TOPOLOGIES OF CONTINUITY FOR CARATHÉODORY PARABOLIC PDES FROM A DYNAMICAL PERSPECTIVE

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Abstract. Systems of non-autonomous parabolic partial differential equations over a bounded domain with nonlinear term of Carathéodory type are considered. Appropriate topologies on sets of Lipschitz Carathéodory maps are defined in order to have a continuous dependence of the mild solutions with respect to the variation of both the nonlinear term and the initial conditions, under different assumptions on the bound-maps of the nonlinearities.

1. Introduction

The study of the topologies of continuity for Carathéodory ordinary differential equations (ODEs for short) is a classical question with important implications in the field of non-autonomous differential equations and in dynamical systems, and their applications in mathematical modelling. In particular, Artstein [1–3], Heunis [6], Miller and Sell [15,16], Neustadt [17], Sell [18,19], and some more references therein, introduced and studied strong and weak topologies of integral type on spaces of Lipschitz Carathéodory functions. These Carathéodory vector fields define a set E, and the convergence of a sequence $(f_n)_{n\geqslant 1}$ with respect to these topologies in E requires the convergence of the integral of the evaluation of the functions f_n, either pointwise in $\mathbb{R}^N$ (topologies $\mathcal{T}_D$ and σ_D), or uniformly on any bounded set of continuous functions (topology $\mathcal{T}_B$). Recently, Longo et al. [10, 11] completed some parts of this theory by introducing the strong and weak topologies $\mathcal{T}_\Theta$ and σ_Θ, where Θ is a suitable set of moduli of continuity. The motivation is that when the set of m-bounds of the functions in E is equicontinuous, then given $j \in \mathbb{N}$ and a compact interval $I \subset \mathbb{R}$, the solutions of the equations for $f \in E$ defined on I and bounded by j admit a common modulus of continuity and are thus included in a compact subset $\mathcal{K}_j^I$ of continuous functions. Therefore, it is possible to define a countable set of moduli of continuity Θ so that the topologies $\mathcal{T}_\Theta$ and σ_Θ, defined by the uniform convergence of the integral of the evaluation of the functions of E on each compact set $\mathcal{K}_j^I$ are of continuity, i.e., if the vector fields $(f_n)_{n\geqslant 1}$ converge to f in E, and the initial data $(x_n)_{n\geqslant 1}$ converge to x in $\mathbb{R}^N$, then the solutions

of the differential equations $y(t, f_n, x_n)$ converge to $y(t, f, x)$ uniformly on compact time intervals.

The references Longo et al. [12, 13] provide an extension of these methods to Carathéodory delay differential equations. To overcome the problem that in general the solutions of the equations do not share the same modulus of continuity as the initial data, it is necessary to introduce hybrid topologies from suitable pairs of topologies of the types defined above which treat the present and the past components of the equations differently. In addition, the conclusions on the new topologies of continuity for delay differential equations are applied to develop dynamical methods to investigate non-autonomous models in populations dynamics.

In this paper we introduce appropriate versions of these topologies on sets E of measurable functions $f : \mathbb{R} \times \overline{U} \times \mathbb{R}^N \to \mathbb{R}^N$ with a Lipschitz variation on the state component, defining Carathéodory parabolic partial differential equations (PDEs for short) on an open and bounded domain $U \subset \mathbb{R}^M$ with a smooth boundary, and prove the continuous variation of the mild solutions of the associated abstract Cauchy problems (ACPs for short) in the space $C(\overline{U}, \mathbb{R}^N)$ with respect to the initial condition and the function $f \in E$. Although this theory follows the same ideas explained above for the case of ordinary and delayed differential equations, new nontrivial technical challenges appear in this setting, as a consequence of the unbounded behavior of the realization of the Laplacian subject to the different boundary conditions.

The paper is organized in four sections. In Section 2 we consider non-autonomous N-dimensional parabolic initial boundary value (IBV for short) problems

$$\begin{cases} \dfrac{\partial y}{\partial t} = \Delta\, y + f(t, x, y)\,, & t > 0\,, \ x \in U, \\ By = 0\,, & t > 0\,, \ x \in \partial U, \\ y(0, x) = z(x)\,, & x \in \overline{U}, \end{cases}$$

with Neumann, Robin or Dirichlet boundary conditions, where the nonlinear term $f(t, x, y)$ belongs to the space $\mathfrak{LC}$ of Lipschitz Carathéodory functions. We consider the infinite dimensional Banach space $X = C(\overline{U}, \mathbb{R}^N)$ if the boundary conditions are of Neumann or Robin type whereas we will take the space $X = C_0(\overline{U}, \mathbb{R}^N)$ if the boundary conditions are of Dirichlet type. We transform the former problem into an ACP in the space X and prove, for each $z \in X$, the existence of a unique mild solution $u(t, f, z)$ satisfying $u(0, f, z) = z$.

In Section 3 the appropriate versions of the classical topologies $\mathcal{T}_B, \mathcal{T}_D, \sigma_D$ on the space $\mathfrak{SC}$ of strong Carathéodory maps, as well as the new topologies in this PDEs context $\mathcal{T}_{\tilde{D}D}$ and $\sigma_{\tilde{D}D}$, are introduced. In the definition of the topologies $\mathcal{T}_\Theta$ and σ_Θ, a suitable set of moduli of continuity $\Theta = (\Theta_1, \Theta_2)$ is now formed by a pair of parametric families of ordered moduli of continuity that, for each compact interval $I \subset \mathbb{R}$ and each $j \in \mathbb{N}$, determine bounded sets $\mathcal{H}_j^I \subset C(I, C(\overline{U}, \mathbb{R}^N))$ with a precise equicontinuous variation on I, and the seminorms are given by the supremum of the integrals of the evaluation of the functions in $\mathfrak{SC}$ along the maps in $\mathcal{H}_j^I$. The reason for this is that the continuity properties that the mild solutions inherit from the equations involve the regularity properties of the semigroup of operators $(e^{tA})_{t \geqslant 0}$ generated by the realization of the Laplacian in X. Then, for a set $E \subset \mathfrak{LC}$ we study the equivalence of some of the previous topologies under different assumptions on the m-bounds and/or the l-bounds of the maps in E.

Finally, Section 4 states precise conditions implying that the previous topologies are topologies of continuity for the parabolic PDEs. Associated to a family $E \subset \mathfrak{LC}$ with L^1_{loc}-equicontinuous m-bounds, with the aim to have a topology of continuity as coarse as possible and to maintain, to some extent, the properties of compactness which held in the ODEs case, we determine a suitable set of moduli of continuity Θ and a suitable set of radii $\mathcal{R}$ that provide subsets $\mathcal{K}^I_j \subset \mathcal{H}^I_j$ satisfying that given a compact interval $I \subset \mathbb{R}$ and $j \in \mathbb{N}$, the mild solutions of the Carathéodory ACPs for $f \in E$ which are bounded by j on I belong to $\mathcal{K}^I_j$. This is crucial to get that the topologies $\mathcal{T}_{\Theta\mathcal{R}}$ and $\sigma_{\Theta\mathcal{R}}$ whose seminorms are given by the supremum of the integral of the evaluation of the functions on the sets $\mathcal{K}^I_j$ are both of them topologies of continuity for the parabolic equations. Since $\mathcal{T}_{\Theta\mathcal{R}} \leqslant \mathcal{T}_\Theta$ and $\sigma_{\Theta\mathcal{R}} \leqslant \sigma_\Theta$, it is obvious that also $\mathcal{T}_\Theta$ and σ_Θ are topologies of continuity. On the other hand, assuming that the family E admits L^1_{loc}-bounded l-bounds, it is proved the continuous variation of the mild solutions $u(t, f, z)$ when f varies on E with respect to any strong topology $\mathcal{T}_D$, $\mathcal{T}_\Theta$ or $\mathcal{T}_B$, all of them equivalent in this case. Finally, the weaker topologies of pointwise type $\mathcal{T}_{\tilde{D}D}$ and $\sigma_{\tilde{D}D}$ are also topologies of continuity in the previous situations, if in addition E has L^1_{loc}-bounded l-bounds with respect to a modulus of continuity $\hat{\theta}$ in the variable x.

2. Mild solutions for Carathéodory parabolic PDEs

In this section we specify the kind of N-dimensional parabolic IBV problems for $y(t, x)$, with Neumann, Robin or Dirichlet boundary conditions, under consideration:

$$\begin{cases} \dfrac{\partial y_i}{\partial t} = \Delta\, y_i + f_i(t, x, y)\,, & t > 0\,, \ x \in U,\ 1 \leqslant i \leqslant N, \\[2mm] By_i := \alpha_i(x)\, y_i + \kappa\, \dfrac{\partial y_i}{\partial n} = 0\,, & t > 0\,, \ x \in \partial U,\ 1 \leqslant i \leqslant N, \\[2mm] y_i(0, x) = z_i(x)\,, & x \in \overline{U},\ 1 \leqslant i \leqslant N, \end{cases} \tag{2.1}$$

where U, the spatial domain, is a bounded, open and connected subset of $\mathbb{R}^M$ ($M \geqslant 1$) with a sufficiently smooth boundary ∂U; Δ is the Laplacian operator on $\mathbb{R}^M$ and the nonlinear terms are given by a map $f : \mathbb{R} \times \overline{U} \times \mathbb{R}^N \to \mathbb{R}^N$ with components $(f_1, \ldots, f_N)$ within a class of so-called *Carathéodory maps*, namely, the class of Lipschitz Carathéodory maps. The problem has Dirichlet boundary conditions if $\kappa = 0$ and $\alpha_i(x) \equiv 1$ for $1 \leqslant i \leqslant N$; Neumann boundary conditions if $\kappa = 1$ and $\alpha_i(x) \equiv 0$ for $1 \leqslant i \leqslant N$; and Robin boundary conditions if $\kappa = 1$ and $\alpha_i : \partial U \to \mathbb{R}$ is a nonnegative sufficiently regular map for $1 \leqslant i \leqslant N$. Recall that $\partial/\partial n$ denotes the outward normal derivative at the boundary. Finally, the initial condition is given by the values of a map $z = (z_1, \ldots, z_N) : \overline{U} \to \mathbb{R}^N$, which we choose to be continuous, with the compatibility condition $z(x) = 0$ for all $x \in \partial U$ in the Dirichlet case.

We define the standard classes of Carathéodory maps. Let L^1_{loc} denote the space of locally integrable real maps defined on $\mathbb{R}$ and let $|\cdot|$ denote the norm on the N-dimensional space $\mathbb{R}^N$. For convenience, we take $|y| = |y_1| + \ldots + |y_N|$. The symbol B_r will denote the closed ball of $\mathbb{R}^N$ centered at the origin with radius r.

We will consider, and denote by $\mathfrak{C}$, the set of Carathéodory functions $f : \mathbb{R} \times \overline{U} \times \mathbb{R}^N \to \mathbb{R}^N$, $(t, x, y) \mapsto f(t, x, y)$ satisfying

(**C1**) f is Borel measurable and

(**C2**) for every compact set $K \subset \mathbb{R}^N$ there exists a nonnegative function $m^K \in L^1_{loc}$, called *m-bound*, such that for almost every $t \in \mathbb{R}$ one has $|f(t, x, y)| \leqslant m^K(t)$ for all $x \in \overline{U}$, $y \in K$.

We introduce the classes of Carathéodory functions which are subsequently used. In all the cases, we identify the maps within a certain class which differ on a negligible set.

Definition 2.1. A function $f \colon \mathbb{R} \times \overline{U} \times \mathbb{R}^N \to \mathbb{R}^N$, $(t, x, y) \mapsto f(t, x, y)$ is said to be *strong Carathéodory*, and we will write $f \in \mathfrak{SC}$, if it satisfies (**C1**), (**C2**) and

(**S**) for almost every $t \in \mathbb{R}$, the function $f(t, \cdot, \cdot)$ is continuous.

Remark 2.2. Note that, if f is Borel measurable and it satisfies (**S**), then (i) f is Lebesgue measurable in t for each fixed $(x, y) \in \overline{U} \times \mathbb{R}^N$ (see, e.g., Lemma 5.1.2 in Cohn [4]) and (ii) it is continuous in (x, y) for almost every fixed $t \in \mathbb{R}$. In fact, the hypotheses are often formulated by requiring (i) and (ii). But using standard arguments of measure theory one can check that, if (i) and (ii) hold, then there exists a Borel measurable map $g \colon \mathbb{R} \times \overline{U} \times \mathbb{R}^N \to \mathbb{R}^N$ satisfying (**S**), and there exists a set $N \subset \mathbb{R}$ of null measure such that f coincides with g on the set $(\mathbb{R} \backslash N) \times \overline{U} \times \mathbb{R}^N$. To give the idea, the map g is built as a pointwise limit of a sequence of Borel measurable maps $f_n := f \chi_{G_n \times \overline{U} \times B_n}$ for an expanding sequence of Borel sets G_n of $\mathbb{R}$ such that $\mathbb{R} \backslash (\cup_{n \geqslant 1} G_n)$ has null measure and the restriction of f to $G_n \times \overline{U} \times B_n$ is continuous for every $n \geqslant 1$.

Definition 2.3. A function $f \colon \mathbb{R} \times \overline{U} \times \mathbb{R}^N \to \mathbb{R}^N$, $(t, x, y) \mapsto f(t, x, y)$ is said to be *Lipschitz Carathéodory*, and we will write $f \in \mathfrak{LC}$, if $f \in \mathfrak{SC}$ and

(**L**) for every compact set $K \subset \mathbb{R}^N$ there exists a nonnegative function $l^K \in L^1_{loc}$, called *l-bound*, such that for almost every $t \in \mathbb{R}$, $|f(t, x, y_1) - f(t, x, y_2)| \leqslant l^K(t)\,|y_1 - y_2|$ for all $x \in \overline{U}$, $y_1, y_2 \in K$.

In particular, for each compact set $K \subset \mathbb{R}^N$, we refer to *the optimal m-bound* and *the optimal l-bound* of f as to the maps defined almost everywhere by

$$m^K(t) = \sup_{x \in \overline{U},\, y \in K} |f(t, x, y)| \quad \text{and}$$

$$l^K(t) = \sup_{\substack{x \in \overline{U},\, y_1, y_2 \in K \\ y_1 \neq y_2}} \frac{|f(t, x, y_1) - f(t, x, y_2)|}{|y_1 - y_2|}, \tag{2.2}$$

respectively. Clearly, for each compact set $K \subset \mathbb{R}^N$ the suprema in (2.2) can be taken for some countable dense subsets of K and $\overline{U}$, respectively, leading to the same definition, which guarantees that the functions defined in (2.2) are measurable. Moreover, we consider a subclass within the class of Lipschitz Carathéodory maps, by fixing a precise continuous behaviour with respect to x, e.g., Hölder continuity or a Lipschitz character.

Definition 2.4. Given a modulus of continuity, i.e., a non-decreasing continuous function $\hat{\theta} \colon \mathbb{R}^+ \to \mathbb{R}^+$ with $\hat{\theta}(0) = 0$, a function $f \colon \mathbb{R} \times \overline{U} \times \mathbb{R}^N \to \mathbb{R}^N$, $(t, x, y) \mapsto f(t, x, y)$ is said to be *Lipschitz Carathéodory with respect to the modulus of continuity* $\hat{\theta}$ *in the variable* x if it satisfies (**C1**), (**C2**), and

($\mathbf{L}_{\hat{\theta}}$) for every compact set $K \subset \mathbb{R}^N$ there is a nonnegative function $l^K \in L^1_{loc}$ such that for almost every $t \in \mathbb{R}$,

$$|f(t, x_1, y_1) - f(t, x_2, y_2)| \leqslant l^K(t)\big(\hat{\theta}(|x_1 - x_2|) + |y_1 - y_2|\big)$$

for all $x_1, x_2 \in \overline{U}$, $y_1, y_2 \in K$.

