Available online at www.sciencedirect.com

ScienceDirect

Journal of Differential Equations 361 (2023) 248–287

Journal of Differential Equations

www.elsevier.com/locate/jde

Uniform stability and chaotic dynamics in nonhomogeneous linear dissipative scalar ordinary differential equations ☆

Juan Campos [a], Carmen Núñez [b,*], Rafael Obaya [b]

[a] *Departamento de Matemática Aplicada, Universidad de Granada, Avenida de la Fuente Nueva S/N, 18071 Granada, Spain*

[b] *Departamento de Matemática Aplicada, Universidad de Valladolid, Paseo del Cauce 59, 47011 Valladolid, Spain*

Received 3 November 2022; revised 18 February 2023; accepted 24 February 2023

Abstract

The paper analyzes the structure and the inner long-term dynamics of the invariant compact sets for the skewproduct flow induced by a family of time-dependent ordinary differential equations of nonhomogeneous linear dissipative type. The main assumptions are made on the dissipative term and on the homogeneous linear term of the equations. The rich casuistic includes the uniform stability of the invariant compact sets, as well as the presence of Li-Yorke chaos and Auslander-Yorke chaos inside the attractor.

MSC: 34D05; 34D45; 37B55; 37A05; 37D05; 37D45; 37B25

Keywords: Nonautonomous ordinary differential equations; Dissipativity and global attractor; Chaotic dynamics; Ergodic theory; Lyapunov exponents

☆ Juan Campos is partly supported by the MINECO-Feder (Spain) project RTI2018-098850-B-I00, and by the Junta de Andalucía (Spain) under projects PY18-RT-2422 and B-FQM-580-UGR20. Carmen Núñez and Rafael Obaya are partly supported Ministerio de Ciencia e Innovación (Spain) under project PID2021-125446NB-100, and by the Universidad de Valladolid under project PIP-TCESC-2020.

* Corresponding author.

E-mail addresses: campos@ugr.es (J. Campos), carmen.nunez@uva.es (C. Núñez), rafael.obaya@uva.es (R. Obaya).

https://doi.org/10.1016/j.jde.2023.02.060

1. Introduction

The mathematical literature collects many different notions of chaos, all of which share a common target: each definition takes into account different properties of the long-term behavior of the system under study, which, combined, imply the unpredictability of the dynamics due to divergence of initially nearby orbits. There are also many different approaches to the concept of stability for dynamical systems, but in this case the subjacent idea is clearer and more globally accepted: initially nearby orbits remain close. Hence it seems correct to say that, at least to some extent, chaos and stability are opposite terms.

This work concerns the long-term dynamics of a quite precise mathematical model for which both situations (chaos and uniform stability) are possible. Our dynamical system is generated by the solutions of the family of nonautonomous (in the sense of time-dependent) scalar dissipative ordinary differential equations

$$x' = a(\omega{\cdot}t)\,x + b(\omega{\cdot}t) + g(\omega{\cdot}t, x)\,, \qquad \omega \in \Omega\,, \tag{1.1}$$

where Ω is a compact metric space, $\sigma\colon \mathbb{R}\times\Omega\to\Omega\,,\,(t,\omega)\mapsto\sigma(\omega{\cdot}t) =: \omega{\cdot}t$ defines a minimal flow on Ω, $a, b\colon \Omega\to\mathbb{R}$ are continuous functions, and $g\colon \Omega\times\mathbb{R}\to\mathbb{R}$ is a smooth dissipative term. The analysis is made under the assumptions $\int_\Omega a(\omega)\,dm \le 0$ for any σ-ergodic measure on Ω and decreasing behavior of g with respect to the state variable x.

We will consider two cases. The first one occurs when the dissipation is negligible as long as the state remains in $[r_1, r_2]$ (since g vanishes at the set $\Omega\times[r_1, r_2]$ with $r_1 < r_2$) and, at the same time, the dissipation is active and dominant with respect to the linear term outside that set of states. Since the restriction of the equation to $\Omega\times[r_1, r_2]$ is linear and nonhomogeneous, we say that (1.1) provides a nonautonomous nonhomogeneous linear dissipative model. This is the case more interesting for our analysis, since the casuistic is richer. The second case, which we will call purely dissipative, occurs when g vanishes exactly at the points of $\Omega\times\{r\}$ (so that $r_1 = r_2 = r$), which in general makes simpler the structure of the attractor.

The family (1.1) generates the skewproduct flow

$$\tau\colon \mathcal{U}\subseteq\mathbb{R}\times\Omega\times\mathbb{R}\to\Omega\times\mathbb{R}\,, \quad (t,\omega,x_0)\mapsto(\omega{\cdot}t, x(t,\omega,x_0))\,,$$

where $\mathcal{I}_{\omega,x_0}\to\mathbb{R}\,,\; t\mapsto x(t,\omega,x_0)$ is the maximal solution of the equation (2.1) corresponding to ω with $x(0,\omega,x_0) = x_0$, and $\mathcal{U}$ is the open set $\bigcup_{(\omega,x_0)\in\Omega\times\mathbb{R}}\mathcal{I}_{\omega,x_0}$.

The analysis of a family of equations like (1.1), or, more generally, of the type $x' = f(\omega{\cdot}t, x)$, is a classical tool in the analysis of a single nonautonomous differential equation $x' = f_0(t,x)$. Under some regularity conditions on f_0 which the translated functions $f_t(s,x) := f_0(t+s, x)$ inherit, the *hull of* f_0, given by the closure in the compact open topology on $C(\mathbb{R}^2,\mathbb{R})$ of the set $\{f_t \mid t\in\mathbb{R}\}$, turns out to be a compact metric space Ω, and the time-translation $\mathbb{R}\times\Omega\to\Omega$, $(t,\omega)\mapsto\omega{\cdot}t := \omega_t$ defines a global continuous flow. By representing $f(\omega,x) := \omega(0,x)$ we obtain a family $x' = f(\omega{\cdot}t, x)$ (i.e., $x' = \omega(t,x)$) which includes the initial equation. The function f_0 is *time-recurrent* if the flow on its hull Ω is minimal, as we assume in this paper. This is for instance the case if f_0 is, roughly speaking, almost periodic in t uniformly in x; but a minimal hull may come from other types of functions. By being a bit more careful in the hull construction, we obtain a family of the type (1.1) if the starting point is $x' = a_0(t)\,x + b_0(t) + g_0(t,x)$. This collective formulation allows us to use techniques of topological dynamics and ergodic theory in the analysis of the long-term behavior of the orbits of the flow τ, which include the graphs of the solutions of the initial equation. In this paper, we choose the (more general) approach of not to assume that Ω is the hull of an initial function.

The dissipative character of τ, due to the hypothesis on g, implies the existence of a global attractor $\mathcal{A}$. Our main objective is the description of the structure and internal dynamics of the compact invariant subsets $\mathcal{K} \subseteq \mathcal{A}$. In some cases, the presence of chaos is precluded: there appear uniformly exponentially stable sets on which the dynamics reproduces that of (Ω, σ), or sets $\mathcal{K}$ which are uniformly (not exponentially) stable. But, in other cases (in the linear dissipative case), we find compact invariant subsets $\mathcal{K} \subseteq \mathcal{A}$ on which the dynamics is highly complex, with the possible occurrence of different types of chaos. This phenomenon (which cannot occur if the functions a, b and g of (1.1) are autonomous or time-periodic) shows that unpredictability can be a natural and expected ingredient in the dynamics of simple nonautonomous mathematical models, which in general are better adapted to the real world than the autonomous ones.

Many of the notions of chaos on an invariant compact subset require a positive upper Lyapunov exponent for the corresponding linearized system, in order to obtain an exponential rate of divergence of the forward orbits starting nearby in the phase space. This behavior is not possible under the assumptions we make on (1.1), which we will precise in Section 3. But some of the notions of chaos do not require this condition. In this paper, we describe some conditions on the function a of the linear part or the equations which imply, in one of the possible dynamical situations, the presence of Li-Yorke chaos and of Auslander-Yorke chaos in a "large part" of the attractor. These notions of chaos are compatible with null upper Lyapunov exponents of the compact invariant sets on which the chaos appears. Roughly speaking, Li-Yorke chaos [25] appears on a compact invariant set when this set contains an uncountable subset of points such that any pair of them is Li-Yorke chaotic; i.e., it gives rise to two orbits which approach each other and separate from each other alternatively on infinitely many intervals of time becoming indistinguishable. The notion of Li-Yorke chaos was introduced in [25] in 1975 for transformations, and it is easily adapted to (semi)flows. The interested reader can find in [5], [2], [24], and the many references therein, some dynamical properties associated to Li-Yorke chaos and its relation with other notions of chaotic dynamics. Auslander-Yorke chaos [3] occurs on topologically transitive flows on compact metric spaces when the flow is sensitive with respect to initial conditions. The idea was trying to capture some representative properties of the notion of turbulence of fluids given by Ruelle and Takens in [33]. The abstract formulation of [3] makes this notion applicable to a much more general dynamical framework. Among the many works devoted to characterize this type of chaos and to analyze its dynamical consequences, as well as to establish connections and differences with other types of chaos, we mention [14] (which is central to our approach in this paper), [15], [27], and references therein.

In the rest of this introduction, we describe the structure of the paper, which is organized in two sections, as well as the main dynamical properties which we prove.

Section 2 is a long preliminary section, divided in seven parts. Its length is due in part to the many different concepts and already known properties needed for the statements and proofs of our main results. First, we recall basic and (more or less) standard notions on topological dynamics, ergodic theory, skewproduct flows, stability, dissipativity and global attractors, exponential dichotomy, Lyapunov exponents, Sacker and Sell spectrum, hyperbolicity of minimal subsets... And then we continue with the description of the less known nonempty set $\mathcal{R}_m$ of those maps $a\colon \Omega \to \mathbb{R}$ which will allow us to detect the presence of chaotic invariant subsets, and with the definitions and basic properties of Li-Yorke chaos (also in measure in the case of a skewproduct flow) and of Auslander-Yorke chaos. The subindex m refers to a σ-invariant measure on which the definition of the set $\mathcal{R}_m$ depends.

The structure of this preliminary section is better described at its first paragraphs. We point out here that, in addition to this large number of notions and already known properties, Section 2 includes the detailed proofs of three new results, fundamental to our purposes. The first one, Theorem 2.14, shows that the sets $\mathcal{R}_m$ are nonempty if the flow on Ω is non periodic, and contain functions with null Sacker and Sell spectrum. The second one, Theorem 2.16, refers to some extra properties of the maps $a \in \mathcal{R}_m$, which will allow us to emphasize that the Li-Yorke chaos which we detect is "quite more chaotic" than what the initial definition requires. We will explain this better in due time. The third result, Theorem 2.26, determines a series of compact subsets which are appropriate to detect the presence of Auslander-Yorke chaos, given by the supports of certain ergodic measures.

The main results of the paper are stated and proved in Section 3, which begins with the precise description of the conditions imposed on the dissipative term g of (1.1): different degrees of smoothness, vanishing set given by $\Omega \times [r_1, r_2]$, dissipativity character, and (strictly or not) decreasing behavior outside $\Omega \times [r_1, r_2]$. The last condition is not needed in our first three results. Theorem 3.2 establishes the existence of a global attractor, which thanks to the minimality assumed of the base flow takes the shape

$$\mathcal{A} = \bigcup_{\omega \in \Omega} \left(\{\omega\} \times [\alpha_{\mathcal{A}}(\omega), \beta_{\mathcal{A}}(\omega)] \right),$$

for two semicontinuous functions $\alpha_{\mathcal{A}}, \beta_{\mathcal{A}} \colon \Omega \to \mathbb{R}$ with τ-invariant graphs. Theorem 3.3 analyzes the properties of two minimal sets, $\mathcal{M}^\alpha$ and $\mathcal{M}^\beta$, associated to the covers of $\mathcal{A}$ (a tool for our main results), and Theorem 3.4 shows that the unique situation in which all the minimal sets have negative Lyapunov exponent is that of existence of a unique minimal set, which is given by the uniformly exponentially stable graph of a continuous function $\eta \colon \Omega \to \mathbb{R}$, and which coincides with the attractor.

With the condition on the monotonicity of g in force from now on, we first analyze the dynamical situation arising when $\int_\Omega a(\omega)\, dm < 0$ for every σ-ergodic measure m: Theorem 3.6 shows that the upper Lyapunov exponent of every minimal sets is negative, so that the situation is that of the end of the previous paragraph.

The rest of the paper analyzes the situation occurring when $\int_\Omega a(\omega)\, dm \le 0$ for every σ-ergodic measure m and there exists one, say $\widetilde{m}$, with $\int_\Omega a(\omega)\, d\widetilde{m} = 0$. Two global dynamical possibilities arise. The first one, which can only occur if $r_1 < r_2$, corresponds to the existence of infinitely many minimal sets. All of them are contained in $\Omega \times [r_1, r_2]$ and are given by the graphs of the functions $\eta_c = c\, \alpha_{\mathcal{A}} + (1-c)\, \beta_{\mathcal{A}}$ for $c \in [0, 1]$, which are continuous; and the union of all these minimal sets, which are uniformly stable, form the global attractor. Theorem 3.10 explores this situation. The second possibility, richer in casuistic, arises when $\mathcal{M}^\alpha = \mathcal{M}^\beta$ is the unique τ-minimal set, which is not necessarily a copy of the base, and which may or may not coincide with the global attractor. In particular, the global attractor is a pinched set; that is, its section over the base reduces to a singleton for at least one element of Ω. These properties and some of their dynamical consequences are described in Theorem 3.11.

When, in addition, the family is linear dissipative and $a \in \mathcal{R}_m$, Li-Yorke chaos and Auslander-Yorke chaos may appear, as we explain in Theorems 3.14 and 3.15. More precisely, if under these conditions the unique minimal set is contained in $\Omega \times [r_1, r_2]$ and at least of one of its covers is at a positive distance from $\Omega \times (\mathbb{R} - [r_1, r_2])$, then the attractor is "strongly" Li-Yorke chaotic, in the following sense: there exists a subset $\Omega_{LY} \subset \Omega$ with full measure m such that, for every $\omega \in \Omega_{LY}$, any two points of $\{\omega\} \times [\alpha_{\mathcal{A}}(\omega), \beta_{\mathcal{A}}(\omega)]$ form a Li-Yorke chaotic pair. Moreover, making use of the above mentioned Theorem 2.16, we explore the internal dynamics in $\mathcal{A}$ in order to confirm the physical observability of the Li-Yorke chaos, and hence its potential relevance in applications. More precisely, we will prove the positive density in $\mathbb{R}$ of two sets of times for m-almost point of the base: those at which the forward orbits associated to every Li-Yorke chaotic pairs (sharing the base point) are "clearly separated", and those at which these orbits are "as close as desired".

Finally, under the same hypotheses, we detect Auslander-Yorke chaos in infinitely many invariant compact subsets $\mathcal{S}_c \subset \mathcal{A}$ for every $c \in [0, 1]$ excepting, perhaps, a particular value c_0. These (also pinched) sets are transitive: they admit a dense forward semiorbit. Besides this, the union $\widetilde{\mathcal{S}}$ of all these sets is a chain recurrent set, supporting an invariant measure $\widetilde{\mu}$, composed by sensitive points, and with a dense subset of generic points. These properties can be understood as a weak version of the classical notion of chaos introduced by Devaney in [11]. In addition, $\widetilde{\mathcal{S}}$ fills an "important part" of $\mathcal{A}$, which shows that also this chaotic phenomenon has physical relevance.

2. Preliminaries

This long preliminary section is organized in seven parts. The first four contain general results, required in Section 3 for the description of the global dynamics for the equations of the Introduction. The last three, less standard, present concepts, known properties, and new results which will be used to analyze the possible presence of chaotic behavior.

The basic concepts and properties of topological dynamics and measure theory, with special focus on skewproduct flows defined from a family of scalar nonautonomous ordinary differential equations, are summarized in the first two subsections, where we will also fix some notation. Good references for their contents are [28], [12], [34,35], [39], [26], [38], and references therein.

As explained in the Introduction, our main results are formulated under different assumptions on the linear homogenous component of the family of equations. In Subsection 2.3 we summarize the required notions and properties concerning exponential dichotomy, Sacker and Sell spectrum, and Lyapunov exponents, which can be found in [10] and [21]. Subsection 2.4 recalls some particular properties of minimal sets for a skewproduct flow in the scalar case, and includes, for the reader's convenience, a proof of a classical result relating the uniform exponential stability of these minimal sets with the sign of their Lyapunov exponents.

In Subsection 2.5 we introduce a set of continuous functions which will provide us with an adequate framework to detect the presence of the two types of chaos mentioned in the Introduction: Li-Yorke chaos, described in Subsection 2.6, and Auslander-Yorke chaos, described in Subsection 2.7. As we mentioned in the introduction, besides basic concepts and known properties, Subsections 2.5 and 2.7 present some new results which we will use in Section 3 but which are valid for a setting more general than that there considered. The contents of these results are explained in the corresponding subsections.

2.1. Basic concepts on flows

Let Ω be a complete metric space, and let dist_Ω be the distance on Ω. A (*real and continuous*) *flow* on Ω is a continuous map $\sigma : \mathbb{R} \times \Omega \to \Omega,\ (t, \omega) \mapsto \sigma(t, \omega)$ such that $\sigma_0 = \mathrm{Id}$ and $\sigma_{s+t} = \sigma_t \circ \sigma_s$ for each $s, t \in \mathbb{R}$, where $\sigma_t(\omega) := \sigma(t, \omega)$. The flow is *local* if the map σ is defined, continuous, and satisfies the previous properties on an open subset of $\mathbb{R} \times \Omega$ containing $\{0\} \times \Omega$.

Let $\mathcal{U} \subseteq \mathbb{R} \times \Omega$ be the domain of the map σ. The set $\{\sigma_t(\omega) \mid (t, \omega) \in \mathcal{U}\}$ is the σ-*orbit* (or simply *orbit*) of the point $\omega \in \Omega$. This orbit is *globally defined* if $(t, \omega) \in \mathcal{U}$ for all $t \in \mathbb{R}$.

Restricting the time to $t \geq 0$ or $t \leq 0$ provides the definition of *forward* or *backward σ-semiorbit*. A Borel subset $\mathcal{C} \subseteq \Omega$ is *σ-invariant* if it is composed by globally defined orbits; i.e., if $\sigma_t(\mathcal{C}) := \{\sigma(t,\omega) \mid \omega \in \mathcal{C}\}$ is defined and agrees with $\mathcal{C}$ for every $t \in \mathbb{R}$. A σ-invariant subset $\mathcal{M} \subseteq \Omega$ is *σ-minimal* (or simply *minimal*) if it is compact and does not contain properly any other compact σ-invariant set; or, equivalently, if each one of the two semiorbits of anyone of its elements is dense in it. The flow (Ω, σ) is *minimal* if Ω itself is minimal. If the semiorbit $\{\sigma_t(\omega_0) \mid t \geq 0\}$ is globally defined and relatively compact, then the *omega limit set* of ω_0, which we represent by $\mathcal{O}_\sigma(\omega_0)$, is given by the points $\omega \in \Omega$ such that $\omega = \lim_{n\to\infty} \sigma_{t_n}(\omega_0)$ for some sequence $(t_n) \uparrow \infty$. This set is nonempty, compact, connected and σ-invariant. By taking sequences $(t_m) \downarrow -\infty$ we obtain the definition of the *alpha limit set* of ω_0, with analogous properties.

Assume now that σ is globally defined. The flow is *equicontinuous* if given $\varepsilon > 0$ there exists $\delta > 0$ such that $\sup_{t\in\mathbb{R}} \operatorname{dist}_\Omega(\sigma_t(\omega_1), \sigma_t(\omega_2)) < \varepsilon$ whenever $\operatorname{dist}_\Omega(\omega_1, \omega_2) < \delta$. If Ω is a compact metric space, equicontinuity is equivalent to *almost periodicity* (as proved in [12]). A flow (Ω, σ) defined on a compact metric space Ω is *chain recurrent* if given $\varepsilon > 0$, $t_0 > 0$, and points ω, $\widetilde{\omega} \in \Omega$, there exist points $\omega_0 := \omega, \omega_1, \ldots, \omega_m := \widetilde{\omega}$ of Ω and real numbers $t_1 > t_0, \ldots, t_{m-1} > t_0$ such that $\operatorname{dist}_\Omega(\sigma_{t_i}(\omega_i), \omega_{i+1}) < \varepsilon$ for $i = 0, \ldots, m-1$. It is easy to check that minimality implies chain recurrence, and that if (Ω, σ) is chain recurrent then Ω is connected.

Let m be a Borel measure on Ω; i.e., a regular measure defined on the Borel sets (any measure appearing in this paper is of this type). The measure is *concentrated on* $\mathcal{B} \subseteq \Omega$ if $m(\Omega - \mathcal{B}) = 0$. Its (*topological*) *support*, $\operatorname{Supp} m$, is the complement of the biggest open set with null measure. In particular, it is contained in any closed set $\mathcal{C}$ on which the measure is concentrated; and if Ω is compact then $\operatorname{Supp} m$ is compact. The measure m is *σ-invariant* if $m(\sigma_t(\mathcal{B})) = m(\mathcal{B})$ for every Borel subset $\mathcal{B} \subseteq \Omega$ and every $t \in \mathbb{R}$. In this case, $\operatorname{Supp} m$ is τ-invariant; and if Ω is minimal, then $\operatorname{Supp} m = \Omega$. Suppose that m is *finite* and *normalized*; i.e., that $m(\Omega) = 1$. Then it is *σ-ergodic* if it is σ-invariant and, in addition, $m(\mathcal{B}) = 0$ or $m(\mathcal{B}) = 1$ for every σ-invariant subset $\mathcal{B} \subseteq \Omega$. The sets of normalized σ-invariant and σ-ergodic measures are represented by $\mathfrak{M}_{\mathrm{inv}}(\Omega, \sigma)$ and $\mathfrak{M}_{\mathrm{erg}}(\Omega, \sigma)$. If Ω is compact, there exists at least an element in $\mathfrak{M}_{\mathrm{erg}}(\Omega, \sigma)$. Any equicontinuous minimal flow (Ω, σ) is *uniquely ergodic*, that is, $\mathfrak{M}_{\mathrm{inv}}(\Omega, \sigma)$ reduces to just one element: a σ-ergodic measure.

Let Ω be a compact metric space. A Borel set $\mathcal{B} \subseteq \Omega$ has *full measure for a measure* $m \in \mathfrak{M}_{\mathrm{inv}}(\Omega, \sigma)$ if $m(\mathcal{B}) = 1$, and it has *complete measure* if $m(\mathcal{B}) = 1$ for any $m \in \mathfrak{M}_{\mathrm{inv}}(\Omega, \sigma)$. A point $\omega_0 \in \Omega$ is *σ-generic* if $\lim_{t\to\infty}(1/t)\int_0^t f(\sigma_s(\omega_0))\,ds$ exists for every $f \in C(\Omega, \mathbb{R})$. In this case, Riesz representation theorem provides a measure $m_{\omega_0} \in \mathfrak{M}_{\mathrm{inv}}(\Omega, \sigma)$ such that $\lim_{t\to\infty}(1/t)\int_0^t f(\sigma_s(\omega_0))\,ds = \int_\Omega f(\omega)\,dm_{\omega_0}$ for every $f \in C(\Omega, \mathbb{R})$. In addition, the sets $\widetilde{\Omega}$ of σ-generic points and the subset $\widetilde{\Omega}_e$ of those for which m_{ω_0} is σ-ergodic are σ-invariant and of complete measure. And given a measure $m \in \mathfrak{M}_{\mathrm{inv}}(\Omega, \mathbb{R})$ and a real function $f \in L^1(\Omega, m)$, there exists a set $\Omega_f \subseteq \widetilde{\Omega}_e$ with $m(\Omega_f) = 1$ such that $f \in L^1(\Omega, m_{\omega_0})$ for every $\omega_0 \in \Omega_f$ and $\int_\Omega f(\omega)\,dm = \int_{\Omega_f}\big(\int_\Omega f(\omega)dm_{\omega_0}\big)\,dm$.

Throughout the paper, $\mathcal{B}_\Omega(\omega_0, \delta) := \{\omega \in \Omega \mid \operatorname{dist}_\Omega(\omega_0, \omega) \leq \delta\}$.

2.2. *Scalar skewproduct flows associated to families of ODEs*

Let (Ω, σ) be a global flow on a compact metric space, and consider the one-dimensional trivial bundle $\Omega \times \mathbb{R}$, which is provided with the structure of a complete metric space by the distance $\operatorname{dist}_{\Omega\times\mathbb{R}}\big((\omega_1, x_1), (\omega_2, x_2)\big) := \operatorname{dist}_\Omega(\omega_1, \omega_2) + |x_1 - x_2|$. The sets Ω and $\mathbb{R}$ are the *base* and the *fiber* of the bundle. The sections of a subset $\mathcal{C} \subseteq \Omega \times \mathbb{R}$, over the base elements are represented as $\mathcal{C}_\omega := \{x \in \mathbb{R} \mid (\omega, x) \in \mathcal{C}\}$.

From now on, and throughout the whole paper, we will represent

$$\omega{\cdot}t := \sigma_t(\omega) = \sigma(t, \omega)\,.$$

Let us consider the scalar family of equations

$$x' = f(\omega{\cdot}t, x) \tag{2.1}$$

for $\omega \in \Omega$, where $f : \mathbb{R} \times \Omega \to \mathbb{R}$ is assumed to be jointly continuous and locally Lipschitz with respect to the state variable x. We will use the notation $(2.1)_\omega$ to refer to the equation of the family corresponding to the point ω, and proceed in an analogous way with the rest of families of equations appearing in the paper.

The family (2.1) allows us to define the map

$$\tau : \mathcal{U} \subseteq \mathbb{R} \times \Omega \times \mathbb{R} \to \Omega \times \mathbb{R}\,, \quad (t, \omega, x_0) \mapsto (\omega{\cdot}t, x(t, \omega, x_0))\,, \tag{2.2}$$

where $\mathcal{I}_{\omega,x_0} \to \mathbb{R}$, $t \mapsto x(t, \omega, x_0)$ is the maximal solution of $(2.1)_\omega$ with initial datum $x(0, \omega, x_0) = x_0$, and $\mathcal{U} := \bigcup_{(\omega,x_0)\in\Omega\times\mathbb{R}} \big(\mathcal{I}_{\omega,x_0} \times \{(\omega, x_0)\}\big)$, an open set. The uniqueness of solutions ensures that $x(s + t, \omega, x_0) = x(s, \omega{\cdot}t, x(t, \omega, x_0))$ whenever the right-hand term is defined, and this property ensures that τ defines a (local or global) flow on $\Omega \times \mathbb{R}$. The properties assumed on f also ensure that $x(t, \omega, x_0)$ varies continuously with respect to ω and x_0, and hence τ is continuous on its domain. If, in addition, f is assumed to be C^1 with respect to x_0, so is the map $(t, \omega, x_0) \mapsto x(t, \omega, x_0)$, as long as it is defined. The uniqueness of solutions also guarantees that τ is *fiber-monotone*; that is, if $x_1 < x_2$ then $x(t, \omega, x_1) < x(t, \omega, x_2)$ for any t in the common interval of definition of both solutions.

