Trabajo Fin de Máster

Máster en Física

Generalized Abelian-Higgs models:

Low energy dynamics in the moduli space

Autor: Sergio Navarro Obregón

Tutores: José M. Fernández Queiruga y Luis M. Nieto Calzada

Abstract

The present work aims to study the dynamics of a particular type of topological soliton called vortex. This solution is present in various theoretical models, depending on the application to be studied. We will specifically consider the Abelian-Higgs model. Taking this model as a basis, we will study various modifications of this model, either by adding new couplings or by including both, magnetic and Higgs impurities. We will compute the metric on a submanifold of the space of field configurations, called the *moduli space*, for each of these generalized models. Such a submanifold will play a crucial role in the study of the dynamics of solitons, since it will provide general properties of the dynamics without the use of full numerical simulations. As in all the previous cases only the translational degrees of freedom will be considered, we will finally qualitatively analyse the dynamics of the vortices in the standard Abelian-Higgs model when some of their internal degrees of freedom have been excited.

Resumen

En el presente trabajo se pretende estudiar la dinámica de un tipo particular de solitón topológico llamado vórtice. Dicha solución aparece en diversos modelos teóricos, dependiendo de la aplicación que se quiera estudiar. Nosotros consideraremos en concreto el modelo de Higgs abeliano. Tomando dicho modelo como base, estudiaremos diversas modificaciones del mismo, bien añadiendo nuevos acoplamientos, bien mediante la inclusión de impurezas, tanto magnéticas como de Higgs. Calcularemos la métrica de una subvariedad del espacio de configuraciones de campo, llamada *espacio de parámetros*, para cada uno de estos modelos generalizados. Tal subvariedad jugará un papel crucial en el estudio de la dinámica de los soltiones, debido a que aportará propiedades generales de la dinámica sin el uso exclusivo de simulaciones numéricas. Como en todos los casos anteriores solo los grados de libertad traslacionales serán considerados, analizaremos finalmente la dinámica de los vórtices en el modelo de Higgs abeliano estándar cuando alguno de sus grados de libertad internos ha sido excitado.

KEYWORDS: Topological solitons, BPS structure, Abelian-Higgs model, Supersymmetry, Higgs and magnetic impurities, Internal degrees of freedom

Contents

1 Introduction to topological solitons

Solitons were introduced into science after J. S. Russell [1] observed in 1834 a hump of water travelling several miles along the Union Canal near Edinburg, Scotland, without changing its shape and speed. That behaviour was characteristic of what we now call *solitary waves*, that is, localised non-singular solutions of a nonlinear field equation whose energy density moves undistorted with constant velocity [2]. When two such waves collided and their profiles were asymptotically restored to their original shapes and speeds, they were called *hydrodynamical solitons*. These definitions have been changing over time, and different classifications are currently given. In the present work, we are interested in the so-called *topological solitons*.

Topological solitons are defined as stable particle-like objects with finite energy and a smooth structure [3], or in more mathematical terms, they are solutions of partial differential equations of nonlinear field theories which are homotopically distinct from the vacuum solution, this being the characteristic that ensures its stability. The topological concepts necessary to understand this topic will be carefully explained in Section 2.1. The main examples of topological solitons are kinks in one dimension (Section 3.1), lumps and vortices in two dimensions (Section 3.2), monopoles and Skyrmions in three dimensions (Section 3.3), as well as instantons in four dimensions. Topological solitons have been employed in a wide variety of disciplines and areas of Physics, such as Condensed matter [4], Cosmology [5], Particle physics [6], Optics, [7] or even Biophysics [8], which confirm the relevance that solitons have been gained in these last decades.

There are two fundamental aspects that must be taken into account when looking for models that can support topological solitons. The first one is related to spontaneous symmetry breaking. For a model to support topological solitons, it needs to possess a structure that allows the symmetry of the theory to be spontaneously broken. Only for this case the model can conceive non-trivial solutions, i.e., different from the vacuum. This topic will be covered in detail in Section 2.3. The second aspect is related to the existence of stable finite energy soliton solutions under a spatial rescaling. Such scaling argument is due to Derrick [11] in 1964. It states that, given a certain field theory where $\boldsymbol{\phi}(\mathbf{x})$ is a finite energy field configuration distinct from vacuum, which is rescaled to $\boldsymbol{\phi}^{(\mu)}(\mathbf{x})$ by applying the map $\mathbf{x} \mapsto \mu\mathbf{x}$, if its associated energy $E\left(\boldsymbol{\phi}^{(\mu)}(\mathbf{x})\right)$ has no stationary points, the corresponding theory does not admit a finite energy static solution of the field equation other than the vacuum. This is related to the fact that topological solitons are stable by definition, so specifically must be stable under spatial scaling, and have a characteristic size.

Interestingly, there exist certain field theories in which the energy enjoys a bound from below that only depends on topological data. This was first noted in Prasad and Sommerfeld's work on the 't Hooft-Polyakov monopole [10] in 1975. A detailed analysis of these phenomenon was accomplished by Bogomolny [9] in 1976. He proposed that the Euler-Lagrange equations of certain field theories can be reduced from second order to first order partial differential equations, provided that the coupling constants take particular values. They consist on first integrals of the Euler-Lagrange equations, and they are equivalent to the zero pressure sector. Then, for such a value, the static intersoliton forces vanish, and the field configuration is an absolute minimum of the energy (within its *topological sector*). Therefore, the solutions of these equations saturate the lower bound of the energy. This is the so-called *BPS bound*, and the first order partial differential equations are referred to as *Bogomolny equations*. When in addition the field theory admits topological solitons in virtue of the two previous arguments, solitons are said to be BPS solitons. All the examples of topological solitons exposed in this work are BPS solitons.

The structure of the present work will be the following: first, the mathematical background of soliton theories is discussed in Section 2, which will be necessary to understand the underlying ideas behind these theories. Section 3 is devoted to the construction of soliton solutions in various models, describing the more relevant issues concerning to each of them. In Section 4 some notions of supersymmetry will be introduced, due to its connection to the study of BPS models. Sections 5 and 6 constitute the original part of the present work, and are orientated to analyse modifications and generalizations of the Abelian-Higgs model, where new couplings and impurities will be included into the Lagrangian. The purpose is to obtain a more general framework to explore the metric of the respective moduli spaces, with the intention of extracting information about the soliton dynamics in the physical situations that these models describe. The significance of this study is that we can gather analytic information about the dynamics of vortices without performing full numerical simulations, which provides mainly phenomenological information. Finally, the conclusions of the work are collected in Section 7.

2 Topological protection and effective dynamics

Section 2.1 aims to give a brief introduction to topology through a formal discussion, focusing on the most significant results necessary to understand the idea of homotopy group. Likewise, an introductory discussion on differential geometry will be covered in Section 2.2. All these results will allow us to accurately connect these ideas with the study of the stability of solitons in Section 2.3, as well as with the treatment of their dynamics through an effective theory exposed in Section 2.4. For this reason, we belief that for a reader unfamiliar with these issues, these two first subsections could be a compilation guide, which would allow him to better grasp the mathematical foundations of our arguments during the present work.

2.1 Topology in Physics

To begin with our discussion, let us start by presenting the concept of topological space.

Definition 2.1.1 *Let X be any set and $\tau = \{U_i | i \in I\}$ denote a certain collection of subsets of X called open sets. The pair (X, τ) is a topological space with τ a topology if τ satisfies the following requirements:*

- $\varnothing, X \in \tau$.
- *If J is any subcollection of I, the family $\{U_j | j \in J\}$ satisfies $\bigcup_{j \in J} U_j \in \tau$.*
- *If K is any finite subcollection of I, the family $\{U_k | k \in K\}$ satisfies $\bigcap_{k \in K} U_k \in \tau$.*

Therewith, we will enunciate the notion of continuity in this more general context.

Definition 2.1.2 *Let us suppose that τ is a topology on X. N is a neighbourhood of a point $x \in X$ if N is a subset of X and N contains some open set U_i to which x belongs.*

Definition 2.1.3 *Let X and Y be topological spaces. A map $\phi : X \to Y$ is continuous at $x_0 \in X$ if for every neighbourhood N of $\phi(x_0)$, $\phi^{-1}(N)$ is a neighbourhood of x_0.*

A map ϕ is said to be continuous in X if ϕ is continuous at every $x_0 \in X$.

Such definition allows us to present the idea of homeomorphism, which intuitively can be understood as two topological spaces being homeomorphic to each other if we can deform one into the other continuously.

Definition 2.1.4 *Let X and Y be topological spaces. Then, a map $\phi : X \to Y$ is a homeomorphism if it is continuous and has an inverse $\phi^{-1} : Y \to X$ which is also continuous. Then, X and Y are said to be homeomorphic.*

Definition 2.1.5 *A property is said to be a topological invariant if it is conserved under a homeomorphism.*

The following definitions will provide the necessary tools to define a manifold.

Definition 2.1.6 *A topological space X is a Hausdorff space if, for an arbitrary pair of distinct $x, x' \in X$, there always exist neighbourhoods N_x of x and $N_{x'}$ of x' such that $N_x \cap N_{x'} = \varnothing$.*

Definition 2.1.7 *A topological space X is second-countable if there exists some countable collection $\mathcal{U} = \{U_i\}_{i=1}^{\infty}$ of open subsets of X such that any open subset of X can be written as a union of elements of some subfamily of $\mathcal{U}$.*

Definition 2.1.8 *A n-dimensional manifold is a second countable Hausdorff space with the property that each point has a neighbourhood that is homeomorphic to an open subset of n-dimensional Euclidean space.*

All the above definitions are essential to introduce the concept of homotopy between two manifolds, which will play a crucial role in our explanation of soliton stability.

Definition 2.1.9 *Let X and Y be two manifolds and ϕ a continuous map between them $\phi : X \to Y$. Let us fix two points $x_0 \in X$ and $y_0 \in Y$, and let us impose that $\phi(x_0) = y_0$. The points x_0 and y_0 are said to be base points, and ϕ is said to be a based map.*

A based map $\phi_1 : X \to Y$ is said to be homotopic to another map ϕ_2 if there is a continuous map

$$\Psi : X \times [0,1] \to Y, \qquad t \in [0,1] \, ,$$

such that $\psi|_{t=0} = \phi_1$ and $\psi|_{t=1} = \phi_2$, along with $\Psi(x_0;t) = y_0 \; \forall t$.

Proposition 2.1.1 *The homotopy is an equivalence relation, and thus all the maps ϕ from $X \to Y$ can be classified into homotopy classes.*

Definition 2.1.10 *The set of homotopy classes of based maps $\phi : \mathbb{S}^n \to Y$, with $n \geq 1$, forms a group called n-th group of homotopy and it is denoted by $\pi_n(Y)$.*

Let $*$ and $\bullet$ be rules endowed on the sets X and Y respectively, inducing in them a certain algebraic structure.

Definition 2.1.11 *A map $\phi : (X, *) \to (Y, \bullet)$ is said to be a homomorphism if*

$$\phi(x_1 * x_2) = \phi(x_1) \bullet \phi(x_2), \qquad \forall x_1, x_2 \in X \; .$$

An important case of homomorphism is the so-called isomorphism, which is a bijective homomorphism. In that case X is said to be isomorphic to Y, and we will denote it by $X \cong Y$.

Theorem 2.1.1 *The n-th homotopy group of the n-dimensional sphere $\mathbb{S}^n$ is isomorphic to $\mathbb{Z}$, i.e.,*

$$\pi_n(\mathbb{S}^n) \cong \mathbb{Z} \ .$$

Therefore, the homotopy classes of the n-th group of homotopy of $\mathbb{S}^n$ are labelled by the integers.

With all this, we only need to introduce the definition of topological degree of a map to show the reason for the stability of solitons in scalar field theories. Nevertheless, first we have to give some fundamental notions of differential geometry to achieve this.

2.2 Differential geometry in Physics

In the above section, we presented the concept of manifold. Nonetheless, we are interested in a precise type of manifold named as *differentiable manifolds*. These differentiable manifolds attract our attention due to their relation with the analysis of the geometrical structure of the so-called *moduli spaces*, or due to their relationship with the definition of the topological degree of a map. Then, we will show the examples that will be useful for our later study. However, we want to emphasise that a detailed definition of all the ideas that will be exposed would need of a great amount of technicalities. Then, only the more relevant ideas will me discussed here. For more information, see [13].

We first present some definitions, and then we will introduce the notion of differentiable structure, which is the necessary property to describe a differentiable manifold.

Definition 2.2.1 *Let $\mathcal{M}$ be a manifold. A local chart on $\mathcal{M}$ is the pair (V, φ) where $V \subset \mathcal{M}$ and φ is a mapping $\varphi : V \to \varphi(V)$ such that $\varphi(V)$ is an open set of $\mathbb{R}^n$ and the mapping is bijective.*

Definition 2.2.2 *Let (V, φ) and $(\widetilde{V}, \phi)$ be two local charts on $\mathcal{M}$ such that $V \cap \widetilde{V} \neq \varnothing$. We will say that the local charts are compatible if*

$$\varphi \circ \phi^{-1} : \phi(V \cap \widetilde{V}) \to \varphi(V \cap \widetilde{V}) \ \textit{is bijective and} \ \mathscr{C}^\infty \ \textit{as a fuction of} \ \mathbb{R}^n \to \mathbb{R}^n.$$

By $\mathscr{C}^\infty$ we denote the infinitely differentiability of the functions.

Definition 2.2.3 *Let $\mathcal{M}$ be a manifold. An atlas on $\mathcal{M}$ is a set $\mathcal{A} = \{V_\alpha, \varphi_\alpha\}$ of local charts with $\alpha \in I$ that fulfil*

- $\bigcup_\alpha V_\alpha = \mathcal{M}$.
- *The local charts are compatible two by two.*

Definition 2.2.4 *Let $\mathcal{A} = \{V_\alpha, \varphi_\alpha\}$ and $\mathcal{A}' = \{\widetilde{V}_\beta, \phi_\beta\}$ be two atlases on $\mathcal{M}$. It is said that two atlases are equivalent if $\mathcal{A} \cup \mathcal{A}'$ is a new atlas. The equivalence among atlases define an equivalence relation.*

Definition 2.2.5 *A differentiable structure $\mathcal{S}$ on $\mathcal{M}$ is the union atlas of all the atlases of a given equivalence class.*

Observation 2.2.1 *When the charts of the atlases consist of holomorphic functions with image on open sets of $\mathbb{C}^n$ instead of $\mathbb{R}^n$, we can similarly define what is called a complex differentiable structure.*

Through this concept we can finally define the notion of differentiable manifold.

Definition 2.2.6 *A differentiable manifold is a pair $(\mathcal{M}, \mathcal{S})$ with $\mathcal{M}$ a manifold and $\mathcal{S}$ a differentiable structure.*

We can endow a differentiable manifold with other structures. One of the most common examples is the Riemannian manifold.

Definition 2.2.7 *A differentiable manifold endowed with a positive definite inner product on each of the individual tangent spaces is called a Riemannian manifold.*

However, there are others less popular structures, but with a hight interest in some mathematical and physical applications. An interesting one is the *Kähler manifold*, that will appear in many occasions along the present work.

Definition 2.2.8 *A s-form ω in a vectorial space V is a tensor s-times covariant that is antisymmetric in all its components. When a differentiable manifold $\mathcal{M}$ is considered, at each point of $\mathcal{M}$ the s-forms are referred to as differentiable s-forms.*

Definition 2.2.9 *A differentiable p-form ω is said to be closed if $d\omega = 0$.*

Definition 2.2.10 *A differentiable manifold $\mathcal{M}$ has a symplectic structure if it is equipped with a closed nondegenerate differential 2-form ω.*

Definition 2.2.11 *A Kähler manifold is a differentiable manifold $\mathcal{M}$ with three mutually compatible structures: a Riemannian structure, a complex differentiable structure, and a symplectic structure.*

Observation 2.2.2 *The relation between the Kähler metric and the Kähler form is through the complex structure: $g(\mathcal{S}X, Y) = \omega(X, Y)$. If $(\mathcal{M}, \omega)$ is Kähler, then about every point $p \in \mathcal{M}$ there exists a neighbourhood N_p and a function $\mathcal{K} \in \mathscr{C}^\infty(N_p, \mathbb{R})$ such that $\omega|_{N_p} = i\partial\overline{\partial}\mathcal{K}$ with $\partial = \sum \frac{\partial}{\partial z_k} dz_k$ and $\overline{\partial} = \sum \frac{\partial}{\partial \bar{z}_k} d\bar{z}_k$. Here $\mathcal{K}$ is called a local Kähler potential.*

To integrate over a differentiable manifolds, it must be orientable, for which the following concept is essential.

Definition 2.2.12 *Let $\mathcal{M}$ be a m-dimensional differentiable manifold. A volume form on $\mathcal{M}$, Ω, is a differential m-form that satisfies*

$$\Omega(p) \neq 0, \qquad \forall p \in \mathcal{M}.$$

A differentiable manifold $\mathcal{M}$ will be orientable if there exists a volume form on it.

