An integrative framework of cooperative advertising with reference price effects

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Abstract

The importance of reference price effects in consumer behavior and marketing decisions is now well established in the literature. However, research on the impact of these effects on cooperative advertising decisions in marketing channels remains very limited. A two-period model is developed to analyze how members of a bilateral monopoly channel should set pricing and advertising decisions in a context where first-period price serves as the reference price of second period. By solving a Stackelberg game where the manufacturer is the leader, nine feasible equilibria are endogenously obtained. These equilibria correspond to different combinations (scenarios) of the respective decisions of the retailer and manufacturer to undertake or not and to support or not local advertising in each period. The profitability of each of these scenarios for the players and their pricing and advertising strategies over time depend, among others, on how sensitive consumers are to price changes over time.

Key words: advertising, cooperative advertising, marketing channel, pricing, reference price

1. Introduction

The literature on cooperative advertising, which studies the share of retailers' advertising expenditures that manufacturers must bear to promote their products locally, has reached maturity. As a result, there is a growing need to integrate previous cooperative advertising work to facilitate the understanding and use of its results (Aust and Buscher, 2014; Jørgensen and Zaccour, 2014). Martín-Herrán and Sigué (2017a) attempt to respond to this need and propose an integrative framework of cooperative advertising, which aims to explain the practices observed in the business world and to integrate some of the findings of previous analytical cooperative advertising models. These authors start from the observation that the literature on cooperative advertising is divided into two modeling streams, static games and dynamic games, the results of which are not always complementary and do no explain common pulsing advertising schedules where advertising activity stops and resumes over time.

Particularly, the static modeling stream, which is also the most dominant, sins by ultra simplification (e.g., Karray, 2013, 2015; Karray and Amin, 2015; Karray and Zaccour, 2006; Xie and Ai, 2006; Yan, 2010; Yan and Pei, 2015). This stream of the literature ignores the long-term effects of marketing activities on consumer demand, leading to suboptimal strategies and profits when these effects exist. The dynamic modeling stream, which relies heavily on differential games, deals with the short and long term effects of marketing activities. Unfortunately, due to simplifying assumptions, the results of this research stream sometimes deviate from business realities (e.g., Jørgensen et al., 2000, 2001, 2003; He et al. 2011; Sigué and Chintagunta, 2009; Zhang et al., 2013, 2015). For instance, the fact that this research stream systematically prescribes constant cooperative advertising support rates over time has been highlighted as a major shortcoming (Jørgensen and Zaccour, 2014; Martín-Herrán and Sigué, 2017b).

Martín-Herrán and Sigué (2017a) therefore proposed a two-period model that copes with the long-term effects of advertising and gives the retailer the option of whether or not to advertise and the manufacturer whether or not to offer cooperative advertising to the retailer in each period. These authors show that at the equilibrium, various advertising arrangements are possible, including those where the manufacturer exclusively supports the retailer over a single period. Their model, however, overlooks reference price effects which are known not only to drive consumer demand in most product categories, but also to affect pricing and cooperative advertising decisions (Lin, 2016; Zhang et al., 2013).

Indeed, it is now well established in the marketing and economics literature that consumers have reference prices or price expectations against which actual prices are compared prior to purchases (Mazumdar et al., 2005). A consumer is considered to gain from a purchase when the reference price of the product is higher than the actual purchase price and the consumer loses from a purchase when the reference price is lower than the actual purchase price (Lin, 2016; Thaler, 1985). Situations where consumers gain from their purchases are believed to be more desirable and to lead to increased consumer demand, while the opposite are believed to negatively impact consumer demand (Zhang et al., 2014). There are several ways to conceptualize and operationalize reference prices in the literature (Briesch et al., 1997). The simplest and most effective way is to consider the reference price as the price of the product at the last purchase occasion (Briesch et al., 1997; Kopalle et al., 1996). This conceptualization implies, among others, that the effects of the reference price must be studied in a dynamic context, which makes it possible to consider the sensitivity of consumers to price variations over time.

