, is nonnegative and, given $t \in [0, T]$, the function $u_{(t)}(a) := u(a, t)$, $a \ge 0$, is such that $u_{(t)}(a) \in \mathcal{A}_\sigma$.

(H2) $\beta \in \mathcal{C}^2([0,\infty)\times D_\beta \times [0,T])$, is nonnegative and, given $(z,t)\in D_\beta \times [0,T]$, the function $\beta_{(z,t)}(a) := \beta(a,z,t)$, $a\geq 0$, is such that $\beta_{(z,t)}(a)\in \mathcal{A}^{\mathrm{b}}_\sigma$, where D_β is a compact neighbourghood of

$$\left\{\int_0^\infty \gamma_\beta(a)\,u(a,t)\,da,\quad 0\leq t\leq T\right\}.$$

(H3) $\mu \in \mathcal{C}^2([0,\infty)\times D_\mu \times [0,T])$, is nonnegative and, given $(z,t)\in D_\mu \times [0,T]$, the function $\mu_{(z,t)}(a) := \mu(a,z,t)$, $a\geq 0$, is such that $\mu_{(z,t)}(a)\in \mathcal{A}^{\mathrm{b}}_\sigma$, where D_μ is a compact neighbourghood of

$$\left\{\int_0^\infty \gamma_\mu(a)\,u(a,t)\,da,\quad 0\leq t\leq T\right\}.$$

(H4) $\gamma_\mu, \gamma_\beta \in \mathcal{A}^{\mathrm{b}}_\sigma$, are nonnegative.

These assumptions should be enough to get second order convergence with the numerical method proposed. Also, they are reasonable requests to make. On the one hand, under the assumptions (H1) at $t=0$, and (H2)–(H4), the problem (1.1)–(1.3) has a unique nonnegative solution, which is global in time (note that we have included some additional restrictions on the hypotheses given in Gurtin and MacCamy [5]). Furthermore, it could be proved that $u\in\mathcal{C}^2(\mathbb{R}^+\times[0,T])$ if an additional second order compatibility condition is satisfied. On the other hand, hypotheses (H1) at $t=0$, and (H2)–(H4), ensure $u(\cdot,t)\in \mathcal{L}^1(\mathbb{R}^+)$.

Once the change of variables is made, the next step is to consider a numerical method to approximate the solution to (2.1)–(2.3) in a fixed time domain $[0,T]$ and the bounded size interval [0,1]. Therefore, if we employ a second order numerical scheme, to reach the optimal rate of convergence, we require some regularity conditions on the functions data (g, μ^*, β^*, γ^*_μ and γ^*_β) and the solution v on their corresponding domains similar as (H1)–(H4), that are obtained from them. That is,

(H5) $v\in\mathcal{C}^2([0,1]\times[0,T])$, is nonnegative.

(H6) $g\in\mathcal{C}^3([0,1])$, is nonnegative and $\int_0^1 g^{-1}(x)\,dx=\infty$.

(H7) $\beta^*\in\mathcal{C}^2([0,1]\times D_{\beta^*}\times[0,T])$, is nonnegative and D_{β^*} is a compact neighbourghood of

$$\left\{\int_0^1 \gamma^*_\beta(x)\,v(x,t)\,dx,\quad 0\leq t\leq T\right\}.$$

(H8) $\mu^*\in\mathcal{C}^2([0,1]\times D_{\mu^*}\times[0,T])$, is nonnegative and D_{μ^*} is a compact neighbourghood of

$$\left\{\int_0^1 \gamma^*_\mu(x)\,v(x,t)\,dx,\quad 0\leq t\leq T\right\}.$$

(H9) $\gamma^*_\mu, \gamma^*_\beta \in\mathcal{C}^2([0,1])$, are nonnegative.

All in all, the proposed change of variable has to satisfy

(H10) $\alpha\in\mathcal{C}^4([0,1))$, $\alpha(0)=0$, $\alpha'(x)>0$, $x\in[0,1)$, $\lim_{x\to1^-}\alpha(x)=\infty$, $\lim_{x\to1^-}\alpha'(x)=\infty$ and $\lim_{x\to1}\alpha'(x)\,u_0(\alpha(x))=0$.

3. Numerical approximation

In this section we introduce a numerical method for the original age-structured model (1.1)–(1.3) through the transformation that converts it into the size-structured model (2.1)–(2.3). The new model could be solved with the usual numerical techniques employed to obtain the solution of size-structured models in the case of a bounded size interval (see Abia et al. [10]). We will approach the solution of the size-structured problem and, finally, we return to the solution of the original problem.

In this work, we will integrate (2.1)–(2.3) by means of a scheme based on the numerical approximation to the solution along the characteristic curves. It is a natural and efficient way to discretize (2.1). To this end, we rewrite it as

$$v_t + g(x)\,v_x = -\big(\mu^*(x, I^*_\mu(t), t) + g'(x)\big)\,v,\quad 0<x<1,\ t>0. \tag{3.1}$$

Now, we denote by $x(t;t^*,x_*)$ the characteristic curve associated with Eq. (3.1) which takes the value x_* at time t^*. It is the solution to the following initial value problem

$$\begin{cases} x'(t;t^*,x_*) &= g(x(t;t^*,x_*)),\quad t\geq t^*,\\ x(t^*;t^*,x_*) &= x_*.\end{cases} \tag{3.2}$$

Then, we define $w(t;t^*,x_*) = v(x(t;t^*,x_*),t)$, $t\geq t^*$, that is the solution of

$$\begin{cases} w'(t;t^*,x_*) &= -\big(\mu^*(x(t;t^*,x_*), I^*_\mu(t), t) + g'(x(t;t^*,x_*))\big)w(t;t^*,x_*),\ t\geq t^*,\\ w(t^*;t^*,x_*) &= v(x_*,t^*).\end{cases} \tag{3.3}$$

The solution along the characteristic curves can be written as

$$w(t;t^*,x_*) = v(x_*,t^*)\ \exp\left\{-\int_{t^*}^{t}\left(\mu^*(x(\tau;t^*,x_*),I_\mu^*(\tau),\tau)+g'(x(\tau;t^*,x_*))\right)d\tau\right\}, t\ge t^* \tag{3.4}$$

Note that the viability of the change of variables proposed in the previous section, is independent of the particular choice of the function α which relates the age variable with the new structuring variable. However, the efficiency of the numerical method used for the approximation of the solution of (2.1)–(2.3), and the corresponding analysis of convergence, would be affected by this election.

Then, when considering a particular choice of α, we will take into account that the problem obtained after the change of variables (2.1)–(2.3) is a typical size-structured population model. Here we propose to employ

$$\alpha(x) = -\frac{1}{K_\alpha}\ \log(1-x), \quad x\in[0,1),$$

(where K_α is a positive parameter that will be fixed), which provides the well-known Bertalanffy's growth rate [11]

$$g(x) = K_\alpha\ (1-x), \quad x\in[0,1]. \tag{3.5}$$

For the numerical integration on the bounded time interval $[0,T]$, we introduce $N\in\mathbb{N}$ as the number of time steps. The time discretization parameter is given by $k=T/N$ and the time levels are represented by $t^n = n\,k$, $0\le n\le N$. Then, we introduce the grid over the size interval. We use the *natural grid* in which the grid points are computed by means of the solution to (3.2). Actually, it can be made analytically due to the simple expression of the growth-rate function (3.5). Therefore, we define x_j, $j\ge 0$ as

$$x_j = 1-e^{-K_\alpha\,k\,j}, \quad j=0,1,2,\dots \tag{3.6}$$

The numerical method requires a finite number of grid nodes $\{x_j\}_{j=0}^{J+1}$. So, we denote as J, the first positive integer such that $1-x_J\le K_1\,k$, with K_1 a suitable positive number independent on k. Taking into account (3.6), J is the first integer that satisfies $J > -\dfrac{\log(K_1\,k)}{K_\alpha\,k}$. Finally, $x_{J+1}=1$. This choice of α (and g) is enough to ensure the existence of x_J [12].

On the other hand, we should note that this grid satisfies $x_j - x_{j-1}\le C\,k$, $j=1,\dots,J+1$, for a fixed positive constant C, independent on k, which is a crucial property for the use of a quadrature, based on these nodes, in order to approximate the nonlocal terms.

We obtain the approximations to the solution at the grid nodes by means of an explicit second order numerical method. We refer to the grid point x_j by a subscript j and to the time level t^n by a superscript n. Let V_j^n be a numerical approximation to $v_j^n = v(x_j,t^n)$, $0\le j\le J+1$, $0\le n\le N$. For simplicity, we use the vectorial representation $\mathbf{V}^n = (V_0^n, V_1^n,\dots,V_{J+1}^n)$, $0\le n\le N$. Therefore, starting from an approximation to the initial data (2.3), for example, the grid restriction of the function $v(x,0)$, the numerical solution at a new time level t^{n+1}, $0\le n\le N-1$, is described in terms of the previous one. Such a general time step is given by the following formulae, that we obtain from the discretization of (3.4)

$$V_{j+1}^{n+1,*} = V_j^n\ \exp\left(-k\left(\mu^*\left(x_j, \mathcal{Q}_k^*\left(\boldsymbol{\gamma}_\mu^*\cdot\mathbf{V}^n\right),t^n\right)-K_\alpha\right)\right), \quad 0\le j\le J-1, \tag{3.7}$$

$$V_{j+1}^{n+1} = V_j^n\ \exp\left(-\frac{k}{2}\left(\mu^*\left(x_j,\mathcal{Q}_k^*(\boldsymbol{\gamma}_\mu^*\cdot\mathbf{V}^n),t^n\right) + \mu^*\left(x_{j+1},\mathcal{Q}_k^*(\boldsymbol{\gamma}_\mu^*\cdot\mathbf{V}^{n+1,*}),t^{n+1}\right)-2\,K_\alpha\right)\right), \quad 0\le j\le J-1, \tag{3.8}$$

and the boundary condition (2.2),

$$V_0^{n+1} = \mathcal{Q}_k^*(\boldsymbol{\beta}^{*,n+1}(\mathbf{V})\cdot\mathbf{V}^{n+1})/K_\alpha, \tag{3.9}$$

where $(\boldsymbol{\gamma}_\mu^*)_j = \gamma_\mu^*(x_j)$, $(\boldsymbol{\gamma}_\beta^*)_j = \gamma_\beta^*(x_j)$, $\boldsymbol{\beta}^{*,n}(\mathbf{V})_j = \beta^*(x_j,\mathcal{Q}_k^*(\boldsymbol{\gamma}_\beta^*\cdot\mathbf{V}^n),t^n)$, $0\le j\le J$, $1\le n\le N$. Finally, we fix $V_{J+1}^{n,*} = V_{J+1}^n = 0$, $0\le n\le N$.