Finally, we are prepared to give the definition of topological degree of a map.

Definition 2.2.13 *Let us regard ϕ as a differentiable map between two compact and without boundary orientable differentiable manifolds $\mathcal{M}$ and $\mathcal{N}$, where the volume form Ω defined on $\mathcal{N}$ is normalized to unity. Now consider $\phi^*(\Omega)$ the pull-back of Ω to $\mathcal{M}$ using the map ϕ. Then, we define*

$$deg\ \phi = \int_{\mathcal{M}} \phi^*(\Omega)$$

as the topological degree of the map ϕ.

The most important example is the map $\phi : \mathbb{S}^n \to \mathbb{S}^n$, that is, a map belonging to $\pi_n(\mathbb{S}^n)$. If we suppose that ϕ is in the k-th homotopy class of $\pi_n(\mathbb{S}^n)$, then deg $\phi = k$. When $n = 1$, we talk about winding number of a map. Naively, we could regard k as the number of times that the domain manifold wraps around the range manifold under the mapping.

Proposition 2.2.1 *If two maps $\phi_1, \phi_2 : X \to Y$ are homotopic, then deg ϕ_1 = deg ϕ_2.*

2.3 Stability of solitons

The stability of the topological solitons requires of the combination of two considerations. The first one is the topological structure of the model and the second one is related to the definition of topological soliton. Recall that we define a topological soliton as a stable particle-like object with finite energy and a smooth structure. As a result, the finiteness of the energy must play an important role in its classification.

For now, suppose that we are considering scalar field theories. Let $\phi : \mathbb{R}^d \to Y$ be a static field configuration. We first have to distinguish between a linear and a nonlinear theory in this context: a linear theory is one in which the kinetic term is quadratic in the Lagrangian, whereas a nonlinear theory is one in which it is not. Soliton solutions in nonlinear theories are known as *textures*. Derrick's argument implies that if a potential term is not present in a linear theory, then the theory does not support stable soliton solutions of finite energy. Thereby, let $U(\phi_1, \cdots, \phi_n)$ be a definite positive potential. To guarantee energy finitude, the field must take values in the vacuum manifold $\mathcal{V}$ at spatial infinity, which is defined as the subset of field values for which the potential function $U(\phi)$ from the theory takes its minimum value. We can just focus on the field configuration ϕ at spatial infinity, i.e. the map defined as $\phi^\infty : \mathbb{S}^{d-1}_\infty \longrightarrow \mathcal{V}$, where both have the same topological character given by $\pi_{d-1}(\mathcal{V})$. Figure 2.1 illustrates this map in the trivial case of $2+1$ dimensions and the famous Mexican hat potential, where $\mathcal{V} = U(1)$. Thus, if we consider two field configurations ϕ and ϕ_{vac} whose topological character is given by the associated fields ϕ^∞ and ϕ^∞_{vac}, in case the degrees of the maps are different, the homotopy classes will differ due to *Proposition* 2.2.1. This ensures the stability of the soliton solutions because it is not possible to continuously transform a soliton solution into the vacuum solution, and temporal evolution is an example of continuous map. However, sometimes this is not enough to guarantee the existence of stable finite energy solutions different from the vacuum, as it happens with global vortices, where it is necessary to couple the scalar field to a gauge field, or as it happens with lumps, where it is necessary to introduce higher derivatives of the scalar field, to avoid the Derrick's theorem. These comments will make more sense in Sections 3.1 and 3.2.2, where we will analyse these topics in more detail.

When we consider nonlinear theories, Derrick's argument shows that a potential is not necessary to guarantee the stability of solitons, although it could be introduced. Instead, to ensure the finiteness of the energy, we only need to impose that the field configuration must tend to an arbitrary constant value y_0 at spatial infinity. Therefore, this base point allows us, in terms of the one-point compactification, to regard the field configurations as based maps $\boldsymbol{\phi} : \mathbb{S}^d \longrightarrow Y$. The topological character of that configuration ϕ is determined by an element of the homotopy group $\pi_d(Y)$. When the homotopy group of the theory is non-trivial for a given manifold Y, arguing again as in *Proposition* 2.2.1, the same conclusion as above are obtained.

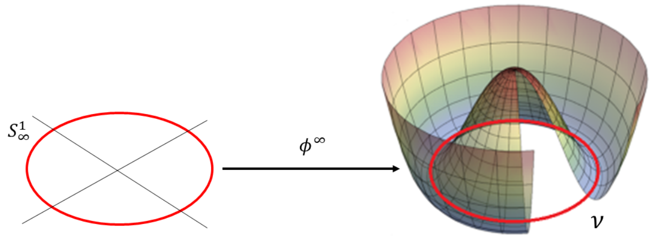

Figure 2.1: Plot of ϕ^∞ between the the circle of infinity $\mathbb{S}^1_\infty$ and the vacuum manifold $\mathcal{V} \cong \mathbb{S}^1$.

In both theories we can note the importance of the existence of spontaneous symmetry breaking to ensure the presence of topological solitons. We need multiple vacua in linear theories so that the vacuum manifold is non-trivial and that the field acquires an arbitrary constant value at infinity in nonlinear theories.

Suppose now a gauge field theory. In this case, the classification of solitons may be associated to another topological invariant different from the degree of a map; this is the so-called *Chern form*. The Chern forms are gauge-invariant differential forms of even power constructed algebraically from the strength field tensor of the theory. Depending on whether the theory is Abelian or non-Abelian, the strength field is defined differently. In an Abelian gauge theory, the strength field tensor is defined as

$$f = da = \sum_{i<j} (\partial_i a_j - \partial_j a_i) dx^i \wedge dx^j, \tag{2.1}$$

whilst in a non-Abelian gauge theory is defined as

$$F = dA + A \wedge A = \sum_{i<j} (\partial_i A_j - \partial_j A_i + [A_i, A_j]) dx^i \wedge dx^j, \tag{2.2}$$

with a and A the respective gauge fields. When the Chern forms are integrated over the manifold $\mathcal{M}$ in which they are defined, the so-called *Chern number* is obtained.

To end this section, let us explain the concept of *topological charge*. The topological charge is any conserved quantity associated with a solution that is conserved independently of the equations of motion (i.e., it is not a conserved Noether charge). It divides the configuration space into sectors named as *topological sectors*. In different models and different dimensions the topological charge is defined by different topological invariants. Thereby, when we talk about topological charge throughout the text, we will be referring to quantities such as the topological degree of a map, the winding number, or the Chern number, depending on the type of theory we are dealing with at that moment. From now on, we will use the term topological charge indistinctly.

2.4 Collective coordinates method: Moduli space dynamics

As it is well known, the fundamental contrast between field dynamics and mechanics is that when we deal with fields, they have an infinite number of degrees of freedom, whereas particles have a finite number of them. For simplicity, it would be interesting to obtain a finite-dimensional approach to field dynamics and have an effective dynamical theory [14]. In this approximation, the topological soliton would be expressed in terms of a few coordinates that are promoted to time-dependent parameters, and that are referred to as *collective coordinates or moduli*, which in certain low speed situations (adiabatic limit motion) could be a good approximation to the soliton dynamics of the full theory [15]. This method is widely known as *collective coordinate method*, and the dynamics is commonly known as *moduli space dynamics*.

The moduli space $\mathcal{M}_N$ is a submanifold of the infinite-dimensional field configuration space $\mathcal{C}$ in the sector of topological charge N. The moduli space is generally curved, and such curvature can be determined by restricting the kinetic term of the full field theory to the moduli space, which provides a positive definite metric. The other term in the moduli space is a potential inherited from the remaining term of the full Lagrangian, which induces static forces that modify the motion in the moduli space. Something remarkable happens when we contemplate a theory with BPS solitons. In that situation, let us consider as initial condition a slow motion tangent to $\mathcal{M}_N$. By conservation of energy, the trajectory of the the system will be constrained by V to lie close to $\mathcal{M}_N$, with V remaining approximately constant and with no direct effect. Therefore, the subsequent motion is geodesic on the manifold and, as a result, fully codified by the metric in $\mathcal{M}_N$. Such geodesic motion is only related to the effect of the curvature of the moduli space. A crucial point is that the moduli space should be smoothly embedded in the field configuration space so that the motion smoothly approximates the true field dynamics [19].

When we list the diverse types of solitons in the next section, we will provide a brief summary of their respective moduli spaces using some of the definitions presented in Section 2.2, and the variety of different manifolds will become visible.

3 Topological solitons in different dimensions

The purpose of this section is to explore some of the best-known theories that allow topological solitons in different spatial dimensions, as well as the context in which each of them appears. In the first place, we will introduce the theory under study with an analysis that will reveal us the presence or not of soliton solutions, followed by the search for their corresponding Bogomolny bound and equations. In addition, we will provide the main features of their solutions. Finally, some of the ideas related to the moduli spaces for each soliton will be mentioned. With the different techniques provided through this section, one could acquire the necessary foundations to study more involved theories. The analysis will cover the kink in 1-spatial dimension (Section 3.1), lumps and vortices in 2-spatial dimensions (Section 3.2), and finally Skyrmions and monopoles in 3-spatial dimensions (Section 3.3).

3.1 Solitons in one spatial dimension: Kinks

Assume the usual Lagrangian density in a $(1 + d)$ scalar field theory with only a real scalar field $\vec{\phi}(t, x_0, x_1, \ldots, x_d)$

$$\mathcal{L} = \frac{1}{2}\partial_\mu\vec{\phi} \cdot \partial^\mu\vec{\phi} - U(\vec{\phi}), \tag{3.1}$$

where $\vec{\phi}:\mathbb{R}^{1,d}\to\mathbb{R}^n$ and $U(\vec{\phi})$ is a potential which we will consider positive definite. The Euler-Lagrange field equations that follow from the Lagrangian density above are

$$\partial_\mu\partial^\mu\phi^a+\frac{\partial U}{\partial\phi^a}=0,\qquad a=1,2,...,n. \tag{3.2}$$

The energy functional

$$E[\vec{\phi}]=\int_{\mathbb{R}^d}\left(\frac{1}{2}\partial_0\vec{\phi}\cdot\partial_0\vec{\phi}+\frac{1}{2}\nabla\vec{\phi}\cdot\nabla\vec{\phi}+U(\vec{\phi})\right)dx^d \tag{3.3}$$

can be split into a kinetic and potential term as follows

$$T=\frac{1}{2}\int_{\mathbb{R}^d}\partial_0\vec{\phi}\cdot\partial_0\vec{\phi}\;dx^d,\qquad V=\int_{\mathbb{R}^d}\left(\frac{1}{2}\nabla\vec{\phi}\cdot\nabla\vec{\phi}+U(\vec{\phi})\right)dx^d\;. \tag{3.4}$$

We know that a field configuration, to have finite energy, requires that the energy drops to zero fast enough as $\mathbf{x}=(x_1,\ldots,x_d)$ approaches spatial infinite, so the field must necessarily reach a minimum of $U(\vec{\phi})$. The vacua of the theory is formed by the constant solutions at the minima of the potential. However, the field may also interpolate between different vacua, and this gives rise different topological sectors that are characterized by the behavior of the field at spatial infinity.

We must first check whether the theory admits soliton solutions or not, for which we will use the argument exposed in Section 1 based on Derrick's theorem. Let us consider that $\vec{\phi}$ is a static field configuration of (3.2). Applying a rescaling map $\mathbf{x}\mapsto\widetilde{\mathbf{x}}=\mu\mathbf{x}$ the energy becomes

$$e[\mu]=\int_{\mathbb{R}^d}\frac{d\widetilde{x}^d}{\mu^d}\left(\frac{1}{2}\mu^2\widetilde{\nabla}\vec{\phi}(\mu\mathbf{x})\cdot\widetilde{\nabla}\vec{\phi}(\mu\mathbf{x})+U(\vec{\phi}(\mu\mathbf{x}))\right)=\mu^{2-d}E_2+\mu^{-d}E_0\;,\quad\widetilde{\nabla}\equiv\frac{\partial}{\partial\widetilde{\mathbf{x}}}\;, \tag{3.5}$$

where E_2 and E_0 represent the terms of the integral along with their respective power of μ that appears when the integral is rescaled and a change of variable is performed. Both, E_2 and E_0, are positive definite. Solitons have a characteristic size, so a field configuration $\vec{\phi}(\mu\mathbf{x})$ solution of the equations of motion corresponds to a stationary point of $e[\mu]$ at $\mu=1$. Minimizing the energy, the condition of stationary point is

$$0=(2-d)\mu^{1-d}E_2-d\,\mu^{-(d+1)}E_0\;, \tag{3.6}$$

and as a consequence, the existence of stationary point depends on the dimensionality of the model. When $d=1$, the stationary point is $\mu_0=\sqrt{E_0/E_2}$, and the analysis of the second derivative reveals that it is a minimum. Conversely, when $d=2$, E_0 must be null everywhere to have a stationary point, so the field always must be in the vacuum state, and when $d\geq 3$, either E_2 or E_0 must be negative, but they are positive by definition. As a result, a fast inspection reveals that there are only static, finite energy solutions, different from the vacuum solution, for a Lagrangian density with the structure given above, in $(1+1)$ dimensions.

For the sake of simplicity, we will restrict our attention to the case in which $\phi:\mathbb{R}^{1,1}\to\mathbb{R}$, so from now on the energy functional will be

$$E[\phi]=\int_{-\infty}^{\infty}\left(\frac{1}{2}\dot{\phi}^2+\frac{1}{2}\phi'^2+U(\phi)\right)dx\;,\quad\dot{\phi}=\partial_t\phi\;,\quad\phi'=\partial_x\phi\;. \tag{3.7}$$

With the equation (3.7) in mind, let us first consider a static field configuration. We can manipulate the expression in such a way that we manage to stablish a bound in the energy from below. With simple algebraic operations it follows that

$$E[\phi]=\int_{-\infty}^{\infty}\left(\frac{1}{\sqrt{2}}\phi'\pm\sqrt{U(\phi)}\right)^2dx\;\mp\int_{-\infty}^{\infty}\left(\sqrt{2U(\phi)}\phi'\right)dx\;. \tag{3.8}$$

Since the first integral is always positive, then

$$E[\phi] \geq \left| \int_{-\infty}^{\infty} \sqrt{2U(\phi)}\phi' dx \right| = \left| \int_{\phi_-}^{\phi_+} \sqrt{2U(\phi)} d\phi \right| , \tag{3.9}$$

where ϕ_+ and ϕ_- denote the values that the field ϕ takes at $x = +\infty$ and $x = -\infty$ respectively. Finally, if we define $U(\phi) = \frac{1}{2}\left(\frac{dW}{d\phi}\right)^2$, inequality (3.8) takes the form

$$E \geq |W(\phi_+) - W(\phi_-)| , \tag{3.10}$$

where $W(\phi)$ is the so-called superpotential. We will discuss the terminology used to refer to $W(\phi)$ in Section 4. This equation also holds for time dependent fields, because the kinetic energy is positive definite. Overall, we accomplished to bound the energy from below just in terms of topological data, and this is precisely the Bogomolny property that we discussed in Section 1. Such a limit from below is reached when the Bogomolny equation is satisfied

$$\phi' = \pm\sqrt{2U(\phi)} , \tag{3.11}$$

which results from vanishing the squared term in (3.8). If a field configuration obeys that equation, it is said to saturate the BPS bound. The solutions of (3.11) are global minima of the energy within a given topological class of fields, so they are critical points of the energy function and thus automatically static solutions of the second order field equation (3.2).