The flow $(\Omega \times \mathbb{R}, \tau)$ is a type of *skewproduct flow on* $\Omega\times\mathbb{R}$ *projecting onto* (Ω, σ). The flow (Ω, σ) is the *base flow* of $(\Omega \times \mathbb{R}, \tau)$. In the linear homogeneous case $f(\omega, x) = a(\omega)\, x$, the flow τ is globally defined and *linear*; that is, the map $\mathbb{R} \to \mathbb{R}$, $x_0 \mapsto x(t, \omega, x_0)$ is defined and linear for all $(t, \omega) \in \mathbb{R} \times \Omega$.

A measurable map $\alpha : \Omega \to \mathbb{R}$ is a τ*-equilibrium* if $\alpha(\omega{\cdot}t) = x(t, \omega, \alpha(\omega))$ for all $t \in \mathbb{R}$ and $\omega \in \Omega$; a τ*-subequilibrium* if $\alpha(\omega{\cdot}t) \le x(t, \omega, \alpha(\omega))$ for all $\omega \in \Omega$ and $t \ge 0$; and a τ-*superequilibrium* if $\alpha(\omega{\cdot}t) \ge x(t, \omega, \alpha(\omega))$ for all $\omega \in \Omega$ and $t \ge 0$. There is a strong connection among sub or superequilibria and upper or lower solutions of the differential equations, which we will explain when required. A set $\mathcal{K} \subset \Omega \times \mathbb{R}$ is a *copy of the base for* τ if it is the graph of a continuous equilibrium α, in which case we write $\mathcal{K} = \{\alpha\}$.

We say that a τ-invariant compact set $\mathcal{K} \subset \Omega \times \mathbb{R}$ projecting over the whole base is *uniformly stable at* $+\infty$ (*on the fiber*) if for any $\kappa > 0$ there exists some $\delta > 0$ such that, if $(\omega, \bar{x}_0) \in \mathcal{K}$ and $(\omega, x_0) \in \Omega \times \mathbb{R}$ satisfy $|\bar{x}_0 - x_0| < \delta$, then $x(t, \omega, x_0)$ is defined for all $t \ge 0$, and in addition $|x(t, \omega, \bar{x}_0) - x(t, \omega, x_0)| \le \kappa$ for $t \ge 0$. Changing $t \ge 0$ by $t \le 0$ provides the definition of *uniformly stable at* $-\infty$ τ-invariant compact set.

A τ-invariant compact set $\mathcal{K} \subset \Omega \times \mathbb{R}$ projecting over the whole base is *uniformly exponentially stable at* $+\infty$ (*on the fiber*) if there exist $\delta > 0$, $\kappa \ge 1$ and $\gamma > 0$ such that, if $(\omega, \bar{x}_0) \in \mathcal{K}$ and $(\omega, x_0) \in \Omega \times \mathbb{R}$ satisfy $|\bar{x}_0 - x_0| < \delta$, then $x(t, \omega, x_0)$ is defined for all $t \ge 0$, and in addition $|x(t, \omega, \bar{x}_0) - x(t, \omega, x_0)| \le \kappa\, e^{-\gamma\, t}\, |\bar{x}_0 - x_0|$ for $t \ge 0$. Changing $t \ge 0$ by $t \le 0$ provides the definition of *uniformly exponentially stable at* $-\infty$ τ-invariant compact set.

Remark 2.1. We want to insist in the fact that our definitions of (exponential or not) uniform stability for skew-product semiflows are not the classical ones for flows, since we do not consider possible variation on the base points: we just refer to variation on the fiber. For further purposes we also point out that, if (Ω,σ) is an equicontinuous flow on a compact metric space, then the whole space is uniformly stable at $\pm\infty$ in the classical sense.

The *Hausdorff semidistance* from $\mathcal{C}_1$ to $\mathcal{C}_2$, where $\mathcal{C}_1,\mathcal{C}_2\subset\Omega\times\mathbb{R}$, is

$$\mathrm{dist}(\mathcal{C}_1,\mathcal{C}_2):=\sup_{(\omega_1,x_1)\in\mathcal{C}_1}\left(\inf_{(\omega_2,x_2)\in\mathcal{C}_2}\big(\mathrm{dist}_{\Omega\times\mathbb{R}}((\omega_1,x_1),(\omega_2,x_2))\big)\right).$$

A set $\mathcal{B}\subset\Omega\times\mathbb{R}$ is said *to attract a set* $\mathcal{C}\subseteq\Omega$ *under* τ if $\tau_t(\mathcal{C})$ is defined for all $t\geq 0$ and, in addition, $\lim_{t\to\infty}\mathrm{dist}(\tau_t(\mathcal{C}),\mathcal{B})=0$. The flow τ is *bounded dissipative* if there exists a bounded set $\mathcal{B}$ attracting all the bounded subsets of $\Omega\times\mathbb{R}$ under τ. And a set $\mathcal{A}\subset\Omega\times\mathbb{R}$ is a *global attractor* for τ if it is compact, τ-invariant, and it attracts every bounded subset of $\Omega\times\mathbb{R}$ under τ.

Finally, given a Borel measure ν on $\Omega\times\mathbb{R}$, the expression $m(\mathcal{B}):=\nu(\mathcal{B}\times\mathbb{R})$ for any Borel subset $\mathcal{B}\subseteq\Omega$ defines a measure m on Ω. We say that ν *projects on* m. It is easy to check that m is σ-invariant if ν is τ-invariant.

2.3. Sacker and Sell spectrum of a family of linear scalar equations

Let (Ω,σ) be a minimal flow on a compact metric space, and let us consider the family of linear differential equations

$$x'=a(\omega{\cdot}t)\,x \tag{2.3}$$

for $\omega\in\Omega$, where $a\colon\Omega\to\mathbb{R}$ is continuous.

Definition 2.2. The family (2.3) has *exponential dichotomy over* Ω if there exist $\kappa\geq 1$ and $\gamma>0$ such that either

$$\exp\int_0^t a(\omega{\cdot}l)\,dl\leq\kappa\,e^{-\gamma t}\quad\text{whenever }\omega\in\Omega\text{ and }t\geq 0 \tag{2.4}$$

or

$$\exp\int_0^t a(\omega{\cdot}l)\,dl\leq\kappa\,e^{\gamma t}\quad\text{whenever }\omega\in\Omega\text{ and }t\leq 0\,. \tag{2.5}$$

Remarks 2.3. 1. Since the base flow (Ω,σ) is minimal, the exponential dichotomy of the family (2.3) over Ω is equivalent to the exponential dichotomy of any of its equations over $\mathbb{R}$: see e.g. Theorem 2 and Section 3 of [34].

2. The family (2.3) has exponential dichotomy over Ω if and only if no one of its equations has a nontrivial bounded solution: see e.g. Theorem 1.61 of [20]. In other words, the property fails if and only if there exists $\widetilde{\omega}\in\Omega$ such that $\sup_{t\in\mathbb{R}}\exp\big(\int_0^t a(\widetilde{\omega}{\cdot}s)\,ds\big)<\infty$.

Definition 2.4. The *Sacker and Sell spectrum* or *dynamical spectrum* of the linear family (2.3) is the set Σ_a of $\lambda \in \mathbb{R}$ such that the family $x' = (a(\omega{\cdot}t) - \lambda)\,x$ does not have exponential dichotomy over Ω.

Note that, in the autonomous case $a(\omega) \equiv a \in \mathbb{R}$, the set Σ_a is given by $\{a\}$.

Definition 2.5. The *lower Lyapunov exponent of the family* (2.3) *for* (Ω, σ) is

$$\gamma^i_\Omega := \inf\left\{ \int_\Omega a(\omega)\,dm \mid m \in \mathfrak{M}_{\mathrm{inv}}(\Omega, \sigma) \right\},$$

and the *upper Lyapunov exponent of the family* (2.3) *for* (Ω, σ) is

$$\gamma^s_\Omega := \sup\left\{ \int_\Omega a(\omega)\,dm \mid m \in \mathfrak{M}_{\mathrm{inv}}(\Omega, \sigma) \right\}.$$

For the reader's convenience, we include a proof of the next well known result.

Theorem 2.6.

(i) *There exist* $m^i, m^s \in \mathfrak{M}_{\mathrm{erg}}(\Omega, \sigma)$ *such that*

$$\gamma^i_\Omega := \int_\Omega a(\omega)\,dm^i \quad \textit{and} \quad \gamma^s_\Omega := \int_\Omega a(\omega)\,dm^s .$$

(ii) *The Sacker and Sell spectrum of the linear family* (2.3) *is* $\Sigma_a = [\,\gamma^i_\Omega, \gamma^s_\Omega\,]$*, and it may be a singleton.*

Proof. The Sacker and Sell spectral theorem [36, Theorem 2] states that, in this scalar case, Σ_a is given by a closed (perhaps degenerate) interval, say $[\lambda_i, \lambda_s]$. Theorem 2.3 of [21] shows that this interval contains $\int_\Omega a(\omega)\,dm$ for all $m \in \mathfrak{M}_{\mathrm{inv}}(\Omega, \sigma)$, as well as the existence of $m^i, m^s \in \mathfrak{M}_{\mathrm{erg}}(\Omega, \sigma)$ such that $\lambda_i := \int_\Omega a(\omega)\,dm^i$ and $\lambda_s = \int_\Omega a(\omega)\,dm^s$. These properties show the assertions. □

Remark 2.7. It is clear $0 \in \Sigma_a$ if and only if the family (2.3) does not have exponential dichotomy over Ω. In addition, Theorem 2.6 ensures that:

- $\Sigma_a \subset (-\infty, 0)$ if and only if the upper Lyapunov exponent of the family (2.3) is negative; or, equivalently, if and only if $\int_\Omega a(\omega)\,dm < 0$ for any $m \in \mathfrak{M}_{\mathrm{inv}}(\Omega, \sigma)$.
- $\Sigma_a \subset (0, \infty)$ if and only if the lower Lyapunov exponent of the family (2.3) is positive; or, equivalently, if and only if $\int_\Omega a(\omega)\,dm > 0$ for any $m \in \mathfrak{M}_{\mathrm{inv}}(\Omega, \sigma)$.

2.4. The minimal subsets of a scalar skewproduct flow induced by a family of scalar ODEs over a minimal base

As in the previous section, (Ω, σ) is a minimal continuous flow on a compact metric space, and this assumption on minimality is fundamental. We will recall in this subsection some properties of the minimal sets for the scalar skewproduct flow $(\Omega \times \mathbb{R}, \tau)$ given by the expression (2.2); that is, given by the solutions of the family (2.1) over Ω. We will also define some types of sets which will be fundamental in the dynamical description of Section 3.

It is very easy to deduce from the minimality of the base flow that any copy of the base is τ-minimal, and that any compact τ-invariant set $\mathcal{K} \subset \Omega \times \mathbb{R}$ projects over the whole base Ω. If, for such a set $\mathcal{K}$, there exists a point $\omega \in \Omega$ such that $\mathcal{K}_\omega$ is a singleton, then $\mathcal{K}$ is a *pinched set*. A minimal pinched set is an *almost automorphic extension of the base*. It turns out that, for our scalar skewproduct flow, any minimal set $\mathcal{M}$ is an almost automorphic extension of the base. To briefly explain this fact, which is proved in Theorem 3.5 of [22], we observe that

$$\mathcal{M} \subseteq \bigcup_{\omega\in\Omega} \big(\{\omega\} \times [\alpha_{\mathcal{M}}(\omega), \beta_{\mathcal{M}}(\omega)]\big) \tag{2.6}$$

where $\alpha_{\mathcal{M}}(\omega) := \inf\{x \in \mathbb{R} \mid (\omega, x) \in \mathcal{M}\}$ and $\beta_{\mathcal{M}}(\omega) := \sup\{x \in \mathbb{R} \mid (\omega, x) \in \mathcal{M}\}$. It is not hard to deduce from the compactness of $\mathcal{M}$ that $\alpha_{\mathcal{M}}$ and $\beta_{\mathcal{M}}$ are lower and upper semicontinuous; from its τ-invariance that they are τ-equilibria; and from its minimality that $\mathcal{M} = \mathrm{closure}_{\Omega\times\mathbb{R}}\{(\omega{\cdot}t, \alpha_{\mathcal{M}}(\omega{\cdot}t)) \mid t \in \mathbb{R}\}$ (resp. $\mathcal{M} = \mathrm{closure}_{\Omega\times\mathbb{R}}\{(\omega{\cdot}t, \beta_{\mathcal{M}}(\omega{\cdot}t)) \mid t \in \mathbb{R}\}$) for any $\omega \in \Omega$, and hence that $\mathcal{M}_\omega = \{\alpha_{\mathcal{M}}(\omega)\}$ (resp. $\mathcal{M}_\omega = \{\beta_{\mathcal{M}}(\omega)\}$) at any point ω at which $\alpha_{\mathcal{M}}$ (resp. $\beta_{\mathcal{M}}$) is continuous. Therefore, $\alpha_{\mathcal{M}}$ and $\beta_{\mathcal{M}}$ have the same (σ-invariant and residual) set $\Omega_{\mathcal{M}} \subseteq \Omega$ of continuity points, which are exactly the points at which both maps coincide; and $\mathcal{M}_\omega$ reduces to a singleton if and only if $\omega \in \Omega_{\mathcal{M}}$. The functions $\alpha_{\mathcal{M}}$ and $\beta_{\mathcal{M}}$ are hence continuous if and only if $\alpha_{\mathcal{M}}(\omega) = \beta_{\mathcal{M}}(\omega)$ for all $\omega \in \Omega$. In other words, $\mathcal{M}_\omega$ reduces to a point for all $\omega \in \Omega$ if and only if $\mathcal{M}$ is a copy of the base: $\mathcal{M} = \{\eta\}$ for $\eta = \alpha_{\mathcal{M}} = \beta_{\mathcal{M}}$, continuous.

Two different τ-minimal sets $\mathcal{M}$ and $\mathcal{N}$ are *fiber-ordered*, in the following sense: if there exist $(\omega_0, x_0) \in \mathcal{M}$ and $(\omega_0, y_0) \in \mathcal{N}$ such that $x_0 < y_0$, then $x < y$ whenever $(\omega, x) \in \mathcal{M}$ and $(\omega, y) \in \mathcal{N}$. To prove this fact, we take a common element $\bar\omega \in \Omega$ such that $\mathcal{M}_{\bar\omega} = \{\bar x\}$ and $\mathcal{N}_{\bar\omega} = \{\bar y\}$ and assume without restriction that $\bar x < \bar y$. Let us reason by contradiction assuming the existence of $(\omega, x) \in \mathcal{M}$ and $(\omega, y) \in \mathcal{N}$ with $x > y$. We look for (t_n) such that $(\bar\omega, \bar x) = \lim_{n\to\infty} \tau(t_n, \omega, x)$ and a suitable subsequence (t_k) such that there exists $\lim_{k\to\infty} \tau(t_k, \omega, y)$. Then, this limit is necessarily $(\bar\omega, \bar y)$, and the fiber-monotonicity of τ ensures that $\bar x \geq \bar y$, which is the sought-for contradiction.

Assume now that f is C^1 with respect to its second argument. Given a τ-minimal set $\mathcal{M}$, we can consider the *linearized flow* on $\mathcal{M} \times \mathbb{R}$, given by the solutions of the family of variational equations

$$z' = f_x(\tau(t, \omega, x_0))\, z \tag{2.7}$$

for $(\omega, x_0) \in \mathcal{M}$, where $f_x := \partial f/\partial x$. A τ-minimal set $\mathcal{M} \subset \Omega \times \mathbb{R}$ is *hyperbolic* if the family (2.7) has exponential dichotomy over $\mathcal{M}$. This last definition is justified by the next result. For the reader's convenience, we give a proof of this well-known fact, concerning hyperbolic minimal sets, which will be crucial in Section 3. The functions $\alpha_{\mathcal{M}}$ and $\beta_{\mathcal{M}}$ are those associated to $\mathcal{M}$ by (2.6). The uniform exponential stability properties are defined in Subsection 2.2.

Proposition 2.8. *Assume that the functions $f, f_x\colon \Omega\times\mathbb{R}\to\mathbb{R}$ are jointly continuous, let $(\Omega\times\mathbb{R},\tau)$ be the flow induced by the family (2.1), and let $\mathcal{M}\subset\Omega\times\mathbb{R}$ be a τ-minimal set. Then,*

(i) *the family (2.7) has exponential dichotomy over $\mathcal{M}$ given by condition (2.4) if and only if $\mathcal{M}$ is a uniformly exponentially stable at $+\infty$ copy of the base: $\mathcal{M}=\{\alpha_{\mathcal{M}}\}=\{\beta_{\mathcal{M}}\}$. In addition, in this case, given $(\omega,x_0)\notin\mathcal{M}$, there exists $\rho>0$ and $t_-<0$ such that $|x(t,\omega,x_0)-\alpha_{\mathcal{M}}(\omega{\cdot}t)|>\rho$ for $t\le t_-$ in the maximal interval of definition of $x(t,\omega,x_0)$.*

(ii) *The family (2.7) has exponential dichotomy over $\mathcal{M}$ given by condition (2.5) if and only if $\mathcal{M}$ is a uniformly exponentially stable at $-\infty$ copy of the base: $\mathcal{M}=\{\alpha_{\mathcal{M}}\}=\{\beta_{\mathcal{M}}\}$. In addition, in this case, given $(\omega,x_0)\notin\mathcal{M}$, there exists $\rho>0$ and $t_+>0$ such that $|x(t,\omega,x_0)-\alpha_{\mathcal{M}}(\omega{\cdot}t)|>\rho$ for $t\ge t_+$ in the maximal interval of definition of $x(t,\omega,x_0)$.*

Proof. (i) Let us fix $(\omega_1,x_1)\in\mathcal{M}$, and assume that the family (2.7) satisfies the corresponding condition (2.4). The hypotheses on f ensure that $f(\omega_1,x)-f(\omega_1,x_1)=f_x(\omega_1,x_1){\cdot}(x-x_1)+r(\omega,x)$, with $\lim_{x\to x_1}|r(\omega,x)|/|x-x_1|=0$. Therefore, the change of variables $y=x-x(t,\omega_1,x_1)$ takes the equation $(2.1)_{\omega_1}$ to

$$y'=f_x(\tau(t,\omega_1,x_1))\,y+\widetilde{r}(\omega_1{\cdot}t,y)\,, \tag{2.8}$$

with $\lim_{x\to 0}|\widetilde{r}(\omega_1,y)|/|y|=0$. Let $y(t,\omega_1,y_0)$ represent the solution of (2.8) with $y(0,\omega_1,y_0)=y_0$, so that $y(t,\omega_1,y_0)=x(t,\omega_1,y_0+x_1)-x(t,\omega_1,x_1)$. Then, condition (2.4) and the First Approximation Theorem (see [16, Theorem III.2.4] and its proof) provide $\delta>0$ such that

$$|y(t,\omega_1,y_0)|\le\kappa\,e^{(-\gamma/2)\,t}|y_0|\qquad\text{for any }\ t\ge 0\ \text{ if }\ |y_0|\le\delta\,. \tag{2.9}$$

In addition, the constant δ can be chosen to satisfy (2.9) for any $\omega_1\in\Omega$.

We take any point $(\omega_1,x_2)\in\mathcal{M}$, and will check that $x_2=x_1$. Recall that any minimal set is pinched. Therefore, we can choose $\widetilde{\omega}$ with $\mathcal{M}_{\widetilde{\omega}}=\{\widetilde{x}\}$. Then, $\lim_{n\to\infty}(\omega_1{\cdot}(-t_n),u(-t_n,\omega_1,x_1))=(\widetilde{\omega},\widetilde{x})$ for a sequence $(t_n)\uparrow\infty$. We take a subsequence (t_k) such that $\lim_{k\to\infty}(\omega_1{\cdot}(-t_k),x(-t_k,\omega_1,x_2))$ exists, and observe that this limit must be $(\widetilde{\omega},\widetilde{x})$, since it belongs to $\mathcal{M}$. Hence, $\lim_{k\to\infty}y(-t_k,\omega_1,x_2-x_1)=\lim_{k\to\infty}(x(-t_k,\omega_1,x_2)-x(-t_k,\omega_1,x_1))=\widetilde{x}-\widetilde{x}=0$. For k large enough to ensure that $|y(-t_k,\omega_1,x_2-x_1)|\le\delta$, (2.9) yields

$$|x_2-x_1|=|y(t_k,\omega_1{\cdot}(-t_k),y(-t_k,\omega_1,x_2-x_1))|\le\kappa\,e^{(-\gamma/2)\,t_k}\,\delta\,.$$

Taking limit as $k\to\infty$ allows us to ensure that $x_2=x_1$, as asserted.

Hence, as explained at the beginning of this Subsection, we can write $\mathcal{M}=\{\eta\}$ for a continuous function $\eta\colon\Omega\to\mathbb{R}$. The continuous flow transformation $(\omega,x)\mapsto(\omega,x-\eta(\omega))$ takes $\mathcal{M}$ to the set $\Omega\times\{0\}$, which is a copy of the base for the flow induced by the family of equations $y'=f_x(\omega{\cdot}t,\eta(\omega{\cdot}t))\,y+\widetilde{r}(\omega{\cdot}t,y)$ for $\omega\in\Omega$. It follows from (2.9) that $\Omega\times\{0\}$ is uniformly exponentially stable, which ensures the analogous property for $\mathcal{M}$ and the initial flow τ. The "only if" part of the first assertion of (i) is proved.

Conversely, let us assume that $\mathcal{M}$ is an exponentially stable copy at $+\infty$ copy of the base. Then, for all $(\omega,x)\in\mathcal{M}$, $|(\partial x/\partial x_0)(t,\omega,x_0)|=\lim_{h\to 0}|x(t,\omega,x_0+h)-x(t,\omega,x_0)|/|h|\le\kappa\,e^{-\gamma t}$ for certain constants $\kappa\ge 1$ and $\gamma>0$, and for all $t\ge 0$. This implies that the family of equations (2.7), defined for $(\omega,x_0)\in\mathcal{M}$, satisfies condition (2.4), and completes the proof of the equivalence stated in (i).

Assume now that we are in the described situation, and let δ, κ and γ be the constants associated to the uniformly exponentially character at $+\infty$ of $\mathcal{M}$. To prove the last assertion in (i) we take $x_0 \neq \alpha_{\mathcal{M}}(\omega)$ and assume for contradiction the existence of $(t_n) \downarrow -\infty$ such that $\lim_{n\to\infty} |x(t_n,\omega,x_0)-\alpha_{\mathcal{M}}(\omega{\cdot}t_n)| = 0$. Thus, for large enough n, $|x(t_n,\omega,x_0)-\alpha_{\mathcal{M}}(\omega{\cdot}t_n)| \le \delta$. But then $|x_0-\alpha_{\mathcal{M}}(\omega)| = |x(-t_n,\omega{\cdot}t_n,x(t_n,\omega,x_0)) - \alpha_{\mathcal{M}}((\omega{\cdot}t_n){\cdot}(-t_n))| \le \kappa\, e^{\gamma\, t_n}\delta$. The contradiction comes from the convergence to 0 of the right-hand term. The proof of (i) is complete.

(ii) The proofs are analogous if the exponential dichotomy is given by (2.5) or the uniform exponential stability occurs at $-\infty$. □

Definition 2.9. The *upper* and *lower Lyapunov exponents* of a τ-minimal set $\mathcal{M} \subset \Omega\times\mathbb{R}$ are the upper and lower Lyapunov exponents of the family of variational equations (2.7) over $\mathcal{M}$.

As a consequence of this definition, Remark 2.3, Theorem 2.6(ii), and Proposition 2.8, we have:

Corollary 2.10. *Assume that the functions $f, f_x\colon \Omega\times\mathbb{R}\to\mathbb{R}$ are jointly continuous, and let $(\Omega\times\mathbb{R},\tau)$ be the flow induced by the family (2.1). If the upper Lyapunov exponent of the τ-minimal set $\mathcal{M}$ is negative, then $\mathcal{M}$ is an exponentially stable at $+\infty$ copy of the base. If its lower Lyapunov exponent is positive, then $\mathcal{M}$ is an exponentially stable at $-\infty$ copy of the base. And, in both cases, $\mathcal{M} = \{\alpha_{\mathcal{M}}\} = \{\beta_{\mathcal{M}}\}$.*

2.5. *The set $\mathcal{R}_m(\Omega)$*

We continue this section of preliminaries by describing a set of continuous maps $a\colon \Omega\to\mathbb{R}$ which will play a crucial role in the description of the occurrence of Li-Yorke chaos and Auslander-Yorke chaos (defined in the next subsections) in one of the dynamical situations which we will consider in Section 3. Most of these properties are (basically) already known; but, to our knowledge, Theorem 2.16 presents a new property. The assumption of minimality of (Ω,σ) is in force.

Definition 2.11. A continuous function $a\colon\Omega\to\mathbb{R}$ *admits a continuous primitive* if there exists a continuous function $h_a\colon\Omega\to\mathbb{R}$ such that $h_a(\omega{\cdot}t)-h_a(\omega) = \int_0^t a(\omega{\cdot}s)\,ds$ for all $\omega\in\Omega$ and $t\in\mathbb{R}$.

Remark 2.12. Note that $\sup_{(t,\omega)\in\mathbb{R}\times\Omega}\left|\int_0^t a(\omega{\cdot}s)\,ds\right| < \infty$ if a admits a continuous primitive, and that Birkhoff's ergodic theorem ensures that $\int_\Omega a(\omega)\,dm = 0$ for any $m\in\mathfrak{M}_{\text{inv}}(\Omega,\sigma)$. It is well-known that if (Ω,σ) is minimal (as in our case) then a admits a continuous primitive if and only if there exists $\widetilde{\omega}\in\Omega$ with $\sup_{t\ge 0}\left|\int_0^t a(\widetilde{\omega}{\cdot}s)\,ds\right| < \infty$ or with $\sup_{t\le 0}\left|\int_0^t a(\widetilde{\omega}{\cdot}s)\,ds\right| < \infty$: a proof is given in [22, Proposition A.1].

Definition 2.13. Given $m\in\mathfrak{M}_{\text{inv}}(\Omega,\sigma)$, $\mathcal{R}_m(\Omega)$ is the set of continuous functions $a\colon\Omega\to\mathbb{R}$ satisfying $\int_\Omega a(\omega)\,dm = 0$ which do not admit a continuous primitive and such that $\sup_{t\le 0}\int_0^t a(\omega{\cdot}s)\,ds < \infty$ for m-a.e. $\omega\in\Omega$.

There are well known examples of quasi-periodic functions $a_0\colon \mathbb{R} \to \mathbb{R}$ giving rise to a hull Ω and a map a in the set $\mathcal{R}_m(\Omega)$ corresponding to the unique ergodic measure on Ω. For example, those described in [19] and in [31]. Our next result shows that it is nonempty whenever the flow is minimal and non periodic. The σ-ergodic measure m_{ω_0} associated to every σ-generic point in the set $\widetilde{\Omega}_e \subseteq \Omega$ (of complete measure) is defined in Subsection 2.1.