In (3.7)–(3.9), $\mathcal{Q}_k^*$ represents a quadrature rule to approximate the integrals over the size interval. In order to save computational effort when computing the quadrature approximations, initially we fix a subgrid of the natural grid $\{x_{j_l}\}_{l=0}^{M+1}$, satisfying $x_{j_0}=0$, $x_{j_{M+1}}=1$, and $C_0\,k\le x_{j_{l+1}}-x_{j_l}\le C_1\,k$, $0\le l\le M+1$, where C_0 and C_1 are positive constants independent of k, for further details we refer to Angulo and López-Marcos [12]. Then, to obtain an explicit numerical method, we choose a quadrature formula based the composite trapezoidal rule, on the subintervals defined by the subgrid, combined with the use of a right-hand rectangular rule on the first interval and a left-hand rectangular rule on the last one. Therefore, the resulting formula is given by

$$\mathcal{Q}_k^*(\mathbf{V}) = x_{j_1}\,V_{j_1} + \sum_{l=1}^{M-1}\frac{x_{j_{l+1}}-x_{j_l}}{2}\left(V_{j_l}+V_{j_{l+1}}\right) + (1-x_{j_M})\,V_{j_M}, \tag{3.10}$$

where $\mathbf{V} = \left(V_0, V_1, \ldots, V_{J+1}\right)$. It keeps the second order of accuracy. Lastly, $\boldsymbol{\gamma}_\mu^* \cdot \mathbf{V}^n$, $\boldsymbol{\gamma}_\mu^* \cdot \mathbf{V}^{n+1,*}$ and $\boldsymbol{\beta}^{*,n+1}(\mathbf{V}) \cdot \mathbf{V}^{n+1}$ represent the componentwise product of the corresponding vectors. By this election, V_0^{n+1} is always well-defined by formula (3.9) as the boundary nodal value V_0^{n+1} does not contribute to the quadrature approximation. We emphasize that, even the quadrature rule were chosen closed at the end $x_{j_0} = 0$, again formula (3.9) will provide implicitly an approximation V_0^{n+1}, for enough small value of the discretization parameter k, because then, the coefficient of the contribution of V_0^{n+1} to the quadrature approximation is decreasing to zero with k.

The formulae previously presented could be described in terms of age by returning to the original variables, and would represent an approximation to the age-structured problem. The points in the age-grid are given by $a_j = \alpha(x_j)$, $0 \le j \le J$. So, it is easy to show that $a_j = j\,k$, $0 \le j \le J$, which are equally spaced (and correlated with time as usual). The condition introduced to finish the computations at the *natural grid* induces that the last point in the age-grid satisfies $a_J > -\dfrac{1}{K_\alpha} \log(K_1\,k)$. Let U_j^n be a numerical approximation to $u(a_j, t^n)$, $0 \le j \le J$, $0 \le n \le N$ and $\mathbf{U}^n = (U_0^n, U_1^n, \ldots, U_J^n)$, $0 \le n \le N$. Thus the numerical method is completed with the following formula

$$U_j^n = K_\alpha\, e^{-K_\alpha\,k\,j} V_j^n,\ 0 \le j \le J,\ 0 \le n \le N. \tag{3.11}$$

4. Convergence

In this section we analyze the numerical method (3.6)–(3.10) that approaches the solution to the associated size-structured model (2.1)–(2.3) presented in Section 3. We finish with the convergence of the transformed values (3.11) to the solution to the age-structured model (1.1)–(1.3).

We begin with a Lemma that establishes the smoothness properties of the associated size-structured problem.

Lemma 1. *Assume that hypotheses (H1)–(H4) hold and* $\alpha(x) = -\dfrac{1}{K_\alpha} \log(1-x)$, $x \in [0,1)$. *Then, the functions data and the solution to (2.1)–(2.3) satisfy (H5)–(H9) when* $3\,K_\alpha < \sigma$.

Proof. Due to the hypotheses (H1)–(H4) and the smoothness of α, we ensure such a smoothness properties when we restrict them to the interval [0,1).

We extend the smoothness properties on the interval [0,1] by means of the definition of each function and derivatives at $x = 1$ as the corresponding limit if there exists.

In the case of the solution to (2.1)–(2.3)

$$\lim_{x\to 1^-} v(x,t) = \lim_{a\to\infty} \frac{u(a,t)}{K_\alpha\, e^{-K_\alpha\, a}} = 0, \quad t > 0,$$

then $v(1,t) = 0$, $t > 0$. The partial derivatives of v with respect to time at $x = 1$ are computed with,

$$\lim_{x\to 1^-} \frac{\partial^n v}{\partial t^n}(x,t) = \lim_{a\to\infty} \frac{\frac{\partial^n u}{\partial t^n}(a,t)}{K_\alpha\, e^{-K_\alpha\, a}} = 0, \quad t > 0, \quad n = 1, 2,$$

then $\dfrac{\partial^n v}{\partial t^n}(1,t) = 0$, $t > 0$, $n = 1, 2$. The first order partial derivative with respect to size at $x = 1$ satisfies

$$\begin{aligned}\lim_{x\to 1^-} v_x(x,t) &= \lim_{x\to 1^-} \frac{u_a(\alpha(x),t) - u(\alpha(x),t)\, g'(x)}{g^2(x)} \\ &= \lim_{a\to\infty} \frac{u_a(a,t) + K_\alpha\, u(a,t)}{K_\alpha^2\, e^{-2\,K_\alpha\, a}} = 0, \quad t > 0,\end{aligned}$$

then $v_x(1,t) = 0$, $t > 0$. And finally, the second orden derivative with respect to size at $x = 1$ is given by

$$\begin{aligned}\lim_{x\to 1^-} v_{xx}(x,t) &= \lim_{x\to 1^-} \frac{u_{aa}(\alpha(x),t) + 3\,K_\alpha\, u_a(\alpha(x),t) + 2\,K_\alpha^2\, u(\alpha(x),t)}{g^3(x)} \\ &= \lim_{a\to\infty} \frac{u_{aa}(a,t) + 3\,K_\alpha\, u(a,t) + 2\,K_\alpha^2\, u(a,t)}{K_\alpha^3\, e^{-3\,K_\alpha\, a}} = 0, \quad t > 0,\end{aligned}$$

then $v_{xx}(1,t) = 0$, $t > 0$. These ensure that $v \in \mathcal{C}^2([0,1] \times [0,T])$.

With similar arguments we derive the smoothness properties of μ^*, β^*, γ_μ^* and γ_β^*. □

The convergence result will be the consequence of a study of the consistency and the nonlinear stability. We shall employ the discretization framework introduced by López-Marcos et al. [13] to analyze the numerical method described by (3.6)–(3.10). We consider that the discretization parameter k takes values in the set $H = \{k > 0 : k = T/N, N \in \mathbb{N}\}$, and for each $k \in H$ we define the vector spaces $\mathcal{X}_k = (\mathbb{R}^{J+1})^{N+1}$ and $\mathcal{Y}_k = (\mathbb{R}^{J+1}) \times \mathbb{R}^N \times (\mathbb{R}^J)^N$. If $\mathbf{V} = \left(V_0, V_1, \ldots, V_J\right) \in \mathbb{R}^{J+1}$, and $\|\mathbf{V}\|_{\infty,J+1} = \max_{0\le j\le J} |V_j|$, we define the following norm on $\mathcal{X}_k$, if $(\mathbf{V}^0, \mathbf{V}^1, \ldots, \mathbf{V}^N) \in \mathcal{X}_k$

$$\|(\mathbf{V}^0, \mathbf{V}^1, \ldots, \mathbf{V}^N)\|_{\mathcal{X}_k} = \max_{0\le n\le N} \|\mathbf{V}^n\|_{\infty,J+1}.$$

In addition, if $\mathbf{Z} \in \mathbb{R}^N$, $\mathbf{W} \in \mathbb{R}^J$, and $\|\mathbf{Z}\|_{\infty,N} = \max_{1\le n\le N} |Z^n|$, $\|\mathbf{W}\|_{\infty J} = \max_{1\le j\le J} |W_j|$, then for $(\mathbf{Z}^0, \mathbf{Z}_0, \mathbf{Z}_*^1, \ldots, \mathbf{Z}_*^N) \in \mathcal{Y}_k$, we define

$$\|(\mathbf{Z}^0, \mathbf{Z}_0, \mathbf{Z}_*^1, \ldots, \mathbf{Z}_*^N)\|_{\mathcal{Y}_k} = \|\mathbf{Z}^0\|_{\infty J+1} + \|\mathbf{Z}_0\|_{\infty,N} + \sum_{n=1}^{N} k\,\|\mathbf{Z}_*^n\|_{\infty J},$$

where $\mathbf{Z}_*^n = (Z_1^n, Z_2^n, \ldots, Z_J^n)$, $1 \le n \le N$. Finally, if $\mathbf{V} = (V_0, V_1, \ldots, V_J)$, we also consider

$$\|\mathbf{V}\|_{1,M+1} = \sum_{l=0}^{M} k\,|V_{j_l}|,$$

where $\{x_{j_l}\}_{l=0}^{M+1}$ are the nodes that define the subgrid.