Now that we have exposed the general framework, we will assume a certain potential $U(\phi)$. Because of its simplicity as well as its importance in many contexts, we will consider the ϕ^4 model. The potential in ϕ^4 model is

$$U(\phi) = \lambda(m^2 - \phi^2)^2 , \tag{3.12}$$

which comes from considering a polynomial of even powers of ϕ^2 up to second order, in addition to imposing that $\lambda > 0$ to allow the presence of minima. Substituting into (3.1) the corresponding Lagrangian density is

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \lambda(m^2 - \phi^2)^2 , \tag{3.13}$$

and the respective field equation is

$$\partial_\mu\partial^\mu\phi - 4\lambda(m^2 - \phi^2)\phi = 0 . \tag{3.14}$$

Clearly, the vacuum values are $\phi = \pm m$, and such values are what the field configurations have to approach at spatial infinity to have finite energy. Since the energy of the solutions is conserved, the field at infinity must always be one of the minima of $U(\phi)$. Moreover, the time evolution of a field is continuous, so as the potential has a discrete set of minima, the field reaches the same vacuum value at $x = \infty$ for all instant of time. The same discussion is applied at $x = -\infty$. Therefore, the space of all finite energy non-singular solutions can be divided into topological sectors, characterised by ϕ_+ and ϕ_-. Then, the topological charge N can be defined thought the expression [3]

$$N = \frac{\phi_+ - \phi_-}{2m} . \tag{3.15}$$

Thereby, N can only take the values $\{-1, 0, 1\}$. If a field has $N = 0$, it is homotopic to the vacuum solution. On the other hand, if $N = 1$, the field is said to be a kink, whilst if $N = -1$, it is said to be an antikink. Let us emphasise here that the topological sectors are not continuously connected, i.e., a solution in a given topological sector cannot decay into a different topological sector. The Bogomolny equation in the ϕ^4 model is

$$\phi' = \pm\sqrt{2\lambda}(m^2 - \phi^2), \tag{3.16}$$

that has the solutions

$$\phi_K = m \tanh\left(\sqrt{2\lambda}m(x - X_0)\right), \qquad \phi_{\overline{K}} = -m \tanh\left(\sqrt{2\lambda}m(x - X_0)\right), \tag{3.17}$$

where X_0 appears as a constant of integration related to the translational invariance of the theory. The first one is the kink solution, whereas the second one is the antikink solution, which are related through reflection. This exemplifies the discrete $\mathbb{Z}_2$ symmetry that the model possesses. The energy density (3.7) associated to both, the kink and antikink, results in

$$\varepsilon = 2\lambda m^4 sech^4\left(\sqrt{2\lambda}m(x - X_0)\right). \tag{3.18}$$

Therefore, integrating in the real line, it is straightforward to confirm that

$$E = \frac{4}{3}m^3\sqrt{2\lambda}\ , \tag{3.19}$$

which is interpreted as the classical rest mass of the kink. A quantum approach to the mass of the kink can be consulted in [2].

In Figure 3.1 we plot the kink solution and the energy density for given values of λ and m. Additionally, we took advantage of translation invariance fixing the kink at the origin. It results that the point where the field ϕ is null, the energy density reaches its maximum, so that point could be consider as the position of the kink. Moreover, remarkably, the energy density profile is exponentially confined in space, which strengthens the particle-like nature of the topological solitons as we mentioned in Section 1.

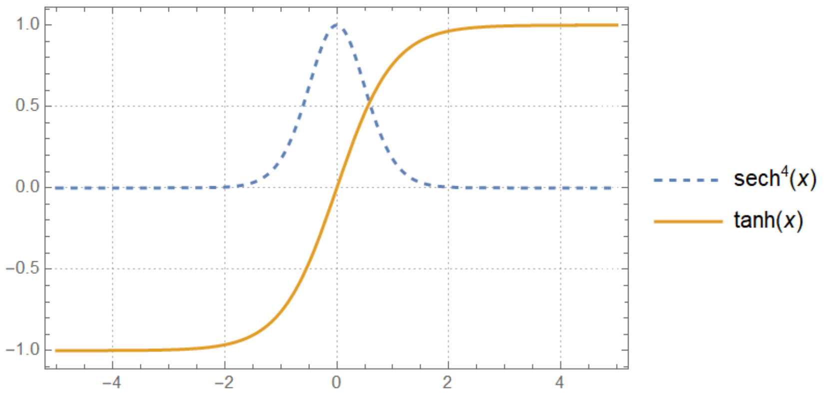

Figure 3.1: Kink solution (solid line) and energy density profile (dashed line) fixed at the origin for $\lambda = \frac{1}{2}$ and $m = 1$.

So far we have not discussed about the dynamical solutions, but certainly a dynamical solution of (3.14) can be obtained simply by applying a Lorentz boost to the kinks (3.17), since the Lagrangian density (3.13) is Lorentz invariant. On the other hand, since the only free parameter in kink solution is the one related to its position, the moduli space is $\mathcal{M}_1 = \mathbb{R}$, and if one promotes the position to a time dependent variable $(X_0 \to X(t))$, the reduced Euler-Lagrange equation has as solution a moving kink with constant speed, which does not solve (3.14) because it does not capture the Lorentz contraction. Consequently, this is a non-relativistic approach only valid at low velocities.

To obtain a relativistic solution we must insert a second parameter as we will see in Section 6.1. The moduli space dynamics gets more involved when we study kink-antikink collisions, since a complex fractal velocity-dependent structure appears in the final state of the collision [18].

To end with the discussion in one spatial dimension, let us remark that similar analysis can be performed in other $(1+1)$ dimensional models, such as the sine-Gordon model. This model is very relevant because is integrable and because field configurations with any number of solitons do appear.

3.2 Solitons in two spatial dimensions: Lumps and vortices

In this section we first intend to discuss a peculiar type of solutions that, technically, are not solitons, since the conformal invariance of the theory where these solutions do appear causes an instability under a spatial rescaling, so they tend to extend to infinity or collapse to a point. Such solutions are called *lumps*, and do appear in the so-called *sigma models*. In addition, we will also introduce another type of soliton in two-dimensions, the *vortex*. These vortices exist in two forms, the so-called *gauge vortices*, whose action in invariant under local gauge transformations, and the so-called *global vortices*, whose action in invariant under global gauge transformations. In this section, we will focus only on the first type, since it will be the starting point of our investigations from Sec. 5.

3.2.1 Lumps

A nonlinear sigma model is a nonlinear scalar field theory in which the fields take values in a target space that corresponds to a curved Riemannian manifold. The most general structure of the nonlinear sigma model in a $(1+d)$-dimensional space time is specified by the following Lagrangian density [12]

$$\mathcal{L} = \frac{1}{4} g_{ab}(\phi) \partial_\mu \phi^a \partial^\mu \phi^b \; , \tag{3.20}$$

where ϕ^a are real scalar fields and g_{ab} is the metric of the Riemannian manifold parametrized by these fields. We are interested in the $O(3)$ nonlinear sigma model, which is the canonical example of nonlinear sigma model. The name $O(3)$ refers to the global rotational symmetry of the target space, corresponding to the unit two-dimensional sphere $\mathbb{S}^2$. Then, we can rewrite (3.20) for such a case by considering that the unit two-dimensional sphere is parametrised through $\vec{\phi} = (\phi^1, \phi^2, \phi^3)$ with the condition $\vec{\phi} \cdot \vec{\phi} = 1$, and introducing the constraint into the Lagrangian of the standard scalar field model by means of a Lagrange multiplier, resulting in

$$\mathcal{L} = \frac{1}{4} \sum_{a=1}^{3} \partial_\mu \phi^a \partial^\mu \phi^a + \lambda \left(\sum_{a=1}^{3} (\phi^a)^2 - 1 \right) . \tag{3.21}$$

Therefore, after elimination of λ, the Euler-Lagrange equation looks like

$$\partial_\mu \partial^\mu \vec{\phi} + \left(\partial_\mu \vec{\phi} \cdot \partial^\mu \vec{\phi} \right) \vec{\phi} = 0 \; . \tag{3.22}$$

The energy of a static field configuration is

$$E = \frac{1}{4} \int_{\mathbb{R}^2} \partial_i \vec{\phi} \cdot \partial_i \vec{\phi} \; d^2 x \; . \tag{3.23}$$

Since the static energy density (3.23) is quadratic in spatial derivatives and, moreover, the space is two-dimensional, a spatial rescaling does not change the energy, which leads to the instability of the solutions mentioned earlier. For this to be finite, ϕ must approach to a constant vector at spatial infinity. We will take here $\vec{\phi}^{\infty} = (0,0,1)$ without loosing generality, although whatever value can be chosen. That makes $\vec{\phi}$ to be a based map. In addition, $\mathbb{R}^2 \cup \{\infty\} \cong S^2$ due to the one-point compactification, so $\vec{\phi}$ can be interpreted as the base map $\vec{\phi} : S^2 \to S^2$, and in terms of this, the relevant homotopy group is $\pi_2(S^2) \cong \mathbb{Z}$. In fact, the usual stereographic projection from the sphere $\mathbb{S}^2$ onto the complex plane allows us to reformulate the $O(3)$ nonlinear sigma model in terms of the complex function

$$R = \frac{(\phi_1 + i\phi_2)}{(1+\phi_3)} \ , \tag{3.24}$$

where, generally, $R = R(z,\bar{z})$ and $z = x^1 + ix^2$ is the complex coordinate in the spatial plane. With this change, the Lagrangian density (3.20) takes the form

$$\mathcal{L} = \frac{\partial_\mu R \partial^\mu \bar{R}}{(1+|R|^2)^2} \ , \tag{3.25}$$

that is referred to as the Lagrangian density of the $\mathbb{C}P^1$ nonlinear sigma model. Note that the metric is indeed that of a sphere of unit radius in terms of the stereographic projection variable, and that it has a Fubini-Study metric form. Using the definitions $\partial_z = \frac{1}{2}(\partial_x - i\partial_y)$ and $\partial_{\bar{z}} = \frac{1}{2}(\partial_x + i\partial_y)$, the energy of a static field configuration (3.23) in terms of $R(z,\bar{z})$ reads as

$$E = 2\int_{\mathbb{R}^2} \frac{|\partial_z R|^2 + |\partial_{\bar{z}} R|^2}{(1+|R|^2)^2} d^2x \ . \tag{3.26}$$

For that map, its topological degree is given by

$$N = \frac{1}{\pi}\int_{\mathbb{R}^2} \frac{|\partial_z R|^2 - |\partial_{\bar{z}} R|^2}{(1+|R|^2)^2} d^2x \ . \tag{3.27}$$

Comparing (3.26) and (3.27) we find the BPS bound

$$E \geq 2\pi|N| \ . \tag{3.28}$$

The Bogomolny equations that saturate the BPS bound are

$$\partial_{\bar{z}} R = 0, \qquad \partial_z R = 0, \tag{3.29}$$

which show that R is a holomorphic function of z or an antiholomorphic function of $\bar{z}$ respectively. The requirement that $R(z)$ has a defined limit as $z \to \infty$ as well as that its energy is finite, makes $R(z)$ a rational function $R(z) = q(z)/p(z)$, where $p(z)$ and $q(z)$ do not have common roots. Remembering that $\vec{\phi}$ satisfies a boundary condition at infinity, the function R must also satisfy a constraint at infinity; we choose it in such a way that $R(\infty) = 0$. The algebraic degree k_{alg} of the rational map is the larger of the degrees of the polynomials $p(z)$ and $q(z)$. Since this number counts, taking into account the multiplicity, the preimages of a given point on the target space manifold, it can be interpreted as the topological degree of the rational map. As an example, we will illustrate the rational map of one lump at the point a, solution that takes the form

$$R(z) = \frac{\lambda e^{i\chi}}{z-a} \ , \tag{3.30}$$

where a is its position, λ is its radius, and χ is an internal phase. The position of the lump is identified with the pole because that is where the energy density reaches its maximum (see Figure 3.2). The parameter λ is called radius because if it is integrated over the disc $|z-a| \leq \lambda$, the topological charge is 1/2, so it is appropriate to identify that parameter with the radius. In case we had a N-lumps solution, the poles of the rational function would be the positions of the lumps, whereas the module and phase of the residues at those poles would be the radii and the internal phases respectively. In contrast to the single lump case, for N coincident lumps at a point, the energy density is zero there and maximal on a circle, giving then a ring shape of radius $\lambda((N-1)/(N+1))^{1/2N}$ (see Figure 3.2).

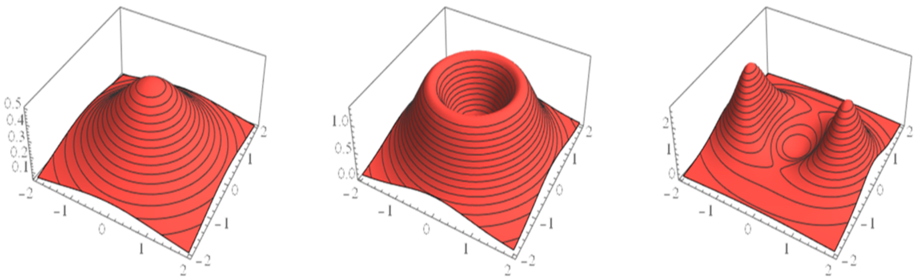

Figure 3.2: Energy density of one lump at the origin (left), two lumps coincident at the origin (middle), and two lumps at two different positions (right) with $\lambda = 2$.

As we have mentioned, Derrick's argument shows that there is no preferred scale even for the $N = 1$. This provoques the lump to shrink into a point or spread at infinity. To remove the instability of the $O(3)$ nonlinear sigma model, we can highlight two simple ways that allow us to accomplish this, which are based on approaches where new terms are added to the Lagrangian density (3.20), although the reasons of stability are completely different.

1. **Baby Skyrme model:** this model presents a Lagrangian density for a given choice of the free parameters that looks like [20]

$$\mathcal{L} = \frac{1}{2}\partial_\mu\vec{\phi}\cdot\partial^\mu\vec{\phi} - \frac{1}{4}\left(\partial_\mu\vec{\phi}\times\partial_\nu\vec{\phi}\right)\cdot\left(\partial^\mu\vec{\phi}\times\partial^\nu\vec{\phi}\right) - U(|\vec{\phi}|), \tag{3.31}$$

where the first term is that of the $O(3)$ nonlinear sigma model, the second is the Skyrme term, and the last is the mass term. The symbol "$\cdot$" represents the scalar product on $\mathbb{S}^2$, and "$\times$" denotes the vector product. With only the Skyrme term the conformal invariance of the theory is broken thanks to the inclusion of higher order terms in the first derivatives. Nevertheless, the potential is needed for soliton solutions to exist, because after scaling $\mathbf{x}\to\mu\mathbf{x}$, the energy looks like $e[\mu] = E_2 + \mu^2 E_4 + \mu^{-2}E_0$ using the notation of (3.5), guaranteeing Derrick's argument the stability under scaling. Note that the nonlinear sigma model term is the only term in (3.31) that can be omitted and that the field configurations remain stable under scaling. That model is the BPS baby Skyrmion model. The most common choice of the potential $U(|\phi|)$ is [12]

$$U = m^2(1-\phi_3), \tag{3.32}$$

preserving the $O(2)$ symmetry of ϕ_1 and ϕ_2 of the $O(3)$ nonlinear sigma model (3.21).

2. **Q-lumps model**: these soliton solutions do appear when the nonlinear $O(3)$ sigma model is modified by adding a potential term of the form $\frac{\alpha^2}{4}(1-\phi_3^2)$ with α a positive constant [21]. When the model is constructed in the $\mathbb{C}P^1$ formulation, the Lagrangian density is no longer (3.25), and takes the form

$$\mathcal{L} = \frac{\partial_\mu R\partial^\mu\bar{R} - \alpha^2|R|^2}{(1+|R|^2)^2}\ . \tag{3.33}$$

The additional term with respect to (3.25) maintains the global $U(1)$ symmetry, where the associated non-topological Noether charge Q that comes from Noether's theorem, contributes to the energy bound as

$$E \geq 2\pi|N| + |\alpha Q| \ . \tag{3.34}$$

In this case, the bound is reached by time-dependent fields rather than static fields, that present the form $R(t,z) = e^{-i\alpha t}R_0(z)$, where $R_0(z)$ is the based rational map with topological index N discussed earlier. The internal spin lifts the degeneracy between solitons of different radii, because the radius is determined by the value of the Noether charge. Consequently, the conservation of the Noether charge ensures that the size of the Q-lumps do not shrink to zero or expand to infinity.

Finally, let's talk about the moduli space of lumps. The moduli space $\mathcal{M}_N$ of based rational maps of degree N is a complex manifold of dimension $2N$, which may be interpreted as the position, radius, and phase of each lump. If the general procedure for obtaining the metric were followed, one would obtain a metric that is not well defined in all directions in $\mathcal{M}_N$. This is because the kinetic energy associated with changing some parameters diverges. They are said to have infinite inertia. For example, such behaviour occurs in $\mathcal{M}_{N=1}$ when we promote the radius or the phase to time-dependent variables. The inspection for the case $N > 1$ suggests that the leading order coefficient of the expansion of $R(z)$ at infinity must be a time-independent variable to guarantee that the divergence is removed. The resulting $(2N-1)$-dimensional submanifold of $\mathcal{M}_N$ is Kähler. This can be seen if the kinetic energy is expressed as

$$T = \int \frac{|\dot{p}(z)q(z) - \dot{q}(z)p(z)|^2}{(|p(z)|^2 + |q(z)|^2)^2}\, d^2x = \dot{u}_\alpha \dot{\bar{u}}_\beta \int \frac{\partial}{\partial u_\alpha}\frac{\partial}{\partial \bar{u}_\beta} \log(|p(z)|^2 + |q(z)|^2)\, d^2x, \tag{3.35}$$

where u_α with $\alpha \in (1, N)$ are the zeros of the polynomial $p(z)$ and u_α with $\alpha \in (N+1, 2N)$ are the zeros of the polynomial $q(z)$. The metric $g_{\alpha\beta}$ is identified with

$$g_{\alpha\beta} = \frac{\partial}{\partial u_\alpha}\frac{\partial}{\partial \bar{u}_\beta} \log(|p(z)|^2 + |q(z)|) = \frac{\partial}{\partial u_\alpha}\frac{\partial}{\partial \bar{u}_\beta}\mathcal{K}, \qquad \mathcal{K} = \log(|p(z)|^2 + |q(z)|^2), \tag{3.36}$$

being $\mathcal{K}$ the Kähler potential, which is very relevant since it describes locally the metric, and it is not always possible to derive it, so the identification of the Kähler potential is of great interest.