We propose an extension of Martín-Herrán and Sigué's (2017a) cooperative advertising model, which incorporates reference price effects. Cooperative advertising, as a financial arrangement whereby a manufacturer agrees to share a percentage of the retailer's local advertising, is intended to increase local advertising and stimulate local demand; while the fact that consumers use a reference price at the time of purchase may stimulate or dampen consumer demand for a product. Consequently, in a context where the reference price is the market price of the previous period, the manufacturer and retailer in a two-member channel must not only set their respective advertising and pricing decisions to maximize their profits in the current period, but they must also take into account the impacts of these decisions on future sales through reference price effects (Zhang et al., 2013) and the long-term effects of advertising (Martín-Herrán and Sigué, 2017a). Indeed, the current advertising decisions of the manufacturer and the retailer affect, among others, the current price and establish the standards against which consumers will compare the future retail price. A product that receives large (low) advertising investments may have a high (low) price and benefit (suffer) from a high (low) reference price in the following period. However, whether the next period retail price is higher or lower than the reference price depends on the period advertising investments and the long-term effects of past advertising. Thus, as Lin (2016) and Zhang et al. (2013) eloquently demonstrated, reference price effects must be taken into account in any attempt to help channel members to make better advertising and promotional decisions. The questions of this research are therefore:

1. How should channel members set channel pricing and cooperative advertising decisions over time when consumers use past prices as reference prices?
2. How should channel pricing and advertising strategies compare over time?
3. What type of advertising arrangements provide the greatest profits to each channel member?

This paper answers these questions in a *two-member channel in which a manufacturer and retailer* set their pricing and advertising decisions over a two-period planning horizon. Particularly, in each period, the manufacturer and retailer respectively set their wholesale and retail prices. Whereas when the retailer decides whether or not to undertake local advertising in each period, the manufacturer must decide whether or not to support retailer advertising. A distinctive feature of this model is that consumers use the first-period retail price as the second-period reference price to gauge whether or not they get better deals. Otherwise, similar to previous two-period games in channel literature, first-period advertising may have positive, negative or no effect on second-period sales (Karray et al., 2017, 2020, 2021; Martín-Herrán and Sigué, 2017a) .

Using the Stackelberg solution concept in a configuration where the manufacturer is the channel leader, we found that the game endogenously has 9 feasible solutions at equilibrium, also called scenarios. These scenarios are essentially different combinations of channel members' advertising decisions over the two periods, including the two most anticipated where the retailer advertises and has cooperative advertising support from the manufacturer in each period (Scenario I) and the retailer advertises in each period and receives no cooperative advertising support (Scenario V). In addition of these two, 7 other scenarios where either the retailer does not advertise or the manufacturer does not provide advertising support in a period were identified (see Proposition 1).

The evolution of player strategies over time critically depends on how sensitive consumers are to price variations over time and the long-term effects of retailer advertising. For instance, when retailer advertising has no long-term effects and consumers are sensitive to price changes over time, the retailer adopts skimming pricing strategies and heavily invests in first-period advertising, with or without cooperative advertising support, to build a strong brand image and benefit from higher profit margins. On the other hand, the manufacturer provides more generous advertising supports in the first period and the evolution of her pricing strategy depends on the remaining parameters. Adding the long-term effects of retailer advertising to the mix may alter these conclusions. As an example, the retailer may find it optimal to adopt a penetration pricing strategy when advertising positively impacts long-term sales and invest heavily in first-period advertising for its ability to expand second-period demand.

The profitability of the different advertising scenarios or arrangements for each channel member depends on the model parameters, including the reference price effects. In most cases, the scenario preferences of the two players diverge, setting the stage for possible channel conflicts. However, their scenario preferences converge when the manufacturer finds it optimal not to offer a cooperative advertising program in any period, while the retailer actively advertises in two periods (Scenario V).