For each $k \in H$, we consider the element $\mathbf{v}_k = (\mathbf{v}^0, \mathbf{v}^1, \ldots, \mathbf{v}^N) \in \mathcal{X}_k$, $\mathbf{v}^n = (v_0^n, v_1^n, \ldots, v_J^n) \in \mathbb{R}^{J+1}$, $v_j^n = v(x_j, t^n)$, $0 \le j \le J$, $0 \le n \le N$, where v is the solution of (2.1)–(2.3). Let R_k be a fixed positive constant and denote by $\mathcal{B}(\mathbf{v}_k, R_k) \subset \mathcal{X}_k$, the open ball with center $\mathbf{v}_k$ and radious R_k.

Next, we introduce the mapping $\mathbf{\Phi}_k : \mathcal{B}(\mathbf{v}_k, R_k) \to \mathcal{Y}_k$ defined by the equations

$$\mathbf{\Phi}_k(\mathbf{W}^0, \mathbf{W}^1, \ldots, \mathbf{W}^N) = (\mathbf{Z}^0, \mathbf{Z}_0, \mathbf{Z}_*^1, \ldots, \mathbf{Z}_*^N), \tag{4.1}$$

$$\mathbf{Z}^0 = \mathbf{W}^0 - \mathbf{V}^0, \tag{4.2}$$

$$Z_{j+1}^{n+1} = \frac{1}{k}\left(W_{j+1}^{n+1} - W_j^n \exp\left(-\frac{k}{2}\left(\mu^*\Big(x_j, \mathcal{Q}_k^*(\boldsymbol{\gamma}_\mu^* \cdot \mathbf{W}^n), t^n\Big) + \mu^*\Big(x_{j+1}, \mathcal{Q}_k^*(\boldsymbol{\gamma}_\mu^* \cdot \mathbf{W}^{n+1,*}), t^{n+1}\Big) - 2K_\alpha\right)\right)\right),$$
$$0 \le j \le J-1, \tag{4.3}$$

$$Z_0^{n+1} = W_0^{n+1} - \mathcal{Q}_k^*(\boldsymbol{\beta}^{*,n+1}(\mathbf{W}) \cdot \mathbf{W}^{n+1})/K_\alpha, \tag{4.4}$$

$0 \le n \le N-1$, where

$$W_{j+1}^{n+1,*} = W_j^n \exp\Big(-k\Big(\mu^*\Big(x_j, \mathcal{Q}_k^*(\boldsymbol{\gamma}_\mu^* \cdot \mathbf{W}^n), t^n\Big) - K_\alpha\Big)\Big), \tag{4.5}$$

$0 \le j \le J-1$, $0 \le n \le N-1$. Then $\mathbf{V}_k = (\mathbf{V}^0, \mathbf{V}^1, \ldots, \mathbf{V}^N) \in \mathcal{X}_k$, is a solution of the scheme (3.6)–(3.10) if and only if

$$\mathbf{\Phi}_k(\mathbf{V}_k) = \mathbf{0}. \tag{4.6}$$

In the following we study the consistency, stability, existence of solutions and convergence of our scheme. In our analysis we shall assume the hypotheses (H5)–(H9). Hence we choose the radius R_k of the open ball $\mathcal{B}(\mathbf{v}_k, R_k) \subset \mathcal{X}_k$ such that, for k sufficiently small, if $(\mathbf{V}^0, \mathbf{V}^1, \ldots, \mathbf{V}^N) \in \mathcal{B}(\mathbf{v}_k, R_k)$ then

$$\mathcal{Q}_k^*(\boldsymbol{\gamma}_\mu^* \cdot \mathbf{V}^n),\ \mathcal{Q}_k^*(\boldsymbol{\gamma}_\mu^* \cdot \mathbf{V}^{n,*}) \in D_{\mu^*}, \text{ and } \mathcal{Q}_k^*(\boldsymbol{\gamma}_\beta^* \cdot \mathbf{V}^n),\ \mathcal{Q}_k^*(\boldsymbol{\gamma}_\beta^* \cdot \mathbf{V}^{n,*}) \in D_{\beta^*}, \tag{4.7}$$

$0 \le n \le N$. At this point, we should prove the second order of accuracy of the cuadrature rule given in (3.10), which is straightforward. In order to demonstrate (4.7), we obtain, as $k \to 0$,

$$|\mathcal{Q}_k^*\big(\boldsymbol{\gamma}_\mu^* \cdot \mathbf{V}^n\big) - I_\mu^*(t^n)| \le |\mathcal{Q}_k^*\big(\boldsymbol{\gamma}_\mu^* \cdot \mathbf{V}^n\big) - \mathcal{Q}_k^*\big(\boldsymbol{\gamma}_\mu^* \cdot \mathbf{v}^n\big)| + |\mathcal{Q}_k^*\big(\boldsymbol{\gamma}_\mu^* \cdot \mathbf{v}^n\big) - I_\mu^*(t^n)| \le \|\gamma_\mu^*\|_\infty R_k + o(1), \quad 0 \le n \le N,$$

and a similar inequality holds for $\mathcal{Q}_k^*(\boldsymbol{\gamma}_\beta^* \cdot \mathbf{V}^n)$. In the case of $\mathcal{Q}_k^*(\boldsymbol{\gamma}_\mu^* \cdot \mathbf{V}^{n,*}) \in D_{\mu^*}$, we prove, with the use of the formula in (4.5), that

$$\begin{aligned}
|V_j^{n,*} - v_j^n| &\le \exp(K_\alpha k)\left|V_{j-1}^{n-1} \exp\Big(-k\,\mu^*\Big(x_{j-1}, \mathcal{Q}_k^*\big(\boldsymbol{\gamma}_\mu^* \cdot \mathbf{V}^{n-1}\big), t_{n-1}\Big)\Big) - v_{j-1}^{n-1} \exp\left(-\int_{t^{n-1}}^{t^n} \mu^*\Big(x\big(\tau; t^{n-1}, x_{j-1}\big), I_\mu^*(\tau), \tau\Big)\, d\tau\right)\right| \\
&\le \exp(K_\alpha k)\Big\{|V_{j-1}^{n-1} - v_{j-1}^{n-1}| \exp\Big(-k\,\mu^*\Big(x_{j-1}, \mathcal{Q}_k^*\big(\boldsymbol{\gamma}_\mu^* \cdot \mathbf{V}^{n-1}\big), t^{n-1}\Big)\Big) \\
&\quad + |v_{j-1}^{n-1}|\,|\exp\Big(-k\,\mu^*\Big(x_{j-1}, \mathcal{Q}_k^*\big(\boldsymbol{\gamma}_\mu^* \cdot \mathbf{V}^{n-1}\big), t^{n-1}\Big)\Big) - \exp\Big(-k\,\mu^*\Big(x_{j-1}, I_\mu^*\big(t^{n-1}\big), t^{n-1}\Big)\Big)| \\
&\quad + |v_{j-1}^{n-1}|\,|\exp\Big(-k\,\mu^*\Big(x_{j-1}, I_\mu^*\big(t^{n-1}\big), t^{n-1}\Big)\Big) - \exp\left(-\int_{t^{n-1}}^{t^n} \mu^*\Big(x\big(\tau; t^{n-1}, x_{j-1}\big), I_\mu^*(\tau), \tau\Big)\, d\tau\right)|\Big\} \\
&\le \exp(K_\alpha k)\,(R_k + o(1)),
\end{aligned}$$

and then

$$\begin{aligned}
|\mathcal{Q}_k^*\big(\boldsymbol{\gamma}_\mu^* \cdot \mathbf{V}^{n,*}\big) - I_\mu^*(t^n)| &\le |\mathcal{Q}_k^*\big(\boldsymbol{\gamma}_\mu^* \cdot \mathbf{V}^{n,*}\big) - \mathcal{Q}_k^*\big(\boldsymbol{\gamma}_\mu^* \cdot \mathbf{v}^n\big)| + |\mathcal{Q}_k^*\big(\boldsymbol{\gamma}_\mu^* \cdot \mathbf{v}^n\big) - I_\mu^*(t^n)| \\
&\le \|\gamma_\mu^*\|_\infty \exp(K_\alpha k)\,(R_k + o(1)) + o(1), \quad 0 \le n \le N,
\end{aligned}$$

and a similar inequality holds for $\mathcal{Q}_k^*(\boldsymbol{\gamma}_\beta^* \cdot \mathbf{V}^{n,*}) \in D_{\beta^*}$. Therefore, Eq. (4.7) is derived if $R_k = o(1)$, as $k \to 0$. From now on, C will denote a positive constant which is independent of k, n $(0 \le n \le N)$ and j $(0 \le j \le J)$; C has possibly different values in different places.

4.1. Consistency

We define the *local discretization error* as

$$\mathbf{l}_k = \mathbf{\Phi}_k(\mathbf{v}_k) \in \mathcal{Y}_k, \tag{4.8}$$

and we say that the discretization (4.1) is *consistent* if, as $k \to 0$,

$$\lim_{k\to 0} \| \mathbf{\Phi}_k(\mathbf{v}_k)\|_{\mathcal{Y}_k} = \lim_{k\to 0} \|\mathbf{l}_k\|_{\mathcal{Y}_k} = 0. \tag{4.9}$$

The next theorem establishes the consistency of the numerical scheme (3.6)–(3.10).