Note that these maps also satisfy (**S**). In particular, for each compact set $K \subset \mathbb{R}^N$, we refer to *the optimal l-bound of f with respect to $\hat{\theta}$* as to the map defined almost everywhere by

$$l^K(t) = \sup_{\substack{x_1, x_2 \in \overline{U}, \, y_1, y_2 \in K \\ x_1 \neq x_2, \, y_1 \neq y_2}} \frac{|f(t, x_1, y_1) - f(t, x_2, y_2)|}{\hat{\theta}(|x_1 - x_2|) + |y_1 - y_2|} \, . \tag{2.3}$$

The aim of this section is to study the existence and uniqueness of mild solutions for the Carathéodory ACPs in an appropriate Banach space associated to the parabolic problems (2.1).

Remark 2.5. Trying to keep a common notation, we will consider the space $X = C(\overline{U}, \mathbb{R}^N)$ of the continuous functions on $\overline{U}$ taking values in $\mathbb{R}^N$ if the boundary conditions are of Neumann or Robin type, whereas we will take the space $X = C_0(\overline{U}, \mathbb{R}^N)$ of the continuous functions on $\overline{U}$ vanishing on the boundary ∂U if the boundary conditions are of Dirichlet type.

Using for simplicity the same notation, when dealing with the Laplacian with Neumann or Robin boundary conditions, for each component $i = 1, \dots, N$ we consider on the Banach space $Y = C(\overline{U})$ of real continuous maps on $\overline{U}$ endowed with the sup-norm, the differential operator $A_i^0 z_i = \Delta z_i$ with domain $D(A_i^0)$ given by

$$\left\{ z_i \in C^2(U) \cap C^1(\overline{U}) \,\Big|\, A_i^0 z_i \in C(\overline{U}), \; \alpha_i(x)\, z_i(x) + \frac{\partial z_i}{\partial n}(x) = 0 \; \forall\, x \in \partial U \right\},$$

whereas if the boundary conditions are of Dirichlet type, then we take the Banach space $Y = C_0(\overline{U})$ and the domain $D(A_i^0) = \left\{ z_i \in C^2(U) \cap C_0(\overline{U}) \mid A_i^0 z_i \in C_0(\overline{U}) \right\}$.

In both cases the closure A_i of A_i^0 in Y is a sectorial operator which generates an analytic semigroup of bounded linear operators $(e^{tA_i})_{t \geqslant 0} \subset \mathcal{L}(Y)$, which is strongly continuous (that is, A_i is densely defined), and e^{tA_i} is compact for all $t > 0$ (see, e.g., Smith [20]). Furthermore, using the theory of positive semigroups of operators, it is deduced that $\|e^{tA_i}\| \leqslant 1$ for all $t \geqslant 0$ (see [20, Corollary 7.2.4]).

At this point, we can consider $A = \Pi_{i=1}^N A_i$ which is a sectorial operator with domain $D(A) = \Pi_{i=1}^N D(A_i)$ on the product Banach space $X = Y^N$ endowed with the norm $\|(z_1, \dots, z_n)\|_1 = \sum_{i=1}^N \|z_i\|$. Then, $\{e^{tA} = \Pi_{i=1}^N e^{tA_i} \mid t \geqslant 0\}$ defines a positive semigroup of operators which satisfies $\|e^{tA}\| \leqslant 1$ for all $t \geqslant 0$ and e^{tA} is compact for all $t > 0$.

For convenience, hereafter we consider the sup-norm on the space X, denoted by $\|z\| = \sup_{x \in \overline{U}} |z(x)|$. As a consequence of the bound in the previous paragraph, now for the operator norm on $\mathcal{L}(X)$ associated to the sup-norm, we have that

$$\|e^{tA}\| \leqslant N \quad \text{for all } t \geqslant 0. \tag{2.4}$$

We will be using this bound later.

Given $f \in \mathfrak{SC}$, for each $t \in \mathbb{R}$ and $z \in X$, let $\tilde{f}(t, z) : \overline{U} \to \mathbb{R}^N$ be defined by

$$\tilde{f}(t, z)(x) = f(t, x, z(x)), \quad x \in \overline{U}. \tag{2.5}$$

With Dirichlet boundary conditions we will assume with no further mention that

$$f(t, x, 0) = 0 \quad \text{for all } t \in \mathbb{R}, \ x \in \partial U,$$

so that for all $t \in \mathbb{R}$ and $z \in X = C_0(\overline{U}, \mathbb{R}^N)$, $\tilde{f}(t, z)(x) = 0$ for all $x \in \partial U$, and provided that $\tilde{f}(t, z)$ is continuous on $\overline{U}$, then $\tilde{f}(t, z) \in C_0(\overline{U}, \mathbb{R}^N)$. Note that, since f satisfies (S), for almost every $t \in \mathbb{R}$ and for all $z \in X$, the map $\tilde{f}(t, z) \in X$. We now list some properties of the map $\tilde{f}$.

Proposition 2.6. *Let $f \in \mathfrak{SC}$ and consider the map $\tilde{f} : \mathbb{R} \times X \to X$ defined for almost every $t \in \mathbb{R}$ and every $z \in X$ by* (2.5). *Then, the following properties hold:*

(i) *There exists a Borel measurable map $\tilde{g} : \mathbb{R} \times X \to X$ and a set of null measure $N \subset \mathbb{R}$ such that $\tilde{f} = \tilde{g}$ on the set $(\mathbb{R} \backslash N) \times X$.*

(ii) *If $m^j \in L^1_{loc}$ is an m-bound of f on the closed ball B_j, then for almost every $t \in \mathbb{R}$,*

$$\|\tilde{f}(t, z)\| \leqslant m^j(t) \in L^1_{loc} \quad \text{for all } z \in X, \ \|z\| \leqslant j. \tag{2.6}$$

(iii) *If $I \subseteq \mathbb{R}$ is an interval and $u \in C(I, X)$, then the map defined almost everywhere $I \to X$, $t \mapsto \tilde{f}(t, u(t))$ is measurable on I.*

(iv) *If in addition $f \in \mathfrak{LC}$, and $l^j \in L^1_{loc}$ is an l-bound of f on B_j, then for almost every $t \in \mathbb{R}$, $\|\tilde{f}(t, z_1) - \tilde{f}(t, z_2)\| \leqslant l^j(t)\|z_1 - z_2\|$ for all $z_1, z_2 \in X$ with $\|z_1\|, \|z_2\| \leqslant j$.*

Proof. As noted in Remark 2.2, one can consider the Borel measurable map g obtained as the pointwise limit of a sequence of Borel measurable maps $f_n = f \chi_{G_n \times \overline{U} \times B_n}$ for an expanding sequence of sets G_n of $\mathbb{R}$ such that $\mathbb{R} \backslash (\cup_{n \geqslant 1} G_n)$ has null measure and the restriction of f to $G_n \times \overline{U} \times B_n$ is continuous for every $n \geqslant 1$. It is not difficult to check that if $\mathcal{B}_n := \{z \in X \mid \|z\| \leqslant n\}$, then $\tilde{f}$ restricted to the Borel measurable set $G_n \times \mathcal{B}_n$ is also continuous. Then, $\tilde{f}_n := \tilde{f} \chi_{G_n \times \mathcal{B}_n}$, $n \geqslant 1$ are Borel measurable maps and their pointwise limit $\tilde{g}$ is a Borel measurable map which satisfies $\tilde{f} = \tilde{g}$ on the set $(\cup_{n \geqslant 1} G_n) \times X$, so that (i) holds.

For (ii), note that by (C2), if $\|z\| \leqslant j$, then for almost every $t \in \mathbb{R}$, $|f(t, x, z(x))| \leqslant m^j(t)$ for all $x \in \overline{U}$. From here, (2.6) holds. Finally, (iii) follows from (i) and the continuity of u on I, and (iv) follows from (L) straightaway. $\qquad \square$

We are in a position to consider a Carathéodory ACP associated to the parabolic problem (2.1) with $f \in \mathfrak{SC}$, that is, an evolution equation in the Banach space X (see Remark 2.5) of Carathéodory type:

$$\begin{cases} u'(t) = A u(t) + \tilde{f}(t, u(t)), & t > 0, \\ u(0) = z. \end{cases} \tag{2.7}$$

The fact that A is an unbounded operator on X prevents the application of the basic theory developed by Holly and Orewczyk [8] for Carathéodory abstract ODEs. In any case, one cannot hope for more than solutions in the sense of Carathéodory. It is standard to search for solutions within the class of the so-called *mild solutions*, that is, continuous solutions of the associated integral equation. This integral equation is a generalization of the variation of constants formula in standard ODEs' linear Cauchy problems to the present abstract setting. It is important to note that, since the variation of $\tilde{f}(t, u(t))$ in t is just measurable, we must deal with the Bochner integral (see, e.g., the classical reference Hille and Phillips [7]), which is an extension of the Lebesgue's integration theory to vector-valued functions. A summary of this

theory can be found in Engel and Nagel [5, Appendix C]. One basic issue is that, if the semigroup of operators $(e^{tA})_{t \geqslant 0}$ is strongly continuous, as it is, and $\tilde{f}(t, u(t))$ is measurable, the integrand in (2.8) is measurable, which is a pre-requisite for the integral to make sense.

Definition 2.7. A map $u \in C([0, \delta], X)$ is a *mild solution* of (2.7) on $[0, \delta]$ if it satisfies

$$u(t) = e^{tA} u(0) + \int_0^t e^{(t-s)A} \tilde{f}(s, u(s)) \, ds, \quad t \in [0, \delta]. \tag{2.8}$$

Being not aware of any appropriate reference for this setting, we offer the basic theory on the existence and uniqueness of mild solutions for the Carathéodory ACP (2.7) following the approach in Lunardi [14] in the case of more regular problems, namely, the case in which $\tilde{f}$ is continuous and satisfies a Lipschitz condition in the second variable. Here we lack the continuity and, in order to have a sufficient Lipschitz-like condition on $\tilde{f}$ to guarantee uniqueness of solutions, we restrict ourselves to nonlinear terms f in the class of Lipschitz Carathéodory maps.

Remarks 2.8. 1. Note that everything that we are doing for the IBV problem (2.1) and the associated ACP (2.7) can be done for a problem defined on a time interval $[a, \infty)$ for each $a \in \mathbb{R}$, just with some small modifications. The problems have been written down with initial condition at $t = 0$ for the sake of simplicity.

2. It is not difficult to check that if $u \in C([a, T], X)$ is a mild solution of the ACP (2.7) with initial condition at a, then u is also a mild solution on any smaller interval $[t_0, T]$ with $a < t_0 < T$, by taking $u(t_0)$ as the initial value at t_0. We just mention that one needs to interchange the continuous operator $e^{(t-t_0)A}, t_0 \leqslant t \leqslant T$ with a Bochner integral and this is justified by, e.g., Theorem 3.7.12 in [7].

We include the proof of the following result for the sake of completeness, but the main arguments are taken from [14, Theorem 7.1.2].

Theorem 2.9. *Let $f \in \mathfrak{LC}$. Then, given any $z_0 \in X$ there exist $r, \delta > 0$ such that, if $z \in X$ with $\|z - z_0\| \leqslant r$, problem* (2.7) *has a unique mild solution u in the space* $C([0, \delta], X)$.

Proof. The proof of existence is based on the contraction mapping theorem. Take an integer $j \geqslant 4 N \|z_0\|$ in such a way that, if $\|z - z_0\| \leqslant j/(4 N) =: r$, then $\sup_{t \geqslant 0} \|e^{tA} z\| \leqslant j/2$ (see (2.4)). Let $m^j(t)$ (resp. $l^j(t)$) be the optimal m-bound (resp. l-bound) of f on B_j and consider the normed space

$$Y_j = \{ u \in C([0, \delta], X) \mid \|u(t)\| \leqslant j \text{ for all } t \in [0, \delta] \},$$

with the sup-norm, where $\delta > 0$ is to be determined. Note that Y_j is the closed ball in $C([0, \delta], X)$ of radius j. Fixed $z \in X$ with $\|z - z_0\| \leqslant r$, define the operator $\Gamma : Y_j \to Y_j$ by

$$\Gamma(u)(t) := e^{tA} z + \int_0^t e^{(t-s)A} \tilde{f}(s, u(s)) \, ds, \quad t \in [0, \delta].$$

First, since $(e^{tA})_{t \geqslant 0}$ is strongly continuous, $t \in [0, \delta] \mapsto e^{tA} z \in X$ is continuous. Then, using (2.4), (2.6) and **(S)**, a combination of the theorem of continuity under the integral sign and the absolute continuity of a Bochner integrable map guarantees that $\Gamma(u) \in C([0, \delta], X)$. To see that $\Gamma(u) \in Y_j$ and that Γ is a contractive map, we need to find the appropriate value of δ. Thanks to the absolute continuity of

the Lebesgue integral, we can choose a $\delta > 0$ such that $\int_0^\delta m^j(s)\,ds \leqslant j/(2\,N)$ and $\int_0^\delta l^j(s)\,ds \leqslant 1/(2\,N)$. Then, for $u \in Y_j$, by Proposition 2.6(ii),

$$\|\Gamma(u)\| \leqslant \sup_{t\in[0,\delta]} \|e^{tA}\,z\| + \sup_{t\in[0,\delta]} \left\| \int_0^t e^{(t-s)A}\,\tilde{f}(s,u(s))\,ds \right\|$$

$$\leqslant \frac{j}{2} + N \int_0^\delta m^j(s)\,ds \leqslant \frac{j}{2} + \frac{j}{2} = j$$

and, for all $u_1, u_2 \in Y_j$, this time by Proposition 2.6(iv),

$$\|\Gamma(u_1) - \Gamma(u_2)\| = \sup_{t\in[0,\delta]} \left\| \int_0^t e^{(t-s)A} \left(\tilde{f}(s,u_1(s)) - \tilde{f}(s,u_2(s)) \right) ds \right\|$$

$$\leqslant N\,\|u_1 - u_2\| \int_0^\delta l^j(s)\,ds \leqslant \frac{1}{2}\,\|u_1 - u_2\|.$$

By the contraction mapping theorem, there exists a unique fixed point of Γ on Y_j. In particular this means that there exists a mild solution on $[0,\delta]$ for the Carathéodory ACP (2.7).

To prove uniqueness, if there were two mild solutions u_1 and u_2 in $C([0,\delta], X)$, define $t_0 = \sup\{t \in [0,\delta] \mid u_1(s) = u_2(s) \text{ for } 0 \leqslant s \leqslant t\}$ and let $z_1 = u_1(t_0) = u_2(t_0)$. Assume by contradiction that $t_0 < \delta$ and note that by the previous reasoning, the problem

$$\begin{cases} u'(t) = A\,u(t) + \tilde{f}(t,u(t)), & t > t_0, \\ u(t_0) = z_1 \end{cases}$$

has a unique mild solution in a set

$$\tilde{Y} = \{u \in C([t_0, t_0 + \varepsilon], X) \mid \|u(t)\| \leqslant k \text{ for all } t \in [t_0, t_0 + \varepsilon]\},$$

for large enough k and small enough $\varepsilon > 0$. Now, both u_1 and u_2 are bounded on $[t_0, \delta]$ and, as recalled in Remark 2.8.2, they are mild solutions of the previous problem on $[t_0, t_0+\varepsilon]$ as far as $t_0+\varepsilon < \delta$, so that $u_1, u_2 \in \tilde{Y}$ for an appropriate choice of k and ε. Since, by the definition of t_0, they are different, we find a contradiction. The proof is finished. $\qquad \square$

In what refers to the maximally defined solution, for each $f \in \mathfrak{LC}$ and $z \in X$ we define $\beta(f,z) = \sup\{\delta > 0 \mid \exists$ a mild solution u_δ on $[0,\delta]\}$ and $u(t) := u_\delta(t)$ if $t \leqslant \delta$. Then, by the uniqueness result, u is well-defined on the interval $I_{f,z} := \bigcup\{[0,\delta] \mid \exists$ a mild solution u_δ on $[0,\delta]\}$ and $\beta(f,z) = \sup I_{f,z}$. Then, we have the following result on the behaviour of the mild solution which resembles the well-known result for standard ODEs.