This paper aims to advance the literature on cooperative advertising, which marginally deals with reference price effects (Aust and Buscher, 2014; Jørgensen and Zaccour, 2014). It departs from works that investigate reference price effects alone and pay no attention to cooperative advertising (e.g., Chen et al., 2020; Crettez et al., 2020; Spann and Prakash, 2022). It is in line with Zhang et al. (2013), *who are the first to study reference price effects in the cooperative advertising literature, and differs from their paper on three notable* *points.* First, we use a two-period game instead of a differential game as they do. As many other differential games in the cooperative literature, they obtain time-constant advertising strategies that cannot account for the different advertising scenarios discussed above. Second, we conceptualize the reference price in the second period as the first-period retail price (Briesch et al., 1997; Popescu and Wu, 2007), while in theirs, the reference price results from a dynamic equation that depends not only on consumer's memory of past prices, but also on channel advertising investments. Third, pricing decisions are exogenous to their model, whereas in ours, players are allowed to set their prices endogenously.

Recently published two-period *cooperative advertising* models, such as Martín-Herrán and Sigué (2017a), Karray et al. (2017, 2021, 2022), offer an alternative to differential game models. However, they only study the long-term effects of advertising in different channel structures and *overlook reference price effects.* We build on this tradition and model the reference price effects to account for the dynamic nature of pricing decisions. So far, these two-period cooperative advertising models have looked at pricing decisions from a short-term perspective *where consumers are not sensitive to price changes over time.*

We show *that combining* reference price effects with the long-term effects of retailer advertising provides a broader and richer perspective of pricing and advertising strategies, *which translates into multiple equilibria or scenarios within a two-member channel.* Failure to take them into account when they exist, as has been done in a large majority of cooperative advertising papers, leads to suboptimal pricing and advertising strategies (Lin, 2016; Zhang et al., 2013).

The remainder of the paper is organized as follows. Section 2 describes the model and discusses its assumptions. Section 3 provides a brief description of the derivation of equilibrium solutions. Section 4 compares the strategies over time. Section 5 discusses player preferences for different advertising scenarios. Section 6 concludes and discusses the theoretical and managerial implications of this research. The detailed derivation of the equilibrium solutions is presented in the Appendix.

2. The model

Consider a bilateral monopoly in which a manufacturer sells a relatively new product to an exclusive retailer for resale to consumers. Both the manufacturer and retailer are in a monopoly situation as they do not face any direct competition or if the competition starts to appear, its effects on the pioneer are still negligible. This assumption is common in the marketing channel literature (e.g., Lin, 2016;Martín-Herrán and Sigué, 2017; Zhang et al., *2013) and, in this case, helps focus on how channel members interact vertically to set their marketing decisions when consumers compare the current price to the past price at the time of purchase.* The two channel members set their pricing and advertising decisions to maximize their respective profits over a two-period planning horizon. Particularly, for each period i, $i \in \{1, 2\}$, the manufacturer sets the wholesale price for the product, w_i, and may support part of the retailer's advertising expenditures at a rate of s_i. To focus on the manufacturer's cooperative advertising program, we assume that the manufacturer does not directly advertise her product or that the effects of direct manufacturer advertising are exogenous to the model and incorporated into the baseline demand of the product, g. Also, in each period i, $i \in \{1, 2\}$, the retailer sets the retail price, p_i, and may undertake local advertising, a_i, to further stimulate the demand of the product.

Consider the following linear demand functions:

$$q_1 = g - p_1 + \alpha a_1, \quad q_2 = g - p_2 - \gamma(p_2 - p_1) + \beta a_1 + \alpha a_2.$$

The second-period demand function is an extension of the the second-period demand, $q_2 = g - p_2 + \beta a_1 + \alpha a_2$, in Martín-Herrán and Sigué (2017a), while the first-period demands in the two papers are identical.