Theorem 2.14. *Assume that the flow (Ω, σ) is minimal and non periodic. Then,*

(i) *$\mathcal{R}_m(\Omega)$ is nonempty for any $m \in \mathfrak{M}_{\rm inv}(\Omega, \sigma)$, and it contains functions a with $\Sigma_a = \{0\}$.*
(ii) *In fact, there exist functions a which belong to $\bigcap_{m\in\mathfrak{M}_{\rm inv}(\Omega,\sigma)} \mathcal{R}_m(\Omega)$, with $\Sigma_a = \{0\}$.*
(iii) *If $a \in \mathcal{R}_m(\Omega)$ for a measure $m \in \mathfrak{M}_{\rm inv}(\Omega, \sigma)$, then there exists at least a measure $\widetilde{m} \in \mathfrak{M}_{\rm erg}(\Omega, \sigma)$ such that $a \in \mathcal{R}_{\widetilde{m}}(\Omega)$. More precisely, $a \in \mathcal{R}_{m_{\omega_0}}(\Omega)$ for m-almost all the measures $m_{\omega_0} \in \mathfrak{M}_{\rm erg}(\Omega, \sigma)$ with $\omega_0 \in \widetilde{\Omega}_e$.*

Proof. (i) Let us fix $m \in \mathfrak{M}_{\rm inv}(\Omega, \mathbb{R})$. We begin by proving an auxiliary result. Let us fix $\omega_0 \in \Omega$, and let us take $\varepsilon > 0$. Then, there exists a continuous function $b_\varepsilon\colon \Omega \to \mathbb{R}$ with $\|b_\varepsilon\|_\Omega := \sup_{\omega\in\Omega} |b_\varepsilon(\omega)| \le \varepsilon$ which admits a continuous primitive $h_{b_\varepsilon}\colon \Omega \to [0, 1]$ with $h_{b_\varepsilon}(\omega_0) = 1$ and $\int_\Omega h_{b_\varepsilon}(\omega)\, dm \le \varepsilon$. In fact, we will construct b_ε and h_{b_ε}. We take $T \ge 2/\varepsilon$, and note that $m(\{\omega_0{\cdot}t \mid t \in [0, T]\}) = 0$, since the flow is non periodic: otherwise we would obtain a σ-orbit with infinite measure, impossible. The regularity of m and Uryshon's Lemma provide a continuous function $c_\varepsilon\colon \Omega \to [0, 1]$ such that $c_\varepsilon(\omega_0{\cdot}t) = 1$ for $t \in [0, T]$ and with $\int_\Omega c_\varepsilon(\omega)\, dm \le \varepsilon$. We define $b_\varepsilon(\omega) := (c_\varepsilon(\omega{\cdot}T) - c_\varepsilon(\omega))/T$ and $h_{b_\varepsilon}(\omega) := (1/T) \int_0^T c_\varepsilon(\omega{\cdot}s)\, ds$, and check that $(h_{b_\varepsilon})'(\omega) := (d/dt)\, h_{b_\varepsilon}(\omega{\cdot}t)|_{t=0}$ coincides with $b_\varepsilon(\omega)$. Clearly, $\|b_\varepsilon\|_\Omega \le 2/T \le \varepsilon$. In addition, $h_{b_\varepsilon} \ge 0$, and $h_{b_\varepsilon}(\omega_0) = (1/T) \int_0^T c_\varepsilon(\omega_0{\cdot}s)\, ds = 1$. Finally, using the σ-invariance of m, we get

$$\int_\Omega h_{b_\varepsilon}(\omega)\, dm = \frac{1}{T}\int_\Omega \int_0^T c_\varepsilon(\omega{\cdot}s)\, ds\, dm = \frac{1}{T}\int_0^T \int_\Omega c_\varepsilon(\omega{\cdot}s)\, dm\, ds$$
$$= \frac{1}{T}\int_0^T \int_\Omega c_\varepsilon(\omega)\, dm\, ds = \int_\Omega c_\varepsilon(\omega)\, dm \le \varepsilon\,,$$

which completes the proof of our initial assertion.

This property allows us to construct a sequence (b_n) of continuous functions with continuous primitives (h_{b_n}) such that $\|b_n\|_\Omega \le 1/2^n$ (so that $\sum_{n=1}^\infty \|b_n\|_\Omega \le 1$), $h_{b_n}(\omega) \in [0, 1]$ for all $\omega \in \Omega$, $\int_\Omega h_{b_n}(\omega)\, dm \le 1/2^n$ (so that $\sum_{n=1}^\infty \int_\Omega h_{b_n}(\omega)\, dm \le 1$), and with $h_{b_n}(\omega_0) = 1$ for a previously fixed $\omega_0 \in \Omega$ and all $n \in \mathbb{N}$. Let us call $\widetilde{h}(\omega) := \sum_{n=1}^\infty h_{b_n}(\omega) \in [0, \infty]$. Lebesgue's monotone convergence theorem shows that $\int_\Omega \widetilde{h}(\omega)\, dm = \sum_{n=1}^\infty \int_\Omega h_{b_n}(\omega)\, dm \le 1$, and hence

$$\widetilde{\Omega} := \{\omega \in \Omega \mid \widetilde{h}(\omega) < \infty\}$$

satisfies $m(\widetilde{\Omega}) = 1$. Note also that $\omega_0 \notin \widetilde{\Omega}$. In addition, $\widetilde{\Omega}$ is σ-invariant: for every $\omega \in \Omega$, $j \in \mathbb{N}$ and $t \in \mathbb{R}$,

$$\sum_{n=1}^{j} h_{b_n}(\omega{\cdot}t) = \sum_{n=1}^{j} h_{b_n}(\omega) + \sum_{n=1}^{j}\int_0^t b_n(\omega{\cdot}s)\,ds \le \sum_{n=1}^{\infty} h_{b_n}(\omega) + |t|\sum_{n=1}^{\infty}\|b_n\|_\Omega\,.$$

Let us define $a := -\sum_{n=1}^{\infty} b_n$, which is a continuous function on Ω. We will check that $a \in \mathcal{R}_m(\Omega)$ and that $\Sigma_a = \{0\}$.

Note that the function h_a defined by $h_a(\omega) := -\sum_{n=1}^{\infty} h_{b_n}(\omega) = -\widetilde{h}(\omega)$ for $\omega \in \widetilde{\Omega}$ and $h_a(\omega) := 0$ for $\omega \notin \widetilde{\Omega}$ satisfies $h_a(\widetilde{\omega}{\cdot}t) - h_a(\widetilde{\omega}) = \int_0^t a(\widetilde{\omega}{\cdot}s)\,ds$ for all $\widetilde{\omega} \in \widetilde{\Omega}$, since $h_{b_n}(\widetilde{\omega}{\cdot}t) - h_{b_n}(\widetilde{\omega}) = \int_0^t b_n(\widetilde{\omega}{\cdot}s)\,ds$. Observe that $\sup_{t\le 0}\int_0^t a(\widetilde{\omega}{\cdot}s)\,ds = \sup_{t\le 0}(h_a(\widetilde{\omega}{\cdot}t) - h_a(\widetilde{\omega})) = \sup_{t\le 0}(\widetilde{h}(\widetilde{\omega}) - \widetilde{h}_a(\widetilde{\omega}{\cdot}t)) \le \widetilde{h}(\widetilde{\omega}) < \infty$ for all $\widetilde{\omega} \in \widetilde{\Omega}$. Let us check that $\inf_{t\le 0}\int_0^t a(\widetilde{\omega}{\cdot}s)\,ds = -\infty$ for all $\widetilde{\omega} \in \widetilde{\Omega}$. We fix $\widetilde{\omega} \in \widetilde{\Omega}$ and $\omega_0 \notin \widetilde{\Omega}$ and look for $(t_k) \downarrow -\infty$ such that $\omega_0 = \lim_{k\to\infty}\widetilde{\omega}{\cdot}(t_k)$. For any $j \in \mathbb{N}$,

$$\int_0^{t_k} a(\widetilde{\omega}{\cdot}s)\,ds = \sum_{n=1}^{\infty}(h_{b_n}(\widetilde{\omega}) - h_{b_n}(\widetilde{\omega}{\cdot}t_k)) \le \sum_{n=1}^{\infty} h_{b_n}(\widetilde{\omega}) - \sum_{n=1}^{j} h_{b_n}(\widetilde{\omega}{\cdot}t_k)\,,$$

and hence $\liminf_{k\to\infty}\int_0^{t_k} a(\widetilde{\omega}{\cdot}s)\,ds \le \sum_{n=1}^{\infty} h_{b_n}(\widetilde{\omega}) - \sum_{n=1}^{j} h_{b_n}(\omega_0)$ for all $j \in \mathbb{N}$. By letting j increase, we check the assertion, which in turn precludes the existence of a continuous primitive for a. Altogether, a satisfies all the conditions of Definition 2.13, and hence $a \in \mathcal{R}_m(\Omega)$.

Finally, note that the map a is the uniform limit of the sequence (s_j), with $s_j := -\sum_{n=1}^{j} b_n$. Each one of the functions s_j has a continuous primitive, and hence $\int_\Omega s_j(\omega)\,d\widetilde{m} = 0$ for every $j \in \mathbb{N}$ and $\widetilde{m} \in \mathfrak{M}_{\rm inv}(\Omega,\sigma)$ (see Remark 2.12). Therefore, $\int_\Omega a(\omega)\,d\widetilde{m} = 0$ for every $\widetilde{m} \in \mathfrak{M}_{\rm inv}(\Omega,\sigma)$, and hence Theorem 2.6 shows that $\Sigma_a = \{0\}$.

(ii) The idea is to repeat the process of (i), but taking functions (b_n) such that $\int_\Omega h_{b_n}(\omega)\,dm \le 1/(2^n)$ for all $m \in \mathfrak{M}_{\rm inv}(\Omega,\mathbb{R})$. Therefore, we must change the initial step in the proof of (i) to show that, given $\omega_0 \in \Omega$ and $\varepsilon > 0$, there exists a continuous function $b_\varepsilon\colon \Omega \to \mathbb{R}$ with $\|b_\varepsilon\|_\Omega \le \varepsilon$ which admits a continuous primitive $h_{b_\varepsilon}\colon \Omega \to [0,1]$ with $h_{b_\varepsilon}(\omega_0) = 1$ and $\int_\Omega h_{b_\varepsilon}(\omega)\,dm \le \varepsilon$ for any $m \in \mathfrak{M}_{\rm inv}(\Omega,\mathbb{R})$. Let us call $\mathcal{J} := \{\omega_0{\cdot}t \mid t \in [0,T]\}$, and look for a continuous function $c\colon \Omega \to [0,1]$ such that $c(\omega) = 1$ for $\omega \in \mathcal{J}$ and $c(\omega) < 1$ for $\omega \notin \mathcal{J}$. Then, the sequence (c^n) decreases pointwisely to the characteristic function of $\mathcal{J}$, and hence Lebesgue's monotone convergence theorem ensures that $\lim_{n\to\infty}\int_\Omega c^n(\omega)\,dm = 0$ for all $m \in \mathfrak{M}_{\rm inv}(\Omega,\sigma)$. We consider the maps $i_n\colon \mathfrak{M}_{\rm inv}(\Omega,\sigma) \to \mathbb{R}\,, m \mapsto \int_\Omega c^n(\omega)\,dm$, which are continuous for the weak* topology of $\mathfrak{M}_{\rm inv}(\Omega,\mathbb{R})$. The space $\mathfrak{M}_{\rm inv}(\Omega,\mathbb{R})$ is compact and metrizable for this topology (see e.g. [39, Theorems 6.4 and 6.5]). The sequence (i_n) decreases to the function 0, and hence Dini's theorem ensures that $0 = \lim_{n\to\infty} i_n$ uniformly on $\mathfrak{M}_{\rm inv}(\Omega,\mathbb{R})$. Therefore, given $\varepsilon > 0$, there exists $n_\varepsilon \in \mathbb{N}$ such that $\int_\Omega c^{n_\varepsilon}(\omega)\,dm \le \varepsilon$ for all $m \in \mathfrak{M}_{\rm inv}(\Omega,\sigma)$. We use c^{n_ε} to construct b_ε and h_{b_ε}, as at the beginning of the proof of (i), and repeat the rest of it to check (ii).

(iii) Let us call $\Omega_-^a := \{\omega \in \Omega \mid \sup_{t\le 0}\int_0^t a(\omega{\cdot}s)\,ds < \infty\}$, with $m(\Omega_-^a) = 1$. Let f be the characteristic function of Ω_-^a. As recalled at the end of Subsection 2.1, there exists a set $\Omega_f \subseteq \widetilde{\Omega}_e$ with $m(\Omega_f) = 1$ such that $m(\Omega_-^a) = \int_\Omega f(\omega)\,dm = \int_{\Omega_f}\big(\int_\Omega f(\omega)dm_{\omega_0}\big)\,dm = \int_{\Omega_f} m_{\omega_0}(\Omega_-^a)\,dm$. This ensures that $m_{\omega_0}(\Omega_-^a) = 1$ for m-almost every $\omega_0 \in \Omega_f$. Let us take one of these points ω_0. Then $\int_\Omega a(\omega)\,dm_{\omega_0} \ge 0$: if, on the contrary, $\widetilde{a} := \int_\Omega a(\omega)\,dm_{\omega_0} < 0$, then Birkhoff's ergodic theorem ensures that $\widetilde{a} = \lim_{t\to-\infty}(1/t)\int_0^t a(\omega{\cdot}s)\,ds$ for m_{ω_0}-almost every $\omega \in \Omega$, which in turn implies $m_{\omega_0}(\Omega_-^a) = 0$, impossible. Now we look for a subset

$\Omega_a \subseteq \widetilde{\Omega}_e$ with $m(\Omega_a)=1$ such that $0=\int_\Omega a(\omega)\,dm=\int_{\Omega_a}\big(\int_\Omega a(\omega)dm_{\omega_0}\big)\,dm$, and conclude that $\int_\Omega a(\omega)\,dm_{\omega_0}$ for m-almost every point. Therefore, a satisfies the conditions of Definition 2.13 for m_{ω_0} for m-almost every $\omega_0\in\widetilde{\Omega}_e$, as asserted. □

From now on, $m\in\mathfrak{M}_{\text{erg}}(\Omega,\sigma)$ is fixed.

The next result summarizes part of the dynamical consequences on the solutions of the family of linear scalar equation $x'=a(\omega{\cdot}t)\,x$, which are $x(t,\omega,x_0)=x_0\exp\big(\int_0^t a(\omega{\cdot}s)\,ds\big)$.

Proposition 2.15. *Let $a\colon\Omega\to\mathbb{R}$ be a continuous function with $\int_\Omega a(\omega)\,dm=0$. The following assertions are equivalent:*

(1) $a\in\mathcal{R}_m(\Omega)$.
(2) *The subset $\Omega^a\subseteq\Omega$ of those points ω such that $\sup_{t\in\mathbb{R}}\int_0^t a(\omega{\cdot}s)\,ds<\infty$, $\inf_{t\le 0}\int_0^t a(\omega{\cdot}s)\,ds=-\infty$, and $\inf_{t\ge 0}\int_0^t a(\omega{\cdot}s)\,ds=-\infty$, is σ-invariant and satisfies $m(\Omega^a)=1$.*
(3) *There exist an upper-semicontinuous function $H_a\colon\Omega\to[0,1]$ and a σ-invariant set $\Omega^a\subseteq\Omega$ with $m(\Omega^a)=1$ such that: $\omega\in\Omega^a$ if and only if $H_a(\omega)>0$; and, for all $\omega\in\Omega$, $H_a(\omega{\cdot}t)=H_a(\omega)\exp\big(\int_0^t a(\omega{\cdot}s)\,ds\big)$ for all $t\in\mathbb{R}$, $\inf_{t\le 0}H_a(\omega{\cdot}t)=0$, and $\inf_{t\ge 0}H_a(\omega{\cdot}t)=0$.*

In addition, the function H_a of point (3) *vanishes at its continuity points.*

Proof. The proof of the equivalences repeats that of [29, Proposition 6.4]: the map

$$H_a(\omega)=\inf_{t\in\mathbb{R}}\frac{1}{\exp\big(\int_0^t a(\omega{\cdot}s)\,ds\big)}$$

satisfies all the assertions of point (3).

Assume now that the semicontinuous function H_a satisfies the properties of (3) and, by contradiction, that $H_a(\omega_0)=\rho>0$ at a continuity point ω_0. Then there is a nonempty open ball $\mathcal{B}:=\mathcal{B}_\Omega(\omega_0,\delta)$ such that $H_a(\omega)>\rho/2$ for any $\omega\in\mathcal{B}$. The minimality of the flow provides values of time $t_1<\ldots<t_p$ such that $\Omega=\sigma_{t_1}(\mathcal{B})\cup\ldots\cup\sigma_{t_p}(\mathcal{B})$, from where it follows easily that H_a is always positive and bounded from below. But this contradicts the last properties mentioned in (3). □

Observe that the previous result shows that the condition $\sup_{t\le 0}\int_0^t a(\omega{\cdot}s)\,ds<\infty$ for m-a.e. $\omega\in\Omega$ in Definition 2.13 can be replaced by $\sup_{t\in\mathbb{R}}\int_0^t a(\omega{\cdot}s)\,ds<\infty$ for m-a.e. $\omega\in\Omega$.

Note also that, if $a\in\mathcal{R}_m(\Omega)$ and H_a and Ω^a are the function and set of (2.15), then the difference between two solutions of the equation of $x'=a(\omega{\cdot}t)\,x$ for $\omega\in\Omega^a$ is

$$x(t,\omega,x_2)-x(t,\omega,x_1)=(x_2-x_1)\exp\Big(\int_0^t a(\omega{\cdot}s)\,ds\Big)=(x_2-x_1)\,\frac{H_a(\omega{\cdot}t)}{H_a(\omega)}\,.$$

The next result shows that for almost every point $\omega\in\Omega^a$, the set of positive values of time at which the forward semiorbits seem to coincide (or are "indistinguishable") has positive lower density; and the same property holds for the set of positive values of time at which the semiorbits are "distinguishable". These facts will be of relevance later, in the analysis of the type of Li-Yorke chaos that we will detect for certain nonhomogeneous linear dissipative scalar equations.

Given a set $\mathcal{C} \subset [0, \infty)$, we define its *lower density* as

$$d_l(\mathcal{C}) = \liminf_{t\to\infty} \frac{1}{t}\, l([0,t]\cap\mathcal{C})\,,$$

where l is the Lebesgue measure on $\mathbb{R}$. Let us take $a \in \mathcal{R}_m(\Omega)$, $\omega \in \Omega^a$, $\varepsilon \in (0,1)$, define

$$\begin{aligned}\mathcal{I}_\varepsilon(\omega) &:= \{t \ge 0 \mid H_a(\omega{\cdot}t)/H_a(\omega) \le \varepsilon\}\,,\\ \mathcal{D}_\varepsilon(\omega) &:= \{t \ge 0 \mid H_a(\omega{\cdot}t)/H_a(\omega) \ge 1-\varepsilon\}\,,\end{aligned} \tag{2.10}$$

and observe that these ones are the sets of values of time we referred to before.

Theorem 2.16. *Assume that $a \in \mathcal{R}_m(\Omega)$, and let Ω^a be the set provided by Proposition 2.15. Then, for every $\varepsilon \in (0,1)$ there exists a subset $\Omega_\varepsilon \subseteq \Omega^a$ with $m(\Omega_\varepsilon) = 1$ such that, for any $\omega \in \Omega_\varepsilon$,*

(i) *the set $\mathcal{I}_\varepsilon(\omega)$ has positive lower density and is relatively dense in $\mathbb{R}^+$.*
(ii) *The set $\mathcal{D}_\varepsilon(\omega)$ has positive lower density.*

Proof. (i) Let us define $\mathcal{C}_n := \{\omega \in \Omega^a \mid H_a(\omega) \ge 1/n\}$. Since $\mathcal{C}_n \subseteq \mathcal{C}_{n+1}$ and $\Omega^a = \bigcup_{n\in\mathbb{N}} \mathcal{C}_n$ (see point (3) of Proposition 2.15), we have $\lim_{n\to\infty} m(\mathcal{C}_n) = 1$, and hence $m(\mathcal{C}_n) > 0$ for $n \ge n_0$. We will work with a fixed $n \ge n_0$. We also fix $\varepsilon \in (0,1)$. Let us take a continuity point ω_0 of H_a, so that $H_a(\omega_0) = 0$ (see again Proposition 2.15), and look for a nonempty open ball $\mathcal{B}_\varepsilon := \mathcal{B}_\Omega(\omega_0, \delta_\varepsilon)$ such that $H_a(\omega) \le \varepsilon/n$ if $\omega \in \mathcal{B}_\varepsilon$. Note that if $\omega \in \mathcal{C}_n$ and $\omega{\cdot}s \in \mathcal{B}_\varepsilon$ then $H_a(\omega{\cdot}s)/H_a(\omega) \le \varepsilon$; that is, $s \in \mathcal{I}_\varepsilon(\omega)$. Since $\mathcal{B}_\varepsilon$ is open and (Ω, σ) is minimal, $m(\mathcal{B}_\varepsilon) > 0$. Birkhoff's ergodic theorem ensures that, for m-almost every $\omega \in \mathcal{C}_n$,

$$\begin{aligned}0 < m(\mathcal{B}_\varepsilon) &= \lim_{t\to\infty} \frac{1}{t}\int_0^t \chi_{\mathcal{B}_\varepsilon}(\omega{\cdot}s)\,ds = \lim_{t\to\infty}\frac{1}{t}\, l(\{s\in[0,t] \mid \omega{\cdot}s \in \mathcal{B}_\varepsilon\})\\ &\le \liminf_{t\to\infty}\frac{1}{t}\, l(\{s\in[0,t] \mid s\in\mathcal{I}_\varepsilon\}) = \liminf_{t\to\infty}\frac{1}{t}\, l([0,t]\cap\mathcal{I}_\varepsilon(\omega)) = d_l(\mathcal{I}_\varepsilon)\,.\end{aligned}$$

This proves the assertion concerning the lower density for m-almost all the elements of $\mathcal{C}_n$, and hence for m-almost all the points of Ω^a.

To check that $\mathcal{I}_\varepsilon(\omega)$ is relatively dense in $\mathbb{R}$, we deduce from the minimality of the base flow and the open character of $\mathcal{B}_\varepsilon$ that there exist positive values of time $t_1 < \cdots < t_p$ such that $\Omega \subset \sigma_{-t_1}(\mathcal{B}_\varepsilon) \cup \ldots \cup \sigma_{-t_p}(\mathcal{B}_\varepsilon)$. In particular, for any $\omega \in \Omega$ there exists $t \in [0, t_p]$ such that $\omega{\cdot}t \in \mathcal{B}_\varepsilon$. We take $\omega \in \Omega^a$. Given $s \in \mathbb{R}^+$, we look for $t \in [0, t_p]$ such that $(\omega{\cdot}s){\cdot}t = \omega{\cdot}(s+t) \in \mathcal{B}_\varepsilon$, which ensures that $\widetilde{s} = s + t \in \mathcal{I}_\varepsilon(\omega)$. This ensures that $\mathcal{I}_\varepsilon(\omega)$ is relatively dense in $\mathbb{R}^+$, and completes the proof of (i).

(ii) Let us define $\eta := \inf\{k \in \mathbb{R} \mid m(\{\omega \in \Omega \mid H_a(\omega) \ge k\}) = 0\} \le 1$ and $\Omega_0 := \{\omega \in \Omega \mid H_a(\omega) \le \eta\}$, and note that $m(\Omega_0) = 1$. We fix $\varepsilon \in (0,1)$, define $\Delta_\varepsilon := \{\omega \in \Omega \mid H_a(\omega) > (1-\varepsilon)\,\eta\}$, and observe that the definition of η ensures that $m(\Delta_\varepsilon) > 0$. Now we take $\omega \in \Omega^a \cap \Omega_0$, and note that the set $\{t \ge 0 \mid \omega{\cdot}t \in \Delta_\varepsilon\}$ is contained in $\mathcal{D}_\varepsilon(\omega)$, since $H_a(\omega{\cdot}t)/H_a(\omega) > (1-\varepsilon)\,\eta/H_a(\omega) \ge (1-\varepsilon)$.

Birkhoff's ergodic theorem ensures that, for m-almost every $\omega \in \Omega^a \cap \Omega_0$ (that is, in a set $\widetilde{\Omega}_\varepsilon$ with $m(\widetilde{\Omega}_\varepsilon) = 1$),

$$0 < m(\Delta_\varepsilon) = \lim_{t\to\infty} \frac{1}{t} \int_0^t \chi_{\Delta_\varepsilon}(\omega{\cdot}s)\, ds = \lim_{t\to\infty} \frac{1}{t}\, l(\{s \in [0,t] \mid \omega{\cdot}s \in \Delta_\varepsilon\})$$

$$\le \liminf_{t\to\infty} \frac{1}{t}\, l(\{s \in [0,t] \mid s \in \mathcal{D}_\varepsilon(\omega)\}) = \liminf_{t\to\infty} \frac{1}{t}\, l([0,t] \cap \mathcal{D}_\varepsilon(\omega)) = d_l(\mathcal{D}_\varepsilon(\omega))\,.$$

This proves (ii). □

Remark 2.17. The set $\mathcal{D}_\varepsilon(\omega)$ of the previous theorem is never relatively dense. To check this, we take $\omega \in \Omega^a$ and $(t_n) \uparrow \infty$ such that $\widetilde{\omega} := \lim_{n\to\infty} \omega{\cdot}t_n = \widetilde{\omega}$ is a continuity point for the semicontinuous function H_a, so that $H_a(\widetilde{\omega}) = 0$. Then, $\lim_{n\to\infty} H_a(\omega{\cdot}(t_n + t)) = \lim_{n\to\infty} H(\omega{\cdot}t_n) \exp\big(\int_0^t a(\omega{\cdot}(t_n + s))\, ds\big) = 0$ uniformly for t in any compact interval of $\mathbb{R}$. Therefore, given $\varepsilon \in (0, 1/2)$ and $t_* > 0$, there exists n_0 such that $H(\omega{\cdot}(t_{n_0} + t)) \le \varepsilon H(\omega)$ for all $t \in [0, t_*]$, so that $[t_{n_0}, t_{n_0} + t_*] \cap \mathcal{D}_\varepsilon(\omega)$ is empty. The assertion follows from the fact that t_* is arbitrarily chosen.

2.6. *Li-Yorke chaos*

As already mentioned, in this paper we will deal with two types of chaos: Li-Yorke (now defined) and Auslander-Yorke (defined in Subsection 2.7). The minimality of the flow (Ω, σ) is not assumed in what follows.

Definition 2.18. Let (Ω, σ) be a continuous flow on a compact metric space. Let ω_1, ω_2 be two points of Ω whose forward σ-semiorbits are globally defined. The points ω_1, ω_2 form a *positively distal pair* for σ if $\liminf_{t\to\infty} \mathrm{dist}_\Omega(\sigma_t(\omega_1), \sigma_t(\omega_2)) > 0$, and a *positively asymptotic pair* if $\limsup_{t\to\infty} \mathrm{dist}_\Omega(\sigma_t(\omega_1), \sigma_t(\omega_2)) = 0$. The points ω_1, ω_2 form *Li-Yorke pair* for the flow if the pair is neither positively distal nor positively asymptotic. A set $\mathcal{S} \subseteq \Omega$ such that every pair of different points of $\mathcal{S}$ form a Li-Yorke pair is called a *scrambled set* for the flow. The flow (Ω, σ) is *Li-Yorke chaotic* if there exists an uncountable scrambled set.