Proposition 2.10. *Let $f \in \mathfrak{LC}$. Assume that $z \in X$ is such that $I_{f,z} = [0,T)$ with $T < \infty$. Then, $t \mapsto \|u(t)\|$ is unbounded on $I_{f,z}$.*

Proof. Assume by contradiction that u is bounded on $[0,T)$. Then, since $u(t)$ satisfies (2.8) on $[0,T)$ and for an appropriate $j \geqslant 1$ and for almost every $s \in [0,T)$, $\|\tilde{f}(s,u(s))\| \leqslant m^j(s) \in L^1([0,T])$, u can be extended with continuity to the closed interval $[0,T]$. But then it is not difficult to check that we can extend the mild solution beyond T, by gluing $u(t)$ on $[0,T]$ to the mild solution of the ACP

$$\begin{cases} v'(t) = A\,v(t) + \tilde{f}(t,v(t)), & t > T, \\ v(T) = u(T) \end{cases}$$

which is defined at least on an interval $[T, T + \delta]$, by Theorem 2.9. This is a contradiction with the fact that $I_{f,z} = [0, T)$ was the maximal interval of definition of the mild solution for f and z. The proof is finished. $\quad\square$

3. STRONG AND WEAK TOPOLOGIES AND TOPOLOGICAL PROPERTIES OF THE BOUND-MAPS

We endow the space $\mathfrak{SC}$ with suitable strong and weak topologies. The terminology that we use is parallel to the one used in the works by Longo et al. [10,11]. As a rule, when inducing a topology on a subspace, we will denote the induced topology with the same symbol. Firstly, we consider some integral-like topologies which are the natural adaptation of the classical ones in the context of Carathéodory ODEs, then we introduce two new topologies $\mathcal{T}_{\tilde{D}D}$ and $\sigma_{\tilde{D}D}$ in this PDEs context, and finally we define the topologies $\mathcal{T}_\Theta$ and σ_Θ associated to adequate sets of moduli of continuity.

As explained in Section 2, linked to the IBV problem (2.1) one builds the Carathéodory ACP (2.7) in the Banach space X with the sup-norm $\|\cdot\|$ (see Remark 2.5). For this reason, in the definitions it naturally appears the sup-norm of maps defined on $\overline{U}$ with values in $\mathbb{R}^N$, where it used to appear the norm in $\mathbb{R}^N$ for standard ODEs. To this end, recall the definition of $\tilde{f}(t, z)$ given in (2.5). Also, given $f \in \mathfrak{SC}$, Proposition 2.6 justifies that all the following definitions make sense.

To simplify the writing, we denote by $\mathcal{B}_j$ the closed ball in $C(\overline{U}, \mathbb{R}^N)$ centered at the null map with radius j, that is, $\mathcal{B}_j = C(\overline{U}, B_j)$, and whenever we write $I = [q_1, q_2]$, $q_1, q_2 \in \mathbb{Q}$, it is understood that $q_1 < q_2$.

Definition 3.1 (Topology $\mathcal{T}_B$). We call $\mathcal{T}_B$ the topology on $\mathfrak{SC}$ generated by the family of seminorms

$$p_{I,j}(f) = \sup_{v \in C(I, \mathcal{B}_j)} \int_I \|\tilde{f}(s, v(s))\| \, ds, \quad f \in \mathfrak{SC}, \tag{3.1}$$

where $I = [q_1, q_2]$, $q_1, q_2 \in \mathbb{Q}$ and $j \in \mathbb{N}$. One has that $(\mathfrak{SC}, \mathcal{T}_B)$ is a locally convex metric space.

Definition 3.2 (Topologies $\mathcal{T}_D$ and σ_D). Let D be a countable and dense subset of $\mathbb{R}^N$. We call $\mathcal{T}_D$ (resp. σ_D) the topology on $\mathfrak{SC}$ generated by the family of seminorms

$$p_{I,y}(f) = \int_I \|\tilde{f}(s, \tilde{y})\| \, ds \quad \left(\text{resp. } \tilde{p}_{I,y}(f) = \left\| \int_I \tilde{f}(s, \tilde{y}) \, ds \right\| = \sup_{x \in \overline{U}} \left| \int_I f(s, x, y) \, ds \right| \right)$$

for each $f \in \mathfrak{SC}$, $y \in D$, and $I = [q_1, q_2]$, $q_1, q_2 \in \mathbb{Q}$, where $\tilde{y}$ stands for the identically equal to y map defined on $\overline{U}$. One has that $(\mathfrak{SC}, \mathcal{T}_D)$ and $(\mathfrak{SC}, \sigma_D)$ are locally convex metric spaces.

Furthermore, in the present PDEs context it makes sense to consider the following so-called *pointwise topologies*, specially useful in applications.

Definition 3.3 (Topologies $\mathcal{T}_{\tilde{D}D}$ and $\sigma_{\tilde{D}D}$). Let $\tilde{D}$ be a countable and dense subset of $U \subset \mathbb{R}^M$ and let D be a countable and dense subset of $\mathbb{R}^N$. We call $\mathcal{T}_{\tilde{D}D}$ (resp. $\sigma_{\tilde{D}D}$) the topology on $\mathfrak{SC}$ generated by the family of seminorms

$$p_{I,x,y}(f) = \int_I |f(s, x, y)| \, ds \quad \left(\text{resp. } \tilde{p}_{I,x,y}(f) = \left| \int_I f(s, x, y) \, ds \right| \right)$$

for each $f \in \mathfrak{SC}$, $x \in \widetilde{D}$, $y \in D$, and $I = [q_1, q_2]$, $q_1, q_2 \in \mathbb{Q}$. One has that $\left(\mathfrak{SC}, \mathcal{T}_{\tilde{D}D}\right)$ and $\left(\mathfrak{SC}, \sigma_{\tilde{D}D}\right)$ are locally convex metric spaces.

Finally, we consider two more metric topologies based on suitable sets of moduli of continuity, trying to adapt the analogous classes proposed and studied in [10] and [11] for ODEs. However, things are more complicated in this PDEs setting and we need to combine two families of moduli of continuity, as in the next definition. The reasons for this precise construction will become clear after the proof of Proposition 4.4.

Definition 3.4 (Suitable set of moduli of continuity)**.** We call *a suitable set of moduli of continuity* every $\Theta = (\Theta_1, \Theta_2)$ formed by two countable sets of non-decreasing continuous functions,

$$\Theta_1 = \left\{ \theta_j^I \in C(\mathbb{R}^+, \mathbb{R}^+) \mid j \in \mathbb{N}, \ I = [q_1, q_2], \ q_1, q_2 \in \mathbb{Q} \right\}$$

such that $\theta_j^I(0) = 0$ for every $\theta_j^I \in \Theta_1$, and the relation of partial order given by

$$\theta_{j_1}^{I_1} \leqslant \theta_{j_2}^{I_2} \quad \text{whenever } I_1 \subseteq I_2 \text{ and } j_1 \leqslant j_2$$

holds, and

$$\Theta_2 = \left\{ \theta_j^{I,\nu} \in C(\mathbb{R}^+, \mathbb{R}^+) \,\middle|\, j, \nu \in \mathbb{N}, \ I = [q_1, q_2], \ q_1, q_2 \in \mathbb{Q}, \ \nu \geqslant 1 \text{ s.t. } q_1 + \frac{1}{\nu} < q_2 \right\}$$

such that $\theta_j^{I,\nu}(0) = 0$ for every $\theta_j^{I,\nu} \in \Theta_2$ and

$$\theta_{j_1}^{I_1,\nu_1} \leqslant \theta_{j_2}^{I_2,\nu_2} \quad \text{whenever } I_1 \subseteq I_2 \,, \ \nu_1 \leqslant \nu_2 \,, \text{ and } j_1 \leqslant j_2 \,, \tag{3.2}$$

with the restriction that $1/\nu_1$ is smaller than the length of the interval I_1.

Definition 3.5 (Topologies $\mathcal{T}_\Theta$ and σ_Θ)**.** Let $\Theta = (\Theta_1, \Theta_2)$ be a suitable set of moduli of continuity. For each $j \in \mathbb{N}$ and $I = [q_1, q_2]$ ($q_1, q_2 \in \mathbb{Q}$), let $\mathcal{H}_j^I$ be the set of functions $v \in C(I, \mathcal{B}_j)$ which satisfy that for every integer $\nu \geqslant 1$ such that $q_1 + 1/\nu < q_2$,

$$\|v(t_1) - v(t_2)\| \leqslant \theta_j^{I,\nu}(|t_1 - t_2|) + \theta_j^I\left(\frac{1}{\nu}\right) \text{ for all } t_1, t_2 \in \left[q_1 + \frac{1}{\nu}, q_2\right].$$

We call $\mathcal{T}_\Theta$ (resp. σ_Θ) the topology on $\mathfrak{SC}$ generated by the family of seminorms

$$p_{I,j}(f) = \sup_{v \in \mathcal{H}_j^I} \int_I \|\tilde{f}(s, v(s))\| \, ds$$

$$\left(\text{resp. } \tilde{p}_{I,j}(f) = \sup_{v \in \mathcal{H}_j^I} \left\| \int_I \tilde{f}(s, v(s)) \, ds \right\| = \sup_{x \in \overline{U}, \, v \in \mathcal{H}_j^I} \left| \int_I f(s, x, v(s)(x)) \, ds \right| \right)$$

for each $f \in \mathfrak{SC}$, $I = [q_1, q_2]$ ($q_1, q_2 \in \mathbb{Q}$), and $j \in \mathbb{N}$. One has that $(\mathfrak{SC}, \mathcal{T}_\Theta)$ and $(\mathfrak{SC}, \sigma_\Theta)$ are locally convex metric spaces.

Notice that, as well as $\mathfrak{SC}$, also $\mathfrak{LC}$ can be endowed with all the previous topologies. Besides, given any dense and countable sets $\widetilde{D} \subset U$ and $D \subset \mathbb{R}^N$, and given any suitable set of moduli of continuity Θ, it is immediate that the following chains of order hold true:

$$\mathcal{T}_{\tilde{D}D} \leqslant \mathcal{T}_D \leqslant \mathcal{T}_\Theta \leqslant \mathcal{T}_B \quad \text{and} \quad \sigma_{\tilde{D}D} \leqslant \sigma_D \leqslant \sigma_\Theta \leqslant \mathcal{T}_\Theta \,. \tag{3.3}$$

In the remaining part of this section we are going to give some definitions and properties on the behaviour of the m-bounds and/or the l-bounds associated to a

set $E \subset \mathfrak{LC}$. We might consider m-bounds just for maps in $\mathfrak{C}$, but since the aim is to study the dependance of mild solutions $u(\cdot, f, z)$ of the abstract problems (2.7) with respect to the map $f \in \mathfrak{LC}$ and the initial condition $z \in X$, we restrict ourselves to $E \subset \mathfrak{LC}$.

Definition 3.6. We say that:

(i) A set $E \subset \mathfrak{LC}$ has L^1_{loc}-*bounded m-bounds (resp. l-bounds [possibly with re-spect to a modulus of continuity $\widehat{\theta}$ in the variable x]), if for every $j \in \mathbb{N}$ there exists a set $S^j \subset L^1_{loc}$ of m-bounds (resp. l-bounds) of the functions of E on $B_j \subset \mathbb{R}^N$, such that S^j is L^1_{loc}-bounded, that is, for every $r > 0$,

$$\sup_{m \in S^j} \int_{-r}^{r} m < \infty.$$

(ii) A set $E \subset \mathfrak{LC}$ has L^1_{loc}-*equicontinuous m-bounds (resp. l-bounds [possibly with respect to a modulus of continuity $\widehat{\theta}$ in the variable x]), if for every $j \in \mathbb{N}$ there exists a set $S^j \subset L^1_{loc}$ of m-bounds (resp. l-bounds) of the functions of E on $B_j \subset \mathbb{R}^N$, such that S^j is L^1_{loc}-equicontinuous, that is, for every $r > 0$ and every $\varepsilon > 0$, there exists a $\delta = \delta(j, r, \varepsilon) > 0$ such that, for all $-r \leqslant t_1 \leqslant t_2 \leqslant r$ with $t_2 - t_1 < \delta$,

$$\sup_{m \in S^j} \int_{t_1}^{t_2} m < \varepsilon.$$

Remark 3.7. According to the previous definitions, the L^1_{loc}-equicontinuity implies the L^1_{loc}-boundedness.

The next result asserts that the existence of L^1_{loc}-bounded or L^1_{loc}-equicontinuous m-bounds and/or l-bounds for a set $E \subset \mathfrak{LC}$ is inherited by all the elements in the closure of E with respect to any of the previously introduced topologies. The proof is omitted as it presents only minor differences with respect to the one of Proposition 2.26 in [9] (which in turn extends Propositions 4.7 and 4.10 in [10]).

Proposition 3.8. *Let $\mathcal{T}$ be any of the introduced topologies.*

(i) *If $E \subset \mathfrak{LC}$ has L^1_{loc}-bounded m-bounds (resp. l-bounds [possibly with respect to a modulus of continuity $\widehat{\theta}$]) then $\mathrm{cls}_{(\mathfrak{LC}, \mathcal{T})}(E)$ has L^1_{loc}-bounded m-bounds (resp. l-bounds).*

(ii) *If $E \subset \mathfrak{LC}$ has L^1_{loc}-equicontinuous m-bounds (resp. l-bounds [possibly with respect to a modulus of continuity $\widehat{\theta}$]), then $\mathrm{cls}_{(\mathfrak{LC}, \mathcal{T})}(E)$ has L^1_{loc}-equicontinuous m-bounds (resp. l-bounds).*

The fact that a set $E \subset \mathfrak{LC}$ has L^1_{loc}-bounded l-bounds has strong implications on the equivalence of some strong topologies when restricted to E. When additional conditions are imposed on E, then some weak topologies turn out to be equivalent, too. We collect the precise results in the next theorem.