Theorem 2. *Assume hypotheses (H5)–(H9) on the functions data and the solution to (2.1)–(2.3). Then, as $k \to 0$, the local discretization error satisfies*

$$\| \mathbf{\Phi}_k(\mathbf{v}_k)\|_{\mathcal{Y}_k} = \|\mathbf{v}^0 - \mathbf{V}^0\|_\infty + O(k^2). \tag{4.10}$$

Proof. We begin with the auxiliary values. They correspond with a first order approximation at the next time level. We employ the first order accuracy of the rectangular quadrature rule, the second order accuracy of $\mathcal{Q}_k^*$, hypotheses (H5), (H8) and (H9), to arrive at

$$\begin{aligned} |v_{j+1}^{n+1} - v_{j+1}^{n+1,*}| &= |v_j^n|\, e^{K_\alpha k} \left| \exp\left(-\int_{t^n}^{t^{n+1}} \mu^*\big(x(\tau;t^n,x_j), I_\mu^*(\tau),\tau\big) d\tau \right) - \exp\left(-k\,\mu^*\big(x_j, \mathcal{Q}_k^*\big(\boldsymbol{\gamma}_\mu^*\cdot \mathbf{v}^n\big), t^n\big)\right) \right| \\ &\le C \left| \int_{t^n}^{t^{n+1}} \mu^*\big(x(\tau;t^n,x_j), I_\mu^*(\tau),\tau\big) d\tau - k\,\mu^*\big(x_j, I_\mu^*(t^n), t^n\big) \right| \\ &\quad + C\,k\,|\mu^*\big(x_j, I_\mu^*(t^n), t^n\big) - \mu^*\big(x_j, \mathcal{Q}_k^*\big(\boldsymbol{\gamma}_\mu^*\cdot \mathbf{v}^n\big), t^n\big)| \\ &\le C\,k^2 + C\,k\,|I_\mu^*(t^n) - \mathcal{Q}_k^*\big(\boldsymbol{\gamma}_\mu^*\cdot \mathbf{v}^n\big)| \\ &\le C\,k^2. \end{aligned} \tag{4.11}$$

Next, we need to bound the goodness of the approach when we combine $\mathcal{Q}_k^*$ with the auxiliary values. Then, the order of convergence of $\mathcal{Q}_k^*$, hypotheses (H5) and (H9), allow us to derive

$$\begin{aligned} |I_\mu^*\big(t^{n+1}\big) - \mathcal{Q}_k^*\big(\boldsymbol{\gamma}_\mu^*\cdot \mathbf{v}^{n+1,*}\big)| &\le |I_\mu^*\big(t^{n+1}\big) - \mathcal{Q}_k^*\big(\boldsymbol{\gamma}_\mu^*\cdot \mathbf{v}^{n+1}\big)| + |\mathcal{Q}_k^*\big(\boldsymbol{\gamma}_\mu^*\cdot \big(\mathbf{v}^{n+1} - \mathbf{v}^{n+1,*}\big)\big)| \\ &\le C\,k^2. \end{aligned} \tag{4.12}$$

Again, the second order of accuracy of the trapezoidal and $\mathcal{Q}_k^*$ rules, hypotheses (H5), (H8) and (H9), and (4.11) and (4.12) imply that

$$\begin{aligned} &\left| \int_{t^n}^{t^{n+1}} \mu^*(x(\tau;t^n,x_j), I_\mu^*(\tau),\tau) d\tau \; - \frac{k}{2} \left(\mu^*(x_j, \mathcal{Q}_k^*(\boldsymbol{\gamma}_\mu^*\cdot \mathbf{v}^n), t^n) + \mu^*(x_{j+1}, \mathcal{Q}_k^*(\boldsymbol{\gamma}_\mu^*\cdot \mathbf{v}^{n+1,*}), t^{n+1}) \right) \right| \\ &\quad \le \left| \int_{t^n}^{t^{n+1}} \mu^*(x(\tau;t^n,x_j), I_\mu^*(\tau),\tau) d\tau \; - \frac{k}{2} \left(\mu^*(x_j, I_\mu^*(t^n), t^n) + \mu^*(x_{j+1}, I_\mu^*(t^{n+1}), t^{n+1}) \right) \right| \\ &\qquad + \frac{k}{2} \left| \mu^*(x_j, I_\mu^*(t^n), t^n) - \mu^*(x_j, \mathcal{Q}_k^*(\boldsymbol{\gamma}_\mu^*\cdot \mathbf{v}^n), t^n) \right| \\ &\qquad + \frac{k}{2} \left| \mu^*(x_{j+1}, I_\mu^*(t^{n+1}), t^{n+1}) - \mu^*(x_{j+1}, \mathcal{Q}_k^*(\boldsymbol{\gamma}_\mu^*\cdot \mathbf{v}^{n+1,*}), t^{n+1}) \right| \\ &\quad \le C\,k^3 + C\,k \left| I_\mu^*(t^n) - \mathcal{Q}_k^*(\boldsymbol{\gamma}_\mu^*\cdot \mathbf{v}^n) \right| + C\,k \left| I_\mu^*(t^{n+1}) - \mathcal{Q}_k^*(\boldsymbol{\gamma}_\mu^*\cdot \mathbf{v}^{n+1,*}) \right| \\ &\quad \le C\,k^3. \end{aligned} \tag{4.13}$$

Now, we denote $\mathbf{\Phi}_k(\mathbf{v}_k) = (\mathbf{L}^0, \mathbf{L}_0, \mathbf{L}^1, \ldots, \mathbf{L}^N)$, thus (4.3), (H5) and (4.13) allow us to obtain,

$$\begin{aligned} |L_{j+1}^{n+1}| &= \frac{1}{k} \left| v_{j+1}^{n+1} - v_j^n \exp\left(-\frac{k}{2} \left(\mu^*(x_j, \mathcal{Q}_k^*(\boldsymbol{\gamma}_\mu^*\cdot \mathbf{v}^n), t^n) + \mu^*(x_{j+1}, \mathcal{Q}_k^*(\boldsymbol{\gamma}_\mu^*\cdot \mathbf{v}^{n+1,*}), t^{n+1}) - 2\,K_\alpha \right) \right) \right| \\ &= \frac{1}{k} |v_j^n|\, e^{K_\alpha k} \left| \exp\left(-\int_{t^n}^{t^{n+1}} \mu^*(x(\tau;t^n,x_j), I_\mu^*(\tau),\tau) d\tau \right) \right. \\ &\qquad \left. - \exp\left(-\frac{k}{2} \left(\mu^*(x_j, \mathcal{Q}_k^*(\boldsymbol{\gamma}_\mu^*\cdot \mathbf{v}^n), t^n) + \mu^*(x_{j+1}, \mathcal{Q}_k^*(\boldsymbol{\gamma}_\mu^*\cdot \mathbf{v}^{n+1,*}), t^{n+1}) \right) \right) \right| \\ &= O(k^2), \end{aligned} \tag{4.14}$$

$0 \le j \le J-1, \quad 0 \le n \le N-1$. Then, we take into account (4.4), the properties of $\mathcal{Q}_k^*$, hypotheses (H5), (H7) and (H9), to arrive at

$$\begin{aligned}
|L_0^n| &= |v_0^n - \mathcal{Q}_k^*(\boldsymbol{\beta}^{*,n}(\mathbf{v}^n)\cdot\mathbf{v}^n)/K_\alpha| \\
&= \frac{1}{K_\alpha}\left|\int_0^1 \beta^*(x, I_\beta^*(t_n), t_n)\, v(x,t_n)\, dx - \mathcal{Q}_k^*(\boldsymbol{\beta}^{*,n}(\mathbf{v}^n)\cdot\mathbf{v}^n)\right| \\
&\le \frac{1}{K_\alpha}\left|\int_0^1 \beta^*(x, I_\beta^*(t_n), t_n)\, v(x,t_n)\, dx - \mathcal{Q}_k^*(\boldsymbol{\beta}_I^{*,n}\cdot\mathbf{v}^n)\right| \\
&\quad + \frac{1}{K_\alpha}\left|\mathcal{Q}_k^*\big(\big(\boldsymbol{\beta}_I^{*,n} - \boldsymbol{\beta}^{*,n}(\mathbf{v}^n)\big)\cdot\mathbf{v}^n\big)\right| \\
&\le C\,k^2 + C\left|I_\beta^*(t^n) - \mathcal{Q}_k^*(\boldsymbol{\gamma}_\beta^*\cdot\mathbf{v}^n)\right| \\
&\le C\,k^2,
\end{aligned} \tag{4.15}$$

$1 \le n \le N$, where we have introduced the following auxiliar notation $(\boldsymbol{\beta}_I^{*,n})_j = \beta^*(x_j, I_\beta^*(t^n), t^n)$, $0 \le j \le J+1$. Thus, the combination of (4.14) and (4.15) proves (4.10). □

4.2. Stability

Another notion that plays an important role in the analysis of our numerical method is *stability*. For each $k \in H$, let R_k be a positive real number or $+\infty$ *(the stability threshold)*. We say that the discretization (4.1) is *stable* for $\mathbf{v}_k$ restricted to the thresholds R_k, if there exist two positive constants k_0 and S *(the stability constant)* such that, for any k in H with $k \le k_0$ the open ball $\mathcal{B}(\mathbf{v}_k, R_k)$ is contained in the domain of $\boldsymbol{\Phi}_k$ and for all $\mathbf{V}_k$, $\mathbf{W}_k \in \mathcal{B}(\mathbf{v}_k, R_k)$

$$\|\mathbf{V}_k - \mathbf{W}_k\|_{\mathcal{X}_k} \le S\,\|\boldsymbol{\Phi}_k(\mathbf{V}_k) - \boldsymbol{\Phi}_k(\mathbf{W}_k)\|_{\mathcal{Y}_k}.$$