3.2.2 Vortices

Field theories with vortices are of two types, global and gauged, so their solutions are called, respectively, *global and gauged vortices*. The basic field theory with vortices is one having a single complex scalar field $\phi(x) = \phi_1(x) + i\phi_2(x)$ that possesses a $U(1)$ internal symmetry. In the global theory we only need the complex scalar field, whilst in the gauged theory, the scalar complex field which receives the name of *scalar Higgs field* is coupled to a $U(1)$ gauge field $a_\mu = (a_0(x), \boldsymbol{a}(x))$. Here we are considering the Abelian-Higgs model

$$\mathcal{L} = \frac{1}{2}\overline{D_\mu \phi} D^\mu \phi - \frac{1}{4} f_{\mu\nu} f^{\mu\nu} + U(\phi\bar{\phi}) \ , \tag{3.37}$$

where $D_\mu\phi = (\partial_\mu - ia_\mu)\phi$ is the covariant gauge derivative and $f_{\mu\nu} = \partial_\mu a_\nu - \partial_\nu a_\mu$ is the field strength. From now on, we will focus on the gauged theory, understanding that one can descend to the global theory by annihilating the gauge terms and exchanging covariant gauge derivatives for ordinary derivatives. The comments corresponding to the global theory will be mentioned when it is necessary to clarify any difference with respect to the gauge vortices. The potential U is usually assumed to be a polynomial of, at most, quadratic or cubic order in $\phi\overline{\phi}$. Taking quadratic order and adjusting the coefficients so that U has minima and $U_{\text{min}} = 0$, U is

$$U = \frac{\lambda}{8}(m^2 - \phi\overline{\phi})^2, \qquad m > 0 \; . \tag{3.38}$$

Therefore, the vacuum manifold $\mathcal{V}$ is the circle $|\phi| = m$, and $\pi(\mathcal{V}) = \mathbb{Z}$. The free parameter λ measures the relative strengths of the attractive scalar force and repulsive magnetic force between vortices. When $\lambda < 1$ the vortices attract each other, whereas when $\lambda > 1$ the vortices repel each other. The $\lambda = 1$ case is the so-called *critical value* where the forces exactly cancel. The resulting Euler-Lagrange field equations are

$$D_\mu D^\mu \phi - \frac{\lambda}{2}(1 - \overline{\phi}\phi)\phi = 0, \tag{3.39}$$

$$\partial_\mu f^{\mu\nu} + \frac{i}{2}(\overline{\phi}D^\nu\phi - \phi\overline{D^\nu\phi}) = 0 \; . \tag{3.40}$$

If we focus on the static fields and consider polar coordinates, the potential looks like

$$V = \frac{1}{2}\int_0^\infty \int_0^{2\pi} \left(\frac{1}{\rho^2} f_{\rho\theta}^2 + \overline{D_\rho\phi}D_\rho\phi + \frac{1}{\rho^2}\overline{D_\theta\phi}D_\theta\phi + \frac{\lambda}{8}(m^2 - \phi\overline{\phi})^2 \right) \rho \; d\rho \; d\theta, \tag{3.41}$$

where $D_\rho = \partial_\rho\phi - ia_\rho\phi$, $D_\theta = \partial_\theta\phi - ia_\theta\phi$ and $f_{\rho\theta} = \partial_\rho a_\theta - \partial_\theta a_\rho = \rho B$. For a field configuration $\{\phi(\boldsymbol{x}), a_i(\boldsymbol{x})\}$ to have finite energy, the field has to satisfy the boundary condition $|\phi| \to m$ as $|\boldsymbol{x}| \to \infty$, as well as $D_\rho\phi$ tends to zero as $|\boldsymbol{x}| \to \infty$, as can be seen in (3.41). The finiteness of the remaining two integrals also implies that $D_\theta\phi \to 0$ and $f_{\rho\theta} \to 0$ as $|\boldsymbol{x}| \to \infty$. Overall, using the so-called *radial gauge* in which $a_\rho = 0$, along with those conditions, the scalar field at infinity results in a map

$$\phi^\infty : S^1_\infty \to S^1 \tag{3.42}$$

that satisfies $\phi^\infty = me^{i\chi^\infty(\theta)}$, as well as $\partial_\theta\chi^\infty = a_\theta^\infty$. Thus, the winding number associated is the integer

$$N = \frac{1}{2\pi}\int_0^{2\pi} \partial_\theta\chi^\infty(\theta) \; d\theta = \frac{1}{2\pi}(\chi^\infty(2\pi) - \chi^\infty(0)) \; , \tag{3.43}$$

that coincides with the first Chern number $c_1 = \frac{1}{2}\int_{\mathbb{R}^2} B \; d^2x$. Instead, in the global theory it is necessary that χ^∞ be a constant to avoid a logarithmic divergence from the ordinary angular partial derivative term when it is integrated. Consequently, this fact restricts the finite energy configurations to belong to the $N = 0$ topological sector.

All known charge one finite energy static solutions of the field equations with $\lambda \neq 1$ have circular symmetry about some point, as well as reflection symmetry. The solutions that present these symmetries are called *vortices*. Such a circular symmetry is contemplated by the combined action of $SO(2)$ group with global phase rotations as follows

$$\left(R(\theta_1), \tilde{R}(\kappa\theta_1)\right) \cdot \left(R(\theta_2), \tilde{R}(\kappa\theta_2)\right) = \left(R(\theta_1 + \theta_2), \tilde{R}(\kappa(\theta_1 + \theta_2))\right), \tag{3.44}$$

where $R \in SO(2)$, $\tilde{R} \in U(1)$ and $\kappa \in \mathbb{N}$. The fact that ϕ has to be invariant under these transformations requires that ϕ possesses the structure $\phi(\rho, \theta) = e^{i\kappa\theta}\phi(\rho)$. Additionally, due to reflection symmetry $\phi(\rho, \theta) = \overline{\phi}(\rho, -\theta)$ the radial term must be a real function. It seems clear that $N = \kappa$. In relation to gauge fields, since the phase rotation is global, it has not effect on such fields, so ultimately gauge fields are just real radial functions where, using the radial gauge, the only component is $a_\theta(\rho)$. When these fields are introduced into the static energy, the resulting reduced expression leads to two coupled Euler-Lagrange equations, giving solutions that near the origin are of the form

$\phi(\rho) = \rho^N F(\rho^2)$ and $a_\theta(\rho) = \rho^2\, G(\rho^2)$ with $F(\rho^2)$ and $G(\rho^2)$ series in ρ^2, whilst as $\rho \to \infty$ the increasing to the vacuum value is exponential and in terms of the modified Bessel functions $K_0(\rho)$ or $K_1(\rho)$.

So far we have made a distinction between $\lambda = 1$ and all other values. The reason is that this value is the limit between Type I and Type II superconductivity and at which a BPS structure can be found. To illustrate it, consider (3.38) with $\lambda = 1$. The static energy can be rewritten as

$$E = \frac{1}{2}\int_{\mathbb{R}^2}\left\{\left(B \mp \frac{1}{2}(1-\phi\bar{\phi})\right)^2 + \left(\overline{D_1\phi} \mp i\overline{D_2\phi}\right)\left(D_1\phi \pm iD_2\phi\right)\right\} d^2x \pm \pi N, \tag{3.45}$$

after using Stoke's theorem. Therefore, since the integrand is non-negative, we have the inequality

$$E \geq \pi|N| \ . \tag{3.46}$$

As usual, the BPS bound saturates if the integrand vanishes, which leads us to the Bogomolny equations

$$D_1\phi \pm iD_2\phi = 0, \tag{3.47}$$

$$B \mp \frac{1}{2}(1-\bar{\phi}\phi) = 0 \ . \tag{3.48}$$

At this point, a significant contribution was made by Taubes [22], who managed to summarise the two Bogomolny equations in a single equation with a redefinition of ϕ via $h = \log|\phi|^2$, obtaining

$$\nabla^2 h + 1 - e^h = 4\pi\sum_{r=1}^{N}\delta^2(\mathbf{x}-\mathbf{X}_r), \tag{3.49}$$

where $\boldsymbol{X}_r$ represents the position of the r-th vortex on the real plane. This expression is determined by assuming the expression of ϕ in terms of h, i.e., $\phi = e^{\frac{1}{2}h+i\chi}$, and introducing that into (3.47). By using the resulting equation, the gauge field components can be trivially written as

$$a_1 = \frac{1}{2}\partial_2 h + \partial_1\chi, \qquad a_2 = -\frac{1}{2}\partial_1 h + \partial_2\chi. \tag{3.50}$$

Hence, rewriting $B = \partial_1 a_2 - \partial_2 a_1$ in terms of the previous identifications, (3.48) reads as (3.49).

Finally, let us sketch some details on the moduli space of the vortices at critical coupling. In the first place, Weinberg proved with an index theorem that the moduli space of vortices had dimension $2N$, and later Samols [16] managed to derive an analytic expression for the metric of such $2N$-dimensional moduli space. For that, it is necessary to define a new field $\eta = \partial_0 \log\phi$ and express the kinetic energy in terms of η and h. A relation between η and the derivative of h with respect to the poles of h can be attained by an approach where another Taubes-like equation is derived for η. Expanding h around the zero Z_j of the Higgs field ϕ, we get

$$h = \log|z-Z_r|^2 + a_r + \frac{1}{2}(b_r(z-Z_r)+\bar{b}_r(\bar{z}-\overline{Z}_r)) + c_r(z-Z_r)^2 + d_r(z-Z_r)(\bar{z}-\overline{Z}_r) + \bar{c}_r(\bar{z}-\overline{Z}_r^2) + \cdots \tag{3.51}$$

and it is easy to see that, in order to obey (3.49), the coefficient d_r has to take the value $d_r = -\frac{1}{4}$, whilst the rest of the terms have no restrictions. Finally, after some complex variable calculus, the metric on the N-vortex moduli space $\mathcal{M}_N$ turns out to be

$$T = \frac{1}{2}\pi\sum_{r,s=1}^{N}\left(\delta_{rs} + 2\frac{\partial\bar{b}_s}{\partial Z_r}\right)\dot{Z}_r\dot{\overline{Z}}_s \ \Rightarrow \ ds^2 = \pi\sum_{r,s=1}^{N}\left(\delta_{rs} + 2\frac{\partial\bar{b}_s}{\partial Z_r}\right)dZ_r d\overline{Z}_s \ . \tag{3.52}$$

A significant feature is that the metric tensor is Kähler (see Section 2.2). In order to see this, one can construct the associated 2-form

$$\omega = \frac{i}{2} \sum_{r,s=1}^{N} g_{rs} dZ_r \wedge d\overline{Z}_s \ , \tag{3.53}$$

for the metric coefficients of (3.52). After a simple algebraic manipulation using the symmetry properties of the coefficients b_s in the expansion of the metric, one can prove that $d\omega = 0$. We will use and generalize this derivation through the following sections.

3.3 Solitons in three dimensions: Skyrmions and monopoles

To end this section we will talk about some of the soliton solutions that appear in three dimensions. Skyrmions do appear in the Skyrme theory [24], which is a topological model of nucleons in which baryons are considered as solitons and where the baryon number coincides with the topological charge. In this picture, pions correspond to linearized fluctuations of the pion field around the vacuum, and it is considered as a low-energy Quantum Chromodynamics effective theory with many quark colours. On the other hand, monopoles are hypothetical isolated magnetic poles that have no yet been experimentally confirmed. In the first place, Dirac established in 1961 a formulation of a monopole (the so-called Dirac monopole) that had at its origin a singularity, whose existence would lead to an explanation for the quantization of electric charge. Later, 't Hooft discovered that non-abelian gauge theories can have magnetic monopole solutions without singularities, and they may be interpreted as soliton solutions.

3.3.1 Skyrmions

The basic field of this model is the Skyrme field $U(\boldsymbol{x}, t)$, which is defined as an unitary $SU(2)$-valued scalar in (3+1) dimensions, which can be written in terms of the following scalar field quartet [24]

$$U = \sigma + i\boldsymbol{\pi} \cdot \boldsymbol{\tau} \ , \tag{3.54}$$

where $\boldsymbol{\pi}$ denotes the triplet of Pauli matrices, $\boldsymbol{\tau}$ alludes to the triplet of pion fields, and σ is an additional field. The unitarity of the Skyrme field imposes the following restriction

$$\sigma^2 + \boldsymbol{\pi} \cdot \boldsymbol{\pi} = \mathbb{I} \ , \tag{3.55}$$

which allows to determine the field σ. The model is defined through the following Lagrangian

$$L = \int \left\{ \frac{F_\pi^2}{16} Tr\left(\partial_\mu U \partial^\mu U^\dagger\right) + \frac{1}{32e^2} Tr\left(\left[\partial_\mu U U^\dagger, \partial_\nu U U^\dagger\right]\left[\partial^\mu U U^\dagger, \partial^\nu U U^\dagger\right]\right)\right\} d^3x \ . \tag{3.56}$$

Here, "[,]" represents the commutator. The quartic term is crucial since it allows stable soliton solutions by invoking Derrick's argument, and because it is the only quartic term that allows Lorentz invariance as well as a Hamiltonian formulation with a positive Hamiltonian. Any other choice of the quartic term excludes either Lorentz invariance or a well-defined Hamiltonian formulation. The parameters F_π (pion decay constant) and e (dimensionless constant) can be scaled away using other units of length and energy, and introducing the current $R_\mu = (\partial_\mu U)U^\dagger \in \mathfrak{su}(2)$, the Lagrangian finally reads as [3]

$$L = \int \left\{ -\frac{1}{2} Tr\left(R_\mu R^\mu\right) + \frac{1}{16} Tr\left(\left[R_\mu, R_\nu\right]\left[R^\mu, R^\nu\right]\right)\right\} d^3x \ . \tag{3.57}$$

One can realize that the pions are massless by introducing (3.54) into (3.57). Consequently, they are the Goldstone bosons of the spontaneously broken chiral symmetry. If (3.57) was supplemented by a symmetry-breaking potential term of the form $m^2 \int Tr(U - \mathbb{I})d^3x$, then the pions would acquire a mass m. The Euler-Lagrange equation associated to (3.57) is

$$\partial_\mu \left(R^\mu + \frac{1}{4} \left[R^\nu, \left[R_\nu, R^\mu \right] \right] \right) = 0, \tag{3.58}$$

that takes the form of a chiral conservation law. For that reason, R_μ is called right-handed chiral current. The Skyrme field U is postulated to satisfy the boundary condition $U(\boldsymbol{x}) \to \mathbb{I}$ as $|\boldsymbol{x}| \to \infty$. That spontaneously breaks the chiral symmetry $(SU(2) \times SU(2))/\mathbb{Z}_2 \cong SO(4)$ to a $SO(3)$ isospin symmetry. Furthermore, the boundary condition implies an one-point compactification of $\mathbb{R}^3$ to $\mathbb{S}^3$. Since the manifold of $SU(2)$ is $\mathbb{S}^3$, then U^∞ is a map between 3-spheres, and therefore the maps fall into homotopy classes indexed by an integer $\pi_3(\mathbb{S}^3) = \mathbb{Z}$, which is denoted by B and identified with the baryon number. As it coincides with the degree of the map U, we conclude that

$$B = -\frac{1}{24\pi^2} \int \varepsilon_{ijk} Tr(R_i R_j R_k) d^3x \ . \tag{3.59}$$

The static energy functional of the Skyrme model looks like

$$E = \int \left\{ -\frac{1}{2} Tr(R_i R_i) - \frac{1}{16} Tr([R_i, R_j] [R_i, R_j]) \right\} \ . \tag{3.60}$$

From (3.59) and (3.60) one deduces the following Bogomolny lower bound

$$E \geq 12\pi^2 |B| \ . \tag{3.61}$$

Nevertheless, the compatibility condition $\partial_\mu \partial_\nu U = \partial_\nu \partial_\mu U$ yields the Maurer-Cartan structure equations [12]

$$\partial_\mu R_\nu - \partial_\nu R_\mu + [R_\mu, R_\nu] = 0, \tag{3.62}$$

and the energy bound is only saturated in case R_i satisfies the equation

$$R_i = \frac{1}{2} \varepsilon_{ijk} R_j R_k \ , \tag{3.63}$$

which is not compatible with (3.62). More geometrically, the reason for the incompatibility is that to satisfy the bound, U would have to be an isometry. Since $\mathbb{R}^3$ is flat and $\mathbb{S}^3$ is curved, then an isometry between the two spaces cannot exist. Hence, the bound is only reached if we consider the vacuum state. Nonetheless, the bound can be attained by non-trivial solutions if the spatial domain is taken to be the 3-sphere of unit radius for U to be a isometry.