The right-hand side of first-period demand (q_1) *has three components. The first component is the positive parameter* g, *which represents the initial demand based on marketing factors exogenous to this model. The second component consists of the first-period retail price* (p_1), *which has a negative impact on demand. For simplicity, the effect of this retail price on demand is normalized to* 1. *The third component is made of the first-period retailer advertising* (a_1) *and the parameter* α, *which represents the positive effect of retailer advertising on current demand (Martín-Herrán and Sigué, 2017a).*

The second-period demand can be rewritten as follows: $q_2 = (g - p_2 + \alpha a_2) - \gamma(p_2 - p_1) + \beta a_1$. *This formulation is made of three components, the first of which is similar to the demand function of the first period. The second component features a positive parameter,* γ, *and the difference between the retail prices of the two periods. The first-period retail price is used as a reference price against which consumers evaluate the second-period retail price.* Consumers consider that the second-period retail price is set at their advantage (disadvantage) when $p_2 - p_1 < 0$ ($p_2 - p_1 > 0$). In other words a decrease in the second-period price is perceived as a price discount (gain) that stimulates the second-period demand, whereas any increase in the second-period price is perceived as a surcharge (loss) and negatively affects the second-period demand (Popescu and Wu, 2007). Thus, the parameter, $0 \leq \gamma \leq 1$, represents consumer sensitivity to price changes over time or the reference price effect, which is also the effect of price discount/surcharge on the second-period demand (Zhang et al., 2014). Higher values of γ mean that consumers are very sensitive to price changes over time, while when consumers are not sensitive to price changes over time or disregard the reference price, $\gamma = 0$. Following Zhang et al.(2013), we assume price discounts and surcharges have symmetric effects. Popescu and Wu (2007) use an alternative specifications in which price discounts (gains) have lower impacts on the demand than price surcharges (losses). *The third component is made of the first-period retailer advertising and the parameter, β, which is the effect of the first-period advertising on the second-period demand. This parameter, also known as the long-term effect of retailer advertising,* depends on the type of local advertising and may have no impact or have a negative or positive impact on second-period demand. Negative long-term effects occur when retailer advertising harms brand image or contributes to advance purchases, reducing the baseline demand of the second period. Conversely, positive long-term effects enhance the brand image and contribute to expanding the baseline demand of the second period. Finally, when retailer advertising does not have any impact on the second-period demand, the parameter β is set to zero.

Observe that in situations where the retailer undertakes local advertising activities in the first period that have no impact on second period demand ($\beta = 0$) and consumers are not sensitive to price changes over time ($\gamma = 0$), the demand functions of the two periods are independent and depend exclusively on the retail price and advertising decisions of each period.

Table 1: Model specification

	Period 1	Period 2
Manufacturer's controls	w_1, s_1	w_2, s_2
Retailer's controls	p_1, a_1	p_2, a_2
Demand functions	$q_1 = g - p_1 + \alpha a_1$	$q_2 = g - p_2 - \gamma(p_2 - p_1) + \beta a_1 + \alpha a_2$
Manufacturer's profits	$M_1 = w_1 q_1 - \frac{1}{2} s_1 a_1^2$	$M_2 = w_2 q_2 - \frac{1}{2} s_2 a_2^2$
Retailer's profits	$R_1 = (p_1 - w_1) q_1 - \frac{1}{2}(1 - s_1) a_1^2$	$R_2 = (p_2 - w_2) q_2 - \frac{1}{2}(1 - s_2) a_2^2$

As is common in marketing channel literature, channel members have no inventory and their production and administration costs are normalized to zero (e.g., Lin, 2016; Martín-Herrán et al., 2010; Yan and Pei, 2015). Conversely, the retailer faces the following cost functions, $\frac{1}{2} a_i^2$, $i \in \{1, 2\}$, for his local advertising activities. This specification of the advertising costs suggests that the marginal costs of advertising are increasing and suits well with linear advertising demand functions. When the manufacturer offers a cooperative advertising arrangement to the retailer, she supports a portion of the retailer's advertising costs given by: $\frac{1}{2}s_i a_i^2$, *where* s_i *is the cooperative advertising support rate set by the manufacturer and* $\frac{1}{2}a_i^2$ *is the total cost of retailer advertising.* Therefore, the retailer's effective advertising costs in the presence of a cooperative advertising arrangement are reduced to: $\frac{1}{2}(1 - s_i)a_i^2$.