Theorem 3.9. *Let $\widetilde{D} \subset U$ and $D \subset \mathbb{R}^N$ be any dense and countable subsets, and let $\Theta = (\Theta_1, \Theta_2)$ be any suitable set of moduli of continuity, according to Definition 3.4. The following assertions hold for a set $E \subset \mathfrak{LC}$:*

(i) *If E has L^1_{loc}-bounded l-bounds, then for any $\mathcal{T}_1, \mathcal{T}_2 \in \{\mathcal{T}_D, \mathcal{T}_\Theta, \mathcal{T}_B\}$,*

$$(E, \mathcal{T}_1) = (E, \mathcal{T}_2) \qquad and \qquad \mathrm{cls}_{(\mathfrak{LC}, \mathcal{T}_1)}(E) = \mathrm{cls}_{(\mathfrak{LC}, \mathcal{T}_2)}(E).$$

(ii) *If E has L^1_{loc}-equicontinuous m-bounds and L^1_{loc}-bounded l-bounds, then*

$$(E, \sigma_D) = (E, \sigma_\Theta) \qquad and \qquad \mathrm{cls}_{(\mathfrak{LC}, \sigma_D)}(E) = \mathrm{cls}_{(\mathfrak{LC}, \sigma_\Theta)}(E).$$

(iii) *If E has L^1_{loc}-bounded l-bounds with respect to a modulus of continuity $\hat{\theta}$ in the variable x, then*

$$(E, \sigma_{\tilde{D}D}) = (E, \sigma_D) \qquad and \qquad \mathrm{cls}_{(\mathfrak{LC}, \sigma_{\tilde{D}D})}(E) = \mathrm{cls}_{(\mathfrak{LC}, \sigma_D)}(E),$$

and, for any $\mathcal{T} \in \{\mathcal{T}_D, \mathcal{T}_\Theta, \mathcal{T}_B\}$,

$$(E, \mathcal{T}_{\tilde{D}D}) = (E, \mathcal{T}) \qquad and \qquad \mathrm{cls}_{(\mathfrak{LC}, \mathcal{T}_{\tilde{D}D})}(E) = \mathrm{cls}_{(\mathfrak{LC}, \mathcal{T})}(E).$$

Proof. (i) The main ideas come from the proof of Theorem 4.12 in [10] in an ODEs context, but we include the proof for the sake of completeness. Due to (3.3), it suffices to prove that if $(f_n)_{n \geq 1}$ is a sequence of elements of E converging to some f in $(\mathfrak{LC}, \mathcal{T}_D)$, then $(f_n)_{n \geq 1}$ converges to f in $(\mathfrak{LC}, \mathcal{T}_B)$ too. Fix a compact interval $I = [q_1, q_2]$ with $q_1, q_2 \in \mathbb{Q}$ and fix a $j \in \mathbb{N}$. For each $n \geq 1$, let $l^j_n \in L^1_{loc}$ be the optimal l-bound of f_n on B_j, and $l^j_f \in L^1_{loc}$ be the one for f, defined as in (2.2). By hypothesis and by Proposition 3.8, there exists a $\rho > 0$ such that

$$\sup_{n \geq 1} \int_I l^j_n(s)\, ds < \rho < \infty \quad \text{and} \quad \int_I l^j_f(s)\, ds < \rho. \tag{3.4}$$

Now, fix $\varepsilon > 0$ and consider $\delta = \varepsilon/(3\rho)$. Since $B_j \subset \mathbb{R}^N$ is compact, and D is dense in $\mathbb{R}^N$, there exist $y_1, \ldots, y_{i_0} \in D$ such that $B_j \subset \bigcup_{i=1}^{i_0} \overset{\circ}{B}_\delta(y_i)$, where $B_\delta(y)$ denotes the closed ball of $\mathbb{R}^N$ of radius δ centered at $y \in \mathbb{R}^N$. Subordinate to the former open cover, there is a continuous partition of unity, that is, there exist continuous functions $\phi_i : \mathbb{R}^N \to [0, 1]$ for $i = 1, \ldots, i_0$ such that

$$\mathrm{supp}(\phi_i) \subset \overset{\circ}{B}_\delta(y_i) \qquad \text{and} \qquad \sum_{i=1}^{i_0} \phi_i(y) = 1 \quad \text{for all } y \in B_j.$$

Let us define the $\mathfrak{SC}$ functions

$$f^*_n(t, x, y) = \sum_{i=1}^{i_0} \phi_i(y)\, f_n(t, x, y_i) \quad \text{and} \quad f^*(t, x, y) = \sum_{i=1}^{i_0} \phi_i(y)\, f(t, x, y_i). \tag{3.5}$$

Then, for each $v \in C(I, \mathcal{B}_j)$, note that $v(t)(x) \in B_j$ for all $t \in I$ and $x \in \overline{U}$, and write

$$\int_I \left\| \tilde{f}_n(t, v(t)) - \tilde{f}(t, v(t)) \right\| dt = \int_I \sup_{x \in \overline{U}} |f_n(t, x, v(t)(x)) - f(t, x, v(t)(x))|\ dt$$

$$\leq \int_I \sup_{x \in \overline{U}} |f_n(t, x, v(t)(x)) - f^*_n(t, x, v(t)(x))|\ dt$$

$$+ \int_I \sup_{x \in \overline{U}} |f^*_n(t, x, v(t)(x)) - f^*(t, x, v(t)(x))|\ dt$$

$$+ \int_I \sup_{x \in \overline{U}} |f^*(t, x, v(t)(x)) - f(t, x, v(t)(x))|\ dt.$$

Let us separately analyze each one of the three terms in the previous sum. As regards the first one, by the construction of the maps ϕ_i we have that for all $n \geqslant 1$,

$$
\int_I \sup_{x \in \overline{U}} |f_n(t, x, v(t)(x)) - f_n^*(t, x, v(t)(x))| \; dt
$$

$$
= \int_I \sup_{x \in \overline{U}} \Big| \sum_{i=1}^{i_0} \phi_i(v(t)(x)) \left(f_n(t, x, v(t)(x)) - f_n(t, x, y_i) \right) \Big| \, dt
$$

$$
\leqslant \int_I \sup_{x \in \overline{U}} \sum_{i=1}^{i_0} \phi_i(v(t)(x)) \, |f_n(t, x, v(t)(x)) - f_n(t, x, y_i)| \; dt \tag{3.6}
$$

$$
\leqslant \int_I \sup_{x \in \overline{U}} \sum_{i=1}^{i_0} \phi_i(v(t)(x)) \, l_n^j(t) \, |v(t)(x) - y_i| \; dt
$$

$$
\leqslant \int_I \sup_{x \in \overline{U}} \sum_{i=1}^{i_0} \phi_i(v(t)(x)) \, l_n^j(t) \, \delta \; dt = \frac{\varepsilon}{3\,\rho} \int_I l_n^j(t) \, dt \leqslant \frac{\varepsilon}{3} \,,
$$

applying (3.4) to get the last inequality. As for the third element in the sum, also the optimal l-bound $l_f^j \in L_{loc}^1$ of f on B_j satisfies the integral bound in (3.4). Therefore, reasoning exactly as in (3.6), we obtain that

$$
\int_I \sup_{x \in \overline{U}} |f^*(t, x, v(t)(x)) - f(t, x, v(t)(x))| \; dt \leqslant \frac{\varepsilon}{3} \,. \tag{3.7}
$$

Finally, for the remaining element in the sum, since $(f_n)_{n \geqslant 1}$ converges to f in $(\mathfrak{LC}, \mathcal{T}_D)$, there exists an $n_0 = n_0(\varepsilon, i_0)$ such that for all $n \geqslant n_0$,

$$
p_{I, y_i}(f_n - f) = \int_I \sup_{x \in \overline{U}} |f_n(t, x, y_i) - f(t, x, y_i)| \; dt < \frac{\varepsilon}{3\,i_0} \quad \text{for all } i = 1, \dots, i_0.
$$

Then, from (3.5) and the fact that $\phi_i(y) \in [0, 1]$ for each $y \in \mathbb{R}^N$ we deduce that, if $n \geqslant n_0$,

$$
\int_I \sup_{x \in \overline{U}} |f_n^*(t, x, v(t)(x)) - f^*(t, x, v(t)(x))| \; dt
$$

$$
\leqslant \sum_{i=1}^{i_0} \int_I \sup_{x \in \overline{U}} |f_n(t, x, y_i) - f(t, x, y_i)| \; dt \leqslant \frac{\varepsilon}{3} \,. \tag{3.8}
$$

Notice that (3.6), (3.7), and (3.8) are independent of $v \in C(I, \mathcal{B}_j)$. Thus, gathering (3.6), (3.7) and (3.8) together, we obtain that $p_{I,j}(f_n - f) \leqslant \varepsilon$ for all $n \geqslant n_0$, for the seminorm $p_{I,j}$ defined in (3.1). Thus, $(f_n)_{n \geqslant 1}$ converges to f in $(\mathfrak{LC}, \mathcal{T}_B)$ and the proof is finished.

(ii) We remark that in this PDEs context we are assuming the L_{loc}^1-equicontinuity of the m-bounds of E, which turns out to be a natural assumption when dealing with weak topologies. Although the main ideas come from the proof of [12, Theorem 2.20(ii)] in a context of diffential equations with delay, the technical details are rather different, so that we give a complete proof.

Once more, since the relations (3.3) hold, it suffices to prove that if $(f_n)_{n \geqslant 1} \subset E$ converges to some f in $(\mathfrak{LC}, \sigma_D)$, then $(f_n)_{n \geqslant 1}$ converges to f in $(\mathfrak{LC}, \sigma_\Theta)$ too. To see it, fix a compact interval $I = [q_1, q_2]$ with $q_1, q_2 \in \mathbb{Q}$ and fix a $j \in \mathbb{N}$. We want to prove that $\tilde{p}_{I,j}(f_n - f) \to 0$ as $n \to \infty$ for the seminorm $\tilde{p}_{I,j}$ given in Definition 3.5.

Namely,

$$\tilde{p}_{I,j}(f_n - f) = \sup_{x \in \overline{U}, \, v \in \mathcal{H}_j^I} \left| \int_I \big(f_n(s, x, v(s)(x)) - f(s, x, v(s)(x)) \big) \, ds \right|. \tag{3.9}$$

We keep the notation introduced before. Moreover, for each $n \geqslant 1$, let $m_n^j \in L_{loc}^1$ be the optimal m-bound of f_n on B_j, and let $m_f^j \in L_{loc}^1$ be the one for f, defined as in (2.2). Recall that the l-bounds satisfy relation (3.4), and by Remark 3.7 and Proposition 3.8, there exists a $\rho_0 > 0$ such that

$$\sup_{n \geqslant 1} \int_I m_n^j(s) \, ds < \rho_0 < \infty \quad \text{and} \quad \int_I m_f^j(s) \, ds < \rho_0. \tag{3.10}$$

The initial arguments are quite similar to the ones in (i). Fixed $\varepsilon > 0$, we consider $\delta = \varepsilon/(3\,\rho)$ for ρ in (3.4), we take a partition of unity just as in (i), and consider the maps in (3.5). Now, for each $v \in \mathcal{H}_j^I$ and each $x \in \overline{U}$ we write

$$\left| \int_I \big(f_n(t, x, v(t)(x)) - f(t, x, v(t)(x)) \big) \, dt \right|$$
$$\leqslant \left| \int_I \big(f_n(t, x, v(t)(x)) - f_n^*(t, x, v(t)(x)) \big) \, dt \right|$$
$$+ \left| \int_I \big(f_n^*(t, x, v(t)(x)) - f^*(t, x, v(t)(x)) \big) \, dt \right| \tag{3.11}$$
$$+ \left| \int_I \big(f^*(t, x, v(t)(x)) - f(t, x, v(t)(x)) \big) \, dt \right|.$$

The first and third terms in the sum are treated similarly as the corresponding ones in (i), using the properties of the partition of unity and bringing the l-bounds into play. Both terms turn out to be less than $\varepsilon/3$ for all $n \geqslant 1$, uniformly for $v \in \mathcal{H}_j^I$ and $x \in \overline{U}$.

The treatment of the second term is more delicate and technical. By (3.5),

$$\left| \int_I \big(f_n^*(t, x, v(t)(x)) - f^*(t, x, v(t)(x)) \big) \, dt \right|$$
$$= \left| \int_I \sum_{i=1}^{i_0} \phi_i(v(t)(x)) \big(f_n(t, x, y_i) - f(t, x, y_i) \big) \, dt \right|. \tag{3.12}$$

The uniform continuity of the functions ϕ_i on the compact set B_j helps to deal with this term. Given $\varepsilon^* = \varepsilon/(18\,i_0\,\rho_0)$, for ρ_0 in (3.10), there exists a $\delta_0 > 0$ such that, if $y, \tilde{y} \in B_j$ satisfy $|y - \tilde{y}| < \delta_0$, then $|\phi_i(y) - \phi_i(\tilde{y})| < \varepsilon^*$ for all $i = 1, \dots, i_0$. Then, we can determine an integer $\nu \geqslant 1$ large enough so that $q_1 + 1/\nu < q_2$ and it satisfies the next two conditions:

(c_1) $\theta_j^I \left(\dfrac{1}{\nu} \right) < \dfrac{\delta_0}{2}$;

(c_2) $\displaystyle\int_{q_1}^{q_1 + 1/\nu} m_n^j(s) \, ds < \dfrac{\varepsilon}{18\,i_0}$ for all $n \geqslant 1$, and $\displaystyle\int_{q_1}^{q_1 + 1/\nu} m_f^j(s) \, ds < \dfrac{\varepsilon}{18\,i_0}$.

Just recall that $\theta_j^I \in \Theta_1$ (see Definition 3.4) is a modulus of continuity, and we are assuming that E has L_{loc}^1-equicontinuous m-bounds and Proposition 3.8 holds. This time considering the modulus of continuity $\theta_j^{I,\nu} \in \Theta_2$, we can find a small enough $h > 0$, with $h \in \mathbb{Q}$ for convenience, so that

(c_3) $\theta_j^{I,\nu}(h) < \dfrac{\delta_0}{2}$.

Then, whenever $v \in \mathcal{H}_j^I$ and $t_1, t_2 \in [q_1 + 1/\nu, q_2]$ with $|t_1 - t_2| \leqslant h$, by Definition 3.5 and conditions (c_1) and (c_3) we have that $\|v(t_1) - v(t_2)\| \leqslant \theta_j^{I,\nu}(|t_1 - t_2|) + \theta_j^I(1/\nu) \leqslant \theta_j^{I,\nu}(h) + \theta_j^I(1/\nu) < \delta_0$. By the uniform continuity of the maps ϕ_i on B_j, for every $x \in \overline{U}$ we have that $|\phi_i(v(t_1)(x)) - \phi_i(v(t_2)(x))| < \varepsilon^*$, for all $i = 1, \ldots, i_0$.

So, the idea is to split the integral in (3.12) over the whole interval $I = [q_1, q_2]$ into a collection of integrals over intervals of length h where we can use the previous bound. Note that in any case we have to separate a first small interval, since the previous bounds only work for $t_1, t_2 \in [q_1 + 1/\nu, q_2]$. This is not a problem, thanks to condition (c_2).