No analytical solutions of (3.58) are known, so the field configurations are obtained by numerical methods where the energy is minimized. When $B = 1$, the minimizer of the energy takes the form of a spherically symmetric Skyrmion called *hedgehog solution*

$$U(\boldsymbol{x}) = \exp(i f(r) \hat{\boldsymbol{x}} \cdot \boldsymbol{\tau}), \tag{3.64}$$

being $\boldsymbol{\pi} = \sin f(r) \hat{\boldsymbol{x}}$ and $\sigma = \cos f(r)$. To ensure the correct baryon number, the boundary conditions $f(0) = \pi$ and $f(\infty)$ are imposed. Solving the corresponding second order ordinary differential equation using a shooting method, the energy associated to this configuration is $E = 12\pi^2 \cdot 1.232$ [12], a value that certainly exceeds the Bogomolny bound. Nevertheless, this ansatz does not represent the minimal energy Skyrmions when $|B| > 1$. Conversely, Skyrmions that minimize the energy for $B > 1$ have other type of symmetries like $B = 2$ $D_{\infty h}$ symmetry, $B = 3$ T_d symmetry, $B = 4$ O_h symmetry, $B = 5$ D_{2h} symmetry or $B = 6$ D_{4d} symmetry, for example. This can be verified if the surfaces of constant baryon density are plotted. Interestingly, there is an hypothesis saying that the structures for higher charge Skyrmions ($B \geq 7$) are fullerene-like [25].

3.3.2 Monopoles

Let us start by introducing the $SU(2)$ Yang-Mills-Higgs theory with the $SU(2)$ gauge potential A_μ and the adjoint Higgs field Φ, both valued in the Lie Algebra $\mathfrak{su}(2)$, where

$$D_\mu\Phi = \partial_\mu\Phi + [A_\mu, \Phi] \quad \text{and} \quad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu] \tag{3.65}$$

are the covariant derivative and the Yang-Mills field tensor respectively. The choice of the basis is the following $\{t^a = i\tau^a : a = 1, 2, 3\}$.

We will consider here a theory with a Lorentz invariant Lagrangian density of the form

$$\mathcal{L} = \frac{1}{8}Tr(F_{\mu\nu}F^{\mu\nu}) - \frac{1}{4}Tr(D_\mu\Phi D^\mu\Phi) - \frac{\lambda}{4}(1-|\Phi|^2)^2 \;\; \text{where} \;\; |\Phi|^2 = -\frac{1}{2}Tr(\Phi^2) \;, \tag{3.66}$$

which spontaneously symmetry breaking to $U(1)$. The boundary condition is $\Phi(0, 0, x^3) \to t^3$ as $x^3 \to \infty$ here. The field equations derived from (3.66) are

$$D_\mu D^\mu\Phi \;=\; \lambda(1-|\Phi|^2)\Phi \;, \qquad D_\mu F^{\mu\nu} \;=\; [D^\nu\Phi, \Phi] \,. \tag{3.67}$$

The static solution representing a magnetic monopole was found independently by 't Hooft and Polyakov using the following ansatz with spherical and reflection symmetry

$$\Phi \;=\; h(r)\frac{x^a}{r}t^a \;, \qquad A_i \;=\; -\frac{1}{2}(1-k(r))\varepsilon_{ija}\frac{x^j}{r}t^a \;. \tag{3.68}$$

Hence, equations (3.67) reduce to

$$\frac{d^2h}{dr^2} + \frac{2}{r}\frac{dh}{dr} \;=\; \frac{2}{r^2}k^2h - \lambda(1-h^2)h \;, \qquad \frac{d^2k}{dr^2} \;=\; \frac{1}{r^2}(k^2-1)k + 4h^2k \;, \tag{3.69}$$

having to be solved numerically. Prasad and Sommerfield obtained an analytic solution for the case $\lambda = 0$ [10], being the solution

$$h(r) \;=\; \coth 2r - \frac{1}{2r} \;, \qquad k(r) \;=\; \frac{2r}{\sinh 2r} \;, \tag{3.70}$$

and 2π the associated energy.

A deeper understanding of this characteristic limit was due to Bogomolny [9]. He observed that the energy of a static field

$$E = -\frac{1}{4}\int \left(Tr(B_iB_i) + Tr(D_i\Phi D_i\Phi)\right) d^3x \tag{3.71}$$

can be rewritten as

$$E = -\frac{1}{4}\int_{\mathbb{R}^3} Tr(B_i + D_i\Phi)(B_i + D_i\Phi)\; d^3x - \int_{\mathbb{S}^2_\infty} b_i\; dS^i \;, \tag{3.72}$$

with $b_i = -\frac{1}{2}Tr(B_i\Phi)$ and $B_i = -\frac{1}{2}\varepsilon_{ijk}F_{jk}$. Since the last integral is quantized as $2\pi N$ [3], then

$$E \geq 2\pi N \;, \tag{3.73}$$

where the condition to saturate the bound is

$$B_i = -D_i\Phi \;. \tag{3.74}$$

Therefore, the monopole solution obtained by Prasad and Sommerfield must saturate the bound to have the 2π predicted energy, hence possessing unity monopole number.

The moduli space approximation for monopoles is extremely involved, so here we will just mention that the moduli space $\mathcal{M}_N$ is a $4N$-dimensional hyper Kähler manifold with the following decomposition

$$\mathcal{M}_N \simeq \mathbb{R}^3 \times \frac{\mathbb{S}^1 \times \widetilde{\mathcal{M}_N^0}}{\mathbb{Z}_N} \ . \tag{3.75}$$

In this metric, $\mathbb{R}^3$ parametrizes the centre of mass coordinate $\mathbf{X}$, $\mathbb{S}^1$ parametrizes the total phase χ, and $\widetilde{\mathcal{M}_N^0}$ is the space of the strongly centred monopoles.

4 Brief introduction to supersymmetry

It is well-known that supersymmetic theories may have BPS sectors. If, at the same time, the model can host topological solitons, they become BPS solitons. This connection suggests that a natural strategy for building models with BPS solitons is to search for the corresponding supersymmetric versions, and then constrain the theory to its bosonic sector. In the following sections we will systematically use supersymmetry as a guiding principle to build BPS models. Below, we briefly discuss the basic ingredients necessary to build supersymmetry using the superspace.

The Coleman-Mandula theorem [26] states that any symmetry group of a quantum field theory has to be locally isomorphic to the direct product of an internal symmetry group and the Poincaré group when the commutator is the bilinear operation in the Lie algebra. That constraints the symmetries of the scattering matrix. Nonetheless, as in every no-go theorem, it is always possible to relax some condition to evade the theorem, in this case, for example, considering a graded Lie algebra whose generators obey commutation and anticommutation relations. For this purpose let's proceed as follows:

Let us introduce a left-handed Weyl spinor generator Q_α together with its right-handed Hermitian conjugate $\overline{Q}_{\dot{\alpha}}$. They will be called *supercharges*. From now on, the symbol α will represent the index of an object that transforms in the spinor representation, whereas $\dot{\alpha}$ will denote the index of an object that transforms in the conjugated spinor representation. The new graded algebra maintains the same commutation relations that existed for the Poincaré algebra, but adds the following ones

$$\begin{aligned}
[M^{\mu\nu}, Q_\alpha] &= (\sigma^{\mu\nu})_\alpha{}^\beta Q_\beta, \qquad && \{Q_\alpha, Q_\beta\} = 0, \\
\left[M^{\mu\nu}, \overline{Q}^{\dot{\alpha}}\right] &= (\sigma^{\mu\nu})^{\dot{\alpha}}{}_{\dot{\beta}} \overline{Q}^{\dot{\beta}}, \qquad && \left\{\overline{Q}_{\dot{\alpha}}, \overline{Q}_{\dot{\beta}}\right\} = 0, \\
\{Q_\alpha, \overline{Q}_{\dot{\alpha}}\} &= 2\sigma^\mu{}_{\alpha\dot{\alpha}} P_\mu, \qquad && [P^\mu, Q_\alpha] = 0.
\end{aligned} \tag{4.1}$$

Here "[,]" denotes a commutator and "{ , }" an anticommutator. The Haag-Lopuszanski-Sohnius theorem [27] shows that the unique graded Lie algebra of symmetries consistent with quantum field theory is the one defined above. This is known as $\mathcal{N}=1$ *supersymmetric algebra*. It is plausible to extend the algebra by introducing more supercharges in what is known as *extended supersymmetry*, but we will focus our explanation on the $\mathcal{N}=1$ supersymmetry. The interpretation of the commutators and anticommutators is simple: the zero anticommutators state that the supercharges are anticommuting objects, whereas the commutators with $M^{\mu\nu}$ show that Q_α and $\overline{Q}_{\dot{\alpha}}$ are spinors, and a consequence of this is that although the irreducible representations of the supersymmetric algebra are labelled by the mass, because $P_\mu P^\mu$ is still a Casimir, $W_\mu W^\mu$ with W_μ the Pauli-Lubanski vector is not, what induces that in a supersymmetric multiplet the particles do not have the same spin.

Since Q_α is a spinor, it is invariant under translations, which explains the null commutator with P^μ. Finally, the most intriguing anticommutation relation is $\{Q_\alpha, \overline{Q}_{\dot{\alpha}}\} = 2\sigma^\mu{}_{\alpha\dot{\alpha}} P_\mu$. As a result, in supersymmetry all states have positive energy, so the ground state is the one with zero energy. It is well known, thanks to E. Witten and D. Olive [28], that when a supersymmetric theory has topological charge, this supersymmetric algebra is not complete, and therefore it is necessary to introduce the topological charge as a central charge.

In special relativity, Lorentz invariance manifests itself in Minkowski space. Similarly, supersymmetry manifests itself in the named superspace, where the coordinates are x^μ, θ_α and $\overline{\theta}^{\dot{\alpha}}$, with x^μ the coordinates of Minkowski space and where θ_α and $\overline{\theta}^{\dot{\alpha}}$ are Grassman-valued spinors. We can define the superspace as

$$\text{Superspace} = G/H = \frac{\text{Super-Poincaré group}}{\text{Lorentz group}} \ .$$

Under a transformation of the form $\exp(ia_\mu P^\mu)$, the Minkowski coordinates translate as we could expect, i.e., $x_\mu \to x_\mu + a_\mu$. The striking transformation occurs when we consider the action due to $\exp(i\varepsilon^\alpha Q_\alpha + i\overline{\varepsilon}_{\dot{\alpha}}\overline{Q}^{\dot{\alpha}})$, that in addition to shift the Grassmann coordinates as $\theta \to \theta + \varepsilon$ and $\overline{\theta} \to \overline{\theta} + \overline{\varepsilon}$, produces a shift in x_μ as $x_\mu \to x_\mu + i\theta\sigma_\mu\overline{\varepsilon} - i\varepsilon\sigma_\mu\overline{\theta}$ due to the anticommutator between Q_α and $\overline{Q}_{\dot{\alpha}}$. In such superspace, one can define a superfield as $Y = Y(x, \theta, \overline{\theta})$. Taylor expanding the superfield in θ and $\overline{\theta}$, we obtain a truncated expansion due to the anticommutation of the Grassmann variables [29]:

$$\begin{aligned} Y(x,\theta,\overline{\theta}) = {} & \phi(x) + \theta^\alpha\psi_\alpha(x) + \overline{\theta}_{\dot{\alpha}}\overline{\chi}^{\dot{\alpha}}(x) + \theta^2 M(x) + \overline{\theta}^2 N(x) \\ & + \theta^\alpha\overline{\theta}^{\dot{\alpha}} V_{\alpha\dot{\alpha}}(x) + \theta^2\overline{\theta}_{\dot{\alpha}}\overline{\theta}^{\dot{\alpha}}(x) + \overline{\theta}^2\theta^\alpha\rho_\alpha(x) + \theta^2\overline{\theta}^2 D(x) \ . \end{aligned} \tag{4.2}$$

In the above expression appear: four complex scalars fields ϕ, M, N and D, two left-handed spinors ψ_α and ρ_α, two right-handed spinors $\bar{\chi}_\alpha$ and $\bar{\lambda}_\alpha$, and a vector $V_{\alpha\dot{\alpha}} = \sigma^\mu_{\alpha\dot{\alpha}} V_\mu$. By analyzing how the superfield transforms under a supersymmetry transformation, we can deduce how each field component of the superfield transforms. We especially emphasise the role played by the higher order term in the Grassmann variables, i.e., the term with the complex scalar field $D(x)$. The reason is the following: suppose we want to construct an action that is manifestly invariant under supersymmetry transformations. Now consider that such action is a function of superfields that we will denote by $K(x, \theta, \overline{\theta})$. The function $K(x, \theta, \overline{\theta})$ will itself be a superfield. Integrated over the whole superspace, the action reads as

$$S = \int d^4x \ d^2\theta \ d^2\overline{\theta} \ K(x, \theta, \overline{\theta}) \ , \tag{4.3}$$

with $K(x, \theta, \overline{\theta})$ a real superfield to ensure that the action is also real. The integration in the Grassmann variables is achieved through the Berezin integral [29], which acts as a derivative. Expanding $K(x, \theta, \overline{\theta})$ as in (4.2), only the higher order term in the Grassmann variables remains, and since it transforms as a total derivative under supersymmetric transformations, the action is invariant if the field drops sufficiently fast to zero. Therefore, any action given by (4.3) is then invariant by construction.

There are four types of superfields depending on the restriction imposed to them: if $\overline{\mathcal{D}}_{\dot{\alpha}}\Phi = 0$, with $\overline{\mathcal{D}}_{\dot{\alpha}} = -\overline{\partial}_{\dot{\alpha}} - i\theta^\alpha\sigma^\mu{}_{\alpha\dot{\alpha}}\partial_\mu$ the anticovariant derivative, then Φ is the chiral superfield; if $\mathcal{D}_\alpha\Psi = 0$ with $\mathcal{D}_{\dot{\alpha}} = \partial_\alpha + i\sigma^\mu{}_{\alpha\dot{\alpha}}\overline{\theta}^{\dot{\alpha}}\partial_\mu$ the covariant derivative, then Ψ is the antichiral superfield; if $V = V^\dagger$, then V is the vector superfield; and if $J = J^\dagger$ and $\mathcal{D}^2 J = \overline{\mathcal{D}}^2 J = 0$, then J is the linear superfield. The most important are the first three because the particles of the chiral multiplet (matter particle and its bosonic superpartner) are embedded in the chiral superfield, and the particles of the vector multiplet (gauge particle and its fermionic superpartner, called *gaugino*) are embedded in the vector superfield.

The chiral superfield in components looks like [29]

$$\begin{aligned}\Phi(x,\theta,\bar{\theta}) =&\phi(x)+\sqrt{2}\theta\psi(x)+\theta^2F(x)+i\theta\sigma^\mu\bar{\theta}\partial_\mu\phi(x)\\&-\frac{i}{\sqrt{2}}\theta^2\partial_\mu\psi(x)\sigma^\mu\bar{\theta}-\frac{1}{4}\theta^2\bar{\theta}^2\partial_\mu\partial^\mu\phi(x)\ ,\end{aligned}\tag{4.4}$$

where ϕ and F are complex scalar fields and ψ_α is a left-handed spinor. The antichiral superfield is its complex conjugate. The simplest example of supersymmetric action involving a chiral superfield is the Wess-Zumino model [30], that was the first example of interacting four-dimensional quantum field theory with supersymmetry. This action reads as

$$\begin{aligned}S_{WZ}&=\int d^4xd\theta^2d\bar{\theta}^2\ \Phi\Phi^\dagger+\int dx^4\left(\int d^2\theta\ W(\Phi)+\int d^2\bar{\theta}\ W^\dagger(\Phi^\dagger)\right)\\&=\int dx^4\ \left(\partial_\mu\bar{\phi}\partial^\mu\phi-i\bar{\psi}\bar{\sigma}^\mu\partial_\mu\psi+F\bar{F}+\left[F\frac{\partial W(\phi)}{\partial\phi}-\frac{1}{2}\frac{\partial^2W(\phi)}{\partial\phi^2}\psi\psi+\text{h.c}\right]\right).\end{aligned}\tag{4.5}$$

Note that F has no kinetic term, revealing that F is a non-dynamic field. Its purpose is to ensure that the action is invariant under supersymmetry for all field configurations, even when the configuration does not obey the equations of motion. Hence, its presence is required to match the number of off-shell bosonic degrees of freedom with the number of fermionic off-shell degrees of freedom. F is usually called *auxiliary field*. Introducing the equations of motion for F and $\bar{F}$, the Wess-Zumino action (4.5) reads as

$$S_{WZ}=\int d^4x\ \left(\partial_\mu\bar{\phi}\partial^\mu\phi-i\bar{\psi}\bar{\sigma}^\mu\partial_\mu\psi-\left|\frac{\partial W}{\partial\phi}\right|^2-\frac{1}{2}\frac{\partial^2W(\phi)}{\partial\phi^2}\psi\psi+\frac{1}{2}\frac{\partial^2\bar{W}(\bar{\phi})}{\partial\bar{\phi}^2}\bar{\psi}\bar{\psi}\right).\tag{4.6}$$

The chiral superfield $W(\Phi)$ is known as *superpotential* and usually is defined a potential $V(\phi,\bar{\phi})$ depending on the complex scalar fields by $V(\phi,\bar{\phi})=\left|\frac{\partial W(\phi)}{\partial\phi}\right|^2$, what explains the terminology used earlier in (3.10). Here a term of the form $\Phi\Phi^\dagger$ was used in the action, but more general constructions could have been used. Such a choice is a particular form of the *Kähler potential* $\mathcal{K}(\Phi,\Phi^\dagger)$, which is defined to be any real superfield that is a function of the chiral and antichiral superfields.