After the initial description of this type of chaos for a certain type of transformations in [25], there have appeared more exigent definitions, like that of Li-Yorke sensitivity in [2]. That is also the case of the next definition, particular for skewproduct flows.

Definition 2.19. Let $(\Omega \times \mathbb{R}, \tau)$ be a skewproduct flow over a minimal base (Ω, σ), and let $\mathcal{K} \subseteq \Omega \times \mathbb{R}$ be a τ-invariant compact set. Then the restricted flow $(\mathcal{K}, \tau)$ is *Li-Yorke fiber-chaotic in measure with respect to* $m \in \mathfrak{M}_{\mathrm{erg}}(\Omega, m)$ if there exists a set $\Omega_0 \subseteq \Omega$ with $m(\Omega_0) = 1$ such that $\mathcal{K}$ contains an uncountable scrambled set of Li-Yorke pairs with first component ω for each $\omega \in \Omega_0$.

Remark 2.20. It is clear that, in the case of skewproduct flow $(\Omega \times \mathbb{R}, \tau)$, a pair of points (ω, x_1), (ω, x_2) (with common first component) form: a positively distal pair if and only if $\liminf_{t\to\infty} |x(t, \omega, x_1) - x(t, \omega, x_2)| > 0$; a positively asymptotic pair if and only if $\limsup_{t\to\infty} |x(t, \omega, x_1) - x(t, \omega, x_2)| = 0$; and a Li-Yorke pair if these two conditions fail.

We point out again that the notion of Li-Yorke fiber-chaos in measure makes only sense in the setting of skewproduct flows. The same happens with the notion of residually Li-Yorke chaotic flow, previously analyzed in [4] and [17]. Li-Yorke chaos for nonautonomous dynamical systems is also the object of analysis in [6], [7] and [29].

2.7. Auslander-Yorke chaos

As in Subsection 2.6, the minimality of the flow (Ω, σ) is not initially required (although we will assume it later to talk about skewproduct flows).

Definition 2.21. Let (Ω, σ) be a continuous flow on a compact metric space. The flow is *topologically transitive* if for any two open subsets $\mathcal{U}$ and $\mathcal{V}$ there exists $t > 0$ such that $\sigma_t(\mathcal{U}) \cap \mathcal{V}$ is nonempty. The flow is *sensitive* or ε*-sensitive* (*with respect to initial conditions*) if there exists $\varepsilon > 0$ such that for any $\omega_1 \in \Omega$ and $\delta > 0$ there exists $\omega_2 \in \mathcal{B}_\Omega(\omega_1, \delta)$ such that $\sup_{t \geq 0} \mathrm{dist}_\Omega(\omega_1{\cdot}t, \omega_2{\cdot}t) > \varepsilon$. The flow is *Auslander-Yorke chaotic* if it is topologically transitive and sensitive.

Remarks 2.22. 1. Observe that this concept of chaos relies deeply on the set we are considering. More precisely, if the restriction of the flow to a σ-invariant compact set $\mathcal{K} \subsetneq \Omega$ is Li-Yorke chaotic, then so is the global flow. But this property is not true for Auslander-Yorke chaos, since neither the transitivity nor the sensitivity on $\mathcal{K}$ are inherited for the containing set Ω.

2. A point $\omega_1 \in \Omega$ is ε*-sensitive* for an $\varepsilon > 0$ if for any $\delta > 0$ there exists $\omega_2 \in \mathcal{B}_\Omega(\omega_1, \delta)$ such that $\sup_{t \geq 0} \mathrm{dist}_\Omega(\omega_1{\cdot}t, \omega_2{\cdot}t) > \varepsilon$. The flow (Ω, σ) is sensitive if there exists a common $\varepsilon > 0$ such every $\omega \in \Omega$ is ε-sensitive, and this definition is also valid for a flow on a complete metric space. A point is *sensitive* if it is ε-sensitive for some $\varepsilon > 0$. A non sensitive point is called *Lyapunov stable*.

3. In the case of a compact metric space Ω, topological transitivity and point transitivity are equivalent, as proved in e.g. [3, Lemma 3]. Point transitivity means the existence of a dense forward semiorbit, which in general is less restrictive. A compact set Ω with a point transitive and sensitive flow was called *chaotic* by Kaplan and Yorke [23]. An exhaustive analysis of the relation among topological transitivity, point transitivity, and many other a priori stronger conditions in more general topological spaces is done in [1, Theorem 1.4].

The next fundamental result is proved by Glasner and Weiss in [14, Theorem 1.3] for the case of a surjective continuous transformation, which provides a discrete-time semiflow; but its proof can be easily adapted to the case of a real flow (see also [15, Proposition 2.4]). Recall that, if Ω is a compact metric space, then the set $\mathfrak{M}_{\mathrm{inv}}(\Omega, \sigma)$ of σ-invariant measures is nonempty, and that topological transitivity and point transitivity are equivalent properties (see Remark 2.22.3): we will simply say *transitivity*. The definition of equicontinuous (or almost periodic) flow on a compact space is given in Subsection 2.1.

Theorem 2.23. *Let (Ω, σ) be a continuous flow on a compact metric space. Assume that the flow is transitive, and that Ω is the support of a measure $m \in \mathfrak{M}_{\mathrm{inv}}(\Omega, \sigma)$. Then,*

(1) *either the flow is minimal and equicontinuous,*
(2) *or it is Auslander-Yorke chaotic.*

That is, in the case of a transitive flow on a compact metric space, uniform stability and Auslander-Yorke chaos are indeed opposite terms (see Remark 2.1 and observe that the sensitivity of a flow precludes its stability).

Corollary 2.24. *Let (Ω, σ) be a continuous flow on a compact metric space. If it is minimal, then it is either equicontinuous or Auslander-Yorke chaotic.*

Recall that an equicontinuous minimal flow is uniquely ergodic, so that the ergodic uniqueness is also a property required to avoid the presence of Auslander-Yorke chaos. On the other hand, point transitivity is equivalent to the fact that Ω is the omega limit set of one of its points. In particular, an Auslander-Yorke chaotic flow is always chain recurrent: see Subsection 2.1 and [37, Section 8].

Let us now talk about Auslander-Yorke chaos for τ-invariant compact subsets of $\Omega \times \mathbb{R}$, where (Ω, σ) is a minimal flow on a compact metric space and $(\Omega \times \mathbb{R}, \tau)$ is the skewproduct flow projecting on (Ω, σ) induced by a family of the type (2.1) given by a continuous function $f : \Omega \times \mathbb{R}$ which is locally Lipschitz with respect to the state variable x.

The papers [18] and [32] describe examples of families of linear equations $x' = a(\omega{\cdot}t)\, x + b(\omega{\cdot}t)$ over an almost periodic (and hence equicontinuous) base flow (Ω, σ) for which there exists just one minimal set $\mathcal{M}$, which in addition is not a copy of the base. Let us take $r_1 < r_2$ such that $\mathcal{M} \subset \Omega \times [r_1, r_2]$, and define $g(x)$ as $(x - r_1)^2$ for $x < r_1$, 0 for $x \in [r_1, r_2]$, and $-(x - r_2)^2$ for $x > r_2$. Then the families $x' = a(\omega{\cdot}t)\, x + b(\omega{\cdot}t) + g(\omega{\cdot}t, x)$, which satisfy all the conditions which we will assume on Section 3, define a flow $(\Omega \times \mathbb{R}, \tau)$ for which (obviously) $\mathcal{M}$ is a τ-minimal set, and the restricted semiflow $(\mathcal{M}, \tau)$ is Auslander-Yorke chaotic (see Remark 2.25.2 below). In Section 3, Theorem 3.15, we will establish conditions under which Auslander-Yorke chaos appears for *infinitely many* τ-invariant compact sets. The proof of the result is strongly based on the next theorem. It describes a branch of τ-invariant compact sets for which the restricted flows satisfy the initial hypotheses in Theorem 2.23, which makes them the suitable sets to detect the presence of Auslander-Yorke chaos. These sets are previously known to coincide with the support of a τ-ergodic measure.

Before stating Theorem 2.26, we explain some basic facts used in its proof.

Remarks 2.25. 1. Theorem 2.23 provides additional information in the context of such a scalar skewproduct flow $(\Omega \times \mathbb{R}, \tau)$, associated to the family (2.1). The target is to analyze the possible presence of Auslander-Yorke chaos for a given τ-invariant compact set $\mathcal{K} \subset \Omega \times \mathbb{R}$ which is transitive and the support of a τ-invariant set. Recall that the base flow (Ω, σ) is always assumed to be minimal, and that any τ-invariant minimal set is an almost automorphic extension of the base: see Subsection 2.4. According to Theorem 2.23, the options for the restricted flow $(\mathcal{K}, \tau)$ are two: either it is an equicontinuous minimal flow, or it is Auslander-Yorke chaotic. Assume that we are in the first case. Then $\mathcal{K}$ (an almost automorphic extension of the base, since it is minimal) is necessarily an equicontinuous copy of the base: see e.g. [38, Theorem A or Part II]. Therefore, in this case, the base is necessarily equicontinuous.

2. In particular, the restricted flow to such a set $\mathcal{K}$ is Auslander-Yorke chaotic whenever either the base flow is not equicontinuous or $\mathcal{K}$ is not a copy of the base.

The omega limit set $\mathcal{O}_\tau(\omega, x)$ of a point $(\omega, x) \in \Omega \times \mathbb{R}$ and the support $\operatorname{Supp} m$ of a τ-invariant measure m are defined in Subsection 2.1, and the notion of τ-equilibrium appears in

Subsection 2.2. The properties required in the function η appearing in the next statement are satisfied, for instance, for the upper and lower cover of a τ-invariant compact set $\mathcal{K} \subset \Omega \times \mathbb{R}$.

Theorem 2.26. *Let (Ω, σ) be a minimal flow on a compact metric space, and let $(\Omega \times \mathbb{R}, \tau)$ be the flow induced by the family of equations (2.1), where $f : \Omega \times \mathbb{R}$ is jointly continuous and locally Lipschitz with respect to the state variable $x \in \mathbb{R}$.*

(i) *Let $\mathcal{M} \subset \Omega \times \mathbb{R}$ be a minimal set. Then, either the base flow (Ω, σ) is equicontinuous and $\mathcal{M}$ is a copy of the base, or the restricted flow $(\mathcal{M}, \tau)$ is Auslander-Yorke chaotic.*

Let us fix $m \in \mathfrak{M}_{\text{erg}}(\Omega, \sigma)$ and let $\eta : \Omega \to \mathbb{R}$ be a bounded Borel function such that

- *there exists a σ-invariant subset Ω_η with $m(\Omega_\eta) = 1$ such that $x(t, \omega, \eta(\omega)) = \eta(\omega{\cdot}t)$ for all $t \in \mathbb{R}$ and $\omega \in \Omega_\eta$,*
- *there exists a continuity point ω_η of η,*
- *the graph of η is contained in a τ-invariant compact set $\mathcal{A} \subset \Omega \times \mathbb{R}$.*

Then,

(ii) $\displaystyle\int_{\mathcal{A}} h(\omega, x)\, d\mu_\eta := \int_{\Omega} h(\omega, \eta(\omega))\, dm$ *for $h \in C(\mathcal{A}, \mathbb{R})$ defines a regular Borel τ-ergodic measure μ_η concentrated on $\mathcal{A}$.*

(iii) *Let us define $\mathcal{S}_\eta := \operatorname{Supp} \mu_\eta$. Then, there exists $\Omega_* \subseteq \Omega_\eta$ with $m(\Omega_*) = 1$ such that $(\bar{\omega}, \eta(\bar{\omega})) \in \mathcal{S}_\eta$ and*

$$\mathcal{S}_\eta = \mathcal{O}_\tau(\bar{\omega}, \eta(\bar{\omega})) \tag{2.11}$$

for all $\bar{\omega} \in \Omega_*$. In particular, $\mathcal{S}_\eta$ is a τ-invariant pinched compact set, the flow $(\mathcal{S}_\eta, \tau)$ is transitive, and the set $\mathcal{X}_\eta \subseteq \mathcal{S}_\eta$ of τ-generic points with forward τ-semiorbit dense in $\mathcal{S}_\eta$ satisfies $\mu_\eta(\mathcal{X}_\eta) = 1$.

(iv) *Either the base flow (Ω, σ) is equicontinuous and $\mathcal{S}_\eta$ is a copy of the base, or the restricted flow $(\mathcal{S}_\eta, \tau)$ is Auslander-Yorke chaotic.*

Proof. (i) According to Corollary 2.24, either $(\mathcal{M}, \tau)$ is Auslander-Yorke chaotic or it is equicontinuous. As explained in Remark 2.25.1, in the second situation, $\mathcal{M}$ is a copy of the base and hence the base flow is equicontinuous.

(ii) This property is a classical result on measure theory, and an easy and nice exercise for the interested reader.

(iii) The τ-invariance and compactness of $\mathcal{S}_\eta$ are general properties which follow from the τ-invariance of μ_η and the compactness of $\mathcal{A}$: see Subsection 2.1. Let us check that $\mathcal{S}_\eta$ is a pinched set which contains a dense forward τ-semiorbit.

Lusin's theorem and the regularity of m provide a compact set $\mathcal{K} \subseteq \Omega_\eta$ with $m(\mathcal{K}) > 0$ such that the restriction $\eta : \mathcal{K} \to \mathbb{R}$ is continuous. Let us define the set

$$\mathcal{K}_* := \{\omega \in \mathcal{K} \mid m(\mathcal{B}_\Omega(\omega, \delta) \cap \mathcal{K}) > 0 \text{ for all } \delta > 0\},$$

which is obviously closed and hence compact. Our first goal is checking that $m(\mathcal{K}-\mathcal{K}_*)=0$. Since m is regular, it is enough to prove that $m(\mathcal{C})=0$ for any compact subset $\mathcal{C}\subseteq\mathcal{K}-\mathcal{K}_*$. For any $\omega_0\in\mathcal{C}$ there exists $\delta_{\omega_0}>0$ such that $m(\mathcal{B}_\Omega(\omega_0,\delta_{\omega_0})\cap\mathcal{K})=0$. The compactness of $\mathcal{C}$ provides a finite number of points $\omega_1,\dots,\omega_m$ such that $\mathcal{C}\subseteq\mathcal{B}_\Omega(\omega_1,\delta_{\omega_1})\cup\cdots\cup\mathcal{B}_\Omega(\omega_m,\delta_{\omega_m})$. Hence, $\mathcal{C}=\mathcal{C}\cap\mathcal{K}\subseteq(\mathcal{B}_\Omega(\omega_1,\delta_{\omega_1})\cap\mathcal{K})\cup\cdots\cup(\mathcal{B}_\Omega(\omega_m,\delta_{\omega_m})\cap\mathcal{K})$, which ensures that $m(\mathcal{C})=0$, as asserted. Consequently, $m(\mathcal{B}_\Omega(\omega,\delta)\cap\mathcal{K}_*)=m(\mathcal{B}_\Omega(\omega,\delta)\cap\mathcal{K})>0$ for any $\omega\in\mathcal{K}_*$ and $\delta>0$.

The compact set $\mathcal{K}_*$ is separable, so that we can find a countable and dense subset $\mathcal{D}:=\{\omega_m\mid m\ge 1\}\subseteq\mathcal{K}_*$. We call $\mathcal{K}_{m,k}:=\mathcal{B}_\Omega(\omega_m,1/k)\cap\mathcal{K}_*$ and observe that $m(\mathcal{K}_{m,k})>0$ for all $m,k\ge 1$, since $\omega_m\in\mathcal{K}_*$. Therefore, Birkhoff's ergodic theorem provides a σ-invariant subset $\Omega_{m,k}\subseteq\Omega_\eta$ with $m(\Omega_{m,k})=1$ such that for any $\omega\in\Omega_{m,k}$ there exists a sequence $(t_n)\uparrow\infty$ with $\omega{\cdot}t_n\in\mathcal{K}_{m,k}$ for any $n\ge 1$.

The set $\Omega_*:=\bigcap_{m\ge 1,\,k\ge 1}\Omega_{m,k}\subseteq\Omega_\eta$ is σ-invariant satisfies $m(\Omega_*)=1$. We fix $\bar\omega\in\Omega_*$ and will check that (2.11) holds and that $(\bar\omega,\eta(\bar\omega))\in\mathcal{S}_\eta$. Before that, observe that these properties ensure that

- the restricted flow $(\mathcal{S}_\eta,\tau)$ is point transitive (and hence topologically transitive, see Remark 2.22.3), since the forward τ-semiorbit of $(\bar\omega,\eta(\bar\omega))$ is dense in $\mathcal{S}_\eta$;
- the set $\mathcal{S}_\eta$ is pinched, since its section over the continuity point ω_η of the map η reduces to the singleton $\{\eta(\omega_\eta)\}$;
- the set of points $\{(\bar\omega,\eta(\bar\omega))\mid\bar\omega\in\Omega_*\}$ has full measure μ_η. Hence $\mu_\eta(\mathcal{X}_\eta)=1$ for the set $\mathcal{X}_\eta$ of statement (iii), since the set of generic points for $(\mathcal{S}_\eta,\tau)$ has complete measure (see Subsection 2.1).

That is, the proof of (iii) will be complete once checked these two assertions.

We begin by observing that the definitions of $\mathcal{K}_*$ and Ω_* provide $t>0$ such that $\bar\omega{\cdot}t\in\mathcal{K}_*$. Since $\bar\omega\in\Omega_*\subseteq\Omega_\eta$, we have $\tau(t,\bar\omega,\eta(\bar\omega))=(\bar\omega{\cdot}t,\eta(\bar\omega{\cdot}t))$. Therefore, $\mathcal{O}_\tau(\bar\omega,\eta(\bar\omega))=\mathcal{O}_\tau(\bar\omega{\cdot}t,\eta(\bar\omega{\cdot}t))$, and $(\bar\omega,\eta(\bar\omega))\in\mathcal{S}_\eta$ if and only if $(\bar\omega{\cdot}t,\eta(\bar\omega{\cdot}t))\in\mathcal{S}_\eta$. Consequently, it is enough to prove the two previous assertions for $\bar\omega\in\mathcal{K}_*$, which we assume from now on.

We first prove that

$$(\widetilde\omega{\cdot}t,\eta(\widetilde\omega{\cdot}t))\in\mathcal{O}_\tau(\bar\omega,\eta(\bar\omega)) \tag{2.12}$$

for all $\widetilde\omega\in\mathcal{K}_*$ and $t\in\mathbb{R}$. To this end, we take $\omega_m\in\mathcal{D}$ and $\varepsilon>0$, and look for $k>1/\varepsilon$ such that, if $\omega\in\mathcal{B}_\Omega(\omega_m,1/k)\cap\mathcal{K}_*$, then $|\eta(\omega_m)-\eta(\omega)|<\varepsilon$. We also look for $(t_n)\uparrow\infty$ such that $\bar\omega{\cdot}t_n\in\mathcal{K}_{m,k}\subseteq\mathcal{B}_\Omega(\omega_m,1/k)$. Thus, $\mathrm{dist}_\Omega(\omega_m,\bar\omega{\cdot}t_n)<1/k<\varepsilon$ and $|\eta(\omega_m)-\eta(\bar\omega{\cdot}t_n)|<\varepsilon$, which proves (2.12) for $\widetilde\omega=\omega_m\in\mathcal{D}$ and $t=0$. The property for all $\widetilde\omega\in\mathcal{K}_*$ and $t=0$ follows from the density of $\mathcal{D}$, the continuity of $\eta\colon\mathcal{K}_*\to\mathbb{R}$, and the closed character of the right set in (2.12). Once this is established, we combine $\mathcal{K}_*\subset\Omega_\eta$ with the τ-invariance of the omega limit in order to deduce (2.12) for all $\widetilde\omega\in\mathcal{K}_*$ and $t\in\mathbb{R}$.

We define $\mathcal{K}_\infty:=\bigcup_{t\in\mathbb{R}}\sigma_t(\mathcal{K}_*)$. The definition of Ω_* ensures that $\Omega_*\subseteq\mathcal{K}_\infty$, and hence $m(\mathcal{K}_\infty)\ge m(\Omega_*)=1$. Note that (2.12) ensures that $(\omega,\eta(\omega))\in\mathcal{O}_\tau(\bar\omega,\eta(\bar\omega))$ whenever $\omega\in\mathcal{K}_\infty$. This property and the regularity of μ_η yield

$$\mu_\eta(\mathcal{O}_\tau(\bar\omega,\eta(\bar\omega)))=\inf\left\{\int_{\mathcal{A}} f(\omega,x)\,d\mu_\eta\;\Big|\; f\in C(\mathcal{A},[0,1])\text{ with } f|_{\mathcal{O}_\tau(\bar\omega,\eta(\bar\omega))}\equiv 1\right\}$$

$$= \inf \left\{ \int_{\mathcal{A}} f(\omega, \eta(\omega))\, dm \mid f \in C(\mathcal{A}, [0,1]) \text{ with } f|_{\mathcal{O}_\tau(\bar\omega, \eta(\bar\omega))} \equiv 1 \right\} \quad (2.13)$$

$$\geq \int_\Omega \chi|_{\mathcal{K}_\infty}(\omega)\, dm = 1\,.$$

(As usual, $\chi|_{\mathcal{B}}$ is the characteristic function of the set $\mathcal{B}$.) Hence, $\mu_\eta(\mathcal{O}_\tau(\bar\omega, \eta((\bar\omega))) = 1$, which ensures that $\mathcal{S}_\eta \subseteq \mathcal{O}_\tau(\bar\omega, \eta(\bar\omega))$.

Let us now check that $\mathcal{O}_\tau(\bar\omega, \eta(\bar\omega)) \subseteq \mathcal{S}_\eta$. We take $(\widetilde\omega, \widetilde x) \in \mathcal{O}_\tau(\bar\omega, \eta(\bar\omega))$ and an open neighborhood $\mathcal{U} \subset \Omega \times \mathbb{R}$ of $(\widetilde\omega, \widetilde x)$, and will prove that $\mu_\eta(\mathcal{U} \cap \mathcal{A}) > 0$. Let us take $\bar t > 0$ such that $(\bar\omega{\cdot}\bar t, \eta(\bar\omega{\cdot}\bar t)) \in \mathcal{U}$. Then $(\bar\omega, \eta(\bar\omega)) \in \mathcal{V} := \tau_{-\bar t}(\mathcal{U})$, which combined with the continuity of $\mathcal{K}_* \to \Omega \times \mathbb{R}\,,\ \omega \mapsto (\omega, \eta(\omega))$ ensures that there exists $\delta > 0$ such that $(\omega, \eta(\omega)) \in \mathcal{V}$ whenever $\omega \in \mathcal{B}(\bar\omega, \delta) \cap \mathcal{K}_*$. Since $m(\mathcal{B}(\bar\omega, \delta) \cap \mathcal{K}_*) > 0$, we conclude as in (2.13) that $\mu_\eta(\mathcal{V} \cap \mathcal{A}) > 0$, which combined with the τ-invariance of the measure ensures that $\mu_\eta(\mathcal{U} \cap \mathcal{A}) = \mu_\eta(\mathcal{V} \cap \mathcal{A}) > 0$. An easy contradiction argument shows that $(\widetilde\omega, \widetilde x) \in \mathcal{S}_\mu$, so that (2.11) is proved for the initially chosen point $\bar\omega \in \mathcal{K}_*$. In turn, (2.11) combined with (2.12) for $\widetilde\omega = \bar\omega$ and $t = 0$ ensures that $(\bar\omega, \eta(\bar\omega)) \in \mathcal{S}_\eta$. This completes the proof of the two assertions, and that of (iii).

(iv) According to Theorem 2.23, either $(\mathcal{S}_\eta, \tau)$ is Auslander-Yorke chaotic or it is equicontinuous and minimal. Remark 2.25.1 completes the proof of (iv). □

3. Dynamics for nonhomogeneous linear dissipative equations

Let (Ω, σ) be a minimal flow on a compact metric space. (This minimality is an important requisite throughout the whole section.) Let $a, b\colon \Omega \to \mathbb{R}$ be continuous functions. Let us consider the family of scalar nonautonomous equations

$$x' = a(\omega{\cdot}t)\, x + b(\omega{\cdot}t) + g(\omega{\cdot}t, x)\,, \qquad \omega \in \Omega \tag{3.1}$$

with nonhomogeneous linear part, under the following conditions on the function $g\colon \Omega \times \mathbb{R} \to \mathbb{R}$ (although not all of them will be always in force):

g1 There exists the partial derivative g_x, and the functions $g,\ g_x\colon \Omega \times \mathbb{R} \to \mathbb{R}$ are continuous.
g2 There exist real numbers $r_1 \le r_2$ such that: $g(\omega, x) = 0$ if $r_1 \le x \le r_2$, $g(\omega, x) > 0$ if $x < r_1$ and $g(\omega, x) < 0$ if $x > r_2$; and $g_x(\omega, r_1) = 0$ for all $\omega \in \Omega$ if $r_1 = r_2$.
g3 $\lim_{x \to \pm\infty}(g(\omega, x)/x) = -\infty$ uniformly on Ω.
g4 $g_x(\omega, x) \le 0$ whenever $x \notin [r_1, r_2]$.
$\tilde{\mathbf{g}}$4 $g_x(\omega, x) < 0$ whenever $x \notin [r_1, r_2]$.

The family (3.1) is said to be *linear dissipative* if $r_1 < r_2$, and *purely dissipative* if $r_1 = r_2$. Theorem 3.2 will justify the use of the term *dissipative* in both cases. In this paper, we are more interested in the linear dissipative case, where we can detect Li-Yorke chaos and Auslander-Yorke chaos. But it is quite easy to complete our analysis in order to include the purely dissipative case, just using at a certain point (in the proof of Theorem 3.11) one result of [30].

As explained in Subsection 2.2, the family (3.1) induces a local continuous flow $(\Omega \times \mathbb{R}, \tau)$, given by

$$\tau : \mathcal{U} \subseteq \mathbb{R} \times \Omega \times \mathbb{R} \to \Omega \times \mathbb{R}\,, \quad (t, \omega, x_0) \mapsto (\omega{\cdot}t, x(t, \omega, x_0))\,,$$

where $t \mapsto x(t, \omega, x_0)$ is the maximal solution of $(3.1)_\omega$ with $x(0, \omega, x_0) = x_0$. In addition, the map $x_0 \mapsto x(t, \omega, x_0)$ is C^1 if **g1** holds.

The associated family of homogeneous linear equations

$$x' = a(\omega{\cdot}t)\, x\,, \tag{3.2}$$

for $\omega \in \Omega$, will play a fundamental role in the proofs of the results. Let us denote $x_l(t, \omega, x_0) := x_0 \exp\big(\int_0^t a(\omega{\cdot}s)\, ds\big)$, and let $(\Omega \times \mathbb{R}, \tau_l)$ be the associated linear flow, so that $\tau_l(t, \omega, x) = (\omega{\cdot}t, x_l(t, \omega, x_0))$.