The retailer's (R_i) and manufacturer's (M_i) profits in period i, $i \in \{1, 2\}$, are given by:

$$R_1 = (p_1 - w_1)q_1 - \frac{1}{2}(1 - s_1)a_1^2, \quad R_2 = (p_2 - w_2)q_2 - \frac{1}{2}(1 - s_2)a_2^2,$$
$$M_1 = w_1 q_1 - \frac{1}{2}s_1 a_1^2, \quad M_2 = w_2 q_2 - \frac{1}{2}s_2 a_2^2.$$

The retailer's profit function in each period (R_i) *has two components. The first component is the retailer's gross profit margin, while the second is the retailer's effective local advertising cost. Similarly, the two components of the right-hand side of the manufacturer's profit function in period* i (M_i) *are respectively the manufacturer's gross profit margin and the share of advertising expenditure of the retailer supported under the cooperative advertising program. This second component is zero when the manufacturer does not offer a cooperative advertising program.*

The manufacturer and retailer set their decision variables so as to maximize their respective profits over the two periods. We consider that both the manufacturer and the retailer give equal importance to the profits made in each period. Therefore, the profits of the two periods are obtained by simply adding the profits of each of the periods: $M = M_1 + M_2$ and $R = R_1 + R_2$.

3. Equilibria

To derive the equilibrium solutions for this game, we use the Stackelberg equilibrium concept and assign the roles of leader and follower to the manufacturer and retailer respectively. We prefer manufacturer leadership, as in many other papers in the literature, as it is proven to benefit both the manufacturer and the retailer (Jørgensen et al., 2001). Also, in business practice, manufacturers usually announce their cooperative advertising programs in advance. The game is played in four stages as follows. First, the manufacturer announces her first-period wholesale price and cooperative advertising participation rate. Second, the retailer reacts to the manufacturer's announcement and sets his first-period retail price and local advertising strategies. Third, considering the retailer first-period strategies, the manufacturer announces her second-period wholesale price and cooperative advertising participation rate. Finally, in the fourth stage, the retailer sets the second-period retail price and local advertising strategies.

Subgame-perfect equilibrium solutions are obtained by solving the game backwards (See Appendix). This means that the retailer's second-period equilibrium strategies are obtained first, followed by the manufacturer's second-period equilibrium strategies. By considering these second-period equilibrium strategies, the retailer's first-period strategies are derived, allowing them to be integrated into the manufacturer's first-period problem, which is solved at the end.

We characterize in the Appendix 12 different scenarios corresponding to 12 different equilibria. We have considered all possible equilibrium solutions, including the interior solution and all corner solutions where the retailer's advertising or/and the manufacturer's cooperative advertising rate in a given period is zero. Particularly, when the retailer advertising is zero, the corresponding cooperative advertising rate is undetermined, and the equilibrium is independent of the value of the cooperative advertising rate. 3 of the 12 initial possible equilibrium solutions turned out to be unfeasible. Conversely, Scenarios III, VI and IX, described below, are always feasible, for any value of $\alpha \in (0,1), \beta \in (-1,1)$ and $\gamma \in \{0.1, 0.25, 0.5, 0.75, 0.9\}$, while for each fixed value of γ, $\gamma \in \{0.1, 0.25, 0.5, 0.75, 0.9\}$, Scenarios I, II, IV, V, VII, and VIII are feasible under some conditions on the parameters α and β. The analysis of the feasibility of the different scenarios has been carried out following this procedure. We fix 6 different values of γ, $\gamma \in \{0.1, 0.25, 0.5, 0.75, 0.9\}$, and consider 100 different values of α in the interval (0, 1) (a mesh of 0.01) and 100 different values of β in the interval (-1, 1) (a mesh of 0.02).