On the other hand, the real superfield in components reads as [29]

$$\begin{aligned}V(x,\theta,\bar{\theta})=&\ C(x)+\theta\chi(x)+\bar{\theta}\bar{\chi}+i\theta^2M(x)-i\bar{\theta}^2\overline{M}(x)+\theta\sigma^\mu A_\mu(x)\\&+\theta^2\bar{\theta}\left(\bar{\lambda}(x)+\frac{i}{2}\bar{\sigma}^\mu\partial_\mu\chi(x)\right)+\bar{\theta}^2\theta\left(\lambda(x)+\frac{i}{2}\sigma^\mu\partial_\mu\bar{\chi}(x)\right)\\&+\frac{1}{2}\theta^2\bar{\theta}^2\left(D(x)-\frac{1}{2}\partial_\mu\partial^\mu C(x)\right),\end{aligned}\tag{4.7}$$

where C and D are two real scalars fields, M is a complex scalar field, χ_α and λ_α are left-handed spinors, and A_μ is a real vector field. As A_μ is a gauge vector field, it must enjoy a gauge transformation, which is accomplished when the real superpotential transforms as the following generalised gauge transformation

$$V\longrightarrow V+i(\Omega-\Omega^\dagger)\ ,\tag{4.8}$$

with Ω a chiral superfield. A simple choice is the one where $C=M=\chi_\alpha=0$, which is called *Wess-Zumino gauge*. If one performs a supersymmetry transformation, the resulting action will not respect the Wess-Zumino gauge, so a compensation transformation is needed as a result. This is achieved if the supersymmetry transformations adds a strength field $F^{\mu\nu}$. Later, in Section 5.2, we will discuss the $U(1)$ supersymmetric gauge theory and see how to build the corresponding supersymmetric gauge action.

The vector superfield can be used to build supersymmetric invariant Yang-Mills actions. To construct the corresponding field strength one needs also the following chiral superfield

$$W_\alpha = -\frac{1}{4}\overline{\mathcal{D}}^{\dot\beta}\overline{\mathcal{D}}_{\dot\beta}\mathcal{D}_\alpha V \ , \tag{4.9}$$

known as *field strength superfield*. Taking into account the chirality of W^α, is it easy to show that the following action is supersymmetric invariant

$$S_{YM} = \frac{1}{4}\int d^4x d^2\theta \left(W^\alpha W_\alpha + \text{h.c.}\right) \ . \tag{4.10}$$

After integration in half of the Grassmann space, one gets

$$S_{YM} = \int d^4x \left(-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - i\psi\sigma^\mu\partial_\mu\bar\psi + \frac{1}{2}D^2\right). \tag{4.11}$$

The auxiliary field D can be eliminated form the action. The remaining terms are the $U(1)$ Yang-Mills term and a Dirac fermion. Later, we will use these techniques to build more exotic supersymmetric models.

5 Generalizations of the Abelian-Higgs model and their moduli space

In the previous sections we have provided the main tools necessary for the study of topological solitons, and in particular, for the study of BPS solitons. From now on, we will focus on the study of a particular soliton, the local vortex. The canonical model that admits vortex type solutions was introduced in Section 3.2, through the Abelian-Higgs model. However, this model admits variations that, without altering the fundamental characteristics of the solutions, give rise to new dynamics. Several modifications of the Abelian-Higgs model have been studied already in the literature. For example, the possibility of finding a BPS structure in these models has been analysed using supersymmetric criteria [34]. The vortex dynamics in the moduli space approximation for different generalised models [31, 32] has been examined, as well as the possibility of finding analytic solutions of the Bogomolny equations [35]. The aim of this section is to follow this line of research, proposing and studying other generalizations of the standard Abelian-Higgs model and obtaining the associated metric of the corresponding moduli space, in order to acquire information about the dynamics of the vortices in these models through the geodesic motion on the moduli space.

5.1 Abelian-Higgs model with a modified Maxwell term

In [31] the authors propose a modification of the Abelian-Higgs model of the form

$$\mathcal{L} = -\frac{1}{4}G(|\phi|)F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}\overline{D_\mu\phi}D^\mu\phi - U(|\phi|), \tag{5.1}$$

with both $G(|\phi|)$ and $U(|\phi|)$ positive definite. Here, $U(|\phi|)$ is a potential, and $G(|\phi|)$ is interpreted as a dielectric term. If one wants to preserve the BPS bound and the Bogomolny equations, not all combinations of $G(|\phi|)$ and $U(|\phi|)$ are possible. It can be shown that, for the following combination of both functions,

$$U(|\phi|) = \frac{\lambda\kappa^2}{8e^2G(|\phi|)}(|\phi|^2 - v^2)^2 \ , \tag{5.2}$$

the BPS structure is preserved. We denote "v" as the vacuum value. It can be shown also that a Bogomolny type bound is saturated [31] by the solutions of the first order equations

$$D_1\phi \pm iD_2\phi = 0, \qquad eF_{12} = \pm\frac{e^2v^2}{2G(|\phi|)}\left(\frac{|\phi|^2}{v^2} - 1\right) . \tag{5.3}$$

They found that only in the special case of considering the standard Abelian-Higgs model, it is possible to reduce the kinetic energy to a contour integral, where no analytical knowledge of the vortices beyond the center of each vortex is therefore needed. This can be verified from the following expression

$$T = \frac{1}{2}\int_{\mathbb{R}^2\backslash\{C_r\}} d^2x \left\{-\frac{1}{2e}\partial_k\left[2G\dot{\chi}\dot{a}_k + \varepsilon_{jk}G\dot{h}\dot{a}_j\right] + \frac{\dot{h}}{2e}\left[\dot{a}_j\varepsilon_{jk}\partial_k G - \dot{G}F_{12}\right]\right\} , \tag{5.4}$$

arguing by Stokes' theorem. In the above expression, the notation from Section 3.2.2 has been used, and C_r denotes a small disk around the zero Z_r. Furthermore, the authors proved that the model (5.1) possesses a kinetic energy whose associated metric on $\mathcal{M}_N$ is simply Kähler when the model reduces to the standard Abelian-Higgs. We will see that further generalizations of the model do not spoil the Kähler structure.

5.2 Abelian-Higgs model with a modified Higgs term

Motivated by the previous section, we aim to discuss a new modification of the Abelian-Higgs model that takes the form

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}G(\phi,\overline{\phi})\overline{D_\mu\phi}D^\mu\phi - V(\phi,\overline{\phi}) \ . \tag{5.5}$$

Note that the modified Higgs part has the structure of a gauge invariant nonlinear sigma model (see Section 3.2.1). As we have mentioned for the case (5.1), a general choice of G and V does not provide a BPS model, but requires a precise combination to achieve it. To show the specific relationship, let us proceed by SUSY considerations using the superfield formulation.

In the literature it is well known that the kinetic term of ϕ can be obtained from the $U(1)$ invariant Kähler potential

$$\mathcal{L}_\phi = \int d^2\theta d^2\overline{\theta}\ \mathcal{K}(\Phi^\dagger e^{2V}\Phi) \ , \tag{5.6}$$

where Φ (resp. $\Phi^\dagger$) is a chiral (resp. antichiral) superfield, whereas V is a vector superfield. After integrating in Grassmann coordinates, one gets in components

$$\mathcal{L}_\phi = G(\phi,\overline{\phi})(D_\mu\overline{\phi}D^\mu\phi + F\overline{F}) + \frac{D}{2}\left(\phi\frac{\partial K}{\partial\phi} + \text{h.c}\right) + \text{fermions} \ , \tag{5.7}$$

where F and D the non-dynamical auxiliary fields and K is the Kähler potential when $\theta = 0$. In addition, one obtains the identification

$$G(\phi,\overline{\phi}) = \partial^2_{\phi,\overline{\phi}}K \ . \tag{5.8}$$

The Maxwell term is also standard and can be generated in the superfield formulation through

$$\mathcal{L}_M = \frac{1}{4}\left(\int d^2\theta W^\alpha W_\alpha + \text{h.c}\right), \tag{5.9}$$

with W_α the field strength superfield. After integrating, we get

$$\mathcal{L}_M = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}D^2 + \text{fermions} \ . \tag{5.10}$$

To allow for non-trivial solitonic solutions, we need a spontaneous symmetry breaking, which will be encoded by the so-called *Fayet-Iliopoulos term*

$$\mathcal{L}_{FI} = -\xi \int d^2\theta d^2\overline{\theta}\ V = -\frac{\xi}{2}D \ . \tag{5.11}$$

Adding the three terms we derive the total Lagrangian. After removing the auxiliary fields F and D we get

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + G(\phi,\overline{\phi})\overline{D_\mu\phi}D^\mu\phi - \frac{1}{2}\left(\frac{\xi}{2} - \frac{1}{2}\phi\frac{\partial K}{\partial\phi} - \text{h.c}\right)^2, \tag{5.12}$$

where

$$V = -\frac{1}{8}\left(\xi - \phi\frac{\partial K}{\partial\phi} - \text{h.c}\right)^2, \tag{5.13}$$

Note that with the choice $K = \frac{1}{2}\phi\overline{\phi}$, $\xi = 1$ the standard Abelian-Higgs model is recovered. We define $W(|\phi|) = \phi\dfrac{\partial K}{\partial\phi} + \overline{\phi}\dfrac{\partial K}{\partial\overline{\phi}}$. The BPS equations of this model are then (for the choice $\xi = 1$)

$$D_1\phi + iD_2\phi = 0, \qquad B - \frac{1}{2}(1 - W(|\phi|)) = 0. \tag{5.14}$$

Now we adapt Samol's calculation to the present situation. After defining $h = \log|\phi|^2$, and using (5.14), the new Taubes' equation is

$$\nabla^2 h + 1 - W(e^h) = 4\pi\sum_{r=1}^{N}\delta^2(\mathbf{x} - \mathbf{X}_r) \ . \tag{5.15}$$

The Gauss' law is modified according to

$$\partial_i\partial_0 a_i + iG(e^h)(\overline{\phi}\partial_0\phi - \phi\partial_0\overline{\phi}) = 0, \tag{5.16}$$

and the kinetic energy in terms of h and $\eta = \partial_0 \log\phi$ as

$$T = \frac{1}{2}\int d^2x(4\partial_z\eta\partial_{\overline{z}}\eta + 2e^h G(e^h)\eta\overline{\eta}) \ . \tag{5.17}$$

If we define $\phi = e^{\frac{1}{2}h + i\chi}$, we obtain the following equation from the modified Gauss' law

$$(\nabla^2 - 2e^h G(e^h))\partial_0\chi = 0, \tag{5.18}$$

whereas from the Taubes equation we get

$$(\nabla^2 - e^h W(e^h))\partial_0 h = 0 \ . \tag{5.19}$$

Thus

$$(\nabla^2 - 2e^h G(e^h))\partial_0 \eta = 0, \tag{5.20}$$

since $W'(e^h) = 2G(e^h)$. Near a zero Z_r of ϕ, one can still write [22] ϕ as

$$\phi = (z - Z_r)e^k \ . \tag{5.21}$$

Following Samols approach [16], no further modifications are necessary, so the kinetic energy and the element of line result in

$$T = \frac{\pi}{2}\sum_{r,s=1}^{N}\left(\delta_{rs} + 2\frac{\partial \bar{b}_s}{\partial Z_r}\right)\dot{Z}_r\dot{\bar{Z}}_s \Longrightarrow ds^2 = \pi\sum_{r,s=1}^{N}\left(\delta_{rs} + 2\frac{\partial \bar{b}_s}{\partial Z_r}\right)d\dot{Z}_r d\dot{\bar{Z}}_s \ , \tag{5.22}$$

that are formally identical to (3.52). The metric in the moduli space is therefore

$$g_{r\bar{s}} = \pi\left(\delta_{rs} + 2\frac{\partial \bar{b}_s}{\partial Z_r}\right). \tag{5.23}$$

It is easy to prove that the 2-form associated to this metric is closed and, as a consequence, the metric in the moduli space is Kähler.

It is important to realize that, although the expression is formally the same as in the standard case, the coefficients $\bar{b}_s$ are not the same because the Higgs field does not satisfy the same BPS equations. Let us clarify this point. All the information about the metric is encoded in the $\bar{b}_s$ coefficients, which in turn are the linear terms in the expansion of h as we showed in (3.51). Now, a BPS solution of (5.14), say (ϕ, a_i), will in general differ from a standard vortex solution. This implies that the dependence of $\bar{b}_s$ in the moduli space coordinate Z_r will be different from the standard case. As a consequence, although the moduli space metric is formally identical to the Abelian-Higgs vortex, one should expect a different geodesic structure and therefore a different dynamics.

On the other hand, it is noteworthy that contrary to what happens in the generalized model with dielectric term [31], here any choice of K produces a Kähler structure in the moduli space. Actually, this is quite natural, since any real function K (Kähler potential) gives a Kähler metric in the target space manifold. This means, in particular, that the generalized model (5.12) inherits all the features of the standard moduli associated to the Kähler structure, namely, that the $\bar{b}_s$ coefficients do not change under a rigid translation of the positions of the vortices. As a consequence, the metric remains invariant. This allows for a splitting of the metric in terms of the center of mass of the vortices and the relative distances, which reduces effectively the degrees of freedom of the moduli. For details see [16]. The explicit computation of the geodesics for a 2-vortex configuration as well as the comparison with full numerical simulations is left for future work as a natural continuation of this Master's thesis.

5.3 Abelian-Higgs model with magnetic impurity

To deform the Lagrangian density (3.37) including magnetic impurities (so referred since it couples to the magnetic field) while preserving half of the BPS structure, the authors in [33] suggested the following model

$$\mathcal{L} = \int\left(-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}\overline{D_\mu\phi}D^\mu\phi - \frac{1}{8}(1 + \sigma_m - |\phi|^2)^2 + \frac{1}{2}\sigma_m B\right)d^2x \ , \tag{5.24}$$

where σ is a static background field with finite L^2 norm. In this case, the Bogomolny equations look like

$$D_1\phi + iD_2\phi = 0, \qquad B - \frac{1}{2}(1+\sigma_m - |\phi|^2) = 0, \tag{5.25}$$

whose solutions saturate the energy bound $E \geq \pi N$. It is straightforward to derive the new Taubes equation, which reads as follows

$$\nabla^2 h + 1 + \sigma_m - e^h = 4\pi \sum_{r=1}^{N} \delta^2(z - Z_r) \ . \tag{5.26}$$

For the vacuum state and for the $N = 1$ vortex, the Taubes equation is numerically solved by an over-relaxing method in [32], where it is argued that a localized impurity also has a localized response. Following the standard derivation by Samols, the kinetic energy and thus the metric of the moduli space are [32]

$$T = \frac{\pi}{2}\sum_{r,s=1}^{N}\left(\delta_{rs}(1+\sigma_m) + 2\frac{\partial \bar{b}_s}{\partial Z_r}\right)\dot{Z}_r\dot{\overline{Z}}_s \to ds^2 = \pi \sum_{r,s=1}^{N}\left(\delta_{rs}(1+\sigma_m) + 2\frac{\partial \bar{b}_s}{\partial Z_r}\right)d\dot{Z}_r d\dot{\overline{Z}}_s \ , \tag{5.27}$$

where the impurity is present even in the case of just a single vortex, revealing that the impurity always has effect on the dynamics of the vortices. The new contribution arises from the coefficient d_r of the expansion of the function $h(z,\overline{z})$ at a zero of ϕ, resulting in $d_r = -\frac{1}{4}(1+\sigma_m)$.