We will begin this section by some general results which require neither the assumption **g4** on g nor any condition on the Sacker and Sell spectrum of the linear family (3.2). More precisely, we establish the existence of global attractor $\mathcal{A}$, in Theorem 3.2, and analyze two minimal sets (which may coincide) determined by the upper and lower covers of $\mathcal{A}$, in Theorem 3.3. Then we show, in Theorem 3.4, that if any τ-minimal set is uniformly exponentially stable at $+\infty$, then there is just one of these sets, which coincides with the global attractor.

The condition **g4** and the assumptions on Σ_a will hence not be in force until Subsections 3.1 and 3.2, where we obtain a much more accurate description of the global dynamics.

Remarks 3.1. We will repeatedly use the next properties.

1. Let us choose $\omega \in \Omega$ and assume that two maps $t \mapsto \alpha(\omega{\cdot}t)$ and $t \mapsto \beta(\omega{\cdot}t)$ are globally defined solutions of the equation $(3.1)_\omega$ with $\alpha(\omega{\cdot}t) \le \beta(\omega{\cdot}t)$ for any $t \in \mathbb{R}$. Assume also that g satisfies **g1** and **g2**, and that $\alpha(\omega{\cdot}t) \le r_2$ and $\beta(\omega{\cdot}t) \ge r_1$ for all $t \in \mathbb{R}$. Then, the map $t \mapsto \beta(\omega{\cdot}t) - \alpha(\omega{\cdot}t)$ is a nonnegative *lower solution* of the linear equation $(3.2)_\omega$ (that is, its derivative satisfies the differential inequality $x' \le a(\omega{\cdot}t)\, x$). This assertion follows from property **g2**, which ensures that $g(\omega{\cdot}t, \beta(\omega{\cdot}t)) \le 0$ and $g(\omega{\cdot}t, \alpha(\omega{\cdot}t)) \ge 0$. A standard comparison argument shows that, in this case, $\beta(\omega{\cdot}t) - \alpha(\omega{\cdot}t) \ge (\beta(\omega) - \alpha(\omega)) \exp\big(\int_0^t a(\omega{\cdot}s)\, ds\big)$ for $t \le 0$ and $\beta(\omega{\cdot}t) - \alpha(\omega{\cdot}t) \le (\beta(\omega) - \alpha(\omega)) \exp\big(\int_0^t a(\omega{\cdot}s)\, ds\big)$ for $t \ge 0$. In particular, if any point $\omega \in \Omega$ satisfies the initial assumption, then the map $\Omega \to \mathbb{R}\,,\ \omega \mapsto \beta(\omega) - \alpha(\omega)$ is a τ_l-subequilibrium. We referred to this type of relation between lower (or upper) solutions and subequilibria (or superequilibria) in Subsection 2.2.

2. Note also that a similar result holds for $t \mapsto c(\beta(\omega{\cdot}t) - \alpha(\omega{\cdot}t))$ if $c > 0$.

3. If, in addition, g satisfies **g4** and $c > 0$, then $t \mapsto c\,(\beta(\omega{\cdot}t) - \alpha(\omega{\cdot}t))$ is a nonnegative lower solution of $x' = a(\omega{\cdot}t)\, x$ independently of the area where their graph is contained, since $c\,(g(\omega{\cdot}t, \beta(\omega{\cdot}t)) - g(\omega{\cdot}t, \alpha(\omega{\cdot}t)) \le 0$.

By repeating the arguments leading to [6, Theorem 16] (see also [8, Section 1.2]), one proves the following fundamental result:

Theorem 3.2. *Assume that g satisfies* **g1** *and* **g3**, *and let $(\Omega \times \mathbb{R}, \tau)$ be the flow induced by the family* (3.1). *Then,*

(i) *the flow τ is bounded dissipative and admits a global attractor*

$$\mathcal{A} = \bigcup_{\omega \in \Omega} \big(\{\omega\} \times [\alpha_{\mathcal{A}}(\omega), \beta_{\mathcal{A}}(\omega)]\big)\,.$$

In particular, any forward τ-semiorbit is globally defined and bounded. In addition, $\alpha_{\mathcal{A}}\colon \Omega \to \mathbb{R}$ and $\beta_{\mathcal{A}}\colon \Omega \to \mathbb{R}$ are respectively lower and upper semicontinuous τ-equilibria; and the sets of continuity points for the functions $\alpha_{\mathcal{A}}$ and $\beta_{\mathcal{A}}$ are residual and σ-invariant.

(ii) *In addition, these functions can be obtained as the limits*

$$\alpha_{\mathcal{A}}(\omega) = \lim_{t\to\infty} x(t, \omega{\cdot}(-t), -\rho_0)\,,$$
$$\beta_{\mathcal{A}}(\omega) = \lim_{t\to\infty} x(t, \omega{\cdot}(-t), \rho_0)\,,$$

where the constant ρ_0 is large enough to guarantee that $a(\omega)\,x + b(\omega) + g(\omega, x) > 0$ whenever $x \le -\rho_0$ and $a(\omega)\,x + b(\omega) + g(\omega, x) < 0$ whenever $x \ge \rho_0$.

(iii) *$\mathcal{A}$ is the union of the all the τ-orbits which are globally defined and bounded.*

In the description of the global dynamics there are two τ-minimal subsets (which may coincide) easily defined from $\mathcal{A}$ which play a fundamental role, and which we describe in the next result. As recalled in Subsection 2.4, given any τ-minimal set $\mathcal{M}$: there exists a residual σ-invariant subset $\Omega_{\mathcal{M}} \subseteq \Omega$ at whose points the functions $\alpha_{\mathcal{M}}$ and $\beta_{\mathcal{M}}$ appearing in the description (2.6) of $\mathcal{M}$ are continuous and take the same value, so that in particular $\mathcal{M}$ is an almost automorphic extension of the base; and $\mathcal{M}$ is a copy of the base if and only if $\alpha_{\mathcal{M}}$ and $\beta_{\mathcal{M}}$ are continuous and coincide everywhere. The fiber-order relation between two τ-minimal sets, denoted as $\mathcal{M} \le \mathcal{N}$ or $\mathcal{M} < \mathcal{N}$, is also described in Subsection 2.4.

Theorem 3.3. *Assume that g satisfies* **g1** *and* **g3***, let $(\Omega \times \mathbb{R}, \tau)$ be the flow induced by the family (3.1), and let $\mathcal{A}$, $\alpha_{\mathcal{A}}$ and $\beta_{\mathcal{A}}$ be provided by Theorem 3.2. Let Ω_c be the residual set of common continuity points of the semicontinuous maps $\alpha_{\mathcal{A}}$ and $\beta_{\mathcal{A}}$. Let us take $\omega_0 \in \Omega_c$ and define*

$$\mathcal{M}^{\alpha} := \text{closure}_{\Omega\times\mathbb{R}}\{(\omega_0{\cdot}t, \alpha_{\mathcal{A}}(\omega_0{\cdot}t)) \mid t \in \mathbb{R}\}\,,$$
$$\mathcal{M}^{\beta} := \text{closure}_{\Omega\times\mathbb{R}}\{(\omega_0{\cdot}t, \beta_{\mathcal{A}}(\omega_0{\cdot}t)) \mid t \in \mathbb{R}\}\,.$$

Then,

(i) *$\mathcal{M}^{\alpha}$ and $\mathcal{M}^{\beta}$ are τ-minimal sets and, for any $\omega \in \Omega_c$, the sections $(\mathcal{M}^{\alpha})_{\omega}$ and $(\mathcal{M}^{\beta})_{\omega}$ are respectively given by the singletons $\{\alpha_{\mathcal{A}}(\omega)\}$ and $\{\beta_{\mathcal{A}}(\omega)\}$. In addition, any τ-minimal set $\mathcal{M}$ satisfies $\mathcal{M}^{\alpha} \le \mathcal{M} \le \mathcal{M}^{\beta}$.*

(ii) *$\mathcal{A}$ is a pinched compact set if and only if there exists $\omega_0 \in \Omega_c$ such that $\alpha_{\mathcal{A}}(\omega_0) = \beta_{\mathcal{A}}(\omega_0)$. In this case, $\Omega_c = \{\omega \in \Omega \mid \alpha_{\mathcal{A}}(\omega) = \beta_{\mathcal{A}}(\omega)\}$.*

Proof. (i) The τ-invariance $\mathcal{M}^{\alpha}$ follows from $(\omega_0{\cdot}t, \alpha_{\mathcal{A}}(\omega_0{\cdot}t)) = \tau(t, \omega_0, \alpha_{\mathcal{A}}(\omega_0))$; and $\mathcal{M}^{\alpha}$ is compact, since $\alpha_{\mathcal{A}}$ is a bounded function. Let us take any $\omega \in \Omega_c$ and $(\omega, x) \in \mathcal{M}^{\alpha}$. Then, $(\omega, x) = \lim_{n\to\infty}(\omega_0{\cdot}t_n, \alpha_{\mathcal{A}}(\omega_0{\cdot}t_n))$ for a sequence (t_n), and the continuity of $\alpha_{\mathcal{A}}$ at ω ensures that $x = \alpha_{\mathcal{A}}(\omega)$. This is, $\mathcal{M}^{\alpha}_{\omega} = \{\alpha_{\mathcal{A}}(\omega)\}$, as asserted. To prove the minimality of $\mathcal{M}^{\alpha}$, we take a τ-minimal subset $\mathcal{M} \subseteq \mathcal{M}^{\alpha}$, so that $\mathcal{M}_{\omega_0} = \mathcal{M}^{\alpha}_{\omega_0} = \{\alpha(\omega_0)\}$. Hence, the definition of $\mathcal{M}^{\alpha}$ ensures that $\mathcal{M}^{\alpha} \subseteq \mathcal{M}$, which shows that they coincide. The arguments are analogous for $\mathcal{M}^{\beta}$. Finally, since any τ-minimal set $\mathcal{M}$ is contained in $\mathcal{A}$, we have $\alpha_{\mathcal{A}}(\omega) \le x \le \beta_{\mathcal{A}}(\omega)$ whenever $\omega \in \Omega$ and $(\omega, x) \in \mathcal{M}$. The last statement in (i) follows easily from here.

(ii) Assume that $\mathcal{A}_{\omega_0}$ is a singleton for a certain point $\omega_0 \in \Omega$, so that $\mathcal{A}_{\omega_0} = \{\alpha_{\mathcal{A}}(\omega_0)\} = \{\beta_{\mathcal{A}}(\omega_0)\}$, and take a sequence (ω_n) with limit ω_0. Any subsequence (ω_k) has, in turn, a subsequence (ω_j) such that there exists $\lim_{j\to\infty} \alpha_{\mathcal{A}}(\omega_j) = x$. The semicontinuity of $\alpha_{\mathcal{A}}$ and $\beta_{\mathcal{A}}$ ensure that $\alpha_{\mathcal{A}}(\omega_0) \le x \le \beta_{\mathcal{A}}(\omega_0)$, and hence $x = \alpha_{\mathcal{A}}(\omega_0)$. This guarantees that $\alpha_{\mathcal{A}}$ is continuous at ω_0. The same argument shows that $\beta_{\mathcal{A}}$ is continuous at ω_0, so that $\omega_0 \in \Omega_c$. In particular, $\{\omega \in \Omega \mid \alpha_{\mathcal{A}}(\omega) = \beta_{\mathcal{A}}(\omega)\} \subseteq \Omega_c$.

Since $\alpha_{\mathcal{A}}$ and $\beta_{\mathcal{A}}$ are τ-equilibria, they agree at $\omega_0{\cdot}t$ for all $t \in \mathbb{R}$. Let us now take $\omega \in \Omega_c$ and a sequence (t_n) with $\lim \omega_0{\cdot}t_n = \omega$. Then $\alpha_{\mathcal{A}}(\omega) = \lim \alpha_{\mathcal{A}}(\omega_0{\cdot}t_n) = \lim \beta_{\mathcal{A}}(\omega_0{\cdot}t_n) = \beta_{\mathcal{A}}(\omega)$. This shows that $\Omega_c \subseteq \{\omega \in \Omega \mid \alpha_{\mathcal{A}}(\omega) = \beta_{\mathcal{A}}(\omega)\}$, and completes the proof of (ii). □

Corollary 2.10 states that a τ-minimal set $\mathcal{M}$ is an exponentially stable at $+\infty$ copy of the base (the graph of the continuous function $\alpha_{\mathcal{M}} = \beta_{\mathcal{M}}$) if and only if its upper Lyapunov exponent is strictly negative. We can add some more information for families of equations of the type (3.1):

Theorem 3.4. *Assume that g satisfies* **g1** *and* **g3**, *and let $(\Omega \times \mathbb{R}, \tau)$ be the flow induced by the family of equations* (3.1). *Then, the following assertions are equivalent:*

(1) *Any τ-minimal set has strictly negative upper Lyapunov exponent.*
(2) *There exists a unique τ-minimal set whose upper Lyapunov exponent is strictly negative.*

Assume that this is the case, let $\mathcal{A}$, $\alpha_{\mathcal{A}}$ and $\beta_{\mathcal{A}}$ be provided by Theorem 3.2, *and let $\mathcal{M}^\alpha$ and $\mathcal{M}^\beta$ be provided by Theorem* 3.3. *Then, the attractor $\mathcal{A}$ is given for the unique τ-minimal set $\mathcal{M}^\alpha = \mathcal{M}^\beta = \{\alpha_{\mathcal{A}}\} = \{\beta_{\mathcal{A}}\}$, and it attracts exponentially any τ-orbit as time increases.*

Proof. Assume that (1) holds. Recall that the existence of a global attractor ensures that any solution is defined and bounded on a positive half-line (see Theorem 3.2(i)), which in turn ensures the existence of its omega limit set. Recall also that (1) ensures that any τ-minimal set $\mathcal{M}$ is a uniformly exponentially stable at $+\infty$ copy of the base: $\mathcal{M} = \{\eta\}$ (see Corollary 2.10). Given one of these sets, we consider its basin of attraction,

$$\mathcal{B}_{\mathcal{M}} := \{(\omega, x_0) \mid \lim_{t\to\infty} |x(t, \omega, x_0) - \eta(\omega{\cdot}t)| = 0\}\,.$$

It is easy to check that $\mathcal{B}_{\mathcal{M}}$ is an open set, and that different τ-minimal sets give rise to disjoint basins of attraction. It is also easy to check that every point (ω, x) belongs to the basin of attraction of a τ-minimal set contained in its omega limit set. Therefore, we can write

$$\Omega \times \mathbb{R} = \bigcup_{\mathcal{M} \text{ is } \tau\text{-minimal}} \mathcal{B}_{\mathcal{M}}\,,$$

which is a disjoint union of open sets. Since $\Omega \times \mathbb{R}$ is connected, we conclude that there exists a unique τ-minimal set: (2) holds. The converse is trivial.

Therefore, $\mathcal{M}^\alpha = \mathcal{M}^\beta$, and is a copy of the base. It follows from the definitions of these sets that the functions $\alpha_{\mathcal{A}}, \beta_{\mathcal{A}} \colon \Omega \to \mathbb{R}$ are continuous and equal, which obviously ensures that $\mathcal{M}^\alpha = \mathcal{M}^\beta = \mathcal{A}$. The last assertion follows easily from the hyperbolicity of $\mathcal{A}$ and the fact that it is contained in the omega limit set of any τ-orbit. The proof is complete. □

We complete this part of general results with a theorem which characterizes the set of common continuity points of $\alpha_{\mathcal{A}}$ and $\beta_{\mathcal{A}}$ in some cases.

Theorem 3.5. *Assume that g satisfies* **g1**, **g2** *and* **g3**, *let $(\Omega\times\mathbb{R},\tau)$ be the flow induced by the family (3.1), let $\mathcal{A}$, $\alpha_{\mathcal{A}}$ and $\beta_{\mathcal{A}}$ be provided by Theorem 3.2, and let Ω_c be the (nonempty) set defined in Theorem 3.3. Assume also that there exists a τ-minimal set $\mathcal{M}\subseteq\Omega\times[r_1,r_2]$. Then,*

(i) *if there exists $\omega_0\in\Omega$ with $\sup_{t\le 0}\int_0^t a(\omega_0{\cdot}s)\,ds=\infty$, then $\omega_0\in\Omega_c$, $\Omega_c=\{\omega\in\Omega\,|\,\alpha_{\mathcal{A}}(\omega)=\beta_{\mathcal{A}}(\omega)\}$, $\mathcal{A}$ is pinched, and $\mathcal{M}=\mathcal{M}^\alpha=\mathcal{M}^\beta$ is the unique τ-minimal set.*

(ii) *Let $\alpha_{\mathcal{M}}$ and $\beta_{\mathcal{M}}$ be defined by (2.6), and assume that $\mathcal{M}\subset\Omega\times[r_1,r_2)$ or $\mathcal{M}\subset\Omega\times(r_1,r_2]$. If there exists $\omega_0\in\Omega$ with $\sup_{t\le 0}\int_0^t a(\omega_0{\cdot}s)\,ds<\infty$, then $\alpha_{\mathcal{A}}(\omega_0)<\beta_{\mathcal{A}}(\omega_0)$ and $\mathcal{M}\subsetneq\mathcal{A}$.*

In particular, if $\mathcal{A}$ is pinched, and if $\mathcal{M}:=\mathcal{M}^\alpha=\mathcal{M}^\beta$ is contained in either $\Omega\times[r_1,r_2)$ or in $\Omega\times(r_1,r_2]$, then $\Omega_c=\{\omega\in\Omega\,|\,\alpha_{\mathcal{A}}(\omega)=\beta_{\mathcal{A}}(\omega)\}=\{\omega\in\Omega\,|\,\sup_{t\le 0}\int_0^t a(\omega{\cdot}s)\,ds=\infty\}$.

Proof. (i) It is enough to prove that $\alpha_{\mathcal{A}}(\omega_0)=\beta_{\mathcal{A}}(\omega_0)$: if so, $\mathcal{A}$ is pinched, and hence Theorem 3.3(ii) proves the remaining assertion. The hypothesis $\mathcal{M}\subseteq\Omega\times[r_1,r_2]$ guarantees the that the conditions of Remark 3.1.1 are fulfilled, and hence $\beta_{\mathcal{A}}(\omega_0{\cdot}t)-\alpha_{\mathcal{A}}(\omega_0{\cdot}t)\ge(\beta_{\mathcal{A}}(\omega_0)-\alpha_{\mathcal{A}}(\omega_0))\exp\big(\int_0^t a(\omega_0{\cdot}s)\,ds\big)$ for $t\le 0$ (see Remark 3.1.1). Since the left-hand term is bounded, it is necessarily $\alpha_{\mathcal{A}}(\omega_0)=\beta_{\mathcal{A}}(\omega_0)$.

(ii) We work in the case $\mathcal{M}\subset\Omega\times[r_1,r_2)$, being the proof analogous in the other case. Recall that $\exp\int_0^t a(\omega_0{\cdot}s)\,ds=x_l(t,\omega_0,1)$, solution of $(3.2)_{\omega_0}$. Let us look for $\varepsilon>0$ such that $\varepsilon\sup_{t\le 0}x_l(t,\omega_0,1)\le r_2-\sup\{\beta_{\mathcal{M}}(\omega)\,|\,\omega\in\Omega\}$, and define $z(t):=\beta_{\mathcal{M}}(\omega_0{\cdot}t)+\varepsilon\,x_l(t,\omega_0,1)$. Then $z(t)$ takes values in $[r_1,r_2]$ for $t\le 0$ (due to $\mathcal{M}\subset\Omega\times[r_1,r_2]$ and to the choice of ε), and hence it solves $(3.1)_{\omega_0}$ in $(-\infty,0]$, where, consequently, it agrees with $x(t,\omega_0,z(0))$. Therefore this last solution of $(3.1)_{\omega_0}$ is globally defined and bounded (see Theorem 3.2(i)), which ensures that $(\omega_0,z(0))\in\mathcal{A}-\mathcal{M}$ (see Theorem 3.2(iii)). This proves (ii).

The final statements of the theorem follow from (i), (ii), and Theorem 3.3(ii). □

3.1. *The case* $\sup\Sigma_a<0$

In the next two subsections, we describe the τ-minimal sets and the possibility of occurrence of chaos for the family of equations (3.1), assuming condition **g4** (or **g̃4**) in two cases which depend on the Sacker and Sell spectrum Σ_a of (3.2) in two cases: $\sup\Sigma_a<0$ and $\sup\Sigma_a=0$. Remark 2.7 explains that the first situation is equivalent to the negative character of the upper of exponential dichotomy of the family (3.2) (which therefore has exponential dichotomy over Ω), and that the second one is equivalent to the null character of that upper Lyapunov exponent (so that the linear family does not have exponential dichotomy).

Let us begin with the case $\sup\Sigma_a<0$. There is not much to say in this situation, in which the conditions assumed on a and g provide a very simple global dynamics:

Theorem 3.6. *Assume that g satisfies* **g1**, **g2**, **g3** *and* **g4**, *let $(\Omega\times\mathbb{R},\tau)$ be the flow induced by the family (3.1), and let $\mathcal{A}$ be the global attractor for τ provided by Theorem 3.2. Assume also that* $\sup\Sigma_a<0$. *Then, $\mathcal{A}$ is a uniformly exponentially stable at $+\infty$ copy of the base which attracts exponentially any τ-orbit as time increases. In particular, $\mathcal{A}$ is the unique τ-minimal set.*

Proof. Recall that, if **g1** holds, the upper Lyapunov of a τ-minimal set $\mathcal{M}$ is

$$\gamma^s_{\mathcal{M}} = \int_{\mathcal{M}} (a(\omega) + g_x(\omega, x))\, d\nu^s_{\mathcal{M}} \tag{3.3}$$

for a suitable τ-invariant measure $\nu^s_{\mathcal{M}}$ on $\Omega \times \mathbb{R}$. Therefore,

$$\gamma^s_{\mathcal{M}} \le \int_{\Omega} a(\omega)\, dm^s_{\mathcal{M}} \le \sup \Sigma_a\,, \tag{3.4}$$

where $m^s_{\mathcal{M}} \in \mathfrak{M}_{\text{inv}}(\Omega, \sigma)$ is the σ-invariant measure onto which $\nu^s_{\mathcal{M}}$ projects. The first inequality follows from (3.3), since conditions **g2** and **g4** ensure that $g_x \le 0$; and the second one from Theorem 2.6. Therefore, $\gamma^s_{\mathcal{M}} < 0$ for any τ-minimal set $\mathcal{M}$ if $\sup \Sigma_a < 0$, and hence the assertions follow from Theorem 3.4. □

3.2. *The case* $\sup \Sigma_a = 0$

This final part is devoted to prove that, as advanced in the Introduction, under the conditions given by **g1**, **g2**, **g3** and $\tilde{\mathbf{g}}\mathbf{4}$ on g, there are just two possible global dynamics for the flow $(\Omega \times \mathbb{R}, \tau)$ induced by (3.1) when $\sup \Sigma_a = 0$, and in one of them we are able to detect chaotic behavior.

We begin by describing a particularly simple condition under which $\sup \Sigma_a = 0$: the existence of a continuous primitive for a: see Definition 2.11. Observe that condition **g3** is not assumed, since the stated properties hold independently of the existence of a global attractor.

Theorem 3.7. *Assume that g satisfies* **g1** *and* **g2** *and that the map a admits a continuous primitive. Let $(\Omega \times \mathbb{R}, \tau)$ be the flow induced by the family* (3.1)*. Then, $\Sigma_a = \{0\}$ and, in addition,*

(i) *any possible τ-minimal set $\mathcal{M}$ contained in $\Omega \times [r_1, r_2]$ is a copy of the base.*
(ii) *If g also satisfies* **g4***, any τ-minimal set $\mathcal{M}$ is a copy of the base.*

Proof. The fact that $\Sigma_a = \{0\}$ follows easily from Theorem 2.6 and Birkhoff's ergodic theorem. Let $h_a\colon \Omega \to \mathbb{R}$ be a continuous primitive of a, and $H_a := e^{h_a}$. Then, for any $\omega \in \Omega$ and $t \in \mathbb{R}$, $H_a(\omega{\cdot}t) = H_a(\omega) \exp\big(\int_0^t a(\omega{\cdot}s)\, ds\big)$. In other words, $H_a(\omega{\cdot}t) = x_l(t, \omega, H_a(\omega))$, solution of $x' = a(\omega{\cdot}t)\, x$. Note also that H_a is positive and bounded from below on Ω.

Let $\alpha_{\mathcal{M}}$ and $\beta_{\mathcal{M}}$ be the maps appearing in the description (2.6) of $\mathcal{M}$. The fundamental points in this proof have been explained in Remark 3.1: if $\mathcal{M}$ is contained in $\Omega \times [r_1, r_2]$ (as we assume in (i)), or if **g4** holds (as in (ii)), then $\beta_{\mathcal{M}}(\omega{\cdot}t) - \alpha_{\mathcal{M}}(\omega{\cdot}t) \le x_l(t, \omega, \beta_{\mathcal{M}}(\omega) - \alpha_{\mathcal{M}}(\omega))$ for any $\omega \in \Omega$ whenever $t \ge 0$. Let us write $\beta_{\mathcal{M}}(\omega) - \alpha_{\mathcal{M}}(\omega) = k(\omega)\, H_a(\omega)$. Then,

$$\begin{aligned} k(\omega{\cdot}t)\, H_a(\omega{\cdot}t) &= \beta_{\mathcal{M}}(\omega{\cdot}t) - \alpha_{\mathcal{M}}(\omega{\cdot}t) \le x_l(t, \omega, \beta_{\mathcal{M}}(\omega) - \alpha_{\mathcal{M}}(\omega)) \\ &= x_l(t, \omega, k(\omega)\, H_a(\omega)) = k(\omega)\, H_a(\omega{\cdot}t) \end{aligned}$$

whenever $\omega \in \Omega$ and $t \ge 0$. It follows easily that the continuous map $t \mapsto k(\omega{\cdot}t)$ is decreasing for any $\omega \in \Omega$. Now we fix any $\omega \in \Omega$ and choose ω_0 in the common set of continuity points of $\alpha_{\mathcal{M}}$ and $\beta_{\mathcal{M}}$, so that $\alpha_{\mathcal{M}}(\omega_0) = \beta_{\mathcal{M}}(\omega_0)$. We look for $(t_n) \downarrow -\infty$ such that

$\omega_0 = \lim_{n\to\infty} \omega{\cdot}t_n$. Then, $\lim_{n\to\infty}(\beta_{\mathcal{M}}(\omega{\cdot}t_n) - \alpha_{\mathcal{M}}(\omega{\cdot}t_n)) = \beta_{\mathcal{M}}(\omega_0) - \alpha_{\mathcal{M}}(\omega_0) = 0$, which since H_a is bounded from below ensures that $\lim_{n\to\infty} k(\omega{\cdot}t_n) = 0$. Consequently, $k(\omega) = 0$, which shows that $\alpha_{\mathcal{M}}(\omega) = \beta_{\mathcal{M}}(\omega)$. The proof is complete. □

In the rest of the results we do not assume the existence of a continuous primitive for a. On the contrary, we will see in Theorem 3.10 that this property is not a hypothesis but a consequence of the first one of the dynamical possibilities for the dynamics described in the Introduction. And the existence of continuous primitive will be precluded in the analysis of the possible occurrence of Li-Yorke chaos and Auslander-Yorke chaos in the second dynamical possibility (in Theorems 3.14 and 3.15): one of our hypotheses there will be precisely the absence of continuous primitive of a.