Theorem 3. *Assume the hypotheses of Theorem 2 and let R_k be a positive constant such that (4.7) holds. Then the discretization (4.2)–(4.6) is stable for $\mathbf{v}_k$ with threshold $R_k = R\,k$.*

Proof. Let $(\mathbf{V}^0, \mathbf{V}^1, \dots, \mathbf{V}^N)$, $(\mathbf{W}^0, \mathbf{W}^1, \dots, \mathbf{W}^N)$ be in the ball $\mathcal{B}(\mathbf{v}_k, R_k)$ of the space $\mathcal{X}_k$. We set

$$\mathbf{E}^n = \mathbf{V}^n - \mathbf{W}^n \in \mathbb{R}^{J+1}, \quad 0 \le n \le N;$$

$$\boldsymbol{\Phi}_k(\mathbf{V}^0, \mathbf{V}^1, \dots, \mathbf{V}^N) = (\mathbf{Z}^0, \mathbf{Z}_0, \mathbf{Z}^1, \dots, \mathbf{Z}^N),$$

$$\boldsymbol{\Phi}_k(\mathbf{W}^0, \mathbf{W}^1, \dots, \mathbf{W}^N) = (\mathbf{S}^0, \mathbf{S}_0, \mathbf{S}^1, \dots, \mathbf{S}^N).$$

There exists a positive constant C such that, for k sufficiently small,

$$|\mathcal{Q}_k^*(\boldsymbol{\gamma}_\beta^*\cdot(\mathbf{V}^n - \mathbf{W}^n))| \le C\|\mathbf{V}^n - \mathbf{W}^n\|_{1,M+1}, \tag{4.16}$$

$$|\mathcal{Q}_k^*(\boldsymbol{\gamma}_\mu^*\cdot(\mathbf{V}^n - \mathbf{W}^n))| \le C\,\|\mathbf{V}^n - \mathbf{W}^n\|_{1,M+1}, \tag{4.17}$$

$$|\mathcal{Q}_k^*(\mathbf{V}^n\cdot\mathbf{W}^n)| \le C\|\mathbf{V}^n\|_{\infty J+1}\|\mathbf{W}^n\|_{1,M+1}, \tag{4.18}$$

$0 \le n \le N$. By (4.3), we can write

$$\begin{aligned}
E_{j+1}^{n+1} &= k\,(Z_{j+1}^{n+1} - S_{j+1}^{n+1}) + e^{K_\alpha k}\left(V_j^n \exp\left(-\frac{k}{2}\Big(\mu^*(x_j, \mathcal{Q}_k^*(\boldsymbol{\gamma}_\mu^*\cdot\mathbf{V}^n), t^n) + \mu^*(x_{j+1}, \mathcal{Q}_k^*(\boldsymbol{\gamma}_\mu^*\cdot\mathbf{V}^{n+1,*}), t^{n+1})\Big)\right)\right. \\
&\quad \left. -W_j^n \exp\left(-\frac{k}{2}\Big(\mu^*(x_j, \mathcal{Q}_k^*(\boldsymbol{\gamma}_\mu^*\cdot\mathbf{W}^n), t^n) + \mu^*(x_{j+1}, \mathcal{Q}_k^*(\boldsymbol{\gamma}_\mu^*\cdot\mathbf{W}^{n+1,*}), t^{n+1})\Big)\right)\right) \\
&= k\,(Z_{j+1}^{n+1} - S_{j+1}^{n+1}) + e^{K_\alpha k}\left\{E_j^n \exp\left(-\frac{k}{2}\Big(\mu^*(x_j, \mathcal{Q}_k^*(\boldsymbol{\gamma}_\mu^*\cdot\mathbf{V}^n), t^n) + \mu^*(x_{j+1}, \mathcal{Q}_k^*(\boldsymbol{\gamma}_\mu^*\cdot\mathbf{V}^{n+1,*}), t^{n+1})\Big)\right)\right. \\
&\quad +W_j^n\left(\exp\left(-\frac{k}{2}\Big(\mu^*(x_j, \mathcal{Q}_k^*(\boldsymbol{\gamma}_\mu^*\cdot\mathbf{V}^n), t^n) + \mu^*(x_{j+1}, \mathcal{Q}_k^*(\boldsymbol{\gamma}_\mu^*\cdot\mathbf{V}^{n+1,*}), t^{n+1})\Big)\right)\right. \\
&\quad \left.\left. - \exp\left(-\frac{k}{2}\Big(\mu^*(x_j, \mathcal{Q}_k^*(\boldsymbol{\gamma}_\mu^*\cdot\mathbf{W}^n), t^n) + \mu^*(x_{j+1}, \mathcal{Q}_k^*(\boldsymbol{\gamma}_\mu^*\cdot\mathbf{W}^{n+1,*}), t^{n+1})\Big)\right)\right)\right\},
\end{aligned} \tag{4.19}$$

$0 \le j \le J-1, \quad 0 \le n \le N-1$. Due to hypothesis (H8), we derive, for k sufficiently small,

$$\exp\left(-\frac{k}{2}\left(\mu^*(x_j, \mathcal{Q}_k^*(\boldsymbol{\gamma}_\mu^* \cdot \mathbf{V}^n), t^n) + \mu^*(x_{j+1}, \mathcal{Q}_k^*(\boldsymbol{\gamma}_\mu^* \cdot \mathbf{V}^{n+1,*}), t^{n+1})\right)\right) \le 1, \tag{4.20}$$

$$e^{K_\alpha k} \le 1 + Ck, \tag{4.21}$$

$0 \le j \le J-1, \quad 0 \le n \le N-1$. Now hypothesis (H8) and (4.17) and (4.18) allow us to obtain the following inequality

$$\begin{aligned}
&\Bigg|\exp\left(-\frac{k}{2}\left(\mu^*(x_j, \mathcal{Q}_k^*(\boldsymbol{\gamma}_\mu^* \cdot \mathbf{V}^n), t^n) + \mu^*(x_{j+1}, \mathcal{Q}_k^*(\boldsymbol{\gamma}_\mu^* \cdot \mathbf{V}^{n+1,*}), t^{n+1})\right)\right) \\
&\quad - \exp\left(-\frac{k}{2}\left(\mu^*(x_j, \mathcal{Q}_k^*(\boldsymbol{\gamma}_\mu^* \cdot \mathbf{W}^n), t^n) + \mu^*(x_{j+1}, \mathcal{Q}_k^*(\boldsymbol{\gamma}_\mu^* \cdot \mathbf{W}^{n+1,*}), t^{n+1})\right)\right)\Bigg| \\
&\le Ck\left(\left|\mu^*(x_j, \mathcal{Q}_k^*(\boldsymbol{\gamma}_\mu^* \cdot \mathbf{V}^n), t^n) - \mu^*(x_j, \mathcal{Q}_k^*(\boldsymbol{\gamma}_\mu^* \cdot \mathbf{W}^n), t^n)\right|\right. \\
&\quad \left.+\left|\mu^*(x_{j+1}, \mathcal{Q}_k^*(\boldsymbol{\gamma}_\mu^* \cdot \mathbf{V}^{n+1,*}), t^{n+1}) - \mu^*(x_{j+1}, \mathcal{Q}_k^*(\boldsymbol{\gamma}_\mu^* \cdot \mathbf{W}^{n+1,*}), t^{n+1})\right|\right) \\
&\le Ck\left(\left|\mathcal{Q}_k^*(\boldsymbol{\gamma}_\mu^* \cdot (\mathbf{V}^n - \mathbf{W}^n))\right| + \left|\mathcal{Q}_k^*\left(\boldsymbol{\gamma}_\mu^* \cdot (\mathbf{V}^{n+1,*} - \mathbf{W}^{n+1,*})\right)\right|\right) \\
&\le Ck\left(\|\mathbf{V}^n - \mathbf{W}^n\|_{1,M+1} + \|\mathbf{V}^{n+1,*} - \mathbf{W}^{n+1,*}\|_{1,M+1}\right),
\end{aligned} \tag{4.22}$$

$0 \le j \le J-1$, $0 \le n \le N-1$. Now, formula in (4.5) allows us to write

$$\begin{aligned}
V_{j+1}^{n+1,*} - W_{j+1}^{n+1,*} &= e^{K_\alpha k}\left(V_j^n \exp\left(-k\mu^*\left(x_j, \mathcal{Q}_k^*\left(\boldsymbol{\gamma}_\mu^* \cdot \mathbf{V}^n\right), t^n\right)\right) - W_j^n \exp\left(-k\mu^*\left(x_j, \mathcal{Q}_k^*\left(\boldsymbol{\gamma}_\mu^* \cdot \mathbf{W}^n\right), t^n\right)\right)\right) \\
&= e^{K_\alpha k}\left\{E_j^n \exp\left(-k\mu^*\left(x_j, \mathcal{Q}_k^*\left(\boldsymbol{\gamma}_\mu^* \cdot \mathbf{V}^n\right), t^n\right)\right) + W_j^n \left(\exp\left(-k\mu^*\left(x_j, \mathcal{Q}_k^*\left(\boldsymbol{\gamma}_\mu^* \cdot \mathbf{V}^n\right), t^n\right)\right)\right.\right. \\
&\quad \left.\left.- \exp\left(-k\mu^*\left(x_j, \mathcal{Q}_k^*\left(\boldsymbol{\gamma}_\mu^* \cdot \mathbf{W}^n\right), t^n\right)\right)\right)\right\},
\end{aligned} \tag{4.23}$$

$0 \le j \le J-1, \quad 0 \le n \le N-1$. Again, due to (H8), we derive, for k sufficiently small,

$$e^{K_\alpha k} \exp\left(-k\mu^*\left(x_j, \mathcal{Q}_k^*(\boldsymbol{\gamma}_\mu^* \cdot \mathbf{V}^n), t^n\right)\right) \le 1 + Ck, \tag{4.24}$$

$0 \le j \le J-1, \quad 0 \le n \le N-1$. We use similar arguments as in (4.22) to arrive at

$$|V_{j+1}^{n+1,*} - W_{j+1}^{n+1,*}| \le (1 + Ck)\,|E_j^n| + Ck\,\|\mathbf{E}^n\|_{1,M+1}, \tag{4.25}$$

$0 \le j \le J-1, \quad 0 \le n \le N-1$, and

$$\|\mathbf{V}^{n+1,*} - \mathbf{W}^{n+1,*}\|_{1,M+1} \le (1 + Ck)\,\|\mathbf{E}^n\|_{1,M+1}, \tag{4.26}$$

$0 \le n \le N-1$. Combining (4.19)–(4.26), we obtain, for k sufficiently small,

$$|E_{j+1}^{n+1}| \le (1 + Ck)|E_j^n| + Ck\,\|\mathbf{E}^n\|_{1,M+1} + k\,|Z_{j+1}^{n+1} - S_{j+1}^{n+1}|, \tag{4.27}$$

$0 \le j \le J-1, \quad 0 \le n \le N-1$.