More precisely, to simplify the writing, for the integer $k_0 \geqslant 0$ such that $q_2 = q_1 + 1/\nu + k_0 h + h_1$, with $0 \leqslant h_1 < h$, let $r_k = q_1 + 1/\nu + k h$ for $k = 0, \ldots, k_0$, and let $r_{k_0+1} = q_2$. First of all, using that $0 \leqslant \phi_i \leqslant 1$ and condition (c_2), we have that for all $v \in \mathcal{H}_j^I$, $x \in \overline{U}$, and $n \geqslant 1$,

$$\left| \int_{q_1}^{q_1 + \frac{1}{\nu}} \sum_{i=1}^{i_0} \phi_i(v(t)(x)) \big(f_n(t, x, y_i) - f(t, x, y_i)\big)\, dt \right|$$

$$\leqslant \int_{q_1}^{q_1 + \frac{1}{\nu}} \sum_{i=1}^{i_0} \big(|f_n(t, x, y_i)| + |f(t, x, y_i)|\big)\, dt$$

$$\leqslant i_0 \int_{q_1}^{q_1 + \frac{1}{\nu}} \big(m_n^j(t) + m_f^j(t)\big)\, dt \leqslant i_0 \left(\frac{\varepsilon}{18\, i_0} + \frac{\varepsilon}{18\, i_0} \right) = \frac{\varepsilon}{9}\, .$$

Secondly, we write

$$\left| \int_{q_1 + \frac{1}{\nu}}^{q_2} \sum_{i=1}^{i_0} \phi_i(v(t)(x)) \big(f_n(t, x, y_i) - f(t, x, y_i)\big)\, dt \right|$$

$$= \left| \sum_{k=0}^{k_0} \int_{r_k}^{r_{k+1}} \sum_{i=1}^{i_0} \big(\phi_i(v(t)(x)) - \phi_i(v(r_k)(x))\big) \big(f_n(t, x, y_i) - f(t, x, y_i)\big)\, dt \right.$$

$$\left. + \sum_{k=0}^{k_0} \sum_{i=1}^{i_0} \phi_i(v(r_k)(x)) \int_{r_k}^{r_{k+1}} \big(f_n(t, x, y_i) - f(t, x, y_i)\big)\, dt \right|$$

$$\leqslant \sum_{k=0}^{k_0} \int_{r_k}^{r_{k+1}} \sum_{i=1}^{i_0} |\phi_i(v(t)(x)) - \phi_i(v(r_k)(x))| \big(m_n^j(t) + m_f^j(t)\big)\, dt$$

$$+ \sum_{k=0}^{k_0} \sum_{i=1}^{i_0} \phi_i(v(r_k)(x)) \left| \int_{r_k}^{r_{k+1}} \big(f_n(t, x, y_i) - f(t, x, y_i)\big)\, dt \right|$$

$$\leqslant \int_{q_1 + \frac{1}{\nu}}^{q_2} i_0\, \varepsilon^* \big(m_n^j(t) + m_f^j(t)\big)\, dt + \sum_{k=0}^{k_0} \sum_{i=1}^{i_0} \left\| \int_{r_k}^{r_{k+1}} \big(\tilde{f}_n(t, \tilde{y}_i) - \tilde{f}(t, \tilde{y}_i)\big)\, dt \right\|$$

$$\leqslant i_0\, \varepsilon^*\, 2\, \rho_0 + \sum_{k=0}^{k_0} \sum_{i=1}^{i_0} \tilde{p}_{I_k, y_i}(f_n - f) = \frac{\varepsilon}{9} + \sum_{k=0}^{k_0} \sum_{i=1}^{i_0} \tilde{p}_{I_k, y_i}(f_n - f)$$

for the intervals with rational endpoints $I_k = [r_k, r_{k+1}]$ for $k = 0, \ldots, k_0$ and the seminorms $\tilde{p}_{I_k, y_i}$ defining the weak topology σ_D (see Definition 3.2). Since by

hypothesis, $f_n \to f$ in this topology, we can find an integer n_0 such that for all $n \geqslant n_0$, $\tilde{p}_{I_k, y_i}(f_n - f) \leqslant \varepsilon/(9\,(k_0+1)\,i_0)$ for each $i = 1, \ldots i_0$ and $k = 0, \ldots, k_0$.

Therefore, bringing everything together, we obtain that for all $n \geqslant n_0$, the so-called second term in the sum (3.11) is less than or equal to $\varepsilon/3$, and this happens uniformly for $v \in \mathcal{H}_j^I$ and $x \in \overline{U}$. In all, we have dealt with the three terms in (3.11). Finally, looking at (3.9) we can affirm that $\tilde{p}_{I,j}(f_n - f) \to 0$ as $n \to \infty$, as we wanted to prove.

(iii) We give all the details for the equivalence of the weak topologies. By (3.3), it suffices to see that $\sigma_D \leqslant \sigma_{\tilde{D}D}$. So, let us assume that $(f_n)_{n \geqslant 1} \subset E$ satisfies that $f_n \to f \in \mathfrak{LC}$ in the topology $\sigma_{\tilde{D}D}$ and let us prove that $f_n \to f$ in the topology σ_D too. Let us fix an interval $I = [q_1, q_2]$ ($q_1, q_2 \in \mathbb{Q}$) and a $y_0 \in D$. By definition,

$$\tilde{p}_{I,y_0}(f_n - f) = \sup_{x \in \overline{U}} \left| \int_I \big(f_n(t, x, y_0) - f(t, x, y_0) \big)\, dt \right|.$$

This time we need to find a partition of unity subordinate to an adequate open cover of $\overline{U}$, so as to be able to work with only a finite collection of points in $\tilde{D}$.

Let us choose a $j \geqslant 1$ so that $y_0 \in B_j$. Once again, we keep the notation introduced before for the optimal l-bounds, l_n^j for $n \geqslant 1$ and l_f^j, but note that this time we are assuming that the maps in E are Lipschitz Carathéodory with respect to a modulus of continuity $\hat{\theta}$ in the variable x (see $(\mathbf{L}_{\hat{\theta}})$) and the optimal l-bounds are defined as in (2.3). We can assume that (3.4) holds.

Now, given $\varepsilon > 0$, we can find a $\delta > 0$ so that $\hat{\theta}(\delta) < \varepsilon/(3\,\rho)$, for ρ the one in (3.4). For this δ, since $\tilde{D}$ is dense in U, and $\overline{U}$ is compact, there exist $x_1, \ldots, x_k \in \tilde{D}$ such that $\overline{U} \subset \bigcup_{i=1}^k \overset{\circ}{B}_\delta(x_i)$, where $B_\delta(x)$ denotes the closed ball of $\mathbb{R}^M$ of radius δ centered at $x \in \mathbb{R}^M$. Subordinate to this open cover, there exists a continuous partition of unity, that is, continuous functions $\varphi_i : \mathbb{R}^M \to [0,1]$ for $i = 1, \ldots, k$ such that

$$\operatorname{supp}(\varphi_i) \subset \overset{\circ}{B}_\delta(x_i) \qquad \text{and} \qquad \sum_{i=1}^k \varphi_i(x) = 1 \quad \text{for all } x \in \overline{U}.$$

Here we define the $\mathfrak{LC}$ functions

$$f_n^*(t, x, y) = \sum_{i=1}^k \varphi_i(x)\, f_n(t, x_i, y) \quad \text{and} \quad f^*(t, x, y) = \sum_{i=1}^k \varphi_i(x)\, f(t, x_i, y).$$

Then, for each $x \in \overline{U}$ we write

$$\left| \int_I \big(f_n(t, x, y_0) - f(t, x, y_0) \big)\, dt \right| \leqslant \left| \int_I \big(f_n(t, x, y_0) - f_n^*(t, x, y_0) \big)\, dt \right|$$
$$+ \left| \int_I \big(f_n^*(t, x, y_0) - f^*(t, x, y_0) \big)\, dt \right| + \left| \int_I \big(f^*(t, x, y_0) - f(t, x, y_0) \big)\, dt \right|.$$

We have a sum of three terms, and the first and the third ones are treated in a similar fashion. We just write down the details for the first term. For each $n \geqslant 1$, by $(\mathbf{L}_{\hat{\theta}})$, (3.4), the properties of the partition of unity, and the choice of δ, we have

that for every $x \in \overline{U}$,

$$\left| \int_I \left(f_n(t, x, y_0) - f_n^*(t, x, y_0) \right) dt \right| = \left| \int_I \sum_{i=1}^k \varphi_i(x) \left(f_n(t, x, y_0) - f_n(t, x_i, y_0) \right) dt \right|$$

$$\leqslant \int_I \sum_{i=1}^k \varphi_i(x) \, l_n^j(t) \, \hat{\theta}(|x - x_i|) \, dt \leqslant \hat{\theta}(\delta) \int_I l_n^j(t) \, dt \leqslant \frac{\varepsilon}{3} \, .$$

As for the second term, for every $x \in \overline{U}$,

$$\left| \int_I \left(f_n^*(t, x, y_0) - f^*(t, x, y_0) \right) dt \right| = \left| \int_I \sum_{i=1}^k \varphi_i(x) \left(f_n(t, x_i, y_0) - f(t, x_i, y_0) \right) dt \right|$$

$$= \left| \sum_{i=1}^k \varphi_i(x) \int_I \left(f_n(t, x_i, y_0) - f(t, x_i, y_0) \right) dt \right| \leqslant \sum_{i=1}^k \tilde{p}_{I, x_i, y_0}(f_n - f)$$

for the seminorms $\tilde{p}_{I, x_i, y_0}$ generating the weak topology $\sigma_{\tilde{D}D}$ (see Definition 3.3). Since $f_n \to f$ in this topology, there exists an n_0 such that for all $n \geqslant n_0$, $\tilde{p}_{I, x_i, y_0}(f_n - f) \leqslant \varepsilon/(3\,k)$ for all $i = 1, \ldots, k$. As a consequence, we have that $\tilde{p}_{I, y_0}(f_n - f) \leqslant \varepsilon$ for every $n \geqslant n_0$, and we are done.

As for the strong topologies, note that by (i), it suffices to prove the equivalence of $\mathcal{T}_{\tilde{D}D}$ and $\mathcal{T}_D$. We skip the details since only minor modifications are needed over the proof just done for the weak topologies. The proof is finished. $\qquad\square$

4. Topologies for the continuous variation of mild solutions

In this section we prove the continuity of the mild solutions $u(\cdot, f, z)$ of the abstract problems (2.7) associated to the parabolic problems (2.1), with respect to the map $f \in \mathfrak{LC}$, under certain topologies, and the initial condition $z \in X$, for $X = C(\overline{U}, \mathbb{R}^N)$ or $X = C_0(\overline{U}, \mathbb{R}^N)$ depending on the boundary conditions (see Remark 2.5).

We will carry out the study, first, for a class $E \subset \mathfrak{LC}$ with L_{loc}^1-equicontinuous m-bounds, introducing a new topology $\sigma_{\Theta \mathcal{R}}$ in Definition 4.6 for an appropriate choice of Θ and $\mathcal{R}$ related to the set E, in an attempt to have a topology of continuity as coarse as possible, as well as to recover, in a precise sense explained later, some properties of compactness which held in the ODEs case. And, second, for a class $E \subset \mathfrak{LC}$ with L_{loc}^1-bounded l-bounds, with respect to any strong topology $\mathcal{T}_D$, $\mathcal{T}_\Theta$ or $\mathcal{T}_B$, all of them equivalent in this case according to Theorem 3.9(i).

In any of the two cases, we will denote by $m_f^j \in L_{loc}^1$ (resp. $l_f^j \in L_{loc}^1$) the optimal m-bound (resp. l-bound) of f on the compact ball $B_j \subset \mathbb{R}^N$. Generally, we will just write m_n^j or l_n^j for the optimal m-bound or l-bound of a map f_n.

4.1. Topologies of continuity when E has L_{loc}^1-equicontinuous m-bounds.
Let us start by considering $E \subset \mathfrak{LC}$ with L_{loc}^1-equicontinuous m-bounds. Associated to the family E, in Proposition 4.4 we determine a suitable set of moduli of continuity Θ as in Definition 3.4 and a suitable set of radii (see Definition 4.1) so that a new topology $\mathcal{T}_{\Theta \mathcal{R}}$, as well as its weak version $\sigma_{\Theta \mathcal{R}}$, are built, and they are both topologies of continuity.

Definition 4.1 (Suitable set of radii)**.** We call *a suitable set of radii* every countable set of positive constants $\mathcal{R} = \{ R_j^I \mid j \in \mathbb{N}, \ I = [q_1, q_2], \ q_1, q_2 \in \mathbb{Q} \}$ such that $R_{j_1}^{I_1} \leqslant R_{j_2}^{I_2}$ whenever $I_1 \subseteq I_2$ and $j_1 \leqslant j_2$.

Let us now introduce a new family of sets $\mathcal{K}_j^I$ which satisfy $\mathcal{K}_j^I \subset \mathcal{H}_j^I$ for the sets $\mathcal{H}_j^I$ given in Definition 3.5. As in the previous section, we denote $\mathcal{B}_j = C(\overline{U}, B_j)$ and we will write $\mathcal{B}_j^X$ to mean $\mathcal{B}_j \cap X$. Also, d denotes the usual distance between a map z and a set of maps F in $C(\overline{U}, \mathbb{R}^N)$, $\mathrm{d}(z, F) := \inf_{\tilde{z} \in F} \|z - \tilde{z}\|$.

Definition 4.2. Let $\Theta = (\Theta_1, \Theta_2)$ be a suitable set of moduli of continuity and let $\mathcal{R}$ be a suitable set of radii. For each $j \in \mathbb{N}$ and $I = [q_1, q_2]$ ($q_1, q_2 \in \mathbb{Q}$), we define $\mathcal{K}_j^I$ as the set of functions $v \in C(I, \mathcal{B}_j)$ which satisfy that, for every interval $J = [r_1, r_2] \subseteq I$ ($r_1, r_2 \in \mathbb{Q}$) and for every integer $\nu \geqslant 1$ such that $r_1 + 1/\nu < r_2$,

(i) $\|v(t_1) - v(t_2)\| \leqslant \theta_j^{J,\nu}(|t_1 - t_2|) + \theta_j^J\left(\dfrac{1}{\nu}\right)$ for all $t_1, t_2 \in \left[r_1 + \dfrac{1}{\nu}, r_2\right]$;

(ii) $\mathrm{d}\big(v(t), e^{1/\nu A} \mathcal{B}_{R_j^J}^X\big) \leqslant \theta_j^J\left(\dfrac{1}{\nu}\right)$ for all $t \in \left[r_1 + \dfrac{1}{\nu}, r_2\right]$.

Remarks 4.3. 1. With the previous definition it is easy to check that, if $v \in \mathcal{K}_j^I$ for some $j \in \mathbb{N}$ and $I = [q_1, q_2]$ ($q_1, q_2 \in \mathbb{Q}$), and $J = [r_1, r_2] \subset I$ ($r_1, r_2 \in \mathbb{Q}$), then the restriction of v to J satisfies $v|_J \in \mathcal{K}_j^J$.

2. One may wonder whether the sets $\mathcal{K}_j^I$ are nonempty, since the construction is rather restrictive. For the specific choice of Θ and $\mathcal{R}$ that will be made in the next result, these sets are nonempty since mild solutions lie therein. To some extent, this is a generalization of what happened in the ODEs context, where the moduli of continuity associated to a set E with L_{loc}^1-equicontinuous m-bounds (see [10, Definition 5.3]) were the appropriate ones for the solutions. As we are going to see, in the present PDEs context some properties of the semigroup of operators $(e^{tA})_{t \geqslant 0}$ come into play in a natural way, and this is the main reason for the combination of a pair of sets of moduli of continuity. Basically, the pair $\Theta = (\Theta_1, \Theta_2)$ depends upon the L_{loc}^1-equicontinuity of the m-bounds, but also upon the boundedness of $(e^{tA})_{t \geqslant 0}$ and the uniform continuity of e^{tA} for t in compact intervals away from 0. This justifies the presence of the additional parameter ν in the set Θ_2.

3. Whereas the sets $\mathcal{K}_j^I$ in the ODEs case were compact, here we can only guarantee that given $I = [q_1, q_2]$ and $j \geqslant 1$, the restriction of the set of maps $\mathcal{K}_j^I$ to each interval $[q_1 + \delta, q_2]$ is compact for every $\delta > 0$, i.e., the set $\{v|_{[q_1 + \delta, q_2]} \mid v \in \mathcal{K}_j^I\}$ is compact. To see it, note that it is closed and apply Arzelà-Ascoli's theorem. From (i) we can deduce the uniform equicontinuity of this set of maps on $[q_1 + \delta, q_2]$ by choosing ν large enough, and from (ii), since the operators $e^{1/\nu A}$ are compact, we can deduce that for each fixed $t \in [q_1 + \delta, q_2]$ the image $v(t)$ of each $v \in \mathcal{K}_j^I$ lies within an arbitrarily small distance from the set $e^{1/\nu A} \mathcal{B}_{R_j^J}^X$ of X, which is precompact, once more by taking ν as big as needed.