An interesting localized impurity limit arises when the impurity approaches a delta function, i.e., an impurity of the form $-4\pi\alpha\delta(z)$ with $\alpha \in \mathbb{N}$. In that case, (5.26) looks like the impurity-free Taubes equation of $N+\alpha$ vortices where α of them are placed at the origin. Therefore, the metric of the moduli space of N vortices in the presence of α delta function impurities is expected to be the submanifold of the moduli space of $N+\alpha$ vortices in the case where α vortices are constrained to lie at $z = 0$. This was proved numerically [32] with great agreement for the case of a 1-vortex with an impurity that is led to be a delta source.

5.4 Abelian-Higgs model with Higgs impurity

It is possible to add another partially BPS preserving impurity in the Abelian-Higgs model although this time, contrary to the case [32], the impurity is coupled to the Higgs field, and thus refers to it as *Higgs impurity*. The half BPS preserving model proposed in [34] with a Higgs impurity is

$$\mathcal{L} = \int d^2x \left(-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}\overline{D_\mu\phi}D^\mu\phi - \frac{1}{8}(1-|\phi|^2)^2\right) + \mathcal{L}_{Higgs} \ , \tag{5.28}$$

being

$$\mathcal{L}_{Higgs} = -\int \frac{1}{2}(\sigma_h\overline{\sigma}_h\phi\overline{\phi} + \overline{\sigma}_h\overline{\phi}D_{\overline{z}}\phi + \sigma_h\phi\overline{D_z\phi})d^2x \ , \tag{5.29}$$

where σ_h is complex-valued and transforms trivially as $\sigma_h \to \sigma_h$ under a gauge transformation. The BPS structure of this model can be traced back to the fact that the impurity term (5.29) is invariant under a supersymmetric transformation [34]. On the other hand, this type of linear coupling in derivatives between the field and the impurity can be a bit exotic. However, it is very similar to the well-known Dzyaloshinskii−Moriya interaction that stabilizes magnetic Skyrmions in two dimensions. In fact, there is a relationship between the solutions of this type of models with impurities and the critical magnetic Skyrmions [34]. The Bogomolny equations are of the form

$$D_1\phi + iD_2\phi + \sigma_h\phi = 0, \qquad B - \frac{1}{2}(1-|\phi|^2) = 0, \tag{5.30}$$

and the solutions of this equations also saturate the bound $E \geq \pi N$.

Through the Bogomolny equations (5.30), we derived the following new Taubes' equation

$$\nabla^2 h + 1 - e^h + 2(\partial_1\sigma_1 + \partial_2\sigma_2) = 4\pi \sum_{r=1}^{N} \delta^2(z - Z_r) \ , \tag{5.31}$$

with σ_1 and σ_2 the real and imaginary part of σ_h respectively. Such a expression is obtained as in (3.49), where in this occasion, we find that the gauge field components (3.50) look like

$$a_1 = \frac{1}{2}\partial_2 h + \partial_1\chi + \sigma_2 \ , \qquad a_2 = -\frac{1}{2}\partial_1 h + \partial_2\chi - \sigma_1 \ . \tag{5.32}$$

Since σ_h is time-independent, the steps to derive the metric on the moduli space of this model are similar to that ones followed by Samols. Remember that a constraint is imposed only on the coefficient d_r of the expansion (3.51) of $h(z, \overline{z})$ around a zero Z_r of the Higgs field ϕ, in order to satisfy the Taubes equation (3.49). For the model (5.28), we obtain that the coefficient d_r is

$$d_r = -\frac{1}{4} - 2(\partial_1\sigma_1 + \partial_2\sigma_2) \ . \tag{5.33}$$

Consequently, we derive that the kinetic energy and the metric associated are

$$T = \frac{\pi}{2} \sum_{r,s=1}^{N} \left(\delta_{rs}(1 + 2(\partial_1\sigma_1 + \partial_2\sigma_2)) + 2\frac{\partial \bar{b}_s}{\partial Z_r} \right) \dot{Z}_r \dot{\overline{Z}}_s \ , \tag{5.34}$$

$$ds^2 = \pi \sum_{r,s=1}^{N} \left(\delta_{rs}(1 + 2(\partial_1\sigma_1 + \partial_2\sigma_2)) + 2\frac{\partial \bar{b}_s}{\partial Z_r} \right) d\dot{Z}_r d\dot{\overline{Z}}_s \ . \tag{5.35}$$

Again, even in the case of just a single vortex, the dynamics is modified by the presence of the (Higgs) impurity. Of course, this is expected since the impurity is nothing but a background field interacting with the vortex which breaks the translational invariance. This implies, in particular, that the vortex does not preserve its shape through the moduli space even in the $N = 1$ configuration. Let us assume that the impurity is exponentially localized near the origin, then one should expect an asymptotically trivial metric (i.e. a constant metric). Note that far from the impurity the BPS equations (5.30) reduce to the standard ones. As the vortex approaches the origin the effect of the impurity cannot be neglected and the change of shape in the vortex translates into accelerations or decelerations. As a consequence, the geodesics associated to the 1-vortex dynamics are not straight lines anymore and therefore the metric is non-trivial. However, it is important to remember that the impurity does not interact statically with the vortex, as the model is BPS. Hence, this repulsion/attraction effect only takes place dynamically.

A detailed analysis of the geodesic of this manifold it is still necessary, but due to the similarities of (5.30) with the BPS equations of the magnetic impurity model (5.25), one should expect similar results to those presented in [32]. This work will be done later on, using numerical simulations.

5.5 Magnetic and Higgs impurities in the AH model

Let's suppose that we have both types of impurities. The Lagrangian density for that situation is

$$\mathcal{L} = \int \left(-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}\overline{D_\mu\phi}D^\mu\phi - \frac{1}{8}(1 + \sigma_m - |\phi|^2)^2 + \frac{1}{2}\sigma_m B \right) d^2x + \mathcal{L}_{Higgs} \ . \tag{5.36}$$

The model (5.36) is still SUSY invariant and preserves the BPS structure [34]. The Bomomolny equations read as

$$D_1\phi + iD_2\phi + \sigma_h\phi = 0, \qquad B - \frac{1}{2}(1+\sigma_m - |\phi|^2) = 0, \tag{5.37}$$

which is just a combination of the Bogomolny equations derive for each impurity case. Through these Bogomolny equations, we derive the following Taubes' equation

$$\nabla^2 h + 1 - e^h + 2(\partial_1\sigma_1 + \partial_2\sigma_2) + \sigma_m = 4\pi \sum_{r=1}^{N} \delta^2(z - Z_r)\ . \tag{5.38}$$

As usual, adapting the Samols calculation to the present situation, we conclude that

$$d_r = -\frac{1}{4}(1+\sigma_m) - \frac{1}{2}(\partial_1\sigma_1 + \partial_2\sigma_2)\ , \tag{5.39}$$

and thereby,

$$T = \frac{\pi}{2} \sum_{r,s=1}^{N} \left(\delta_{rs}(1+\sigma_m + 2(\partial_1\sigma_1 + \partial_2\sigma_2)) + 2\frac{\partial \bar{b}_s}{\partial Z_r} \right) \dot{Z}_r \dot{\bar{Z}}_s\ , \tag{5.40}$$

$$ds^2 = \pi \sum_{r,s=1}^{N} \left(\delta_{rs}(1+\sigma_m + 2(\partial_1\sigma_1 + \partial_2\sigma_2)) + 2\frac{\partial \bar{b}_s}{\partial Z_r} \right) d\dot{Z}_r d\dot{\bar{Z}}_s\ . \tag{5.41}$$

The first aspect that is noteworthy is that for $\sigma_m + 2(\partial_1\sigma_1 + \partial_2\sigma_2) = 0$ the structure of (5.41) reduces formally to the standard metric (3.52). This suggests a sort of cancelation between the Higgs and magnetic impurities with respect to the metric structure in such a way that the general properties of the standard moduli space remain (since the metric has the same structure). However, the geodesics of the moduli are not the same, as the dependence of $\bar{b}_s$ in Z_r still gets modified by the presence of impurities. A numerical study of the moduli is left for a future work.

6 Moduli space approximation with internal degrees of freedom

In this section we intend to study the moduli space dynamics for the 1-vortex including not only translational degrees of freedom (the standard moduli space discussed so far) but also internal degrees of freedom, which allows the vortex to change its shape. Once the shape of the 1-vortex has been modified, the new configuration is not BPS, so it does not obey the Taubes' equation and the Samols approach is not applicable as in the procedures performed in Section 5. In Section 6.1 we show a first example of collective coordinates method with internal degrees of freedom in the case of the ϕ^4 model, to familiarize ourselves with the notion of internal structure, its obtaining, and the coupling it can have with the translational degrees of freedom. Then, in Section 6.2, we suggest a perturbation of the BPS solution in terms of a single shape mode, which will allow us to identify the unperturbed terms with the metric derived by Samols and the rest of the contributions with the influence of the change of shape.

6.1 Collective coordinates method for the ϕ^4: A toy example

In Section 3.1 we had the opportunity to study the ϕ^4 model, where the Lagrangian there exposed takes the simple form

$$L = \int \left(\frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}(1-\phi^2)^2 \right) dx \tag{6.1}$$

for a given rescaling of the time, space and of the field in terms of the parameters λ and m from (3.12). The standard degree of freedom is the position of the kink, due to the translational invariance of the theory. For that reason, let's consider the ϕ^4 kink solution and promote the position of the kink to a time-dependent variable $X_0 \to X(t)$

$$\phi_k(x,t) = \tanh(x - X(t)) \ . \tag{6.2}$$

In this case, the effective Lagrangian that appears by introducing (6.2) into (6.1) is simply

$$L_{eff} = -\frac{4}{3} + \frac{2}{3}X'(t)^2 \ , \tag{6.3}$$

so the associated Euler-Lagrange equation is $X''(t) = 0$, whose solution is a motion at constant velocity $X(t) = x_0 + vt$. Obviously, this solution is not relativistic because it does not include the Lorentz contraction, so the lower the velocity, the better this description will be. However, the following could still be done instead of (6.2): let us propose the so-called *Derrick mode ansatz*

$$\phi_k(x,t) = \tanh\left(\frac{x - X(t)}{a(t)}\right) , \tag{6.4}$$

where a new degree of freedom has been inserted which can be interpreted as the size of the kink. Introducing this new ansatz back into (6.1), the effective Lagrangian obtained is

$$L_{eff} = -\frac{2}{3a(t)} - \frac{2a(t)}{3} - \frac{a'(t)^2}{3a(t)} + \frac{\pi^2 a'(t)^2}{18a(t)} + \frac{2X'(t)^2}{3a(t)} \ . \tag{6.5}$$

If one calculates the Euler-Lagrange equations associated to these two degrees of freedom, it is trivial to confirm that a possible combination is

$$X(t) = X_0 + vt, \qquad a(t) = \sqrt{1 - v^2} \ , \tag{6.6}$$

that is, a Lorentz boosted kink that contains the desired Lorentz contraction, so this simple model can describe relativistic solutions. One could say that this is a expected possible solution because of the Lorentz invariance of the Lagrangian (6.1). Notably, however, the ansatz (6.4) also contains the non-trivial solution

$$X = X_0 \ , \qquad a(t) = 1 + \epsilon \cos\left(\sqrt{\frac{12}{\pi^2 - 6}}\, t\right) , \tag{6.7}$$

which describe a stilled kink with its size oscillating slightly (with a small amplitude ϵ) with respect to its characteristic size.

Alternatively, let us consider this other ansatz

$$\tanh(x - X(t)) + a(t)\eta_s(x - X(t)), \qquad \eta_s(x - X(t)) = \frac{3}{2}\frac{\tanh(x - X(t))}{\cosh(x - X(t))}, \tag{6.8}$$

where η_s denotes the so-called *shape mode* of the kink. The spectral structure of the kink is formed by this shape mode, a zero mode, and a continuum of scattering states. Such a spectral structure comes from the Schrödinger-like eigenvalue problem of the linearised equations of motion of the ϕ^4 model. Then, the combination (6.8) is an exact solution of the linearised equations of motion if $a(t)$ is very small with respect to the scale of the kink mass. This time, the ansatz is given by the perturbed kink solution through the shape mode with amplitude $a(t)$. Introducing (6.8) into (6.1), one obtains

$$L_{eff} = -\frac{4}{3} - \frac{9a(t)^2}{4} + \frac{3}{4}a'(t)^2 + \frac{2}{3}X(t)'^2 + \frac{3}{8}\pi a(t)X'(t)^2 + \frac{21}{20}a(t)^2X'(t)^2 \ . \tag{6.9}$$

In case that the kink was stilled, the Euler-Lagrange equations for $a(t)$, at leading order, would give the equation of an harmonic oscillator with a frequency $\sqrt{3}$, value that is not very away from the frequency of (6.7). The main difference is then that the ansatz (6.8) contains the exact spectral structure whereas the ansatz (6.4) does not, although on the contrary it allows for Lorentz contraction. When the terms beyond quadratic order are considered, there is a coupling between the two dynamical parameters, so the energy is exchanged between the translational and the internal mode. This produces an involved dynamics in which the kink, instead of following a motion at constant speed, alternates time intervals in which it accelerates with time intervals in which it decelerates (see Figure 6.1). Sometimes this motion is called *wobbling motion*.

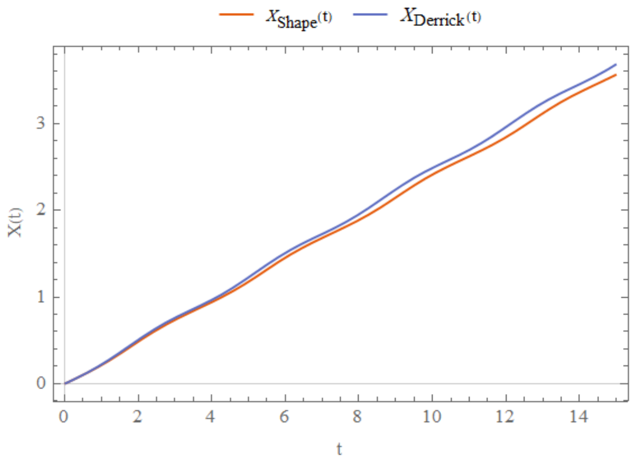

Figure 6.1: The blue curve is the position of the kink with the Derrick mode ansatz ($X_0 = 0$, $X_0' = 0.2$, $a_0 = 0.8$, $a_0' = 0$), and the red curve is the position of the kink with the shape mode ansatz ($X_0 = 0$, $X_0' = 0.2$, $a_0 = 0.1$, $a_0' = 0$).

The comparison of the two ansatze reveals the degree of agreement between them, having a good degree of approximation. The difference starts to increase as time increases. The reason for this is that the oscillation frequency is not the same for the shape mode as it is for Derrick mode. Therefore, since energy is exchanged between the translational and the oscillatory mode, the phase difference between the ansatze leads to an advance from one solution to the other, specifically, from Derrick mode to the shape mode.

6.2 Vortex dynamics with internal degrees of freedom

Now that we are familiarized with the inclusion of internal degrees of freedom in the collective coordinates method for the ϕ^4 kink, let us apply the same approach to the vortex. The treatment is more cumbersome this time because now we do not have only a single scalar field, but also a gauge field. Therefore, we need two time-dependent parameters $A(t)$ and $B_i(t)$ to introduce the effect of the internal degrees of freedom. This would allow us to study the vortex in situations where it is excited. Some qualitative results can be expected to coincide with those reported in Section 6.1. For example, an exchange of energy between the translational and internal degrees of freedom is expected to occur, appearing a wobbling motion, i.e., a motion where the vortex accelerates and decelerates and where the energy goes back and forth between the kinetic energy and the internal energy.