Now, if $n \le j$, by a recursive argument

$$|E_{j+1}^{n+1}| \le C\,|E_{j-n}^0| + Ck\sum_{m=0}^{n} \|\mathbf{E}^m\|_{1,M+1} + Ck\sum_{m=0}^{n} |Z_{j+1-m}^{n+1-m} - S_{j+1-m}^{n+1-m}|, \tag{4.28}$$

and, when $N-1 \ge n > j$,

$$|E_{j+1}^{n+1}| \le C\,|E_0^{n-j}| + Ck\sum_{m=n-j}^{n} \|\mathbf{E}^m\|_{1,M+1} + Ck\sum_{m=0}^{j} |Z_{j+1-m}^{n+1-m} - S_{j+1-m}^{n+1-m}|. \tag{4.29}$$

Furthermore, Eq. (4.4), hypotheses (H7), (H9) and (4.16)–(4.18), allow us to achieve

$$\begin{aligned}
|E_0^{n+1}| &\le |Z_0^{n+1} - S_0^{n+1}| + \left|\left(\mathcal{Q}_k^*\left(\boldsymbol{\beta}^{*,n+1}\left(\mathbf{V}^{n+1}\right) \cdot \mathbf{V}^{n+1}\right) - \mathcal{Q}_k^*\left(\boldsymbol{\beta}^{*,n+1}\left(\mathbf{W}^{n+1}\right) \cdot \mathbf{W}^{n+1}\right)\right)/K_\alpha\right| \\
&\le |Z_0^{n+1} - S_0^{n+1}| + C\left|\mathcal{Q}_k^*\left(\boldsymbol{\beta}^{*,n+1}\left(\mathbf{V}^{n+1}\right) \cdot \mathbf{E}^{n+1}\right)\right| \\
&\quad + C\left|\mathcal{Q}_k^*\left(\left(\boldsymbol{\beta}^{*,n+1}\left(\mathbf{V}^{n+1}\right) - \boldsymbol{\beta}^{*,n+1}\left(\mathbf{W}^{n+1}\right)\right) \cdot \mathbf{W}^{n+1}\right)\right| \\
&\le |Z_0^{n+1} - S_0^{n+1}| + C\,\|\boldsymbol{\beta}^{*,n+1}\left(\mathbf{V}^{n+1}\right)\|_{\infty,J+1}\|\mathbf{E}^{n+1}\|_{1,M+1} \\
&\quad + C\,\|\left(\boldsymbol{\beta}^{*,n+1}\left(\mathbf{V}^{n+1}\right) - \boldsymbol{\beta}^{*,n+1}\left(\mathbf{W}^{n+1}\right)\right)\|_{\infty,J+1}\,\|\mathbf{W}^{n+1}\|_{1,M+1} \\
&\le |Z_0^{n+1} - S_0^{n+1}| + C\left(\|\mathbf{E}^{n+1}\|_{1,M+1} + \left|\mathcal{Q}_k^*\left(\boldsymbol{\gamma}_\beta^* \cdot \left(\mathbf{V}^{n+1} - \mathbf{W}^{n+1}\right)\right)\right|\right) \\
&\le |Z_0^{n+1} - S_0^{n+1}| + C\|\mathbf{E}^{n+1}\|_{1,M+1},
\end{aligned} \tag{4.30}$$

$0 \le n \le N-1$.

Next, multiplying $|E^n_{j_l}|$ by k and summing in l, $0 \le l \le M$, we derive by means of (4.27)–(4.30)

$$\begin{aligned}\|\mathbf{E}^n\|_{1,M+1} &= \sum_{j_l\ge n} k\,|E^n_{j_l}| + \sum_{0<j_l<n} k\,|E^n_{j_l}| + k\,|E^n_0| \le C\sum_{j_l\ge n} k\left\{|E^0_{j_l-n}| + k\sum_{m=0}^{n-1}\|\mathbf{E}^m\|_{1,M+1} + k\sum_{m=0}^{n-1}|Z^{n-m}_{j_l-m} - S^{n-m}_{j_l-m}|\right\}\\ &\quad + C\sum_{0<j_l<n} k\left\{|E^{n-j_l}_0| + k\sum_{m=n-j_l}^{n-1}\|\mathbf{E}^m\|_{1,M+1} + k\sum_{m=0}^{j_l-1}|Z^{n-m}_{j_l-m} - S^{n-m}_{j_l-m}|\right\} + k\left\{|Z^n_0 - S^n_0| + C\,\|\mathbf{E}^n\|_{1,M+1}\right\}\\ &\le C\left\{\|\mathbf{E}^0\|_{\infty J+1} + \sum_{m=0}^{n} k\,\|\mathbf{E}^m\|_{1,M+1} + \sum_{m=1}^{n} k\,\|\mathbf{Z}^m - \mathbf{S}^m\|_{\infty J} + \sum_{m=1}^{n} k\,|Z^m_0 - S^m_0|\right\},\end{aligned}$$

$1 \le n \le N$. Using the discrete Gronwall lemma, it follows that

$$\begin{aligned}\|\mathbf{E}^n\|_{1,M+1} &\le C\left\{\|\mathbf{E}^0\|_{\infty J+1} + \sum_{m=1}^{n} k\,|Z^m_0 - S^m_0| + \sum_{m=1}^{n} k\,\|\mathbf{Z}^m - \mathbf{S}^m\|_{\infty J}\right\}\\ &\le C\left\{\|\mathbf{E}^0\|_{\infty J+1} + \|\mathbf{Z}_0 - \mathbf{S}_0\|_{\infty,N} + \sum_{m=1}^{n} k\,\|\mathbf{Z}^m - \mathbf{S}^m\|_{\infty J}\right\}.\end{aligned} \tag{4.31}$$

Now, on the one hand, if $n \le N-1$, $n \le j \le J-1$, then by means of (4.28) and (4.31), we arrive at

$$|E^{n+1}_{j+1}| \le C\left\{\|\mathbf{E}^0\|_{\infty J+1} + \|\mathbf{Z}_0 - \mathbf{S}_0\|_{\infty,N} + \sum_{m=1}^{n+1} k\,\|\mathbf{Z}^m - \mathbf{S}^m\|_{\infty J}\right\}. \tag{4.32}$$

On the other hand, when $N-1 \ge n > j \ge 0$, taking into account (4.29) and (4.31) we obtain

$$|E^{n+1}_{j+1}| \le C\,|E^{n-j}_0| + C\left\{\|\mathbf{E}^0\|_{\infty J+1} + \|\mathbf{Z}_0 - \mathbf{S}_0\|_{\infty,N} + \sum_{m=1}^{n+1} k\,\|\mathbf{Z}^m - \mathbf{S}^m\|_{\infty J}\right\}. \tag{4.33}$$

And, finally, by (4.30) and (4.32) we derive for the boundary terms

$$|E^{n+1}_0| \le |Z^{n+1}_0 - S^{n+1}_0| + C\left\{\|\mathbf{E}^0\|_{\infty J+1} + \|\mathbf{Z}_0 - \mathbf{S}_0\|_{\infty,N} + \textstyle\sum_{m=1}^{n+1} k\,\|\mathbf{Z}^m - \mathbf{S}^m\|_{\infty J}\right\}, \tag{4.34}$$

$0 \le n \le N-1$. Thus by (4.32)–(4.34), we achieve

$$\|\mathbf{E}^n\|_{\infty} \le C\left\{\|\mathbf{E}^0\|_{\infty J+1} + \|\mathbf{Z}_0 - \mathbf{S}_0\|_{\infty,N} + \sum_{m=1}^{n} k\,\|\mathbf{Z}^m - \mathbf{S}^m\|_{\infty J}\right\},$$

$0 \le n \le N$. Therefore

$$\|(\mathbf{E}^0, \dots, \mathbf{E}^N)\|_{\mathcal{X}_h} \le C\,\|((\mathbf{V}-\mathbf{W})^0, (\mathbf{Z}-\mathbf{S})_0, (\mathbf{Z}-\mathbf{S})^1, \dots, (\mathbf{Z}-\mathbf{S})^N)\|_{\mathcal{Y}_h}.$$

□

4.3. Existence and convergence

We say that the discretization (4.1) is *convergent* if there exists $k_0 > 0$ such that for each k in H with $k \le k_0$, Eq. (4.6) has a solution $\mathbf{V}_k$ for which,

$$\lim_{k\to 0}\|\mathbf{v}_k - \mathbf{V}_k\|_{\mathcal{X}_k} = 0.$$

We define the *global discretization error* as

$$\mathbf{e}_k = \mathbf{v}_k - \mathbf{V}_k \in \mathcal{X}_k.$$

To derive the existence and convergence of numerical solutions of (3.6)–(3.10), we shall use a result of the general discretization framework introduced by López-Marcos et al. [13].