The next result establishes general properties of the minimal sets. As in the previous one, condition **g3** is not assumed, since the description of the attractor is postponed. In particular, we check that a τ-minimal set which is not a copy of the base, if it exists, is contained in $\Omega \times [r_1, r_2]$ (and hence requires $r_1 < r_2$: such a minimal set cannot exist in the purely dissipative case if $\sup \Sigma_a = 0$).

Theorem 3.8. *Assume that g satisfies* **g1**, **g2** *and* **$\tilde{\text{g}}$4**, *and let $(\Omega \times \mathbb{R}, \tau)$ be the flow induced by the family (3.1). Assume also that $\sup \Sigma_a = 0$, let $\mathcal{M}$ be a τ-minimal set, let $\alpha_{\mathcal{M}}, \beta_{\mathcal{M}} : \Omega \to \mathbb{R}$ be the semicontinuous τ-equilibria associated to $\mathcal{M}$ by (2.6), and let $\Omega_{\mathcal{M}}$ be the set of their common continuity points. Then,*

(i) *there exists $\omega \in \Omega_{\mathcal{M}}$ such that $\alpha_{\mathcal{M}}(\omega) < r_1$ if and only if there exists $(\omega, x) \in \mathcal{M}$ with $x < r_1$. In this case, $\mathcal{M}$ is a uniformly exponentially stable at $+\infty$ copy of the base: $\mathcal{M} = \{\alpha_{\mathcal{M}}\} = \{\beta_{\mathcal{M}}\}$.*
(ii) *There exists $\omega \in \Omega_{\mathcal{M}}$ such that $\beta_{\mathcal{M}}(\omega) > r_2$ if and only if there exists $(\omega, x) \in \mathcal{M}$ with $x > r_2$. In this case, $\mathcal{M}$ is a uniformly exponentially stable at $+\infty$ copy of the base: $\mathcal{M} = \{\alpha_{\mathcal{M}}\} = \{\beta_{\mathcal{M}}\}$.*

Consequently, if $\mathcal{M}$ is not a copy of the base, then $r_1 < r_2$ and $\mathcal{M} \subset \Omega \times [r_1, r_2]$. In addition,

(iii) *$\mathcal{M} \subseteq \Omega \times [r_1, r_2]$ if its upper Lyapunov exponent is 0.*
(iv) *If $\mathcal{M} \subseteq \Omega \times [r_1, r_2]$ and either $r_1 < r_2$ or $r_1 = r_2$ and $g_x(\omega, r_1) = 0$ for all $\omega \in \Omega$, then the upper Lyapunov exponent of $\mathcal{M}$ is 0.*

Proof. We have seen in the proof of Theorem 3.6 that conditions **g1**, **g2** and **g4** guarantee (3.4), which in turn ensures that the upper Lyapunov exponent of any τ-minimal set $\mathcal{M}$ is $\gamma^s_{\mathcal{M}} \le 0$ if $\sup \Sigma_a = 0$, as we assume in this subsection. This fact will be used in what follows.

(i) Recall that $\mathcal{M} = \text{closure}_{\Omega\times\mathbb{R}}\{(\omega{\cdot}t, \alpha_{\mathcal{M}}(\omega{\cdot}t)) \mid t \in \mathbb{R}\}$, where ω is any point in $\Omega_{\mathcal{M}}$: see Subsection 2.4. The first assertion in (i) follows easily from here. Now we will prove that $\mathcal{M}$ has negative upper Lyapunov exponent $\gamma^s_{\mathcal{M}}$. Recall that the upper Lyapunov exponent is given by (3.3) for a suitable τ-invariant measure $\nu^s_{\mathcal{M}}$, whose support is, due to minimality, the whole of $\mathcal{M}$. Let us take $(\omega_0, x_0) \in \mathcal{M}^\alpha$ with $x_0 < r_1$. Property **$\tilde{\text{g}}$4** ensures that $g_x(\omega_0, x_0) = -\rho < 0$, so that **g1** ensures the existence of an open set $\mathcal{B}$ of $\Omega \times \mathbb{R}$ with $\mathcal{B} \cap \mathcal{M}$ non empty and on which g_x is less that $-\rho/2$. Since $\mathcal{B}$ is open and $\text{Supp}\, \nu^s_{\mathcal{M}} = \mathcal{M}$, we have $\nu^s_{\mathcal{M}}(\mathcal{B} \cap \mathcal{M}^\alpha) > 0$. Using this fact and the property $g_x \le 0$ everywhere, we obtain $\int_{\mathcal{M}} g_x(\omega, x)\, d\nu^s_{\mathcal{M}} \le \int_{\mathcal{B}\cap\mathcal{M}} g_x(\omega, x)\, d\nu^s_{\mathcal{M}} \le (-\rho/2)\, \nu^s_{\mathcal{M}}(\mathcal{B} \cap \mathcal{M}^\alpha) < 0$, and hence $\gamma^s_{\mathcal{M}} < \int_\Omega a(\omega)\, dm_{\mathcal{M}}$, where $m_{\mathcal{M}} \in \mathfrak{M}_{\text{inv}}(\Omega, \sigma)$ is the measure onto which $\nu^s_{\mathcal{M}}$ projects. Definition 2.5 and Theorem 2.6 show that $\sup \Sigma_a = 0$ yields $\int_\Omega a(\omega)\, dm_{\mathcal{M}} \le 0$, and hence $\gamma^s_{\mathcal{M}} < 0$. Corollary 2.10 shows that $\mathcal{M}$ is a uniformly exponentially stable at $+\infty$ copy of the base, which in turn ensures that $\alpha_{\mathcal{M}}$ is continuous and equal to $\beta_{\mathcal{M}}$, and that its graph is $\mathcal{M}$.

(ii) The proof of this point is analogous, and the consequence of (i) and (ii) is clear.

(iii)&(iv) Properties (i) and (ii) prove point (iii). To prove (iv), we take a τ-minimal set $\mathcal{M} \subseteq \Omega \times [r_1, r_2]$. Theorem 2.6 ensures the existence of $m^s \in \mathfrak{M}_{\text{erg}}(\Omega, \sigma)$ such that $\int_\Omega a(\omega)\, dm^s = 0$. Let us define ν^s from m^s by $\int_{\Omega\times\mathbb{R}} f(\omega, x)\, d\nu^s := \int_\Omega f(\omega, \alpha_{\mathcal{M}}(\omega))\, dm^s$ for $f : \Omega \times \mathbb{R} \to \mathbb{R}$ continuous. It is easy to check that ν^s is τ-invariant with $\nu^s(\mathcal{M}) = 1$ (i.e., $\nu^s \in \mathfrak{M}_{\text{inv}}(\mathcal{M}, \tau)$), and that it projects onto m^s. Since, under the conditions in (iv) (see **g2**), $g_x \equiv 0$ on $\Omega \times [r_1, r_2]$, we have

$$\int_{\mathcal{M}} (a(\omega) + g_x(\omega, x))\, d\nu^s = \int_\Omega a(\omega)\, dm^s = 0\,,$$

and, since $\gamma^s_{\mathcal{M}} \le 0$ for any τ-minimal set $\mathcal{M}$, we deduce that $\gamma^s_{\mathcal{M}} = 0$. □

The next result plays a fundamental role in the analysis of the occurrence of Li-Yorke chaos in the second dynamical situation, carried-on in Theorem 3.14. It establishes conditions under which the attractor is m-almost contained in $\Omega \times [r_1, r_2]$ (i.e., $\mathcal{A}_\omega \subseteq [r_1, r_2]$ for m-almost every $\omega \in \Omega$), where $m \in \mathfrak{M}_{\text{erg}}(\Omega, \sigma)$ satisfies $\int_\Omega a(\omega)\, dm = 0$. Recall once again that Theorem 2.6 guarantees the existence of such a measure when $\sup \Sigma_a = 0$. Now, for the sake of generality, we simply assume that $0 \in \Sigma_a$ and that m exists. The result is valid for the linear dissipative and purely dissipative cases.

Theorem 3.9. *Assume that g satisfies* **g1**, **g2** *and* **g3**, *let $(\Omega \times \mathbb{R}, \tau)$ be the flow induced by the family (3.1), and let $\mathcal{A}$ be the global attractor for τ provided by Theorem 3.2. Assume also that $0 \in \Sigma_a$ and that $a \in \mathcal{R}_m(\Omega)$, where $m \in \mathfrak{M}_{\text{erg}}(\Omega, \sigma)$ satisfies $\int_\Omega a(\omega)\, dm = 0$. And assume finally that there exists a minimal $\mathcal{M} \subseteq \Omega \times [r_1, r_2]$. Then, the σ-invariant set*

$$\Omega_l := \{\omega \in \Omega^a \mid r_1 \le \alpha_{\mathcal{A}}(\omega{\cdot}t) \le \beta_{\mathcal{A}}(\omega{\cdot}t) \le r_2 \text{ for all } t \in \mathbb{R}\} \tag{3.5}$$

satisfies $m(\Omega_l) = 1$.

Proof. The ideas are taken from [6, Theorem 35] and [7, Theorem 5.8]. Note that it is enough to check that the two σ-invariant sets

$$\Omega_\alpha := \{\omega \in \Omega \mid \text{there exists } t \in \mathbb{R} \text{ such that } \alpha_{\mathcal{A}}(\omega{\cdot}t) < r_1\}\,,$$

$$\Omega_\beta := \{\omega \in \Omega \mid \text{there exists } t \in \mathbb{R} \text{ such that } \beta_{\mathcal{A}}(\omega{\cdot}t) > r_2\}$$

have null measure. We will reason with Ω_β, being the argument similar in the case of Ω_α. Let us assume for contradiction that $m(\Omega_\beta) > 0$. This provides $s > 0$ such that $\Omega_{\beta,s} := \{\omega \in \Omega \mid \text{there exists } t \in \mathbb{R} \text{ with } \beta_{\mathcal{A}}(\omega{\cdot}t) > r_2 + s\} \subseteq \Omega_\beta$ has positive measure. We call $\Omega^+_{\beta,s} := \{\omega \in \Omega \mid \text{there exists } t > 0 \text{ with } \beta_{\mathcal{A}}(\omega{\cdot}t) > r_2 + s\} \subseteq \Omega_\beta$.

We use Lusin's theorem to find a compact set $\mathcal{K} \subset \Omega_{\beta,s}$ with positive measure such that the restrictions of $\beta_{\mathcal{A}}$ and $\alpha_{\mathcal{A}}$ to $\mathcal{K}$ are continuous. Note that $\alpha_{\mathcal{A}}(\omega) \ne \beta_{\mathcal{A}}(\omega)$ whenever $\omega \in \mathcal{K}$, since the hypothesis $\mathcal{M} \subseteq \Omega \times [r_1, r_2]$ and the definition of Ω_β provide, for any $\omega \in \Omega_\beta$, a time $t \in \mathbb{R}$ such that $\alpha_{\mathcal{A}}(\omega{\cdot}t) \le r_2 < \beta_{\mathcal{A}}(\omega{\cdot}t)$. We will use this property later. Birkhoff's ergodic theorem ensures that for m-a.e. $\omega \in \Omega$ there exists $(t_n) \uparrow \infty$ such that $\omega{\cdot}t_n \in \mathcal{K}$, and the regularity of the measure provides a new compact set $\mathcal{C}$ with positive measure with the previous property. Our next goal is proving that $\mathcal{C} \subset \Omega^+_{\beta,s}$. First we check the existence of $\widetilde{t} > 0$ such that for any $\omega \in \mathcal{K}$ there exists $t \in [-\widetilde{t}, \widetilde{t}]$ with $\beta_{\mathcal{A}}(\omega{\cdot}t) > r_2 + s$. This follows easily from the equality $\beta_{\mathcal{A}}(\omega{\cdot}t) = x(t, \omega, \beta_{\mathcal{A}}(\omega))$, the continuity of $\beta_{\mathcal{A}}|_{\mathcal{K}}$ and the compactness of $\mathcal{K}$. Now we take $\omega \in \mathcal{C}$, look for $t_n > \widetilde{t}$ such that $\omega{\cdot}t_n \in \mathcal{K}$, and look for $t \in [-\widetilde{t}, \widetilde{t}]$ such that $\beta_{\mathcal{A}}((\omega{\cdot}t_n){\cdot}t) > r_2 + s$. Since $(\omega{\cdot}t_n){\cdot}t = \omega{\cdot}(t_n + t)$ and $t_n + t > 0$, we conclude that $\omega \in \Omega^+_{\beta,s}$, as asserted.

Let us fix $\omega \in \mathcal{C}$ and $(t_n) \uparrow \infty$ such that $\omega{\cdot}t_n \in \mathcal{K}$ for all $n \in \mathbb{N}$, and such that there exists $\widetilde{\omega} := \lim_{n\to\infty} \omega{\cdot}t_n$ (so that $\widetilde{\omega} \in \mathcal{K}$). We will check that $\lim_{n\to\infty} \exp\big(\int_0^{t_n} a(\omega{\cdot}s)\, ds\big) = \infty$, or, equivalently, that $\lim_{n\to\infty} x_l(t_n, \omega, \beta_{\mathcal{A}}(\omega) - \alpha_{\mathcal{A}}(\omega)) = \infty$. Before that, observe that this fact contradicts Proposition 2.15(2), since $m(\mathcal{C}) > 0$, and hence it completes the proof.

As established in Remark 3.1.2, the fact that $M \subseteq \Omega \times [r_1, r_2]$ ensures that any $c > 0$ determines the lower solution $t \mapsto c\,(\beta_{\mathcal{A}}(\omega{\cdot}t) - \alpha_{\mathcal{A}}(\omega{\cdot}t))$ for the linear equation $z' = a(\omega{\cdot}t)\, z$, and hence that $x_l(t_n, \omega, \beta_{\mathcal{A}}(\omega) - \alpha_{\mathcal{A}}(\omega)) \ge \beta_{\mathcal{A}}(\omega{\cdot}t_n) - \alpha_{\mathcal{A}}(\omega{\cdot}t_n) > \inf_{\omega \in \mathcal{K}} (\beta_{\mathcal{A}}(\omega) - \alpha_{\mathcal{A}}(\omega)) > 0$. (This is the point in which we use $\alpha_{\mathcal{A}}(\omega) < \beta_{\mathcal{A}}(\omega)$ for $\omega \in \mathcal{K}$.) Let us assume for contradiction that, for a suitable subsequence (t_k), we have $\lim_{k\to\infty} x_l(t_k, \omega, \beta_{\mathcal{A}}(\omega) - \alpha_{\mathcal{A}}(\omega)) = c_0\,(\beta_{\mathcal{A}}(\widetilde{\omega}) - \alpha_{\mathcal{A}}(\widetilde{\omega}))$, finite, and hence $c_0 > 0$. We take $t_{\widetilde{\omega}}$ such that $\beta_{\mathcal{A}}(\widetilde{\omega}{\cdot}t_{\widetilde{\omega}}) > r_2 + s$, so that $(d/dt)(\beta_{\mathcal{A}}(\widetilde{\omega}{\cdot}t_{\widetilde{\omega}}) - \alpha_{\mathcal{A}}(\widetilde{\omega}{\cdot}t_{\widetilde{\omega}})) < a(\widetilde{\omega}{\cdot}t_{\widetilde{\omega}})(\beta_{\mathcal{A}}(\widetilde{\omega}{\cdot}t_{\widetilde{\omega}}) - \alpha_{\mathcal{A}}(\widetilde{\omega}{\cdot}t_{\widetilde{\omega}}))$. This ensures the existence of $\varepsilon > 0$ and $t_* > t_{\widetilde{\omega}}$ such that $(c_0 + \varepsilon)(\beta_{\mathcal{A}}(\widetilde{\omega}{\cdot}t_*) - \alpha_{\mathcal{A}}(\widetilde{\omega}{\cdot}t_*)) < x_l(t_*, \widetilde{\omega}, c_0\,(\beta_{\mathcal{A}}(\widetilde{\omega}) - \alpha_{\mathcal{A}}(\widetilde{\omega})))$. In turn, the last inequality and the definition of c_0 provide a point t_{k_0} of the sequence with $(c_0 + \varepsilon)(\beta_{\mathcal{A}}(\omega{\cdot}(t_{k_0} + t_*)) - \alpha_{\mathcal{A}}(\omega{\cdot}(t_{k_0} + t_*))) < x_l(t_*, \omega{\cdot}t_{k_0}, x_l(t_{k_0}, \omega, \beta_{\mathcal{A}}(\omega) - \alpha_{\mathcal{A}}(\omega))) = x_l(t_* + t_{k_0}, \omega, \beta_{\mathcal{A}}(\omega) - \alpha_{\mathcal{A}}(\omega))$. Now, we write $t_k = t_* + t_{k_0} + s_k$ with $s_k > 0$ for large enough k. Then,

$$
\begin{aligned}
&x_l(t_k, \omega, \beta_{\mathcal{A}}(\omega) - \alpha_{\mathcal{A}}(\omega)) \\
&\qquad = x_l(s_k, \omega{\cdot}(t_* + t_{k_0}), x_l(t_* + t_{k_0}, \omega, \beta_{\mathcal{A}}(\omega) - \alpha_{\mathcal{A}}(\omega))) \\
&\qquad > x_l(s_k, \omega{\cdot}(t_* + t_{k_0}), (c_0 + \varepsilon)(\beta_{\mathcal{A}}(\omega{\cdot}(t_{k_0} + t_*)) - \alpha_{\mathcal{A}}(\omega{\cdot}(t_{k_0} + t_*)))) \\
&\qquad \ge (c_0 + \varepsilon)(\beta_{\mathcal{A}}(\omega{\cdot}t_k) - \alpha_{\mathcal{A}}(\omega{\cdot}t_k))\,.
\end{aligned}
$$

We have used again Remark 3.1.2 for the last inequality. Taking limits as $k \to \infty$, we get $c_0 \ge c_0 + \varepsilon$. This is the sought-for contradiction. The proof is complete. □

Let us finally describe the two dynamical possibilities in the case $\sup \Sigma_a = 0$, as well as the cases in which we can ensure the occurrence of Li-Yorke chaos and Auslander-Yorke chaos. The first possibility, now analyzed, occurs if and only if the maps $\alpha_{\mathcal{A}}$ and $\beta_{\mathcal{A}}$ of Theorem 3.2 coincide at no point of Ω. The second one, which occurs when $\alpha_{\mathcal{A}}$ and $\beta_{\mathcal{A}}$ coincide at (at least) one point of Ω, is studied in Theorem 3.11. And the situations in which we are able to detect Li-Yorke chaos and Auslander-Yorke chaos are described in Theorems 3.14 and 3.15, which fit in the second dynamical possibility.

Theorem 3.10. *Assume that g satisfies* **g1**, **g2**, **g3** *and* $\tilde{\mathbf{g}}\mathbf{4}$, *let $(\Omega \times \mathbb{R}, \tau)$ be the flow induced by the family (3.1), let $\mathcal{A}$, $\alpha_{\mathcal{A}}$ and $\beta_{\mathcal{A}}$ be provided by Theorem 3.2, and let Ω_c, $\mathcal{M}^\alpha$ and $\mathcal{M}^\beta$ be defined in Theorem 3.3. Assume also that $\sup \Sigma_a = 0$, and that there exists $\omega_0 \in \Omega_c$ such that $\alpha_{\mathcal{A}}(\omega_0) < \beta_{\mathcal{A}}(\omega_0)$. Then, $\mathcal{M}^\alpha < \mathcal{M}^\beta$. In addition,*

(i) $r_1 < r_2$*: we are necessarily in the linear dissipative case.*
(ii) *The map a has a continuous primitive.*
(iii) $\Omega = \Omega_c$, $\mathcal{M}^\alpha = \{\alpha_\mathcal{A}\}$ *and* $\mathcal{M}^\beta = \{\beta_\mathcal{A}\}$.
(iv) *Any* τ*-minimal set is the graph* $\mathcal{M}^c$ *of the continuous map* $c\alpha_\mathcal{A} + (1-c)\beta_\mathcal{A}$ *for a* $c \in [0, 1]$*, and has zero upper Lyapunov exponent.*
(v) $\mathcal{A} = \bigcup_{c\in[0,1]} \mathcal{M}^c \subseteq \Omega \times [r_1, r_2]$*, and hence the restriction of* τ *to* $\mathcal{A}$ *is linear and uniformly stable at* $\pm\infty$.

Proof. The definitions of $\mathcal{M}^\alpha$ and $\mathcal{M}^\beta$ ensure that they are different, so that they are fiber-ordered (see Subsection 2.4): $\mathcal{M}^\alpha < \mathcal{M}^\beta$. The main step of this proof is showing that both of them are contained in $\Omega \times [r_1, r_2]$. Let us assume for the moment being that this is the case, and let us see how to deduce all the assertions of the theorem.

The existence of two different minimal sets contained in $\Omega \times [r_1, r_2]$ yields $r_1 < r_2$, which is property (i). Let us take $\omega \in \Omega$, $(\omega, x_\alpha) \in \mathcal{M}^\alpha$ and $(\omega, x_\beta) \in \mathcal{M}^\beta$. Then the map $t \mapsto x(t, \omega, x_\beta) - x(t, \omega, x_\alpha)$ solves $x' = a(\omega{\cdot}t)\, x$, and it is bounded and also positively bounded from below. This implies that all the solutions of $x' = a(\omega{\cdot}t)\, x$ are bounded, for every $\omega \in \Omega$, and hence a has a continuous primitive: see Remark 2.12. This proves (ii). Theorem 3.7(i) shows that $\mathcal{M}^\alpha$ and $\mathcal{M}^\beta$ are copies of the base: the graphs of $\alpha_\mathcal{A}$ and $\beta_\mathcal{A}$, respectively. Therefore, (iii) holds. Now, let us take $c \in [0, 1]$. It is easy to check that $t \mapsto c\, \alpha_\mathcal{A}(\omega{\cdot}t) + (1 - c)\, \beta_\mathcal{A}(\omega{\cdot}t) = \beta_\mathcal{A}(\omega{\cdot}t) + c\, (\alpha_\mathcal{A}(\omega{\cdot}t) - \beta_\mathcal{A}(\omega{\cdot}t))$ satisfies $x' = a(\omega{\cdot}t)\, x + b(\omega{\cdot}t)$. Since its graph remains in $\Omega \times [r_1, r_2]$, where g vanishes, we conclude that $x(t, \omega, c\, \alpha_\mathcal{A}(\omega) + (1-c)\, \beta_\mathcal{A}(\omega)) = c\alpha_\mathcal{A}(\omega{\cdot}t) + (1 - c)\beta_\mathcal{A}(\omega{\cdot}t)$. That is, the graph of $c\, \alpha_\mathcal{A} + (1 - c)\, \beta_\mathcal{A}$ is τ-invariant, and hence it determines a copy of the base: a τ-minimal set $\mathcal{M}^c$. And there are no more τ-minimal sets, as Theorem 3.3 implies: any other one should be below $\mathcal{M}^\alpha$ or above $\mathcal{M}^\beta$, impossible. Theorem 3.8(iii) shows that the upper Lyapunov exponent of $\mathcal{M}^c$ is 0, which completes the proof of (iv). Finally, the decomposition of $\mathcal{A}$ stated in (v) is an easy consequence of (iv) and the definition of $\mathcal{M}^\alpha$ and $\mathcal{M}^\beta$; the linearity follows from $\mathcal{A} \subset \Omega \times [r_1, r_2]$; and the uniform stability at $\pm\infty$ of the set $\mathcal{A}$ for the flow $(\mathcal{A}, \tau)$ follows from the linearity.

So that the proof will be complete once we show that $\mathcal{M}^\alpha,\ \mathcal{M}^\beta \subset \Omega \times [r_1, r_2]$. We work with $\mathcal{M}^\beta$, assuming for contradiction that this is not the case. It follows from Theorem 3.8(i)&(ii) that $\mathcal{M}^\beta$ is a copy of the base; i.e., $\mathcal{M}^\beta = \{\beta_\mathcal{A}\}$. Then, there exists at least a τ-minimal set $\mathcal{M}$ contained in $\Omega \times [r_1, r_2]$: if not, and according to Theorem 3.8(i)&(ii), any τ-minimal set has strictly negative upper Lyapunov exponent; and hence Theorem 3.4 ensures that there exists only one τ-minimal set, which is not the case. Theorem 3.3(i) ensures that $\mathcal{M}^\alpha \le \mathcal{M} \le \mathcal{M}^\beta$, so that $\beta_\mathcal{A} \ge r_1$. Therefore, since $\mathcal{M}^\beta \not\subset \Omega \times [r_1, r_2]$, $\mathcal{M} < \mathcal{M}^\beta$ and there exists $\bar\omega \in \Omega$ with $\beta_\mathcal{A}(\bar\omega) > r_2$. We will make use of these facts a few lines below.