Proposition 1. *At the equilibrium, there are 9 feasible solutions that correspond to the scenarios described in Table 2.*

Table 2: Feasible scenarios at equilibrium

	Description
Scenario I	*The retailer advertises and the manufacturer supports retailer advertising in each period.*
Scenario II	*The retailer advertises in each period and the manufacturer supports retailer advertising only in the second period.*
Scenario III	*The retailer only advertises in the second period and the manufacturer offers advertising support in each period.*
Scenario IV	*The retailer advertises in each period and the manufacturer support retailer advertising only in the first period.*
Scenario V	*The retailer advertises in each period and the manufacturer offers no advertising support.*
Scenario VI	*The retailer only advertises in the second period and the manufacturer offers advertising support only in the first period.*
Scenario VII	*The retailer only advertises in the first period and the manufacturer offers advertising support in each period.*
Scenario VIII	*The retailer only advertises in the first period and the manufacturer offers advertising support only in the second period.*
Scenario IX	*The retailer does not advertise in both periods and the manufacturer offers advertising support in each period.*

Any of these 9 scenarios or equilibria can be implemented, depending on whether or not it offers the greatest profits to the manufacturer, as the channel leader, in a given area of the parameter space. *Observe that the availability of a cooperative advertising program from the manufacturer during any given period does not imply that the retailer will advertise during that period or, if he does, that he will participate in such a program.* Channel member scenario preferences will be discussed in a later section.

3.1. Comparisons of strategies over time

The goal of this subsection is to have an understanding on how taking into account the reference price affects player strategies over time. To achieve this goal, we compared the changes in the strategies for the 9 feasible scenarios when the values of the parameter γ are fixed ($\gamma \in \{0, 0.5, 0.9\}$) and the parameters $\alpha \in (0,1)$ and $\beta \in (-1,1)$ vary. These comparisons give consistent results and for parsimony, we only report the results of Scenarios

I and V, which are, respectively, more likely to be preferred by the manufacturer and the retailer, respectively. For each value of γ, we generate a grid for different values of α and β such that there are 100 different values of α in the interval (0, 1) and 100 different values of β in the interval (-1, 1). For each of these 100x100 pairs (α, β) which belong to the region where each scenario is feasible, we compare the players' strategies in the first and second periods.

3.2. Scenario I: The retailer advertises and the manufacturer supports retailer advertising in each period.

Scenario I corresponds to the interior solution, where the retailer advertises with the support of the manufacturer's cooperative advertising in each of the two periods. The following two propositions summarize the results of our analysis.

Proposition 2. *When consumers are not sensitive to price changes over time ($\gamma = 0$), the strategies of the players in the two periods of Scenario I compare as follows:*

For any $\alpha \in (0, 1)$,

1. If $\beta \in (0, 1)$, then $w_1^I < w_2^I,\ \ p_1^I < p_2^I,\ \ a_1^I > a_2^I,\ \ s_1^I > s_2^I$.
2. If $\beta = 0$, then $w_1^I = w_2^I,\ \ p_1^I = p_2^I,\ \ a_1^I = a_2^I,\ \ s_1^I = s_2^I$.
3. If $\beta \in (-1, 0)$, then $a_1^I < a_2^I,\ \ s_1^I < s_2^I$, and p_1^I and w_1^I can be greater or lower than p_2^I and w_2^I, respectively.