We proceed by a perturbation approach, assuming the following ansatz

$$\phi(\mathbf{x}-\mathbf{X}(t)) = \phi^{BPS}(\mathbf{x}-\mathbf{X}(t)) + A(t)\eta(\mathbf{x}-\mathbf{X}(t)), \tag{6.10}$$

$$a_i(\mathbf{x}-\mathbf{X}(t)) = a_i^{BPS}(\mathbf{x}-\mathbf{X}(t)) + B_i(t)\xi_i(\mathbf{x}-\mathbf{X}(t)), \tag{6.11}$$

which consist of a perturbation of the BPS solution by the internal modes $\eta(z-Z(t))$ and $\xi_i(z-Z(t))$ through, respectively, a small amplitude $A(t)$ and $B_i(t)$ with respect to the vortex mass. The reason for $A(t)$ and $B(t)$ to be small is that, in such a case, the ansatze (6.10) and (6.11) are exact solutions of the linearised equations of motion. Therefore, at sufficiently small amplitudes, the ansatz is very close to satisfy the Bogomolny equations. This means that, if such a mode is excited, it will not decay rapidly and will have an important influence on the vortex dynamics. Vortices, as opposed to kinks, have several internal modes, not just one. As a first approximation, we will consider that only a particular internal degree of freedom has been excited. Assuming for a moment that $\mathbf{X}(t) = \mathbf{X}_0$ the second order perturbed action reads

$$S = S_0(\phi^{BPS}, a_i^{BPS}) + \frac{1}{2}\int d^2x dt\ \Omega^\dagger \mathcal{D}\Omega\ , \tag{6.12}$$

where $\mathcal{D}$ is the second order matrix perturbation operator and Ω denotes collectively the perturbations on the BPS solution (for the explicit computation of $\mathcal{D}$ see for example [36]). Since by construction $(\eta(\mathbf{x}), \eta^*(\mathbf{x}), \xi_i(\mathbf{x}))$ are internal modes of the vortex they solve the second order eigenvalue equation

$$\mathcal{M}\begin{pmatrix}\eta(\mathbf{x})\\ \eta^*(\mathbf{x})\\ \xi_i(\mathbf{x})\end{pmatrix} = \omega^2\begin{pmatrix}\eta(\mathbf{x})\\ \eta^*(\mathbf{x})\\ \xi_i(\mathbf{x})\end{pmatrix}, \tag{6.13}$$

where $\mathcal{M}$ is the spatial part of the operator $\mathcal{D}$, i.e.,

$$\mathcal{D} = \begin{pmatrix}-\frac{1}{2}\partial_t^2 & 0 & 0\\ 0 & -\frac{1}{2}\partial_t^2 & 0\\ 0 & 0 & -\delta^{ij}\partial_t^2\end{pmatrix} - \mathcal{M}\ . \tag{6.14}$$

Now, let us assume that $\Omega = (\eta(\mathbf{x})A(t), \eta^*(\mathbf{x})A^*(t), \xi_i(\mathbf{x})B_i(t))$, then the second order perturbed action reads simply

$$S = S_0(\phi^{BPS}, a_i^{BPS}) + \frac{1}{2}\left(\dot{A}(t)\dot{A}^*(t) + \dot{B}_i(t)\dot{B}_i(t) - \omega^2\left(A(t)A^*(t) + B_i(t)B_i(t)\right)\right), \tag{6.15}$$

where we have used the normalization condition for the internal modes

$$\int d^2x\ \eta(\mathbf{x})\eta^*(\mathbf{x}) = 1, \qquad \int d^2x\ \xi_i(\mathbf{x})\xi_i(\mathbf{x}) = 1. \tag{6.16}$$

Let us assume now that $\mathbf{X}$ depends on time. Then, the zero-th order action splits into kinetic and potential part

$$S_0 = T - V\ . \tag{6.17}$$

The kinetic part is just what we got in Section 3.2.2, and in the 1-vortex case is simply

$$T = \frac{\pi}{2}\dot{\bar{Z}}\dot{Z}, \tag{6.18}$$

i.e. the vortex moves at constant velocity. Since we are considering a BPS configuration, the potential V is approximately the energy of the vortex E_1. Finally, at second order in Z, A and B_i we have

$$S = -E_1 + \frac{\pi}{2}\dot{\bar{Z}}\dot{Z} + \frac{1}{2}\left(\dot{A}(t)\dot{A}^*(t) + \dot{B}_i(t)\dot{B}_i(t) - \omega^2\left(A(t)A^*(t) + B_i(t)B_i(t)\right)\right) + \mathcal{O}(3), \tag{6.19}$$

At this order, we note that translational and internal degrees of freedom are not coupled and, as a consequence, the interpretation of the effective action (6.19) is simple: the vortex simply translates, information that is encoded in the variable Z, at constant velocity, and with its excited internal modes, represented by η and ξ_i, oscillating at constant frequency ω

$$Z(t) = Z^{(0)} + vt, \qquad A(t) = A^{(0)}\cos(\omega t), \qquad B_i(t) = B_i^{(0)}\cos(\omega t). \tag{6.20}$$

The coupling between internal and translation degrees of freedom appears beyond quadratic order. This allows for an interchange of energy between them. The time dependence on $\mathbf{X}$, generates in the second order perturbed action, higher order couplings between $\mathbf{X}(t)$, the translational degrees of freedom and $(A(t), B_i(t))$, the internal ones. The perturbation has now the following form $\Omega = (\eta(\mathbf{x}-\mathbf{X}(t))A(t), \eta^*(\mathbf{x}-\mathbf{X}(t))A^*(t), \xi_i(\mathbf{x}-\mathbf{X}(t))B_i(t))$. By inserting this ansatz in the second order action and integrating by parts in the time derivative we get finally

$$\begin{aligned} S_{3,4} = \ & \frac{1}{4}C_1^4\dot{\bar{Z}}\dot{Z}|A|^2 + \frac{1}{4}C_2^4\dot{Z}^2|A|^2 - \frac{i}{2}C_1^3\dot{A}\dot{Z}\bar{A} - \frac{i}{2}C_2^3\dot{\bar{A}}\dot{Z}A + \text{c.c.} \\ & + \frac{1}{4}D_{1,i}^4\dot{\bar{Z}}\dot{Z}B_i^2 + \frac{1}{4}D_{2,i}^4\dot{Z}^2B_i^2 - iD_{1,i}^3\dot{B}_i\dot{Z}B_i + \text{c.c.} \end{aligned} \tag{6.21}$$

The coefficients that appear in $S_{3,4}$ are given by

$$\begin{aligned} C_1^4 &= \int d^2x \left(\partial_x\eta\partial_x\bar{\eta} + \partial_y\eta\partial_y\bar{\eta}\right), \\ C_2^4 &= \int d^2x \left(\partial_x\bar{\eta} - i\partial_y\bar{\eta}\right)\left(\partial_x\eta - i\partial_y\eta\right), \\ C_1^3 &= \int d^2x \left(\partial_y\bar{\eta} + i\partial_x\bar{\eta}\right)\eta, \\ C_2^3 &= \int d^2x \left(\partial_y\eta + i\partial_x\eta\right)\bar{\eta}, \\ D_{1,i}^4 &= \int d^2x \left(\partial_x\xi_i\partial_x\xi_i + \partial_y\xi_i\partial_y\xi_i\right), \\ D_{2,i}^4 &= \int d^2x \left(\partial_y\xi_i + i\partial_x\xi_i\right)\left(\partial_y\xi_i + i\partial_x\xi_i\right), \\ D_{1,i}^3 &= \int d^2x \left(\partial_y\xi_i + i\partial_x\xi_i\right)\xi_i. \end{aligned}$$

These numerical coefficients depend on the chosen internal mode. We do not intend here to make a detailed analysis of the problem, but, even without the knowledge of the internal modes, it is possible to extract some information from the action (6.21). The first obvious consequence we may read directly form (6.21) is that the Kähler structure of the moduli space has been broken by the coupling to the internal modes, as one can see from the Z^2 and $\bar{Z}^2$ terms in the quartic action. Secondly, if we assume that there is no back-reaction of the internal modes on the translation mode, the action (6.21) simply couples the coordinate Z ($\bar{Z}$) to periodic functions of frequency ω (the frequency of the internal mode considered). This translates into a sort of "wavy" trajectories of the vortex, similar to what we found in the kink case. Finally, the most interesting situation requires configurations of N-vortices, each of which could have an internal mode excited. A detailed analysis of such configuration is an ambitious program that requires a numerical analysis of the internal modes as well as the geodesics. These results may provide new relevant insights in the study of vortex scattering.

7 Conclusions

In the present work, we have systematically studied different generalizations of the standard Abelian-Higgs model (3.37). In the literature, previous attempts has been analysed. For example, in [31], the authors inspected the case in which a field dependent factor couples to the Maxwell term (5.1). Inspired by this type of modification of a particular term of the Lagrangian density, we have explored the possibility of constructing a modification of the Abelian-Higgs model where a field dependent factor appears coupled, but to the Higgs term (5.5). For this, we have used the superfield formalism, and we have tried to derive a supersymmetric formulation where the bosonic sector of the theory coincides with the modified Abelian-Higgs model under study. In [31], the authors found that when the Maxwell term is generalized by introducing a field dependent factor, the metric on the moduli space associated to that model is only Kähler for the trivial choice of the pre-factor (5.4), i.e., the pure Abelian-Higgs model (3.37). However, the model that we have proposed admits, for whatever Kähler potential that we assume, a metric on the moduli space that is always Kähler (5.22). This is true by construction, due to the form in which is integrated over the superspace, that induces sigma model terms where the metric on the target space manifold is Kähler by definition. Moreover, the explicit expression is formally the same as in the standard case (3.52). However, we have noted that the geodesic motion is not the same, since the Higgs field satisfies other Bogomolny equations (5.14), and therefore, the $\bar{b}_s$ coefficients are not the same either. We conclude that, as the metric is Kähler, the properties derived from the Abelian-Higgs model regarding the Kähler structure, also hold in our case. As we have mentioned, such properties result in great simplifications on the computation of the moduli space metric.

We have also investigated the case in which the model is modified by the inclusion of impurities, which breaks the translational symmetry of the theory. In [33], the authors proposed a model with a localised impurity that is coupled to the magnetic field (magnetic impurity), and that is a half-BPS preserving soliton-impurity model (5.24). Such a model was studied by [32], and the authors found that the metric on the moduli space changes non-trivially even for a single vortex (5.27). Another possibility of half-BPS preserving soliton-impurity model was conceived in [34], where now, the impurity is coupled to the Higgs field (Higgs impurity). Taking the model proposed in [34] as a basis, we have attempted to extract dynamics information for that model (5.28). Our calculations suggest that the metric on the moduli space (5.35) for that model, is also modified in a non-trivial way even for a single vortex once more. In addition, we expect that the presence of the impurity changes the shape of the vortex, resulting in a repulsion/attraction effect when the vortex moves. At this point, we faced the case in which magnetic and Higgs impurities do appear (5.36) as a more general scenario. We noted something that in principal is not obvious. It is very striking that the metric that we found, i.e. (5.41), enjoys a sort of combination between both types of impurities that simplifies the metric structure, in such a way that the general properties of the standard moduli space remains, when actually these impurities are introduced via completely different couplings. Furthermore, the geodesics are altered due to the modified dependence of $\bar{b}_s$ with the zeros Z_r of the scalar field, as a consequence of the presence of impurities. There are many directions in which one may continue this work, and a comparison of all these results with full numeric simulations is left for the future.

So far, in all the previous models, only the translational degrees of freedom have been taken into account. This fact has motivated us to generalize the standard moduli space for a single vortex, by including internal degrees of freedom through a perturbation approach of the BPS configuration with one of the possible shape modes that the vortex possesses (6.10)-(6.11). We have found the time evolution of the translational degree of freedom $Z(t)$, as well as that of the perturbation amplitudes $A(t)$ and $B_i(t)$, at second order in perturbation theory. We have noted that, at quadratic order in

Z, A and B_i, the translational and internal modes are not coupled, so the vortex moves in straight line at constant speed with the internal modes excited (6.20). It turns out that the couplings appear when terms beyond quadratic order are considered (6.21). As we had expected, some qualitative characteristics are the same as in the kink example. For example, the wobbling motion due to the energy exchange between the translational mode and the internal modes is obtained. We hope to derive a generalization of this approach to an arbitrary number of vortices in forthcoming work, where each vortex could have a different internal mode excited. These results may provide new relevant insights in the study of vortex scattering.

Last but not least, we emphasise the relevance of all the conclusions derived from this work, since we have extracted general properties of the dynamics for these generalised models, and such properties may be important to analyse the vortex-scattering processes. Moreover, the conclusions have been derived without the use of full numerical simulations, which sometimes do not provide enough information to describe certain behaviours, and that might be extremely computationally expensive in many cases.

References

[1] A. Scott, *The encyclopedia of nonlinear science*, Routledge, New York (2004).

[2] R. Rajaraman, *Solitons and instantons: An introduction to solitons and instantons in quantum field theory*, North-Holland Publishing Company, Amsterdam (1982).

[3] N. S. Manton and P. M. Sutcliffe, *Topological solitons*, Cambridge University Press, Cambridge (2004).

[4] A.R.Bishop, J.A.Krumhansl, S.E.Trullinger, *Solitons in condensed matter: A paradigm*, Physica D: Nonlinear Phenomena **1**, 1 (1980) [DOI].

[5] A. Vilenkin and E. P. S. Shellard, *Cosmic stings and other topological defects*, Cambridge University Press, Cambridge (1994).

[6] J. K. Perring and T. H. R. Skyrme, *A model unified field equation*, Nuclear Phys. **31**, 550 (1962) [DOI].

[7] L. F. Mollenauer and J. P. Gordon, *Solitons in optical fibers- Fundamentals and applications*, Academic Press, Cambrigde (2006).

[8] L. V. Yakushevich, *Nonlinear Physics of DNA*, Wiley-VCH, Weinheim (2004).

[9] E. B. Bogomolny, *Stability of classical solutions*, Sov. J. Nucl. Phys. **24**, 449 (1976).

[10] M. K. Prasad and C. M. Sommerfield, *Exact classical solution for the t́ Hooft monopole and the Julia-Zee dyon*, Phys. Rev. Lett. **35**, 760 (1975) [DOI].

[11] G. H. Derrick, C*omments on nonlinear wave equations as models for elementary particles*, J. Math. Phys. **5**, 1252 (1964) [DOI].

[12] Y. M. Shnir, *Topological and non-topological solitons in scalar field theories*, Cambridge University Press (2018).

[13] M. Nakahara, Geometry, *Topology and Physics*, Second Edition, Institute of Physics Publishing (2003).

[14] N. S. Manton, *An effective Lagrangian for solitons*, Nucl. Phys. B **150**, 397 (1979) [DOI].

[15] N. S. Manton, *A remark on the scattering of BPS monopoles*, Phys. Lett. B **110**, 54 (1982). [DOI].

[16] T. M. Samols, *Vortex scattering*, Commum. Math. Phys. **145**, 149 (1992) [DOI].

[17] N. S. Manton, *The force between 't Hooft-Polyakov monopoles*, Nucl. Phys. B, **126**, 525 (1977). [DOI].

[18] C. Adam, N. S. Manton, K. Oles, T. Romanczukiewicz, A. Wereszczynski, *Relativistic Moduli Space for Kink Collisions*, Phys. Rev. D **105**, 065012 (2022) [DOI].

[19] N. S. Manton, K. Oleś, T. Romańczukiewicz, and A. Wereszczyński, *Kink moduli spaces: Collective coordinates reconsidered*, Phys. Rev. D **103**, 025024 (2021) [DOI].

[20] B. M. A. G. Piette, B. J. Schroers and W. J. Zakrzewski, *Dynamics of baby Skyrmions*, Nucl. Phys. B **439**, 205 (1995). [DOI].

[21] R. A. Leese, *Q-lumps and their interactions*, Nucl. Phys. B **366**, 283 (1991) [DOI].

[22] A. Jaffe and C. Taubes, Vortices and monopoles, Birkhäuser (1980).

[23] E. J. Weinberg, *Multivortex Solutions of the Ginzburg-landau Equations*, Phys. Rev. D **19**, 3008 (1979) [DOI].

[24] T. H. R. Skyrme, A non-linear field theory, Proc. R. Soc. Lond. A **260**, 127 (1961) [DOI].

[25] R. A. Battye and P. M. Sutcliffe, *Skyrmions, fullerenes and rational maps*, Rev. Math. Phys. **14**, 29 (2002) [DOI].

[26] S. Coleman and J. Manudla , *All Possible Symmetries of the S-Matrix*, Phys. Rev. **159**, 1251 (1967) [DOI].

[27] R. Haag, J. T. Lopuszański and M. Sohnius, *All possible generators of supersymmetries of the S-matrix*, Nucl. Phus. B **88**, 275 (1975) [DOI].

[28] E. Witten and D. Olive, *Supersymmetry algebras that include topological charges*, Phys. Lett. B **78**, 97 (1978) [DOI].

[29] J. Wess and J. Bagger, *Supersymmetry And Supergravity*, Princeton University Press (1992).

[30] J. Wess and B. Zumino, *Supergauge transformations in four dimensions*, Nucl. Phys. B **70**, 39 (1974) [DOI].

[31] W. G. Fuertes and J. Mateos Guilarte, *Low-energy vortex dynamics in Abelian Higgs systems*, Eur. Phys. J. C **9**, 535 (1999) [DOI]

[32] A. Cockburn, S. Krush and A. Muhamed, *Dynamics of vortices and magnetic impurities*, J. Math. Phys. **58**, 063509 (2017) [DOI]

[33] D. Tong and K. Wong, *Vorteces and Impurities*, J. High Energ. Phys. **2014**, 90 (2014) [DOI].

[34] C. Adam, J.M. Queiruga and A. Wereszczynski, *BPS soliton-impurity models and supersymmetry*, J. High Energ. Phys. **2019**, 164 (2019) [DOI]

[35] D. Bazeia, L. Losano, M. A. Marques and R. Menezes, *Analytic vortex solution in generalized model of the Maxwell-Higgs type*, Phys. Lett. B **778**, 22 (2018) [DOI]

[36] M. Goodband and M. Hindmarsh, *Bound states and instabilities of vortices*, Phys. Rev. D **52**, 4621 (1995) [DOI]