Now we will check that $\sup_{t\ge 0} x_l(t, \omega, 1) = \infty$ for any $\omega \in \Omega$. A similar argument has been used in the proof of Theorem 3.9. Let $\alpha_\mathcal{M}$ be the map appearing in the description (2.6) of the minimal set $\mathcal{M} \subset \Omega \times [r_1, r_2]$, and note that $\alpha_\mathcal{M} < \beta_\mathcal{A}$. We take a point $\widetilde\omega \in \Omega$ of continuity of $\alpha_\mathcal{M}$ (and, of course, of $\beta_\mathcal{A}$), and look for $(t_n) \uparrow \infty$ such that $\lim_{n\to\infty} \omega{\cdot}t_n = \widetilde\omega$. Remark 3.1.3 ensures that any $c > 0$ determines the lower solution $t \mapsto c\, (\beta_\mathcal{A}(\omega{\cdot}t) - \alpha_\mathcal{M}(\omega{\cdot}t))$ of the linear equation $z' = a(\omega{\cdot}t)\, z$, and hence that $x_l(t_n, \omega, \beta_\mathcal{A}(\omega) - \alpha_\mathcal{M}(\omega)) \ge \beta_\mathcal{A}(\omega{\cdot}t_n) - \alpha_\mathcal{M}(\omega{\cdot}t_n) > \inf_{\omega\in\Omega}(\beta_\mathcal{A}(\omega) - \alpha_\mathcal{M}(\omega)) > 0$. Let us assume (for contradiction) that $\lim_{k\to\infty} x_l(t_k, \omega, \beta_\mathcal{A}(\omega) - \alpha_\mathcal{M}(\omega)) = c_0\, (\beta_\mathcal{A}(\widetilde\omega) - \alpha_\mathcal{M}(\widetilde\omega)) < \infty$ for certain subsequence (t_k), so that $c_0 > 0$. Since $\beta_\mathcal{A}$ is continuous, $\beta_\mathcal{A}(\bar\omega) > r_2$, and (Ω, σ) is minimal, we can find $t_0 > 0$ such that $\beta_\mathcal{A}(\widetilde\omega{\cdot}t_0) > r_2$. This property ensures that $(d/dt)(\beta_\mathcal{A}(\widetilde\omega{\cdot}t_0) - \alpha_\mathcal{M}(\widetilde\omega{\cdot}t_0)) < a(\widetilde\omega{\cdot}t_0)(\beta_\mathcal{A}(\widetilde\omega{\cdot}t_0) - \alpha_\mathcal{M}(\widetilde\omega{\cdot}t_0))$. The continuity of both maps through the σ-orbit of $\widetilde\omega$ ensures that $(d/dt)(\beta_\mathcal{A}(\widetilde\omega{\cdot}t) - \alpha_\mathcal{M}(\widetilde\omega{\cdot}t)) <$

$a(\widetilde{\omega}{\cdot}t_0)(\beta_{\mathcal{A}}(\widetilde{\omega}{\cdot}t)-\alpha_{\mathcal{M}}(\widetilde{\omega}{\cdot}t))$ if t is close enough to t_0. Therefore, there exists $t_*>t_0$ such that $c_0(\beta_{\mathcal{A}}(\widetilde{\omega}{\cdot}t_*)-\alpha_{\mathcal{M}}(\widetilde{\omega}{\cdot}t_*))<x_l(t_*-t_0,\widetilde{\omega}{\cdot}t_0,c_0\,(\beta_{\mathcal{A}}(\widetilde{\omega}{\cdot}t_0)-\alpha_{\mathcal{M}}(\widetilde{\omega}{\cdot}t_0)))$. We take $\varepsilon>0$ with

$$\begin{aligned}(c_0+\varepsilon)&(\beta_{\mathcal{A}}(\widetilde{\omega}{\cdot}t_*)-\alpha_{\mathcal{M}}(\widetilde{\omega}{\cdot}t_*))\\&<x_l(t_*-t_0,\widetilde{\omega}{\cdot}t_0,c_0\,(\beta_{\mathcal{A}}(\widetilde{\omega}{\cdot}t_0)-\alpha_{\mathcal{M}}(\widetilde{\omega}{\cdot}t_0)))\\&\le x_l(t_*-t_0,\widetilde{\omega}{\cdot}t_0,x_l(t_0,\widetilde{\omega},c_0\,(\beta_{\mathcal{A}}(\widetilde{\omega})-\alpha_{\mathcal{M}}(\widetilde{\omega}))))\\&=x_l(t_*,\widetilde{\omega},c_0\,(\beta_{\mathcal{A}}(\widetilde{\omega})-\alpha_{\mathcal{M}}(\widetilde{\omega})))\,.\end{aligned}$$

The second inequality follows again from Remark 3.1.3. This strict inequality combined with $\widetilde{\omega}=\lim_{t\to\infty}\omega{\cdot}t_n$ and with the definition of c_0 allows us to take a point t_{k_0} of the sequence with $(c_0+\varepsilon)(\beta_{\mathcal{A}}(\omega{\cdot}(t_{k_0}+t_*))-\alpha_{\mathcal{M}}(\omega{\cdot}(t_{k_0}+t_*)))<x_l(t_*,\omega{\cdot}t_{k_0},x_l(t_{k_0},\omega,\beta_{\mathcal{A}}(\omega)-\alpha_{\mathcal{M}}(\omega)))=x_l(t_*+t_{k_0},\omega,\beta_{\mathcal{A}}(\omega)-\alpha_{\mathcal{M}}(\omega))$. Now, we write $t_k=t_*+t_{k_0}+s_k$ with $s_k>0$ for large enough k. Then,

$$\begin{aligned}x_l(t_k,&\,\omega,\beta_{\mathcal{A}}(\omega)-\alpha_{\mathcal{M}}(\omega))\\&=x_l(s_k,\omega{\cdot}(t_*+t_{k_0}),x_l(t_*+t_{k_0},\omega,\beta_{\mathcal{A}}(\omega)-\alpha_{\mathcal{M}}(\omega)))\\&>x_l(s_k,\omega{\cdot}(t_*+t_{k_0}),(c_0+\varepsilon)(\beta_{\mathcal{A}}(\omega{\cdot}(t_{k_0}+t_*))-\alpha_{\mathcal{M}}(\omega{\cdot}(t_{k_0}+t_*))))\\&\ge(c_0+\varepsilon)(\beta_{\mathcal{A}}(\omega{\cdot}t_k)-\alpha_{\mathcal{M}}(\omega{\cdot}t_k))\,.\end{aligned}$$

We have used once more Remark 3.1.3 in the last inequality. Recall that $\beta_{\mathcal{A}}-\alpha_{\mathcal{M}}$ is continuous at $\widetilde{\omega}$. Taking limits as $k\to\infty$ we get $c_0\ge c_0+\varepsilon$, impossible. The conclusion is that $\lim_{n\to\infty}x_l(t_n,\omega,\beta_{\mathcal{A}}(\omega)-\alpha_{\mathcal{M}}(\omega))=\infty$, and hence that $\lim_{n\to\infty}x_l(t_n,\omega,1)=\infty$, as asserted.

The previous contradiction shows that $\sup_{t\ge0}x_l(t,\omega,1)=\infty$ for all $\omega\in\Omega$. However, since $0\in\Sigma_a$, there exists at least a point $\widetilde{\omega}\in\Omega$ such that $\sup_{t\in\mathbb{R}}x_l(t,\widetilde{\omega},1)<\infty$: see Remarks 2.7 and 2.3.2. This new contradiction shows that our initial assumption cannot hold. That is, $\mathcal{M}^\beta$ is contained in $\Omega\times[r_1,r_2]$. An analogous argument shows that also $\mathcal{M}^\alpha$ is contained in $\Omega\times[r_1,r_2]$. □

The definitions of set of complete measure and of chain recurrent flow, appearing in the next statement, are given in Subsection 2.1.

Theorem 3.11. *Assume that g satisfies* **g1**, **g2**, **g3** *and* $\tilde{\mathbf{g}}\mathbf{4}$, *let $(\Omega\times\mathbb{R},\tau)$ be the flow induced by the family (3.1), let $\mathcal{A}$, $\alpha_{\mathcal{A}}$ and $\beta_{\mathcal{A}}$ be provided by Theorem 3.2, and let Ω_c, $\mathcal{M}^\alpha$ and $\mathcal{M}^\beta$ be defined in Theorem 3.3. Assume also that $\sup\Sigma_a=0$, and that $\mathcal{A}$ is a pinched set. Then, $\Omega_c=\{\omega\in\Omega\,|\,\alpha_{\mathcal{A}}(\omega)=\beta_{\mathcal{A}}(\omega)\}$, and $\mathcal{M}:=\mathcal{M}^\alpha=\mathcal{M}^\beta$ is the unique τ-minimal set. In addition,*

- (i) *$\mathcal{M}\not\subset\Omega\times[r_1,r_2]$ if and only if there exists $\omega\in\Omega_c$ such that $\alpha_{\mathcal{A}}(\omega)<r_1$ or $\beta_{\mathcal{A}}(\omega)>r_2$. In this case, $\mathcal{A}=\mathcal{M}$ is a uniformly exponentially stable at $+\infty$ copy of the base: the graph of the continuous function $\alpha_{\mathcal{A}}=\beta_{\mathcal{A}}$.*
- (ii) *If $\mathcal{M}\subseteq\Omega\times[r_1,r_2]$, then its upper Lyapunov exponent is 0.*
- (iii) *If $\mathcal{M}\subseteq\Omega\times[r_1,r_2]$ and the map a has a continuous primitive, then $\mathcal{A}=\mathcal{M}=\{\alpha_{\mathcal{A}}\}=\{\beta_{\mathcal{A}}\}$. In addition, in this case, $\inf\{\alpha_{\mathcal{A}}(\omega)\,|\,\omega\in\Omega\}=r_1$ and $\sup\{\alpha_{\mathcal{A}}(\omega)\,|\,\omega\in\Omega\}=r_2$.*
- (iv) *If $r_1=r_2=:r$ and $\mathcal{M}=\{r\}$, then Ω_c has complete measure.*

(v) *If* $r_1 < r_2$*, and either* $\mathcal{M} \subset \Omega \times [r_1, r_2)$ *or* $\mathcal{M} \subset \Omega \times (r_1, r_2]$*, then the map a does not admit a continuous primitive,* $\mathcal{M} \subsetneq \mathcal{A}$*, and* $\Omega_c \subsetneq \Omega$*.*
(vi) *The restricted flow* $(\mathcal{A}, \tau)$ *is chain recurrent.*

Proof. The equality $\Omega_c = \{\omega \in \Omega \,|\, \alpha(\omega) = \beta(\omega)\}$ is proved in Theorem 3.3(ii). Hence, clearly, for all $\omega \in \Omega_c$, the points $(\omega, \alpha_{\mathcal{A}}(\omega)) = (\omega, \beta_{\mathcal{A}}(\omega))$ belong to any τ-minimal set, and this fact combined with the definition of $\mathcal{M}^\alpha$ (see Theorem 3.3) ensures that $\mathcal{M} := \mathcal{M}^\alpha$ contains any τ-minimal set. Hence, it is the unique one.

(i)&(ii) These assertions follow immediately from Theorem 3.8.

(iii) We repeat step by step the proof of Theorem 3.7, working with the map $\beta_{\mathcal{A}} - \alpha_{\mathcal{A}}$ instead of $\beta_{\mathcal{M}} - \alpha_{\mathcal{M}}$. The conclusion is that $\alpha_{\mathcal{A}}$ and $\beta_{\mathcal{A}}$ are equal, so that they are continuous and determine the copy of the base $\mathcal{A} = \mathcal{M}$. The last assertion in (iii) is trivial if $r_1 = r_2$ and follows from (v) (which will be proved independently) if $r_1 < r_2$.

(iv) This assertion follows from (iii) if a has a continuous primitive: in this case, $\alpha_{\mathcal{A}} = \beta_{\mathcal{A}} \equiv r$ and $\Omega_c = \Omega$. Assume that this is not the case. Then, the change of variables $y = x - r$ takes (3.1) to the purely dissipative family $y' = a(\omega{\cdot}t)\, y + g(\omega{\cdot}t, y + r)$ with linear homogeneous part, for which $\Omega \times \{0\}$ is the unique minimal set and the attractor is $\bigcup_{\omega\in\Omega}\{\omega\} \times [\alpha_{\mathcal{A}}(\omega) - r, \beta_{\mathcal{A}}(\omega) - r]$. Following the arguments of [30, Theorem 5.10], we prove that the set of points ω at which $\alpha_{\mathcal{A}}(\omega) - r = \beta(\omega) - r = 0$ has complete measure. In fact, [30] is devoted to scalar parabolic partial differential equations, but ours can be understood as one of that type; and also a symmetric condition is assumed there on g, but this condition does not imply differences in the arguments we refer to, which can be repeated for $\alpha_{\mathcal{A}} - r$ and for $\beta_{\mathcal{A}} - r$. Therefore, coming back to our initial family, we have that $\alpha_{\mathcal{A}}$ and $\beta_{\mathcal{A}}$ coincide (and take the value r) on a set of complete measure which, as seen at the beginning of the proof, is Ω_c.

(v) Since $0 \in \Sigma_a$, there exists $\bar\omega \in \Omega$ such that $\sup_{t\in\mathbb{R}} \int_0^t a(\bar\omega{\cdot}s)\, ds < \infty$: see Remarks 2.7 and 2.3.2. Theorem 3.5(ii) ensures that $\mathcal{M} \subsetneq \mathcal{A}$ and that $\alpha_{\mathcal{A}}(\bar\omega) < \beta_{\mathcal{A}}(\bar\omega)$, which ensures that $\Omega_c \subsetneq \Omega$. Theorem 3.5 also shows that $\Omega_c = \{\omega \in \Omega \,|\, \sup_{t\le 0} \int_0^t a(\omega{\cdot}s)\, ds = \infty\}$, which precludes the existence of continuous primitive for a (since Ω_c is nonempty). This completes the proof of (v).

(vi) Let us fix two points (ω, x) and $(\widetilde\omega, \widetilde x)$ in $\mathcal{A}$, $\varepsilon > 0$, $t_0 > 0$, and consider three cases which exhaust the possibilities.

If $(\widetilde\omega, \widetilde x) \in \mathcal{M}$, then it belongs to the omega limit set of $(\omega_1, x_1) := \tau(t_0, \omega, x)$, and hence there exists $t_1 > t_0$ such that $\mathrm{dist}_{\Omega\times\mathbb{R}}(\tau(t_1, \omega_1, x_1), (\widetilde\omega, \widetilde x)) < \varepsilon$. The definition of chain recurrence is fulfilled for the chain $(\omega_0, x_0) := (\omega, x)$, (ω_1, x_1) and $(\omega_2, x_2) := (\widetilde\omega, \widetilde x)$ (and the times t_0 and t_1).

Assume that $(\omega, x) \in \mathcal{M}$. We call $(\omega_1, x_1) := \tau(t_0, \omega, x) \in \mathcal{M}$, choose $t_1 > t_0$, and observe that $\tau(t_1, \omega_1, x_1)$ belongs to the alpha limit set of $(\widetilde\omega, \widetilde x)$, since it belongs to $\mathcal{M}$. We take $t_2 > t_0$ such that $\mathrm{dist}_{\Omega\times\mathbb{R}}(\tau(-t_2, \widetilde\omega, \widetilde x), \tau(t_1, \omega_1, x_1)) < \varepsilon$ and call $(\omega_2, x_2) := \tau(-t_2, \widetilde\omega, \widetilde x)$, so that $\tau(t_2, \omega_2, x_2) = (\widetilde\omega, \widetilde x)$. The definition of chain recurrence is fulfilled for the chain $(\omega_0, x_0) := (\omega, x)$, (ω_1, x_1), (ω_2, x_2) and $(\omega_3, x_3) := (\widetilde\omega, \widetilde x)$ (and the times t_0, t_1, and t_2).

Finally, if none of the points belongs to $\mathcal{M}$, we construct the chain from (ω, x) to $(\bar\omega, \bar x)$ through any point $(\bar\omega, \bar x) \in \mathcal{M}$. This completes the proofs of (vi) and of the theorem. □

Remark 3.12. In the purely dissipative case considered in point (iv), the set Ω_c can be Ω (and hence the attractor agrees with $\{r\}$). This is the simplest situation. But it is also possible that $\Omega_c \subsetneq \Omega$, in which case the dynamics is much more complex An example of this is given by the family obtained by the hull procedure (explained in the Introduction) from the equation $x' = (1/2)(a(t)\,x - x^3)$, where $a(t) = \widetilde{a}(-t)$ for an almost periodic function $\widetilde{a}\colon \mathbb{R} \to \mathbb{R}$ with zero mean value and whose integral $\int_0^t \widetilde{a}(s)\,ds$ grows like t^μ as t increases, for some $0 < \mu < 1$. The interested reader is referred to [7, Example 5.13] for the details, as well as for references in which functions $\widetilde{a}$ with the required properties are constructed.

Remark 3.13. By reviewing the proof of Theorem 3.11(vi), we observe that the property is general: any flow on a compact metric space admitting a unique minimal set is chain recurrent.

The framework of the next theorems, concerning the presence of chaos, is that of point (v) of the previous one. In particular, it requires the family (3.1) to be in the linear dissipative case (i.e., $r_1 < r_2$). The nonempty set $\mathcal{R}_m(\Omega)$ is described in Subsection 2.5: Theorem 2.14 ensures the existence of functions $a \in \mathcal{R}_m(\Omega)$ with $\Sigma_a = 0$ for any $m \in \mathfrak{M}_{\text{inv}}(\Omega, \sigma)$. Recall that when $\mathcal{A}$ is pinched, there exists just one τ-minimal set: see Theorem 3.11. The scope of the properties stated in these two results is analyzed after their proofs.

Theorem 3.14. *Assume that g satisfies* **g1**, **g2**, **g3** *and* **$\tilde{\mathbf{g}}$4**, *let $(\Omega \times \mathbb{R}, \tau)$ be the flow induced by the family (3.1), let $\mathcal{A}$, $\alpha_\mathcal{A}$ and $\beta_\mathcal{A}$ be provided by Theorem 3.2, and let $\Omega_c \subseteq \Omega$ be the residual set provided by Theorem 3.3. Assume also that $r_1 < r_2$ and that*

- *$\mathcal{A}$ is a pinched set, and the only τ-minimal set $\mathcal{M}$ is contained either in $\Omega \times [r_1, r_2)$ or in $\Omega \times (r_1, r_2]$;*
- *$\sup \Sigma_a = 0$ and $a \in \mathcal{R}_m(\Omega)$ for a measure $m \in \mathfrak{M}_{\text{erg}}(\Omega, \sigma)$.*

Then,

(i) *$m(\Omega_c) = 0$, and the restricted flow $(\mathcal{A}, \tau)$ is chain recurrent and Li-Yorke chaotic in measure with respect to m. More precisely, the σ-invariant set subset $\Omega_{LY} \subseteq \Omega$ of points ω such that the section $\mathcal{A}_\omega$ is a nondegenerate interval and the set $\{\omega\} \times \mathcal{A}_\omega$ is scrambled, satisfies $\Omega_{LY} \subseteq \Omega - \Omega_c$ and $m(\Omega_{LY}) = 1$.*

(ii) *For every $\varepsilon \in (0, 1)$ there exists a subset $\Omega_\varepsilon \subseteq \Omega_{LY}$ with $m(\Omega_\varepsilon) = 1$ such that, for any $\omega \in \Omega_\varepsilon$, the set*

$$\{t \geq 0 \,|\, |x(t,\omega,x_2) - x(t,\omega,x_1)| \leq \varepsilon\, |x_2 - x_1| \;\text{ if }\; (\omega,x_1), (\omega,x_2) \in \mathcal{A}\}$$

has positive lower density and is relatively dense in $\mathbb{R}^+$; and the set

$$\{t \geq 0 \,|\, |x(t,\omega,x_2) - x(t,\omega,x_1)| \geq (1-\varepsilon)\, |x_2 - x_1| \;\text{ if }\; (\omega,x_1), (\omega,x_2) \in \mathcal{A}\}$$

has positive lower density.

Proof. (i) The chain recurrence of $(\mathcal{A}, \tau)$ is guaranteed by Theorem 3.11(vi). Let us take $\omega \in \Omega - \Omega_c$ and a pair of points $(\omega, x_1), (\omega, x_2) \in \mathcal{A}$ with $x_1 \neq x_2$, choose $\omega_0 \in \Omega_c$, recall that $\mathcal{A}_{\omega_0} = \{x_0\}$, and choose a suitable sequence (t_n) such that $\lim_{n\to\infty} \omega{\cdot}t_n = \omega_0$ and there exist the two limits $x(t_n, \omega, x_1)$ and $x(t_n, \omega, x_2)$. These limits must coincide with x_0, so that any pair of points of $\mathcal{A}$ sharing the first component form a non positively distal pair.

To look for non positively asymptotic pairs requires some more work. Recall that, if $a \in \mathcal{R}_m(\Omega)$, then $\sup_{t\in\mathbb{R}} \exp\big(\int_0^t a(\widetilde{\omega}{\cdot}s)\,ds\big) < \infty$ whenever ω belongs to a σ-invariant set $\Omega^a \subset \Omega$ with $m(\Omega^a) = 1$: see Proposition 2.15. Theorem 3.5(ii) ensures that $\Omega^a \subseteq \Omega - \Omega_c$, and hence $m(\Omega_c) = 0$. Theorem 3.9 shows that also the σ-invariant set Ω_l defined by (3.5) satisfies $m(\Omega_l) = 1$. It is clear that $\Omega_l \subseteq \Omega^a$, since the restriction of the flow to $\Omega_l \times \mathbb{R}$ is linear. Let us take $(\omega, x_1), (\omega, x_2) \in \mathcal{A}$ with $\omega \in \Omega_l$. Then, $t \mapsto x(t,\omega,x_1) - x(t,\omega,x_2)$ solves $x' = a(\omega{\cdot}t)\,x$, and hence

$$\begin{aligned} |x(t,\omega,x_1) - x(t,\omega,x_2)| &= |x_l(t,\omega,x_1 - x_2)| \\ &= |x_1 - x_2| \exp \int_0^t a(\omega{\cdot}s)\,ds = |x_1 - x_2|\, \frac{H_a(\omega{\cdot}t)}{H_a(\omega)}\,, \end{aligned} \tag{3.6}$$

where $H_a\colon \Omega \to [0,1]$ is the bounded function associated to a by Proposition 2.15. Lusin's theorem and the regularity of m provide a compact set $\mathcal{K} \subset \Omega_l$ with positive measure such that the restriction of H_a to $\mathcal{K}$ is continuous, and Birkhoff's ergodic theorem provides a set $\Omega^0 \subseteq \Omega$ with $m(\Omega^0) = 1$ such that, if $\omega \in \Omega^0$, then there exists $(t_n) \uparrow \infty$ such that $\omega{\cdot}t_n \in \mathcal{K}$ for all $n \in \mathbb{N}$. In particular, $\Omega^0 \subseteq \Omega^l$, since Ω^l is σ-invariant. Since H_a is globally bounded and strictly positive at the points of $\Omega^a \supseteq \Omega_l$ (see again Proposition 2.15), and continuous on $\mathcal{K} \subset \Omega_l$, we conclude that there exists $\kappa_\omega > 0$ such that, whenever $\omega \in \Omega^0$ and x_1, x_2 are two different points of the nondegenerate interval $\mathcal{A}_\omega$,

$$|x(t_n,\omega,x_1) - x(t_n,\omega,x_2)| \geq \kappa_\omega\, |x_1 - x_2| > 0$$

for a sequence $(t_n) \uparrow \infty$. This shows that (ω, x_1) and (ω, x_2) form a non positively asymptotic pair.

Altogether, we have proved that the set $\mathcal{A}_\omega$ is a nondegenerate interval with $\{\omega\} \times \mathcal{A}_\omega$ scrambled for m-almost all $\omega \in \Omega$. The σ-invariance of the set $\Omega_{LY} \supseteq \Omega^0$ formed by these points is a clear consequence of the definition of scrambled set, and this completes the proof of (i).

(ii) Equality (3.6) and the definitions (2.10) of the sets $\mathcal{I}_\varepsilon(\omega)$ and $\mathcal{U}_\varepsilon(\omega)$ associated to the function a show that, if $\omega \in \Omega_{AY} \cap \Omega_l$, then

$$\begin{aligned} &t \in \mathcal{I}_\varepsilon(\omega),\ (\omega,x_1),\ (\omega,x_2) \in \mathcal{A} \Rightarrow |x(t,\omega,x_1) - x(t,\omega,x_2)| \leq \varepsilon\, |x_1 - x_2|\,, \\ &t \in \mathcal{D}_\varepsilon(\omega),\ (\omega,x_1),\ (\omega,x_2) \in \mathcal{A} \Rightarrow |x(t,\omega,x_1) - x(t,\omega,x_2)| \geq (1-\varepsilon)\, |x_1 - x_2|\,. \end{aligned}$$

Therefore, Theorem 2.16 proves (ii). □

Theorem 3.15. *Assume the same hypotheses as in Theorem 3.14, and let Ω_l be the σ-invariant set with $m(\Omega_l) = 1$ defined by (3.5). Let us define $\eta_c = c\,\alpha_{\mathcal{A}} + (1-c)\,\beta_{\mathcal{A}}$ for $c \in [0,1]$. Then,*

(i) $\displaystyle\int_{\mathcal{A}} h(\omega,x)\,d\mu_c := \int_\Omega h(\omega,\eta_c(\omega))\,dm$ *for $h \in C(\mathcal{A},\mathbb{R})$ defines a regular Borel τ-ergodic measure μ_c concentrated on $\mathcal{A}$.*

Let us define $\mathcal{S}_c := \operatorname{Supp} \mu_c$ for $c \in [0,1]$. Then,

(ii) *there exists a σ-invariant set $\Omega_{AY} \subseteq \Omega_l$ with $m(\Omega_{AY}) = 1$ such that $(\bar\omega, \eta_c(\bar\omega)) \in \mathcal{S}_c$ and*

$$\mathcal{S}_c = \mathcal{O}_\tau(\bar\omega, \eta_c(\bar\omega)) \tag{3.7}$$

for all $\bar\omega \in \Omega_{AY}$ and $c \in [0, 1]$. In particular, $(\mathcal{S}_c, \tau)$ is a transitive flow on a pinched compact set for any $c \in [0, 1]$.

(iii) *One of the following situations holds:*

(1) *$\mathcal{M}$ is a copy of the base, in which case there exists just a $c_0 \in [0, 1]$ such that $\mathcal{M} = \mathcal{S}_{c_0}$, and the restricted flow $(\mathcal{S}_c, \tau)$ is Auslander-Yorke chaotic for any $c \in [0, 1]$, $c \neq c_0$.*

(2) *$\mathcal{M}$ is not a copy of the base, in which case the restricted flow $(\mathcal{S}_c, \tau)$ is Auslander-Yorke chaotic for any $c \in [0, 1]$.*

(iv) *$\widetilde{\mathcal{S}} := \bigcup_{c\in[0,1]} \mathcal{S}_c$ is a compact τ-invariant subset of $\mathcal{A}$, all its points are sensitive, the restricted flow $(\widetilde{\mathcal{S}}, \tau)$ is chain recurrent, $\mathcal{A}_\omega = \widetilde{\mathcal{S}}_\omega$ for every $\omega \in \Omega_{AY} \cup \Omega_c$, $\widetilde{\mathcal{S}} := \operatorname{Supp}\widetilde\eta$ for the measure $\widetilde\eta \in \mathfrak{M}_{\rm inv}(\mathcal{A}, \tau)$ given by $\int_{\mathcal{A}} h(\omega, x)\, d\widetilde\mu := \int_0^1 \int_\Omega h(\omega, \eta_c(\omega))\, dm\, dc$ for $h \in C(\mathcal{A}, \mathbb{R})$, and there exists a dense τ-invariant subset $\widetilde{\mathcal{X}} \subseteq \widetilde{\mathcal{S}}$ of τ-generic points.*

Proof. (i) Theorem 3.9 allows us to assert that

$$\begin{aligned} x(t, \omega, \eta_c(\omega)) &= x(t, \omega, c\,\alpha_{\mathcal{A}}(\omega) + (1-c)\,\beta_{\mathcal{A}}(\omega)) \\ &= x_l(t, \omega, c\,\alpha_{\mathcal{A}}(\omega) + (1-c)\,\beta_{\mathcal{A}}(\omega)) \\ &= c\, x_l(t, \omega, \alpha_{\mathcal{A}}(\omega)) + (1-c)\, x_l(t, \omega, \beta_{\mathcal{A}}(\omega)) \\ &= c\,\alpha_{\mathcal{A}}(\omega{\cdot}t) + (1-c)\,\beta_{\mathcal{A}}(\omega{\cdot}t) = \eta_c(\omega{\cdot}t) \end{aligned}$$

for all $t \in \mathbb{R}$ and $\omega \in \Omega_l$. Therefore, η_c satisfies the conditions assumed in Theorem 2.26, whose point (ii) proves (i).