Proposition 4.4. *Let $E \subset \mathfrak{LC}$ have L_{loc}^1-equicontinuous m-bounds. Then, there is an associated suitable set of moduli of continuity $\Theta = (\Theta_1, \Theta_2)$, with $\Theta_1 = \big\{\theta_j^I \in C(\mathbb{R}^+, \mathbb{R}^+) \mid j \in \mathbb{N}, \ I = [q_1, q_2], \ q_1, q_2 \in \mathbb{Q}\big\}$ and $\Theta_2 = \big\{\theta_j^{I,\nu} \in C(\mathbb{R}^+, \mathbb{R}^+) \mid j, \nu \in \mathbb{N}, \ I = [q_1, q_2], \ q_1, q_2 \in \mathbb{Q}, \ \nu \geqslant 1 \ \text{s.t.} \ q_1 + 1/\nu < q_2\big\}$, and there is an associated suitable set of radii $\mathcal{R} = \{R_j^I \mid j \in \mathbb{N}, \ I = [q_1, q_2], \ q_1, q_2 \in \mathbb{Q}\}$ such that, if $\mathcal{K}_j^I$ are the sets given in Definition 4.2, and $u(\cdot, f, z)$ is the mild solution of the problem*

$$\begin{cases} u'(t) = A\, u(t) + \tilde{f}(t, u(t)), & t > q_1, \\ u(q_1) = z, \end{cases}$$

on $I = [q_1, q_2]$, $q_1, q_2 \in \mathbb{Q}$, for certain $f \in E$ and $z \in X$, and for some $j \geqslant 1$, $\|u(t, f, z)\| \leqslant j$ for all $t \in I$, then $u(\cdot, f, z) \in \mathcal{K}_j^I$.

Proof. Let $u(t) := u(t, f, z)$, $t \in I = [q_1, q_2]$. Trying to define a modulus of continuity in t, for $t_1, t_2 \in I$ with $t_1 < t_2$ we write, having in mind (2.8) and Remark 2.8.1,

$$
\begin{aligned}
u(t_2) - u(t_1) &= e^{(t_2 - q_1)A} z + \int_{q_1}^{t_2} e^{(t_2 - s)A} \tilde{f}(s, u(s)) \, ds - e^{(t_1 - q_1)A} z \\
&\quad - \int_{q_1}^{t_1} e^{(t_1 - s)A} \tilde{f}(s, u(s)) \, ds = \left(e^{(t_2 - q_1)A} - e^{(t_1 - q_1)A} \right) z \\
&\quad + \int_{q_1}^{t_1} \left(e^{(t_2 - s)A} - e^{(t_1 - s)A} \right) \tilde{f}(s, u(s)) \, ds + \int_{t_1}^{t_2} e^{(t_2 - s)A} \tilde{f}(s, u(s)) \, ds \,.
\end{aligned}
\tag{4.1}
$$

Since the semigroup of operators $(e^{tA})_{t \geqslant 0}$ is analytic, the map $t \in (0, \infty) \mapsto e^{tA} \in \mathcal{L}(X)$ is in particular norm continuous. Now, note that, to bound the term $(e^{(t_2 - q_1)A} - e^{(t_1 - q_1)A}) z$, we have to keep away from q_1. The reason is that, even if the semigroup of operators is strongly continuous, we are considering initial conditions z which vary in a bounded set of X, $\|z\| \leqslant j$, which is not a compact set in X, so that we cannot obtain a modulus of continuity at q_1 which is valid for all such z. What is true is that, writing $\|(e^{(t_2 - q_1)A} - e^{(t_1 - q_1)A}) z\| \leqslant j \|e^{(t_2 - q_1)A} - e^{(t_1 - q_1)A}\|$, given any integer $\nu \geqslant 1$ such that $q_1 + 1/\nu < q_2$, we can determine a modulus of continuity for $t_1, t_2 \in [q_1 + 1/\nu, q_2]$ by the uniform continuity of $t \mapsto e^{tA}$ on compact intervals $[1/\nu, T]$, for each $T > 1/\nu$. Namely, the map

$$
\theta^{I, \nu}(s) := \sup_{\substack{t \in [\frac{1}{\nu}, q_2 - q_1] \\ r \in [0, s]}} \|e^{tA} - e^{(t+r)A}\|, \quad s \geqslant 0
\tag{4.2}
$$

is a modulus of continuity and $\|(e^{(t_2 - q_1)A} - e^{(t_1 - q_1)A}) z\| \leqslant j \, \theta^{I, \nu}(t_2 - t_1)$ for all $q_1 + 1/\nu \leqslant t_1 < t_2 \leqslant q_2$. Besides, it is easy to check that, if $\nu_1 \leqslant \nu_2$ and $I_1 \subseteq I_2$ with the restriction that $1/\nu_1$ is smaller than the length of the interval I_1, then $\theta^{I_1, \nu_1} \leqslant \theta^{I_2, \nu_2}$. This is independent of $f \in E$ and $z \in X$ with $\|z\| \leqslant j$.

Now we tackle the integral terms. Taking relations (2.4) and (2.6) into account, the last term in (4.1) is bounded for each $q_1 \leqslant t_1 < t_2 \leqslant q_2$ as follows,

$$
\begin{aligned}
\left\| \int_{t_1}^{t_2} e^{(t_2 - s)A} \tilde{f}(s, u(s)) \, ds \right\| &\leqslant \int_{t_1}^{t_2} \|e^{(t_2 - s)A}\| \|\tilde{f}(s, u(s))\| \, ds \\
&\leqslant N \int_{t_1}^{t_2} m_f^j(s) \, ds \leqslant \theta_j^I(t_2 - t_1) \,,
\end{aligned}
$$

where θ_j^I is a slight modification of the modulus of continuity associated to E given in Definition 5.3 in [10] in an ODEs context. Precisely, here we take for convenience

$$
\theta_j^I(s) := 2 N \sup_{t \in I, g \in E} \int_t^{t+s} m_g^j, \quad s \geqslant 0 \,.
\tag{4.3}
$$

The fact that this is a modulus of continuity is due to the hypothesis that the family E has L_{loc}^1-equicontinuous m-bounds. This is again valid for all $f \in E$ and $z \in X$ with $\|z\| \leqslant j$, as far as the solution $u(t)$ keeps bounded by j on the interval I.

Finally, we deal with the remaining integral term in (4.1). We split the integral in two. For each $t_1 < t_2$ in $[q_1 + 1/\nu, q_2]$ we write

$$\left\| \int_{t_1 - \frac{1}{\nu}}^{t_1} \left(e^{(t_2 - s)A} - e^{(t_1 - s)A} \right) \tilde{f}(s, u(s)) \, ds \right\| \leqslant 2N \int_{t_1 - \frac{1}{\nu}}^{t_1} m_f^j(s) \, ds \leqslant \theta_j^I \left(\frac{1}{\nu} \right).$$

Finally, whenever $t_1 > q_1 + 1/\nu$, we bound the term

$$\left\| \int_{q_1}^{t_1 - \frac{1}{\nu}} \left(e^{(t_2 - s)A} - e^{(t_1 - s)A} \right) \tilde{f}(s, u(s)) \, ds \right\|$$

$$\leqslant \sup_{s \in [q_1, t_1 - \frac{1}{\nu}]} \left\| e^{(t_2 - s)A} - e^{(t_1 - s)A} \right\| \sup_{g \in E} \int_I m_g^j \leqslant \theta^{I,\nu}(t_2 - t_1) \sup_{g \in E} \int_I m_g^j ,$$

(4.4)

for $\theta^{I,\nu}$ given in (4.2). Recall that E has L_{loc}^1-bounded m-bounds by Remark 3.7. Bringing everything together, we can conclude that appropriate moduli of continuity $\theta_j^{I,\nu} \in C(\mathbb{R}^+, \mathbb{R}^+)$ for each $j \in \mathbb{N}$, $I = [q_1, q_2]$, $q_1, q_2 \in \mathbb{Q}$, and $\nu \geqslant 1$ such that $q_1 + 1/\nu < q_2$ can be determined, so that the order condition (3.2) holds and $\|u(t_1) - u(t_2)\| \leqslant \theta_j^{I,\nu}(|t_1 - t_2|) + \theta_j^I(1/\nu)$ for all $t_1, t_2 \in [q_1 + 1/\nu, q_2]$. More specifically,

$$\theta_j^{I,\nu}(s) := j \, \theta^{I,\nu}(s) + \theta_j^I(s) + \theta^{I,\nu}(s) \sup_{g \in E} \int_I m_g^j , \quad s \geqslant 0 .$$

The previous arguments work identically if we consider any other interval $J = [r_1, r_2] \subset I$ ($r_1, r_2 \in \mathbb{Q}$) and every integer $\nu \geqslant 1$ such that $r_1 + 1/\nu < r_2$, by recalling Remark 2.8.2 and writing $u(t) = e^{(t - r_1)A} u(r_1) + \int_{r_1}^t e^{(t-s)A} \tilde{f}(s, u(s)) \, ds$, for $t \in [r_1, r_2]$. Therefore, (i) in Definition 4.2 holds for $u(t)$.

As for (ii), we have to determine a suitable set of radii $\mathcal{R}$ according to Definition 4.1, so that for every interval $J = [r_1, r_2] \subseteq I$ and for every $\nu \geqslant 1$ such that $r_1 + 1/\nu < r_2$ it holds that $\mathrm{d}\left(u(t), e^{1/\nu A} \mathcal{B}_{R_j^J}^X\right) \leqslant \theta_j^J(1/\nu)$ for all $t \in [r_1 + 1/\nu, r_2]$. As before, it suffices to argue for $I = [q_1, q_2]$ and $\nu \geqslant 1$ such that $q_1 + 1/\nu < q_2$.

Since $e^{(t+s)A} = e^{tA} e^{sA}$ for all $t, s \geqslant 0$, and $e^{1/\nu A} \in \mathcal{L}(X)$, for each $t \in [q_1 + 1/\nu, q_2]$ we can write

$$u(t) = e^{\frac{1}{\nu} A} \left(e^{(t - q_1 - \frac{1}{\nu})A} z + \int_{q_1}^{t - \frac{1}{\nu}} e^{(t - s - \frac{1}{\nu})A} \tilde{f}(s, u(s)) \, ds \right)$$

$$+ \int_{t - \frac{1}{\nu}}^t e^{(t-s)A} \tilde{f}(s, u(s)) \, ds .$$

Now, by (2.4) and (2.6) we can bound

$$\left\| e^{(t - q_1 - \frac{1}{\nu})A} z + \int_{q_1}^{t - \frac{1}{\nu}} e^{(t - s - \frac{1}{\nu})A} \tilde{f}(s, u(s)) \, ds \right\| \leqslant N j + N \sup_{g \in E} \int_I m_g^j =: R_j^I .$$

Before we proceed, note that the constants $R_j^I > 0$ defined in this way satisfy that $R_{j_1}^{I_1} \leqslant R_{j_2}^{I_2}$ whenever $I_1 \subseteq I_2$ and $j_1 \leqslant j_2$. Thus, the set $\mathcal{R} = \{R_j^I \mid j \in \mathbb{N}, \ I = [q_1, q_2], \ q_1, q_2 \in \mathbb{Q}\}$ is a suitable set of radii associated to the family E.

Finally, it is immediate to conclude that

$$\mathrm{d}\left(u(t), e^{\frac{1}{\nu} A} \mathcal{B}_{R_j^I}^X\right) \leqslant \left\| \int_{t - \frac{1}{\nu}}^t e^{(t-s)A} \tilde{f}(s, u(s)) \, ds \right\| \leqslant \theta_j^I \left(\frac{1}{\nu} \right),$$

for the modulus in (4.3), as we wanted. Therefore, $u(\cdot, f, z) \in \mathcal{K}_j^I$ and the proof is finished. $\qquad\square$

Remark 4.5. From now on, we will refer to $\Theta = (\Theta_1, \Theta_2)$ and $\mathcal{R}$ as *the associated set of moduli of continuity and set of radii*, respectively, for a family $E \subset \mathfrak{LC}$ with L_{loc}^1-equicontinuous m-bounds.

We can now define the new topologies $\mathcal{T}_{\Theta\mathcal{R}}$ and $\sigma_{\Theta\mathcal{R}}$.

Definition 4.6 (Topologies $\mathcal{T}_{\Theta\mathcal{R}}$ and $\sigma_{\Theta\mathcal{R}}$). Let $E \subset \mathfrak{LC}$ have L_{loc}^1-equicontinuous m-bounds and let $\Theta = (\Theta_1, \Theta_2)$ and $\mathcal{R}$ be the associated sets of moduli of continuity and radii, respectively. For each $j \in \mathbb{N}$ and $I = [q_1, q_2]$ ($q_1, q_2 \in \mathbb{Q}$), let $\mathcal{K}_j^I$ be the set of functions defined in Definition 4.2. We call $\mathcal{T}_{\Theta\mathcal{R}}$ (resp. $\sigma_{\Theta\mathcal{R}}$) the topology on $\mathfrak{SC}$ generated by the family of seminorms

$$p_{I,j}(f) = \sup_{v \in \mathcal{K}_j^I} \int_I \left\| \tilde{f}(s, v(s)) \right\| ds$$

$$\left(\text{resp. } \tilde{p}_{I,j}(f) = \sup_{v \in \mathcal{K}_j^I} \left\| \int_I \tilde{f}(s, v(s))\, ds \right\| = \sup_{x \in \overline{U}, v \in \mathcal{K}_j^I} \left| \int_I f(s, x, v(s)(x))\, ds \right| \right)$$

for each $f \in \mathfrak{SC}$, $I = [q_1, q_2]$ ($q_1, q_2 \in \mathbb{Q}$), and $j \in \mathbb{N}$.

One can prove that $(\mathfrak{SC}, \mathcal{T}_{\Theta\mathcal{R}})$ and $(\mathfrak{SC}, \sigma_{\Theta\mathcal{R}})$ are locally convex metric spaces.

Lemma 4.7. *Let $E \subset \mathfrak{LC}$ have L_{loc}^1-equicontinuous m-bounds, let $\Theta = (\Theta_1, \Theta_2)$ and $\mathcal{R}$ be the associated sets of moduli of continuity and radii, respectively, and consider the associated topology $\sigma_{\Theta\mathcal{R}}$. Let $(f_n)_{n\geqslant 1} \subset \mathfrak{SC}$ be such that $f_n \to f$ in $(\mathfrak{SC}, \sigma_{\Theta\mathcal{R}})$ as $n \to \infty$, and take $j \geqslant 1$, $I = [q_1, q_2]$ with $q_1, q_2 \in \mathbb{Q}$. Then, for all $r_1, r_2 \in \mathbb{Q}$ with $q_1 \leqslant r_1 \leqslant r_2 \leqslant q_2$,*

$$\lim_{n \to \infty} \sup_{v \in \mathcal{K}_j^I} \left\| \int_{r_1}^{r_2} \left(\tilde{f}_n(s, v(s)) - \tilde{f}(s, v(s)) \right) ds \right\| = 0.$$

Proof. Let $J = [r_1, r_2] \subset I$. It suffices to note that, by Remark 4.3.1, for each $v \in \mathcal{K}_j^I$,

$$\left\| \int_{r_1}^{r_2} \left(\tilde{f}_n(s, v(s)) - \tilde{f}(s, v(s)) \right) ds \right\| \leqslant \tilde{p}_{J,j}(f_n - f) \to 0 \quad \text{as } n \to \infty.$$

$\qquad\square$

We are in a position to state the first continuity result with respect to the weak topology $\sigma_{\Theta\mathcal{R}}$. The main arguments in the proof are the same as the ones in Theorem 3.8(i) in [11], but technical difficulties arise due to the presence of the semigroup of operators $(e^{tA})_{t\geqslant 0}$. Recall that $I_{f,z}$ denotes the maximal interval of definition of the mild solution $u(t, f, z)$ of problem (2.7). We refer the reader to Section 2 for more details.