Theorem 4. *Assume that (4.1) is consistent and stable with thresholds R_k. If $\mathbf{\Phi}_k$ is continuous in $\mathcal{B}(\mathbf{v}_k, R_k)$ and $\|\mathbf{l}_k\|_{\mathcal{Y}_k} = o(R_k)$ as $k \to 0$, then:*

1. *for k sufficiently small, the discrete Eq. (4.6) possess a unique solution in $\mathcal{B}(\mathbf{v}_k, R_k)$,*
2. *as $k \to 0$, the solutions converge and $\|\mathbf{e}_k\|_{\mathcal{X}_k} = O(\|\mathbf{l}_k\|_{\mathcal{Y}_k})$.*

Now the existence and convergence are immediately obtained by means of the consistency (Theorem 2), the stability (Theorem 3) and Theorem 4. We emphasize that this theorem establishes the existence of a unique solution of the nonlinear system of equations for the approximation derived through the discretization of the problem. Therefore, the analysis can be extended even though the quadrature rule (3.10) were closed at $x_{j_0} = 0$, once we establish the consistency and stability properties of the new numerical scheme.

Theorem 5. *Under the hypotheses of Theorem 3, let the numerical initial condition* $\mathbf{V}^0$ *be such that*

$$\|\mathbf{V}_0 - \mathbf{v}_0\|_{\infty,J+1} = o(R_k),$$

as $k \to 0$. *Then, for* k *sufficiently small, there exists a unique solution*

$$(\mathbf{V}^0, \mathbf{V}^1, \ldots, \mathbf{V}^N)$$

in the ball $\mathcal{B}(\mathbf{v}_k, R)$ *of* $\mathcal{X}_k$, *of Eqs. (3.6)–(3.10) and*

$$\max_{0\le n\le N} \|\mathbf{V}^n - \mathbf{v}^n\|_{J+1,\infty} = O(\|\mathbf{V}^0 - \mathbf{v}^0\|_{J+1,\infty} + k^2).$$

Note that, in particular, if $\mathbf{V}^0$ is taken as the grid restriction $\mathbf{v}^0$ of the initial condition (2.3), then our scheme is second order accurate.

Now, we present the convergence theorem of the solution to the original age-structured model.

Theorem 6 (Convergence)**.** *Assume the hypotheses of Lemma 1 and let the numerical initial condition* $\mathbf{U}^0$ *be such that*

$$\max_{0\le j\le J} \left\{ \frac{\left|U_j^0 - u_0(a_j)\right|}{K_\alpha\, e^{-K_\alpha\, a_j}} \right\} = o(R_k).$$

Then, for sufficiently small k, the numerical method defined by (3.6)–(3.11) has a unique solution $(\mathbf{U}^0, \mathbf{U}^1, \ldots, \mathbf{U}^N)$ *and*

$$\max_{0\le n\le N} \left\{\|\mathbf{U}^n - \mathbf{u}^n\|_{\infty,J+1}\right\} = \mathcal{O}\left(\max_{0\le j\le J} \left\{ \frac{\left|U_j^0 - u_0(a_j)\right|}{K_\alpha\, e^{-K_\alpha\, a_j}} \right\} + k^2 \right), \tag{4.35}$$

where $\mathbf{u}^n = (u_0^n, u_1^n, \ldots, u_J^n) \in \mathbb{R}^{J+1}$, $u_j^n = u(a_j, t^n)$, $0 \le j \le J$, $0 \le n \le N$. *where u is the solution of (1.1)–(1.3).*

Proof. Lemma 1 ensures the smoothness properties (H5)–(H9) of the solution and the functions data of (2.1)–(2.3). Under the hypotheses in Theorems 3 and 5 concludes that the numerical solution of the size-structured associated problem satisfies,

$$\max_{0\le n\le N} \|\mathbf{V}^n - \mathbf{v}^n\|_{\infty,J+1} = O(\|\mathbf{V}^0 - \mathbf{v}^0\|_{\infty,J+1} + k^2). \tag{4.36}$$

Now, the relationship between the solution of the age-structured problem and the size-structured associated model $u(a,t) = K_\alpha\, e^{-K_\alpha\, a}\, v(1 - e^{-K_\alpha\, a}, t)$ and $U_j^n = K_\alpha\, e^{-K_\alpha\, a_j}\, V_j^n$, $0 \le j \le J$, $0 \le n \le N$, derives

$$U_j^n - u_j^n = K_\alpha\, e^{-K_\alpha\, a_j}\, (V_j^n - v_j^n), \quad 0 \le j \le J, \quad 0 \le n \le N.$$

Finally, Eq. (4.36) allow us to conclude with (4.35). □

The hypothesis that the initial condition has to satisfy in Theorem 6 means that the difference between the approximated and exact initial condition has to decay to zero faster than $(\alpha^{-1})'(a) = K_\alpha\, e^{-K_\alpha\, a}$, when a is large. This is immediately satisfied if u_0 has a compact support (which is usual in most real cases). Note that, in particular, if $\mathbf{U}^0$ is taken as the grid restriction $\mathbf{u}^0$ of the initial condition (1.3), then our scheme is second order accurate.

5. Numerical results

We have carried out numerical experiments with the scheme (3.7)–(3.11) in a theoretical test problem which presents meaningful nonlinearities.

Let $c \in \mathbb{R}^+$. We choose the age-dependent mortality rate as $\mu(a,z,t) = c^2\, z$, the age-specific birth rate as

$$\beta(a,z,t) = \frac{4\, c^3\, z\, a\, e^{-c\, a}}{(1 + c\, z)^2}\, \frac{(2 + e^{-c\, t})^2}{1 + e^{-c\, t}},$$

and the weight functions $\gamma_\mu(a) = \gamma_\beta(a) = 1$. Thus, the solution to (1.1)–(1.3) is

$$u(a,t) = \frac{e^{-c\, a}}{1 + e^{-c\, t}}.$$

The numerical integration of this problem is carried out in the time interval [0,10]. Since the exact solution is known, we are able to show quantitatively the efficiency of our numerical method.

Table 1
Theoretical experiment. Errors and numerical order, $c = 1$, $K_\alpha = 1$, $K_1 = 0.1$.

k	e_k	s
5e−2	8.403361e−3	
2.5e−2	2.210079e−3	1.93
1.25e−2	5.671056e−4	1.96
6.25e−3	1.436652e−4	1.98
3.125e−3	3.615675e−5	1.99
1.5625e−3	9.069523e−6	2.00
7.8125e−4	2.271191e−6	2.00
3.90625e−4	5.682751e−7	2.00
1.953125e−4	1.421285e−7	2.00
9.765625e−5	3.553959e−8	2.00

Table 2
Biological experiment. Development of Nicholson's blowflies model [16].

Class	Egg	Larva	Pupa	Inmature	Adult
Duration (days)	0.6	5.0	5.9	4.1	
Per capita egg production (egg/day)	0	0	0	0	$8.5\, e^{-z/600}$
Per capita death rate (day^{-1})	0.07	0.004	0.003	0.0025	0.27

In Table 1, we show the global error

$$e_k = \max_{0 \le n \le N} \|\mathbf{u}^n - \mathbf{U}^n\|_{\infty, J+1},$$

and the numerical order s as computed from

$$s = \frac{\log(e_{2k}/e_k)}{\log(2)}.$$

In the following experiment, we have employed as parameter values: $c = 1$, $K_\alpha = 1$, $K_1 = 0.1$. The subgrid has been generated, at the beginning of the procedure from the theoretical *natural grid*, as follows. We have considered the first node as $x_{j_0} = x_0 = 0$, and a new subgrid node has been selected from the *natural grid* (see Angulo and López-Marcos [12] for details) when finding a node whose difference with the last choice has been larger than $0.5 \max_{1 \le j \le J} \{x_{j+1} - x_j\}$. Finally, the last point x_{j_M} has been considered when $1 - x_{j_M} < 0.5 \max_{1 \le j \le J} \{x_{j+1} - x_j\} + K_1 k$, and $x_{j_{M+1}} = x_{J+1} = 1$.

The results in Table 1 confirm the expected second order of convergence. This case also exhibits that the hypotheses we need in the proof of the Theorem 6 are restrictive because K_α and σ do not satisfy the relationship $3 K_\alpha < \sigma$, therefore the numerical scheme behaves better than expected there.

The procedure introduced facilitates the simulation of more complex examples in which the interest is to discover the long time behaviour of the model. In the following, we will present an example which is based on a biologically more realistic experiment. We test the numerical method with the classical Nicholson's blowflies model [14] where population is divided into four life cycle stages. Our aim is to ascertain if the numerical solution exhibits the same qualitative and quantitative behaviour as suggested by the experimental field data collected in Nicholson [14]. The birth and death rates required as inputs were given in Sulsky [15]

$$\beta(a, z, t) = \begin{cases} 0, & a < \tau, \\ 8.5\, e^{-z/600}, & a \ge \tau, \end{cases} \tag{5.1}$$

$$\mu(a, z, t) = \begin{cases} 0.07, & a < 0.6, \\ 0.004, & 0.6 \le a < 5.6, \\ 0.003, & 5.6 \le a < 11.5, \\ 0.0025, & 11.5 \le a < \tau, \\ 0.27, & \tau \le a, \end{cases} \tag{5.2}$$

where the competition for resources is produced by the adult population $I_\beta(t) = \int_\tau^\infty u(a, t)\, da$, and τ represents the beginning of the mature adult stage. The value used in Sulsky [15], Gurney et al. [16] was $\tau = 15.6$. Main information data are taken from Gurney et al. [16] and listed in Table 2. This selection of vital rates does not accomplish the regularity requirements assumed in the convergence theorem, however the numerical experiments reported are in good agreement with the qualitative behaviour known from the biological model.