(ii) By reviewing the proof of Theorem 2.26(iii), we first check that we can take as starting point a compact set $\mathcal{K} \subset \Omega_l$ such that $\alpha_{\mathcal{A}}, \beta_{\mathcal{A}} : \mathcal{K} \to \mathbb{R}$ are continuous, so that also $\eta_c : \mathcal{K} \to \mathbb{R}$ is continuous for all $c \in [0, 1]$. Second, we observe that this property combined with the σ-invariance of Ω_l ensures that the set $\Omega_{AY} := \Omega_*$ constructed from $\mathcal{K}$ turns out to be common for all $c \in [0, 1]$, and is contained in Ω_l. Therefore, the assertions in (ii) follow from Theorem 2.26(iii).

(iii) Assume that $\mathcal{M}$ is a copy of the base. Let Ω_{LY} be provided by Theorem 3.14. We fix $\bar\omega \in \Omega_{LY} \cap \Omega_{AY}$ (which has measure 1, as (ii) and Theorem 3.14(i) ensure), and observe that $\alpha_{\mathcal{A}}(\bar\omega) < \beta_{\mathcal{A}}(\bar\omega)$. Hence, there exists a unique $c_0 \in [0, 1]$ such that $\mathcal{M}_{\bar\omega} = \{\eta_{c_0}(\bar\omega)\}$. Then, (3.7) ensures that $\mathcal{S}_{c_0} = \mathcal{O}_\tau(\bar\omega, \eta_{c_0}(\bar\omega)) \subseteq \mathcal{M}$, so that $\mathcal{S}_{c_0} = \mathcal{M}$. In addition, if $c \in [0, 1] - \{c_0\}$, then $(\bar\omega, \eta_c(\bar\omega)) \in \mathcal{S}_c - \mathcal{M}$, and hence $\mathcal{S}_c \supsetneq \mathcal{M}$: these sets $\mathcal{S}_c$ are not copies of the base. In the case that $\mathcal{M}$ is not a copy of the base, $\mathcal{S}_c \supseteq \mathcal{M}$ is not a copy of the base for any $c \in [0, 1]$. According to Remark 2.25.2, the restricted flows $(\mathcal{S}_c, \tau)$ are Auslander-Yorke chaotic whenever $\mathcal{S}_c$ is not a copy of the base. This proves the assertions in (iii). (Incidentally, note that, in the first situation, $(\mathcal{S}_{c_0}, \tau)$ is also Auslander-Yorke chaotic unless the base flow (Ω, σ) is equicontinuous.)

(iv) Let us check that $\widetilde{\mathcal{S}} := \bigcup_{c\in[0,1]} \mathcal{S}_c$ is closed. We fix $\bar\omega \in \Omega_{AY}$ and take $(\omega_0, x_0) := \lim_{n\to\infty}(\omega_n x_n)$ with $(\omega_n, x_n) \in \mathcal{S}_{c_n} = \mathcal{O}_\tau(\bar\omega, \eta_{c_n}(\bar\omega))$. Let us take a subsequence (c_j) of (c_n)

such that there exists $c_0 := \lim_{j\to\infty} c_j$. We will prove that $(\omega_0, x_0) \in \mathcal{O}_\tau(\bar\omega, \eta_{c_0}(\bar\omega))$. We call $k := \sup_{\omega\in\Omega} |\alpha_{\mathcal{A}}(\omega) - \beta_{\mathcal{A}}(\omega)|$ and note that $\sup_{\omega\in\Omega} |\eta_{c_j}(\omega) - \eta_{c_0}(\omega)| = k\,|c_j - c_0|$. For each $j \in \mathbb{N}$, we look for $t_j > 0$ such that $\mathrm{dist}_{\Omega\times\mathbb{R}}\big((\omega_j, x_j), (\bar\omega{\cdot}t_j, \eta_{c_j}(\bar\omega{\cdot}t_j))\big) < 1/j$. Then, $\mathrm{dist}_\Omega\big(\omega_0, \bar\omega{\cdot}t_j\big) \le \mathrm{dist}_\Omega\big(\omega_0, \omega_j\big) + \mathrm{dist}_\Omega\big(\omega_j, \bar\omega{\cdot}t_j\big)$, with limit 0; and $|x_0 - \eta_{c_0}(\bar\omega{\cdot}t_j)| \le |x_0 - \eta_{c_j}(\bar\omega{\cdot}t_j)| + |\eta_{c_j}(\bar\omega{\cdot}t_j) - \eta_{c_0}(\bar\omega{\cdot}t_j)| \le |x_0 - \eta_{c_j}(\bar\omega{\cdot}t_j)| + k\,|c_j - c_0|$, also with limit 0. That is, $(\omega_0, x_0) = \lim_{j\to\infty}(\bar\omega{\cdot}t_j, \eta_{c_0}(\bar\omega{\cdot}t_j)) \in \mathcal{O}_\tau(\bar\omega, \eta_{c_0}(\bar\omega)) = \mathcal{S}_{c_0} \subseteq \widetilde{\mathcal{S}}$.

Therefore, $\widetilde{\mathcal{S}}$ is closed, and hence, since $\widetilde{\mathcal{S}} \subseteq \mathcal{A}$, it is a compact pinched set. Observe that if we are in the situation (1) of point (iii), then $\widetilde{\mathcal{S}} = \bigcup_{c\in[0,1]-\{c_0\}} \mathcal{S}_c$, since $\mathcal{S}_{c_0} = \mathcal{M} \subset \mathcal{S}_c$ for any $c \in [0,1]$. Therefore, all the points of $\widetilde{\mathcal{S}}$ are sensitive (see Remark 2.22.2). Clearly, $\widetilde{\mathcal{S}}$ is τ-invariant, since each set $\mathcal{S}_c$ is τ-invariant. Consequently, it is chain recurrent: see Remark 3.13. If $\omega \in \Omega_{AY}$ then $\eta_c(\omega) \in \widetilde{\mathcal{S}}_\omega$ for any $c \in [0,1]$, and $\mathcal{A}_\omega = [\alpha_{\mathcal{A}}(\omega), \beta_{\mathcal{A}}(\omega)] = [\eta_0(\omega), \eta_1(\omega)] \subseteq \widetilde{\mathcal{S}}_\omega \subseteq \mathcal{A}_\omega$, so that the sections coincide. If $\omega \in \Omega_c$, then $\mathcal{A}_\omega$ is a singleton, and hence $\mathcal{A}_\omega = \widetilde{\mathcal{S}}_\omega$ also in this case.

Note now that $\displaystyle\int_{\mathcal{A}} h(\omega,x)\,d\widetilde{\mu} = \int_0^1 \int_{\mathcal{A}} h(\omega,x)\,d\eta_c\,dc$ for $h \in C(\mathcal{A},\mathbb{R})$. It is easy to deduce from this property that $\widetilde{\mu}$ is a τ-invariant (regular) measure, and from the regularity that $\widetilde{\mu}(\mathcal{K}) \ge \int_0^1 \mu_c(\mathcal{K})\,dc$ for any compact set $\mathcal{K} \subset \mathcal{A}$. In particular, $\widetilde{\mu}(\widetilde{\mathcal{S}}) = 1$, which in turn ensures that $\mathrm{Supp}\,\widetilde{\mu} \subseteq \widetilde{\mathcal{S}}$. To check that $\mathrm{Supp}\,\widetilde{\mu} \supseteq \widetilde{\mathcal{S}}$, we assume for contradiction that $\mathcal{U} := \widetilde{\mathcal{S}} - \mathrm{Supp}\,\widetilde{\mu}$ is nonempty, choose $(\omega_0, x_0) \in \mathcal{U}$ look for an open set $\mathcal{V} \subset \Omega\times\mathbb{R}$ such that $\mathcal{U} = \mathcal{V} \cap \widetilde{\mathcal{S}}$, and take $\delta > 0$ such that $\mathcal{B}_{\Omega\times\mathbb{R}}((\omega_0, x_0), \delta) \subseteq \mathcal{V}$. Now we look for $c_0 \in [0,1]$ such that $(\omega_0, x_0) \in \mathcal{S}_{c_0}$, take $\bar\omega \in \Omega_{AY}$, and look for $t > 0$ such that $\mathrm{dist}_{\Omega\times\mathbb{R}}\big((\omega_0, x_0), (\bar\omega{\cdot}t, \eta_{c_0}(\bar\omega{\cdot}t))\big) < \delta/2$. Then, $\mathrm{dist}_{\Omega\times\mathbb{R}}\big((\omega_0, x_0), (\bar\omega{\cdot}t, \eta_c(\bar\omega{\cdot}t))\big) < \delta/2 + |\eta_{c_0}(\bar\omega{\cdot}t) - \eta_c(\bar\omega{\cdot}t)| < \delta$ if c is close enough to c_0, so that $(\omega_0, x_0) \in \mathcal{U}_c := \mathcal{V} \cap \mathcal{S}_c$ for these values of c. Therefore, $\mu_c(\mathcal{U}) \ge \mu_c(\mathcal{U}_c) > 0$ for a set of values of c with positive Lebesgue measure, which ensures that $\mu(\mathcal{U}) > 0$, impossible.

It remains to prove that the subset of τ-generic points of $\widetilde{\mathcal{S}}$ is dense. Let us take an open set $\mathcal{U}$ of $\widetilde{\mathcal{S}}$, so that $\widetilde{\mu}(\mathcal{U}) > 0$. Since the set $\widetilde{X} \subseteq \widetilde{\mathcal{S}}$ of τ-generic points has complete measure, $\widetilde{\mu}(\widetilde{\mathcal{X}} \cap \mathcal{U}) > 0$, and hence there are generic points in $\mathcal{U}$. Clearly the subset of generic points of $\widetilde{\mathbb{S}}$ is τ-invariant. The proof is complete. □

Note that, unlike the set Ω_{LY} of Theorem 3.14, the set Ω_{AY} of Theorem 3.15 depends on the measure m of its statement.

Let us make a short analysis of the previous results. Regarding Li-Yorke chaos, we point out again that the set of Li-Yorke pairs that we detect is incomparably larger than what Definition 2.18 requires. More precisely, as Theorem 3.14(i) proves, for m-almost every point of Ω we obtain a scrambled set which can be identified with a nondegenerate interval, incomparably larger than a simply uncountable set.

Moreover, Theorem 3.14(ii) shows that, for m-almost every point $\omega \in \Omega$, the set of positive values of time at which the forward τ-semiorbits of points in $\{\omega\}\times\mathcal{A}_\omega$ seem to coincide (or are "indistinguishable") has positive density in $\mathbb{R}^+$; and the same property holds for the set of positive values of time at which the semiorbits are "distinguishable".

Observe also that, under the hypotheses of Theorem 3.14, a function $a \in C(\Omega,\mathbb{R})$ may belong to the set $\mathcal{R}_{\widetilde{m}}(\Omega)$ for a measure $\widetilde{m} \in \mathfrak{M}_{\mathrm{erg}}(\Omega,\sigma)$ different from m. This is in fact the case whenever $\widetilde{m}(\{\omega\in\Omega \mid \sup_{t\le 0}\int_0^t a(\omega{\cdot}s)\,ds < \infty\}) = 1$, since this property combined with $\int a(\omega)\,d\widetilde{m} \le 0$ (in turn guaranteed by Theorem 2.6) and Birkhoff's ergodic theorem ensures that $\int a(\omega)\,d\widetilde{m} = 0$. Therefore, for each one of these measures, $\widetilde{m}(\Omega_{LY}) = 1$, where Ω_{LY} is the set provided by Theorem 3.14. Similarly, if $\widetilde{m} \in \mathfrak{M}_{\mathrm{inv}}(\Omega,\sigma)$ and $\widetilde{m}(\{\omega \in \Omega \,|\, \sup_{t\le 0}\int_0^t a(\omega{\cdot}s)\,ds < \infty\}) = 1$, we have $\widetilde{m}(\Omega_{LY}) = 1$, as deduced from the decomposition of $\widetilde{m}$ in σ-ergodic measures described in Subsection 2.1. In some cases, Ω_{LY} is a set of complete measure: see Theorem 2.14(ii).

These properties show the physical observability, both in time and state, of the type of Li-Yorke chaos that we detect on the global attractor.

Coming now to the Auslander-Yorke chaos detected in almost all (or all) set $\mathcal{S}_c = \mathrm{Supp}\,\mu_c$, Theorem 2.26(iii) shows that $\mathcal{S}_c$ contains a τ-invariant subset $\mathcal{X}_c$ with full measure μ_c composed of μ_c-generic points with dense forward τ-semiorbits. Since the orbit of a generic point is composed by generic points, the orbit of each point of $\mathcal{X}_c$ provides a dense subset of $\mathcal{S}_c$ of generic points. The natural extension of periodic point for autonomous or time-periodic systems to non periodic ones is that of generic point. Hence, as indicated in [14], the type of chaos detected on the sets $\mathcal{S}_c$ extends the classical one of [11] (which requires transitivity, sensitivity, and the existence of a dense set of periodic points).

Besides this, as Theorem 3.15(iv) shows, the union $\widetilde{\mathcal{S}}$ of all these possibly Auslander-Chaotic sets (perhaps one of them is not) is composed by sensitive (not Lyapunov-stable: see Remark 2.22.2) points; although it is not transitive, it is chain recurrent, which according to [13, Theorem A] (see also [9]) ensures that it is an *abstract omega limit set* (that is, $(\widetilde{\mathcal{S}},\tau)$ is topologically conjugate to the restriction of a flow on a compact space to one of its omega limit sets); it is the support of a τ-invariant measure; and it has a dense subset of τ-generic points. One can also consider these facts enough to talk about a certain type of chaos on $\mathcal{S}$, again opposed to the idea of stability, and again related to the idea of [11]. Finally, $\widetilde{\mathcal{S}}$ fills an “important part” of $\mathcal{A}$. More precisely, $\widetilde{\mathcal{S}}_\omega = \mathcal{A}_\omega$ in a σ-invariant residual set of points with full measure m: the set $\Omega_c \cup \Omega_{AY}$. This property shows that also this chaotic phenomenon has physical relevance. Observe also that $(\widetilde{\mathcal{S}},\tau)$ is Li-Yorke chaotic in measure, since for every ω in the set $\Omega_{LY}\cap\Omega_{AY}$ (with full measure m), $\widetilde{\mathcal{S}}_\omega = \mathcal{A}_\omega$, and hence $\{\omega\}\times\widetilde{\mathcal{S}}_\omega$ is a scrambled set: see Theorems 3.14(i) and 3.15(iv).

Let us finally recall that there are functions in $C(\Omega,\mathbb{R})$ which satisfy the hypotheses required on a in Theorems 3.14 and 3.15 (namely, $\sup\Sigma_a = 0$ and $a \in \mathcal{R}_m(\Omega)$ for a measure $m \in \mathfrak{M}_{\mathrm{erg}}(\Omega,\sigma)$), and that the set of functions a satisfying these two conditions coincides with that of the functions a such that $\sup\Sigma_a = 0$ and $a \in \mathcal{R}_m(\Omega)$ for a measure $m \in \mathfrak{M}_{\mathrm{inv}}(\Omega,\sigma)$. Theorem 2.14 proves these assertions.

We complete the paper with an easy extension of [15, Corollary 4.5], which refers to a quasiperiodically forced map $f\colon \mathbb{S}^1\times[a,b]\to\mathbb{S}^1\times[a,b]$ inducing the discrete semiflow $(\mathbb{S}^1\times[a,b],\phi)$ given by $\phi(n,\omega,x) := f^n(\omega,x)$. The authors establish the sensitivity of $(\mathbb{S}^1\times[a,b],\phi)$ under certain conditions which the next result adapts to our setting.

Proposition 3.16. *Assume that g satisfies* **g1**, **g2** *and* **g3**, *let $(\Omega\times\mathbb{R},\tau)$ be the flow induced by the family (3.1), and let $\mathcal{A}$, $\alpha_{\mathcal{A}}$ and $\beta_{\mathcal{A}}$ be provided by Theorem 3.2. Assume also that $\mathcal{A}$ is pinched, and that the semicontinuous functions $\alpha_{\mathcal{A}},\beta_{\mathcal{A}}\colon\Omega\to\mathbb{R}$ are not continuous. Then, the flow $(\Omega\times\mathbb{R},\tau)$ is sensitive.*

Proof. The result is trivial if (Ω,σ) is sensitive. So, there is not restriction in assuming that this is not the case, which according to Corollary 2.24 and Definition 2.21 means that (Ω,σ) is equicontinuous.

Let $\widetilde{\omega}$ be a continuity point for $\beta_{\mathcal{A}}$. Given $(\omega_0,x_0)\in\Omega\times\mathbb{R}$ with $x_0 \ge \beta_{\mathcal{A}}(\omega_0)$, we look for $(t_n)\uparrow\infty$ with $\widetilde{\omega} = \lim_{n\to\infty}\omega_0{\cdot}t_n$ and such that there exists $\widetilde{x} := \lim_{n\to\infty} x(t_n,\omega_0,x_0) \ge \beta_{\mathcal{A}}(\widetilde{\omega})$. Then $(\widetilde{\omega},\widetilde{x}) \in \mathcal{O}_\tau(\omega_0,x_0) \subseteq \mathcal{A}$, and hence $\widetilde{x} = \beta_{\mathcal{A}}(\widetilde{\omega})$. This ensures that $\inf_{t\ge 0}|x(t,\omega_0,x_0) -$

$\beta_{\mathcal{A}}(\omega_0{\cdot}t)| = 0$. The arguments of [15, Lemma 4.4], which can be adapted to our setting thanks to the equicontinuity of the base flow, provide $\varepsilon_{\beta_{\mathcal{A}}} > 0$ such that all the points (ω, x) above $\mathcal{A}$, i.e., with $x > \beta_{\mathcal{A}}(\omega)$, are $\varepsilon_{\beta_{\mathcal{A}}}$-sensitive (see Remark 2.22.2). An analogous argument provides $\varepsilon_{\alpha_{\mathcal{A}}} > 0$ such that all the points (ω, x) below $x < \alpha_{\mathcal{A}}(\omega)$, are $\varepsilon_{\alpha_{\mathcal{A}}}$-sensitive. Given $\varepsilon := \min(\varepsilon_{\alpha_{\mathcal{A}}}, \varepsilon_{\beta_{\mathcal{A}}})$, we define $\mathcal{T}_\varepsilon \subseteq \Omega \times \mathbb{R}$ as the set of points (ω, x) such that for any $\delta > 0$ there exist two points (ω_1, x_1), $(\omega_2, x_2) \in \mathcal{B}_{\Omega\times\mathbb{R}}((\omega, x), \delta)$ such that $\sup_{t\geq 0} \text{dist}_{\Omega\times\mathbb{R}}(\tau(t, \omega_1, x_1), \tau(t, \omega_2, x_2)) > \varepsilon$. It is easy to check that $\mathcal{T}_\varepsilon$ is closed: if $(\omega, x) = \lim_{n\to\infty}(\omega_n, x_n)$ for a sequence $(\omega_n, x_n) \in \mathcal{T}_\varepsilon$ and $\delta > 0$, we take n_0 with $(\omega_{n_0}, x_{n_0}) \in \mathcal{B}_{\Omega\times\mathbb{R}}((\omega, x), \delta/2)$ and apply the property defining $\mathcal{T}_\varepsilon$ to (ω_{n_0}, x_{n_0}) and $\delta/2$. Clearly, $\mathcal{T}_\varepsilon$ contains all the ε-sensitive points. Therefore, $(\Omega \times \mathbb{R}) - \mathcal{A} \subset \mathcal{T}_\varepsilon$, and hence $(\Omega \times \mathbb{R}) - \mathcal{T}_\varepsilon \subset \mathcal{A}$. But the unique open set contained in a pinched set is the empty one, so that $\mathcal{T}_\varepsilon = \Omega \times \mathbb{R}$. The proof is completed by checking that any point in $\mathcal{T}_\varepsilon$ is $\varepsilon/2$-sensitive. □

Observe that if the attractor $\mathcal{A}$ is a pinched set, then $\alpha_{\mathcal{A}}$ (or $\beta_{\mathcal{A}}$) is continuous if and only if the unique τ-minimal set $\mathcal{M}$ is given by its graph. Consequently, if the base flow (Ω, σ) is equicontinuous and if $\mathcal{A}$ is pinched, then the flow $(\Omega \times \mathbb{R}, \tau)$ is sensitive at least in these two cases:

– $\mathcal{M}$ is not a copy of the base;
– $r_1 < r_2$, $\mathcal{M} \subset \Omega \times (r_1, r_2)$, and $\sup_{t\leq 0} \int_0^t a(\omega_0{\cdot}s)\, ds < \infty$ for a point $\omega_0 \in \Omega$: as seen in the proof of Theorem 3.5(ii), in this case the points $(\omega, \alpha_{\mathcal{A}}(\omega))$ and $(\omega, \beta_{\mathcal{A}}(\omega))$ do not belong to $\mathcal{M}$ whenever ω belongs to the nonempty set $\Omega - \Omega_c$.

Data availability

No data was used for the research described in the article.

Acknowledgment

Massimo Tarallo died while we were beginning to discuss some of the results included in this article. His contribution is fundamental in this work. His friendship, irreplaceable.

The authors thank an anonymous reviewer for the careful reading of the paper and for some suggestions to improve its presentation.

References

[1] E. Akin, J.D. Carlson, Conceptions of topological transitivity, Topol. Appl. 159 (2012) 2815–2830.
[2] E. Akin, S. Kolyada, Li-Yorke sensitivity, Nonlinearity 16 (4) (2003) 1421–1433.
[3] J. Auslander, J.A. Yorke, Interval maps, factors of maps, and chaos, Tohoku Math. J. (2) 32 (2) (1980) 177–188.
[4] K. Bjerklöv, R. Johnson, Minimal subsets of projective flows, Discrete Contin. Dyn. Syst., Ser. B 9 (3–4) (2008) 493–516.
[5] F. Blanchard, E. Glasner, S. Kolyada, A. Maass, On Li-Yorke pairs, J. Reine Angew. Math. 547 (2002) 51–68.
[6] T. Caraballo, J. Langa, R. Obaya, Pullback, forwards and chaotic dynamics in 1-D non-autonomous linear-dissipative equations, Nonlinearity 30 (1) (2017) 274–299.
[7] T. Caraballo, J. Langa, R. Obaya, A.M. Sanz, Global and cocycle attractors for non-autonomous reaction-diffusion equations. The case of null upper Lyapunov exponent, J. Differ. Equ. 265 (2018) 3914–3951.
[8] A. Carvalho, J. Langa, J. Robinson, Attractors for Infinite-Dimensional Non-autonomous Dynamical Systems, Appl. Math. Sci., vol. 182, Springer, 2013.
[9] C. Conley, The gradient structure of a flow: I, Ergod. Theory Dyn. Syst. 8* (1988) 11–26.
[10] W.A. Coppel, Dichotomies in Stability Theory, Lecture Notes in Math., vol. 629, Springer-Verlag, Berlin, Heidelberg, New York, 1978.

[11] R. Devaney, An Introduction to Chaotic Dynamical Systems, Addison-Wesley, New York, 1989.
[12] R. Ellis, Lectures on Topological Dynamics, Benjamin, New York, 1969.
[13] J.E. Franke, J.F. Selgrade, Abstract ω-limit sets, chain recurrent sets, and basic sets for flows, Proc. Am. Math. Soc. 60 (1976) 309–316.
[14] S. Glasner, B. Weiss, Sensitive dependence on initial conditions, Nonlinearity 6 (1993) 1067–1075.
[15] P. Glendinning, T.H. Jäger, G. Keller, How chaotic are strange non-chaotic attractors?, Nonlinearity 19 (9) (2006) 2005–2022.
[16] J.K. Hale, Ordinary Differential Equations, Wiley-Interscience, New York, 1969.
[17] W. Huang, Y. Yi, Almost periodically forced circle flows, J. Funct. Anal. 257 (3) (2009) 832–902.
[18] R. Johnson, A linear, almost periodic equation with an almost automorphic solution, Proc. Am. Math. Soc. 82 (2) (1981) 199–205.
[19] R. Johnson, Bounded solutions of scalar, almost periodic linear equations, Ill. J. Math. 25 (4) (1981) 632–643.
[20] R. Johnson, R. Obaya, S. Novo, C. Núñez, R. Fabbri, Nonautonomous Linear Hamiltonian Systems: Oscillation, Spectral Theory and Control, Developments in Mathematics, vol. 36, Springer, 2016.
[21] R. Johnson, K. Palmer, G.R. Sell, Ergodic theory of linear dynamical systems, SIAM J. Math. Anal. 18 (1987) 1–33.
[22] À. Jorba, C. Núñez, R. Obaya, J.C. Tatjer, Old and new results on strange nonchaotic attractors, Int. J. Bifurc. Chaos 17 (11) (2007) 3895–3928.
[23] J. Kaplan, J.A. Yorke, Preturbulence: a regime observed in a fluid flow model of Lorenz, Commun. Math. Phys. 67 (1979) 93–108.
[24] S.F. Kolyada, Li-Yorke sensitivity and other concepts of chaos, Ukr. Math. J. 56 (8) (2004) 1242–1257.
[25] T.-Y. Li, J.A. Yorke, Period three implies chaos, Am. Math. Mon. 82 (10) (1975) 985–992.
[26] R. Mañé, Ergodic Theory and Differentiable Dynamics, Springer-Verlag, Berlin, Heidelberg, New York, 1987.
[27] A. Miller, Unpredictable points and stronger versions of Ruelle–Takens and Auslander–Yorke chaos, Topol. Appl. 253 (2019) 7–16.
[28] V. Nemytskii, V. Stepanoff, Qualitative Theory of Differential Equations, Princeton University Press, Princeton, NJ, 1960.
[29] C. Núñez, R. Obaya, Li-Yorke chaos in nonautonomous Hopf bifurcation patterns – I, Nonlinearity 32 (10) (2019) 3940–3980.
[30] R. Obaya, A.M. Sanz, Non-autonomous scalar linear-dissipative and purely dissipative parabolic PDEs over a compact base flow, J. Differ. Equ. 285 (5) (2021) 714–750.
[31] R. Ortega, M. Tarallo, Massera's theorem for quasi-periodic differential equations, Topol. Methods Nonlinear Anal. 19 (2002) 39–61.
[32] R. Ortega, M. Tarallo, Almost periodic linear differential equations with non-separated solutions, J. Funct. Anal. 237 (2006) 402–426.
[33] D. Ruelle, F. Takens, On the nature of turbulence, Commun. Math. Phys. 20 (1971) 167–192.
[34] R.J. Sacker, G.R. Sell, Existence of dichotomies and invariant splittings for linear differential systems I, J. Differ. Equ. 15 (1974) 429–458.
[35] R.J. Sacker, G.R. Sell, Lifting Properties in Skew-Products Flows with Applications to Differential Equations, Mem. Amer. Math. Soc., vol. 190, Amer. Math. Soc., Providence, 1977.
[36] R.J. Sacker, G.R. Sell, A spectral theory for linear differential systems, J. Differ. Equ. 27 (1978) 320–358.
[37] J.F. Selgrade, Isolated invariant sets for flows on vector bundles, Trans. Am. Math. Soc. 203 (1975) 359–390.
[38] W. Shen, Y. Yi, Almost Automorphic and Almost Periodic Dynamics in Skew-Product Semiflows, Mem. Amer. Math. Soc., vol. 647, Amer. Math. Soc., Providence, 1998.
[39] P. Walters, An Introduction to Ergodic Theory, Springer, New York, 1982.