Theorem 4.8. *Let $E \subset \mathfrak{LC}$ have L_{loc}^1-equicontinuous m-bounds, associated suitable set of moduli of continuity Θ, and associated suitable set of radii $\mathcal{R}$. Assume that $(f_n)_{n\geqslant 1} \subset E$ converges to f in $(\mathfrak{LC}, \sigma_{\Theta\mathcal{R}})$ and $(z_n)_{n\geqslant 1} \subset X$ converges to $z \in X$, as $n \to \infty$. Then, the mild solutions $u_n(t) := u(t, f_n, z_n)$ converge uniformly as $n \to \infty$ to the mild solution $u(t) := u(t, f, z)$ on every time-interval $[0, T] \subset I_{f,z}$.*

Proof. For a fixed $T > 0$ such that $[0, T] \subset I_{f,z}$, define

$$0 < \rho := 1 + \max \left\{ \left(\|z_n\| \right)_{n \geqslant 1}, \, \sup\{\|u(t, f, z)\| \mid t \in [0, T]\} \right\}. \tag{4.5}$$

Now, for each $n \geqslant 1$ we consider the map $v_n : [0, T] \to X$ defined by

$$v_n(t) := \left\{ \begin{array}{ll} u_n(t), & \text{if } t \in [0, T_n], \\ u_n(T_n), & \text{if } t \in [T_n, T], \end{array} \right.$$

where $T_n := \sup\{t \in [0, T] \mid \|u_n(s)\| \leqslant \rho \text{ for all } s \in [0, t]\}$. Note that by (4.5) and the continuity of u_n it follows that $T_n > 0$ for every $n \geqslant 1$. Fix an integer j such that $j > \rho$ and recall that we denote by $\mathcal{B}_j$ the closed ball of radius j in $C(\overline{U}, \mathbb{R}^N)$.

We affirm that the set $\{v_n(\cdot) \mid n \geqslant 1\}$, which is contained in $C([0, T], \mathcal{B}_j)$ by construction, is relatively compact. To see it, by Arzelà-Ascoli's theorem, we have to check the equicontinuity at any $t \in [0, T]$ as well as the relative compactness of the set of maps $\{v_n(t) \mid n \geqslant 1\} \subset C(\overline{U}, \mathbb{R}^N)$ for each fixed $t \in [0, T]$.

In what respects to equicontinuity, for each $n \geqslant 1$, if $t_1, t_2 \in [T_n, T]$, then $v_n(t_2) - v_n(t_1) = 0$, so that we only have to worry about $t_1, t_2 \in [0, T_n]$. In this case, for $t_1 < t_2$, similarly to the writing in (4.1), here we have

$$v_n(t_2) - v_n(t_1) = \left(e^{t_2 A} - e^{t_1 A} \right) z_n$$
$$+ \int_0^{t_1} \left(e^{(t_2 - s)A} - e^{(t_1 - s)A} \right) \tilde{f}_n(s, v_n(s)) \, ds + \int_{t_1}^{t_2} e^{(t_2 - s)A} \tilde{f}_n(s, v_n(s)) \, ds. \tag{4.6}$$

Note that the set $K = \{(z_n)_{n \geqslant 1}, z\}$ is a compact set in X. With the strong continuity of $(e^{tA})_{t \geqslant 0}$, we can assert that the map $[0, T] \times K \to X$, $(t, \tilde{z}) \mapsto e^{tA} \tilde{z}$ is uniformly continuous on $[0, T] \times K$ (see, e.g., [5, Lemma I.5.2]). This implies that the term $\left\| \left(e^{t_2 A} - e^{t_1 A} \right) z_n \right\|$ becomes as small as wanted provided that $t_2 - t_1$ is small enough, uniformly on $[0, T_n]$ and $n \geqslant 1$.

About the integral terms in (4.6), the last one is treated as the corresponding one in the proof of Proposition 4.4, using that $\|e^{tA}\| \leqslant N$ for all $t \in I = [0, T]$ and that all the maps $f_n \in E$ and E has L^1_{loc}-equicontinuous m-bounds. That is, fixed an $\varepsilon > 0$ we find a $\delta > 0$ such that $\theta_j^I(\delta) < \varepsilon$, for the modulus θ_j^I in (4.3) (which makes sense whether $T \in \mathbb{Q}$ or not) and then, $\left\| \int_{t_1}^{t_2} e^{(t_2 - s)A} \tilde{f}_n(s, v_n(s)) \, ds \right\| < \varepsilon$ for all $n \geqslant 1$, provided that $0 < t_2 - t_1 \leqslant \delta$. Now, for the first integral term, we take $t_1^* := \max(0, t_1 - \delta)$ which equals 0 if $t_1 \leqslant \delta$, and equals $t_1 - \delta$ if $t_1 > \delta$, and just as before, for all $n \geqslant 1$,

$$\left\| \int_{t_1^*}^{t_1} \left(e^{(t_2 - s)A} - e^{(t_1 - s)A} \right) \tilde{f}_n(s, v_n(s)) \, ds \right\| \leqslant 2N \int_{t_1^*}^{t_1} m_n^j(s) \, ds < \varepsilon.$$

Finally, provided that $t_1 > \delta$, we bound the remaining integral on $[0, t_1 - \delta]$ just as in (4.4) and we apply the uniform continuity of e^{tA} on $[\delta, T]$ and the fact that E has L^1_{loc}-bounded m-bounds. In all, we can affirm that we have proved the uniform equicontinuity of the sequence $\{v_n(\cdot) \mid n \geqslant 1\}$ on $[0, T]$.

It remains to check the precompactness of the set $\{v_n(t) \mid n \geqslant 1\} \subset X$ for each fixed $t \in [0, T]$. If $t = 0$, then $\{v_n(0) \mid n \geqslant 1\} = \{z_n \mid n \geqslant 1\}$ which by hypothesis is relatively compact. If $t > 0$, it suffices to check that given any $\varepsilon > 0$ we can determine a precompact set which is within a distance less than ε from the former set. But, no matter if $T \in \mathbb{Q}$ or not, by Proposition 4.4 we can affirm that

$(v_n)_{n \geqslant 1} \subset \mathcal{K}_j^{[0,T]}$, so that in particular the maps $(v_n)_{n \geqslant 1}$ satisfy (ii) in Definition 4.2 and this is enough for our purposes (see also Remark 4.3.3).

As a consequence, we can assume without loss of generality that $v_n(t) \to v(t)$ uniformly for $t \in [0,T]$ as $n \to \infty$, for a continuous map v. Now, define T_0 by

$$T_0 := \sup\{t \in [0,T] \mid \|v(s)\| < \rho - 1/2 \text{ for all } s \in [0,t]\} \tag{4.7}$$

and note that $T_0 > 0$ because of (4.5), $\|z_n\| \to \|z\|$ as $n \to \infty$, $v(0) = z$ and v is continuous. Besides, since v_n converges uniformly to v on $[0,T]$, there exists an $n_0 \geqslant 1$ such that $\|v_n(t)\| \leqslant \rho - 1/4$ for all $t \in [0,T_0]$ and $n \geqslant n_0$. By the definition of the maps v_n, this means that $v_n(t) = u_n(t) = u(t, f_n, z_n)$ for all $t \in [0,T_0]$ and $n \geqslant n_0$. Therefore, to finish the proof it suffices to check first that $v(t) = u(t,f,z)$ for $t \in [0,T_0]$, and afterwards that $T_0 = T$.

To see that $v(t)$ is actually the mild solution for f and z on $[0,T_0]$, write

$$v_n(t) = e^{tA} z_n + \int_0^t e^{(t-s)A} \tilde{f}_n(s, v_n(s)) \, ds, \quad t \in [0,T_0], \; n \geqslant n_0.$$

We already know that $v_n(t) \to v(t)$ and $e^{tA} z_n \to e^{tA} z$ as $n \to \infty$, in fact uniformly for $t \in [0,T_0]$. So, if we prove that for each $t \in (0,T_0]$,

$$\left\| \int_0^t e^{(t-s)A} \tilde{f}_n(s, v_n(s)) \, ds - \int_0^t e^{(t-s)A} \tilde{f}(s, v(s)) \, ds \right\| \to 0 \quad \text{as } n \to \infty, \tag{4.8}$$

we will be done with the first claim. Let us separately study these two terms:

$$a_n := \left\| \int_0^t e^{(t-s)A} \tilde{f}_n(s, v_n(s)) \, ds - \int_0^t e^{(t-s)A} \tilde{f}(s, v_n(s)) \, ds \right\|$$

and

$$b_n := \left\| \int_0^t e^{(t-s)A} \tilde{f}(s, v_n(s)) \, ds - \int_0^t e^{(t-s)A} \tilde{f}(s, v(s)) \, ds \right\|.$$

We start with the second one, since it is easier to tackle. Let l_f^j be the optimal l-bound of f on B_j and apply (2.4) and Proposition 2.6(iv) to write

$$b_n \leqslant \int_0^t \|e^{(t-s)A}\| \, \|\tilde{f}(s, v_n(s)) - \tilde{f}(s, v(s))\| \, ds$$

$$\leqslant N \int_0^t l_f^j(s) \, \|v_n(s) - v(s)\| \, ds$$

$$\leqslant N \sup_{s \in [0,T_0]} \|v_n(s) - v(s)\| \int_0^{T_0} l_f^j(s) \, ds \to 0 \quad \text{as } n \to \infty,$$

since v_n converges uniformly to v on $[0,T]$.

The first term a_n is more delicate. By Proposition 3.8(ii), keeping the agreed notation for the optimal m-bounds, we know that fixed an $\varepsilon > 0$, there is a $\delta = \delta([0,T_0], j, \varepsilon, N) > 0$ such that whenever $t_1, t_2 \in [0,T_0]$ satisfy $|t_1 - t_2| \leqslant \delta$,

$$\int_{t_1}^{t_2} m_n^j, \int_{t_1}^{t_2} m_f^j < \frac{\varepsilon}{2N} \quad \text{for all } n \geqslant n_0,$$

and besides, we can take a $\rho_0 > 0$ such that

$$\int_0^{T_0} m_n^j, \int_0^{T_0} m_f^j < \rho_0 \quad \text{for all } n \geqslant n_0.$$

Now, on the interval $[\delta, T_0]$, e^{tA} is uniformly continuous. Thus, we can take an $h > 0$ with the restrictions $h < \delta$ and $h \in \mathbb{Q}$ for convenience, so that

$$\left\| e^{t_1 A} - e^{t_2 A} \right\| < \frac{\varepsilon}{2\rho_0} \quad \text{for all } t_1, t_2 \in [\delta, T_0] \text{ with } |t_1 - t_2| < h \,.$$

The first situation we consider is immediate: if $t \in (0, T_0]$ satisfies $t < \delta$, then for all $n \geqslant n_0$, thanks to the L^1_{loc}-equicontinuous m-bounds,

$$a_n \leqslant \int_0^t \left\| e^{(t-s)A} \right\| \left\| \tilde{f}_n(s, v_n(s)) - \tilde{f}(s, v_n(s)) \right\| ds \leqslant N \left(\int_0^t m_n^j + \int_0^t m_f^j \right) < \varepsilon \,.$$

If this is not the case, that is, if $t \geqslant \delta$, then h is the appropriate length of a subinterval where things go well. We write $t - \delta = k_0 h + h_1$ where $k_0 \in \mathbb{N}$ and $0 \leqslant h_1 < h$. Then, to get the idea, we split the integral on $[0, t]$ as follows:

$$\int_0^t = \sum_{k=0}^{k_0-1} \int_{kh}^{(k+1)h} + \int_{k_0 h}^{t-\delta} + \int_{t-\delta}^{t}$$

where we understand that the sum is empty if $k_0 = 0$, so that there are always k_0 terms in the sum. Note that, since $t - \delta - k_0 h = h_1 < h < \delta$, the last two integrals can be treated as the one in the first situation just considered ($t < \delta$). For the remaining sum, note that for $s \in [0, k_0 h]$, $t - s \geqslant t - k_0 h = \delta + h_1 \geqslant \delta$. Then, we do as follows:

$$\left\| \sum_{k=0}^{k_0-1} \int_{kh}^{(k+1)h} e^{(t-s)A} \left(\tilde{f}_n(s, v_n(s)) - \tilde{f}(s, v_n(s)) \right) ds \right\|$$

$$= \left\| \sum_{k=0}^{k_0-1} \int_{kh}^{(k+1)h} \left(e^{(t-s)A} - e^{(t-kh)A} \right) \left(\tilde{f}_n(s, v_n(s)) - \tilde{f}(s, v_n(s)) \right) ds \right.$$

$$\left. + \sum_{k=0}^{k_0-1} e^{(t-kh)A} \int_{kh}^{(k+1)h} \left(\tilde{f}_n(s, v_n(s)) - \tilde{f}(s, v_n(s)) \right) ds \right\|$$

$$\leqslant \sum_{k=0}^{k_0-1} \int_{kh}^{(k+1)h} \frac{\varepsilon}{2\rho_0} \left(m_n^j(s) + m_f^j(s) \right) ds$$

$$+ N \sum_{k=0}^{k_0-1} \left\| \int_{kh}^{(k+1)h} \left(\tilde{f}_n(s, v_n(s)) - \tilde{f}(s, v_n(s)) \right) ds \right\|$$

$$\leqslant \frac{\varepsilon}{2\rho_0} \int_0^{k_0 h} \left(m_n^j(s) + m_f^j(s) \right) ds$$

$$+ N \sum_{k=0}^{k_0-1} \sup_{w \in \mathcal{K}_j^I} \left\| \int_{kh}^{(k+1)h} \left(\tilde{f}_n(s, w(s)) - \tilde{f}(s, w(s)) \right) ds \right\|$$

$$\leqslant \varepsilon + N \sum_{k=0}^{k_0-1} \sup_{w \in \mathcal{K}_j^I} \left\| \int_{kh}^{(k+1)h} \left(\tilde{f}_n(s, w(s)) - \tilde{f}(s, w(s)) \right) ds \right\| \,,$$

for the interval $I = [0, k_0 h]$ with $k_0 h \in \mathbb{Q}$. Note that once more Proposition 4.4 justifies that $v_n \in \mathcal{K}_j^I$, for every $n \geqslant n_0$. At this point we only have to note that for each fixed $t \geqslant \delta$ we have a finite number of terms in the sum, namely $k_0 = k_0(t)$, and each of them tends to 0 as $n \to \infty$, by Lemma 4.7. After this, we can conclude that $v(t) = u(t, f, z)$, the mild solution for f and z, on $[0, T_0]$.