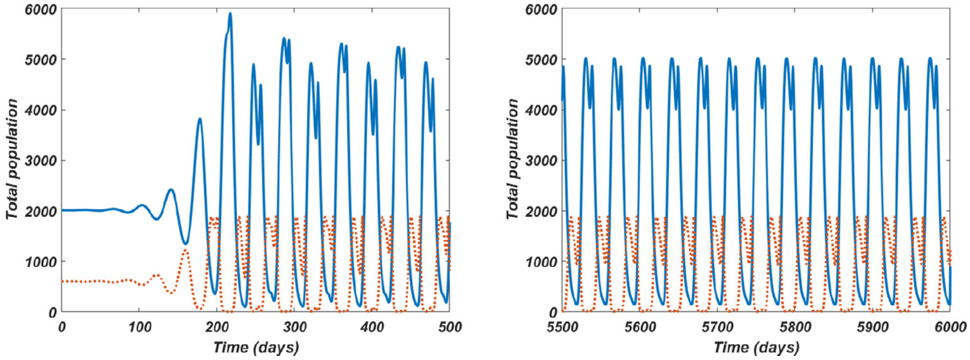

Fig. 1. Numerical simulation. Adult and egg production vs. time. Parameters: $k = 0.01$, $K_1 = 0.1$, $K_\alpha = 0.1$ and $\tau = 15.6$. Adult population ($I_\beta(t)$) solid (blue) line, egg production ($u(0,t)$) dotted (red) line. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

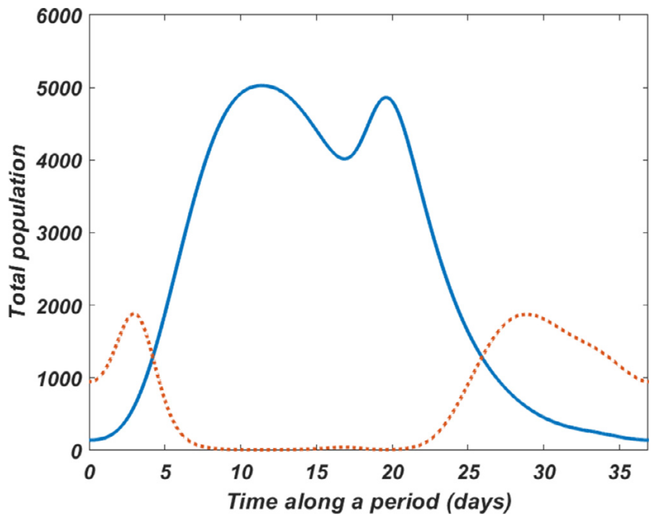

Fig. 2. Numerical simulation. Adult and egg production vs. time in one period. Parameters: $k = 0.01$, $K_1 = 0.1$, $K_\alpha = 0.1$ and $\tau = 15.6$. Adult population solid (blue) line, egg production dotted (red) line. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

We generate a numerical simulation with $k = 0.01$, $K_1 = 0.1$ and $K_\alpha = 0.1$. The nodes in the discretization are chosen to match the discontinuity points to minimize their impact. We employ the nontrivial steady state as the initial condition

$$u_0(a) = C \begin{cases} e^{-0.07\,a}, & a < 0.6, \\ e^{-0.004\,a-0.0396}, & 0.6 \le a < 5.6, \\ e^{-0.003\,a-0.0452}, & 5.6 \le a < 11.5, \\ e^{-0.0025\,a-0.05095}, & 11.5 \le a < \tau, \\ e^{-0.27\,a+0.2675\,\tau-0.05095}, & \tau \le a, \end{cases} \tag{5.3}$$

where $C = 162\,B\,\log\left(\frac{8.5}{0.27\,B}\right)$, $B = e^{0.0025\cdot\tau+0.05095}$, and the original parameter value for the maturation threshold is $\tau = 15.6$. In Fig. 1, we present the evolution of the adult and newborns populations with time. We observe the behaviour of the solution. In this case the equilibrium state is unstable which means that small changes in such a solution destabilize it. It happens quickly (after approximately 100 days). A periodical behaviour appears, as it was predicted in Sulsky [15], Gurney et al. [16], in which fluctuations must be due to the delay in the reproduction term. In Fig. 2 we show the evolution along a whole period of oscillation (the shape of the asymptotically stable periodic solution after the dynamics have been stabilized). The value of the period is computed numerically as 36.93 days with a maximum adult population of 5027 and a minimum of 130 (the period is in accordance with the experimental data which is 38 ± 1.5 days, although with lower population quantities, the maximum adult population is 7500 ± 500 and the minimum is 270 ± 120). The period appears to be halved with respect to the original works [15,16]. It must be due to an unsuitable numerical simulation made in such references that would not achieve the right dynamics completely. The behaviour of this dynamical system changes when the value of τ increases. The asymptotic stationary state is unstable for each applicable value of τ ($\tau > 11.5$) and the period of the stable

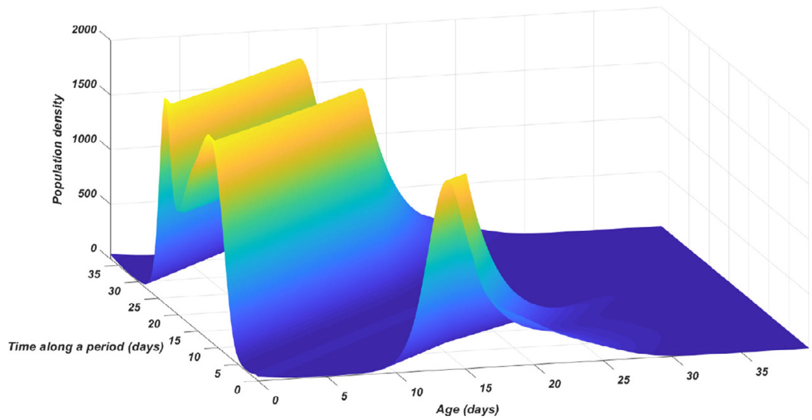

Fig. 3. Numerical simulation. Population density in one period. Parameters: $k = 0.01$, $K_1 = 0.1$, $K_\alpha = 0.1$ and $\tau = 15.6$.

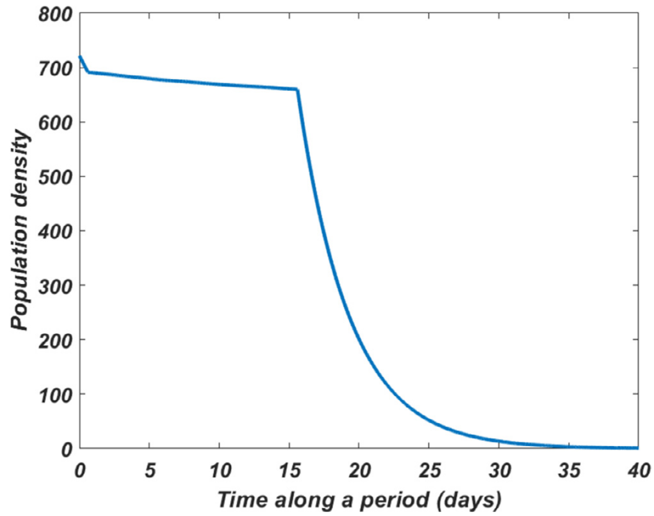

Fig. 4. Numerical simulation. Age distribution (average). Parameters: $k = 0.01$, $K_1 = 0.1$, $K_\alpha = 0.1$ and $\tau = 15.6$.

periodic orbit doubled at a bifurcation point that is very close to the key value, $\tau = 15.6$. Henceforth, it is necessary to be careful with the numerical simulation because it could be misinterpreted.

In addition, we also plot the evolution of the age-distribution in a period of fluctuation (Fig. 3) and the average density of the population (Fig. 4). This last figure shows the same shape as the histogram in Nicholson [14], however the quantities are larger (although the adult population size is similar 2439 vs. 2520 as in the Nicholson's report).

6. Conclusions

The selection of the age as a physiological parameter to structure a population and to describe its dynamics involves the election of the life-span. In this work we propose a numerical procedure to approximate the solution to an age-structured population model with an unbounded life-span. This is a problem which has been theoretically and numerically studied in the past and in addition new models have emerged with this setting [9].

We have completely analyzed a new numerical method with second order of convergence in which, for the first time, the unbounded age domain is not truncated previously. So far, the numerical integration of the model was performed in a finite integration time $[0, T]$ and an initial condition with bounded-domain $[0, A]$, that eventually becomes a bounded-domain problem $[0, A + T]$ which approaches the unbounded one. By using an artificial new independent variable, the numerical method proposed can be reinterpreted as a discretization of the model in a truncated age-interval whose length increases adaptively to infinity as the discretization parameter decreases to zero. The analysis establishes second order of convergence that is experimentally confirmed. Also, we show that the hypotheses in the analysis are too restrictive because the numerical method behaves well under lower regularity requirements on the vital functions.

The more interesting fact of this numerical approximation is that it can be used in a long time integration of the model. Thus the numerical method could be used to carry out a study on the existence of steady states and the stability of the equilibria and, when applicable, it allows us to obtain a representation of stable limit cycles. We have applied this approach to the Nicholson's blowflies model and we have obtained similar results as predicted by other authors, however, with a significant difference as the stable limit cycle approached is in accordance with original experimental values. This problem could have more interesting dynamics than shown in previous works which remain unexplored.

Finally, the numerical method proposed to study this model invites us to rethink and question other problems that originally we proposed with unbounded life-span but they were simplified with a suitable truncated age-interval [9,17].

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