To finish the proof, it suffices to check that $T_0 = T$. If not, by (4.7) and the continuity of v, we would have that $\|v(T_0)\| = \rho - 1/2$, which falls in contradiction with the definition of ρ in (4.5). Therefore, $T_0 = T$ and the proof is finished. $\square$

Corollary 4.9. *Since $\sigma_{\Theta\mathcal{R}} \leqslant \sigma_\Theta$ and $\sigma_{\Theta\mathcal{R}} \leqslant \mathcal{T}_{\Theta\mathcal{R}} \leqslant \mathcal{T}_\Theta \leqslant \mathcal{T}_B$, whenever $E \subset \mathfrak{LC}$ has L^1_{loc}-equicontinuous m-bounds, as a corollary of the previous theorem we obtain the corresponding continuity result when $(f_n)_{n\geqslant 1}$ converges to f in any of the topological spaces $(\mathfrak{LC}, \sigma_\Theta)$, $(\mathfrak{LC}, \mathcal{T}_{\Theta\mathcal{R}})$, $(\mathfrak{LC}, \mathcal{T}_\Theta)$ or $(\mathfrak{LC}, \mathcal{T}_B)$.*

4.2. **Topologies of continuity when E has L^1_{loc}-bounded l-bounds.** In this section, we consider a class $E \subset \mathfrak{LC}$ with L^1_{loc}-bounded l-bounds, and prove the continuity result of mild solutions with respect to f when any strong topology $\mathcal{T}_D, \mathcal{T}_\Theta$ or $\mathcal{T}_B$ is considered, all of which are equivalent in this case, as seen in Theorem 3.9. Before we state the theorem, we include a technical result which will be used in the proof.

Lemma 4.10. *Let $E \subset \mathfrak{LC}$ with L^1_{loc}-bounded l-bounds and let D be any dense and countable subset of $\mathbb{R}^N$. Assume that $(f_n)_{n\geqslant 1} \subset E$ converges to f in $(\mathfrak{LC}, \mathcal{T}_D)$ as $n \to \infty$. Then, the family $(f_n)_{n\geqslant 1}$ has L^1_{loc}-bounded m-bounds too.*

Proof. Consider the closed ball B_j in $\mathbb{R}^N$ and take a $y_0 \in B_j \cap D$. Then, for each $n \geqslant 1$, for almost every $t \in \mathbb{R}$, and for all $x \in \overline{U}$ and $y \in B_j$, $|f_n(t, x, y)| \leqslant |f_n(t, x, y) - f_n(t, x, y_0)| + |f_n(t, x, y_0)| \leqslant l^j_n(t)\,|y - y_0| + |f_n(t, x, y_0)| \leqslant 2\,j\,l^j_n(t) + \|\tilde{f}_n(t, \tilde{y}_0)\|$. Since the l-bounds $(l^j_n)_{n\geqslant 1}$ are L^1_{loc}-bounded, it suffices to check that given any interval $I = [q_1, q_2]$ with $q_1, q_2 \in \mathbb{Q}$, $\sup_{n\geqslant 1} \int_I \|\tilde{f}_n(t, \tilde{y}_0)\|\,ds < \infty$. With this purpose, we write $\|\tilde{f}_n(t, \tilde{y}_0)\| \leqslant \|\tilde{f}_n(t, \tilde{y}_0) - \tilde{f}(t, \tilde{y}_0)\| + \|\tilde{f}(t, \tilde{y}_0)\|$. Since $f_n \to f$ in $\mathcal{T}_D$, $p_{I, y_0}(f_n - f) = \int_I \|\tilde{f}_n(s, \tilde{y}_0) - \tilde{f}(s, \tilde{y}_0)\|\,ds \to 0$ as $n \to \infty$, and given any $\varepsilon > 0$, $p_{I, y_0}(f_n - f) < \varepsilon$ for all n greater than a certain n_0. Since a finite number of maps do not matter, the result follows straightaway. $\square$

Theorem 4.11. *Let $E \subset \mathfrak{LC}$ have L^1_{loc}-bounded l-bounds and let D be any dense and countable subset of $\mathbb{R}^N$. Assume that $(f_n)_{n\geqslant 1} \subset E$ converges to f in $(\mathfrak{LC}, \mathcal{T}_D)$ and $(z_n)_{n\geqslant 1} \subset X$ converges to $z \in X$, as $n \to \infty$. Then, the mild solutions $u_n(t) := u(t, f_n, z_n)$ converge uniformly as $n \to \infty$ to the mild solution $u(t) := u(t, f, z)$ on every time-interval $[0, T] \subset I_{f,z}$.*

Proof. The main ideas come from the proof of [10, Theorem 5.8(i)] in an ODEs context. Nevertheless, there are some technical differences due to the new infinite dimensional scenario. The proof follows the same outline as that of Theorem 4.8. Once the maps $v_n : [0, T] \to X$ have been defined on an interval $[0, T] \subset I_{f,z}$, we have to check that they form a relatively compact set in $C([0, T], X)$. For the equicontinuity, whenever $0 \leqslant t_1 < t_2 \leqslant T_n$, we have (4.6) for $v_n(t_2) - v_n(t_1)$. The term $(e^{t_2 A} - e^{t_2 A})\,z_n$ therein is treated identically. Using (2.4), we bound the term

$$\left\| \int_{t_1}^{t_2} e^{(t_2 - s)A}\,\tilde{f}_n(s, v_n(s))\,ds \right\| \leqslant N \int_{t_1}^{t_2} \|\tilde{f}_n(s, v_n(s))\|\,ds\,,$$

and

$$\int_{t_1}^{t_2} \|\tilde{f}_n(s, v_n(s))\|\,ds \leqslant \int_{t_1}^{t_2} \|\tilde{f}_n(s, v_n(s)) - \tilde{f}(s, v_n(s))\|\,ds + \int_{t_1}^{t_2} m^j_f(s)\,ds\,. \quad (4.9)$$

By Theorem 3.9(i), $f_n \to f$ as $n \to \infty$ also in the topology $\mathcal{T}_B$. Therefore, taking an interval $I = [0, q_2]$ with $T \leqslant q_2 \in \mathbb{Q}$, we have that

$$p_{I,j}(f_n - f) = \sup_{w \in C(I, \mathcal{B}_j)} \int_I \| \tilde{f}_n(s, w(s)) - \tilde{f}(s, w(s)) \| \, ds \to 0 \quad \text{as } n \to \infty \, .$$

Then, fixed any $\varepsilon > 0$, and (if necessary) extending the maps $v_n(t)$ to the interval $[0, q_2]$ just by $v_n(T_n)$ for $t \in [T, q_2]$, it is clear that $v_n \in C(I, \mathcal{B}_j)$ for every $n \geqslant 1$ and therefore there exists an integer n_0 such that for all $n > n_0$, and for all $0 \leqslant t_1 < t_2 \leqslant T_n$,

$$\int_{t_1}^{t_2} \| \tilde{f}_n(s, v_n(s)) - \tilde{f}(s, v_n(s)) \| \, ds \leqslant \int_I \| \tilde{f}_n(s, v_n(s)) - \tilde{f}(s, v_n(s)) \| \, ds < \frac{\varepsilon}{2\,N} \, .$$

The absolute continuity of the integral on $[0, T]$ for the integrable maps $m_f^j(s)$ and $\| \tilde{f}_n(s, v_n(s)) - \tilde{f}(s, v_n(s)) \|$ for $n = 1, \ldots, n_0$ permits us to determine a $\delta > 0$ so that the integral of any of these maps over a subinterval in $[0, T]$ of length less than or equal to δ, is less than $\varepsilon/(2\,N)$. In all, we have found a $\delta = \delta(\varepsilon, T)$ such that, if $0 < t_2 - t_1 \leqslant \delta$, then $\| \int_{t_1}^{t_2} e^{(t_2 - s)A} \tilde{f}_n(s, v_n(s)) \, ds \| < \varepsilon$ for all $n \geqslant 1$.

Once this $\delta > 0$ has been determined, note that $t_1^* := \max(0, t_1 - \delta)$ equals 0 if $t_1 \leqslant \delta$, and equals $t_1 - \delta$ if $t_1 > \delta$. Then, from the previous argument, we can assert that for all $n \geqslant 1$,

$$\left\| \int_{t_1^*}^{t_1} \left(e^{(t_2 - s)A} - e^{(t_1 - s)A} \right) \tilde{f}_n(s, v_n(s)) \, ds \right\| \leqslant 2\,N \int_{t_1^*}^{t_1} \| \tilde{f}_n(s, v_n(s)) \| \, ds < 2\,\varepsilon \, .$$

Finally, provided that $t_1 > \delta$, we still have to treat the term

$$\left\| \int_0^{t_1 - \delta} \left(e^{(t_2 - s)A} - e^{(t_1 - s)A} \right) \tilde{f}_n(s, v_n(s)) \, ds \right\|$$

$$\leqslant \sup_{s \in [0, t_1 - \delta]} \left\| e^{(t_2 - s)A} - e^{(t_1 - s)A} \right\| \int_0^T m_n^j(s) \, ds \, .$$

Here we apply the uniform continuity of e^{tA} on $[\delta, T]$ and Lemma 4.10. Summing up, we have the uniform equicontinuity on $[0, T]$ for the maps $(v_n)_{n \geqslant 1}$.

About the relative compactness of the set $\{ v_n(t) \mid n \geqslant 1 \}$ for a fixed $t \in [0, T]$, as in the proof of Theorem 4.8 we note that for $t = 0$ it is known by hypothesis, and if $t > 0$ we can look for a precompact set within an arbitrarily small distance from the former set. So, let us fix a $t > 0$ and an $\varepsilon > 0$. Apply the previous argument after (4.9) to determine a $0 < \delta = \delta(\varepsilon, t)$ with $\delta < t$ and so that $\| \int_{t - \delta}^t e^{(t - s)A} \tilde{f}_n(s, v_n(s)) \, ds \| < \varepsilon$ for all $n \geqslant 1$. Then, similarly to what has been done in the last part of the proof of Proposition 4.4, we write for each $n \geqslant 1$,

$$v_n(t) = e^{tA} z_n + \int_0^t e^{(t - s)A} \tilde{f}_n(s, v_n(s)) \, ds$$

$$= e^{\delta A} \left(e^{(t - \delta)A} z_n + \int_0^{t - \delta} e^{(t - s - \delta)A} \tilde{f}_n(s, v_n(s)) \, ds \right)$$

$$+ \int_{t - \delta}^t e^{(t - s)A} \tilde{f}_n(s, v_n(s)) \, ds \, .$$

Using once more Lemma 4.10, it is now easy to determine an $R > 0$ so that

$$d\big(v_n(t), e^{\delta A}\mathcal{B}_R^X\big) < \varepsilon \quad \text{for all } n \geqslant 1\,,$$

and since the operator $e^{\delta A}$ is compact and the set $\mathcal{B}_R^X$ is bounded, the set $e^{\delta A}\mathcal{B}_R^X$ is precompact and we are done. Then, by Arzelà-Ascoli's theorem, we can assume without loss of generality that $v_n \to v$ uniformly on $[0, T]$ as $n \to \infty$.

Finally, we define T_0 as in (4.7) and we check that $v(t)$ is the mild solution for the problem for f and z on $[0, T_0]$. For that, it suffices to prove relation (4.8). As in the cited proof, we consider the terms named a_n and b_n. The treatment of b_n is just the same, whereas the treatment of a_n is much easier now. We just need to bound

$$a_n := \bigg\| \int_0^t e^{(t-s)A}\,\tilde{f}_n(s, v_n(s))\,ds - \int_0^t e^{(t-s)A}\,\tilde{f}(s, v_n(s))\,ds \bigg\|$$

$$\leqslant N \int_0^t \big\| \tilde{f}_n(s, v_n(s)) - \tilde{f}(s, v_n(s)) \big\|\,ds \leqslant N\,p_{I,j}(f_n - f) \to 0 \quad \text{as } n \to \infty$$

due to the convergence of f_n to f in the topology $\mathcal{T}_B$.

The proof is finished as in Theorem 4.8, by checking that $T_0 = T$. $\qquad\square$

We finish the paper with a corollary that follows from the continuity results Theorems 4.8 and 4.11, combined with Theorem 3.9 on the equivalence of different topologies in some cases.

Corollary 4.12. *If $E \subset \mathfrak{LC}$ has L_{loc}^1-bounded l-bounds with respect to a modulus of continuity $\hat{\theta}$ in the variable x, then $\mathcal{T}_{\tilde{D}D}$ is the coarsest or simplest strong topology which is good for the continuity of the mild solutions. If, in addition, E has L_{loc}^1-equicontinuous m-bounds, then $\sigma_{\tilde{D}D}$ is the coarsest weak topology of continuity.*

References

[1] Z. Artstein, Topological dynamics of an ordinary differential equation, *J. Differential Equations* **23** (1977), 216–223.

[2] Z. Artstein, Topological dynamics of ordinary differential equations and Kurzweil equations, *J. Differential Equations* **23** (1977), 224–243.

[3] Z. Artstein, The limiting equations of nonautonomous ordinary differential equations, *J. Differential Equations* **25** (1977), 184–202.

[4] D.L. Cohn, *Measure Theory*. Second Edition, Birkhäuser, 2013.

[5] K.-J. Engel, R. Nagel, *One-Parameter Semigroups for Linear Evolution Equations*, GTM **194**, Springer-Verlag, New York, 2000.

[6] A.J. Heunis, Continuous dependence of the solutions of an ordinary differential equation, *J. Differential Equations* **54** (1984), 121–138.

[7] E. Hille, R.S. Phillips, *Functional Analysis and Semigroups*, Amer. Math. Soc. Coll. Publ., vol. **31**, Amer. Math. Soc., 1957.

[8] K. Holly, J. Orewczyk, Applications of the Carathéodory theorem to PDEs, *Ann. Polon. Math.* **73** (1) (2000), 1–27.

[9] I.P. Longo, Topologies of continuity for Carathéodory differential equations with applications in non-autonomous dynamics, PhD Thesis, Universidad de Valladolid, 2018.

[10] I.P. Longo, S. Novo, R. Obaya, Topologies of L_{loc}^1 type for Carathéodory functions with applications in non-autonomous differential equations, *J. Differential Equations* **263** (2017), 7187–7220.

[11] I.P. Longo, S. Novo, R. Obaya, Weak topologies for Carathéodory differential equations: continuous dependence, exponential dichotomy and attractors, *J. Dyn. Diff. Equat.* **31** (2019), 1617–1651.

[12] I.P. Longo, S. Novo, R. Obaya, Topologies of continuity for Carathéodory delay differential equations with applications in non-autonomous dynamics, *Discrete Contin. Dyn. Syst.* **39** (9) (2019), 5491–5520.

[13] I.P. Longo, S. Novo, R. Obaya, Monotone skew-product semiflows for Carathéodory differential equations and applications, *J. Dyn. Diff. Equat.* **34** (2022), 2997–3033.

[14] A. Lunardi, *Analytic Semigroups and Optimal Regularity in Parabolic Problems*, Progress in Nonlinear Differential Equations and Their Applications Vol. **16**, Birkhäuser, Basel, Boston, Berlin, 1995.

[15] R.K. Miller, G. Sell, Volterra Integral Equations and Topological Dynamics, *Mem. Amer. Math. Soc.*, no. 102, Amer. Math. Soc., Providence, 1970.

[16] R.K. Miller, G. Sell, Existence, uniqueness and continuity of solutions of integral equations, *Ann. Math. Pura Appl.* **80** (1968), 135–152; Addendum: ibid. **87** (1970), 281–286.

[17] L.W. Neustadt, On the solutions of certain integral-like operator equations. Existence, uniqueness and dependence theorems, *Arch. Rational Mech. Anal.* **38** (1970), 131–160.

[18] G. Sell, Compact sets of nonlinear operators, *Funkcial. Ekvac.* **11** (1968), 131–138.

[19] G. Sell, *Topological Dynamics and Ordinary Differential Equations*, Van Nostrand-Reinhold, London, 1971.

[20] H.L. Smith, *Monotone Dynamical Systems. An Introduction to the Theory of Competitive and Cooperative Systems*, AMS Mathematical Surveys and Monographs, Vol. **41**, AMS, Providence, 1995.

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