Proposition 3. *When consumers are sensitive to price changes over time ($\gamma > 0$), particularly when $\gamma = 0.5$ or $\gamma = 0.9$, the strategies of the players in the two periods of Scenario I compare as follows:*

For any $\alpha \in (0, 1)$,

1. If $\beta \in (0, 1)$, then $a_1^I > a_2^I,\ \ s_1^I > s_2^I$, and p_1^I and w_1^I can be greater or lower than p_2^I and w_2^I, respectively.
2. If $\beta = 0$, then $w_1^I > w_2^I,\ \ p_1^I > p_2^I,\ \ a_1^I > a_2^I,\ \ s_1^I > s_2^I$.
3. If $\beta \in (-1, 0)$, then $w_1^I > w_2^I,\ \ p_1^I > p_2^I$, and a_1^I and s_1^I can be greater or lower than a_2^I and s_2^I, respectively.

The findings of the above two propositions critically depend on the nature of the long-term effect of the retailer advertising (β). In Proposition 2, where consumers are not sensitive to price changes over time, if retailer advertising has no impacts on long-term sales ($\beta = 0$), the two players adopt the same pricing and advertising strategies over the two periods. This is expected as the games in the two periods are independent. On the other hand, when first-period retailer advertising expands second period demand ($\beta > 0$), the retailer invests more in advertising and received more cooperative advertising support from the manufacturer in the first period to benefit from advertising carryover effects, which expand the second-period baseline demand and allow for second-period price increases. A penetration pricing strategy is adopted to take advantage of increased demand due to previous advertising. Conversely, when first-period retailer advertising damages the second-period demand ($\beta < 0$), it pays for the retailer to invest more and for the manufacturer to provide more cooperative advertising support in the second period to mitigate the negative effect of first-period retailer advertising. Channel prices may increase or decrease over time depending on the values of parameters α and β, as illustrated in Figure 1.

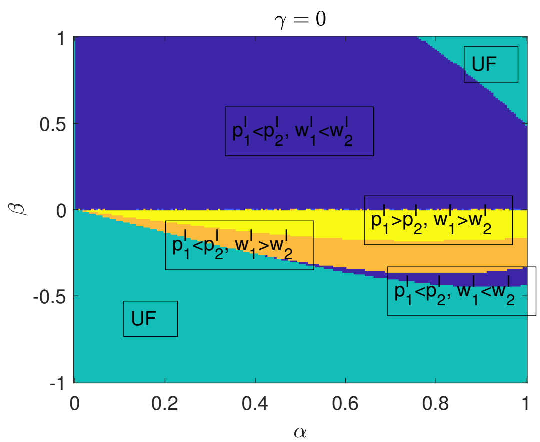

Figure 1: Case $\gamma = 0$. Scenario I: Comparison retail and wholesale prices for the first and second periods.

Proposition 3 states that the qualitative results of the comparisons of the strategies for $\gamma = 0.5$ and $\gamma = 0.9$ are similar. Thus, the magnitude of the positive reference price effect does not substantially change the reported results. Indeed, when retailer advertising has no long-term effects ($\beta = 0$), the only way first-period decisions can affect second-period sales is through pricing. In such a context, it pays for the two channel members to adopt skimming pricing strategies and to invest more in advertising in the first period. For the two channel members, offering price discounts in the second period helps increase consumer demand to compensate for declining profit margins. Conversely, when retailer advertising positively impacts second-period demand ($\beta > 0$), the retailer invests more in advertising and receives more cooperative advertising support from the manufacturer in the first period to benefit from advertising carryover effects. Whether or not channel members' should lower or increase prices over time depends on the values of the model parameters as illustrated in

Figure 2 (left) for $\gamma = 0.5$. Finally, when retailer advertising negatively impacts second-period demand ($\beta < 0$), second-period discounted prices are more desirable to meet weaker second-period baseline demand (skimming pricing strategies). Whether the retailer should invest more in advertising or whether the manufacturer should provide a more generous cooperative advertising in a given period depends on the values of the model parameters, as shown in Figure 2 (right) for $\gamma = 0.5$.

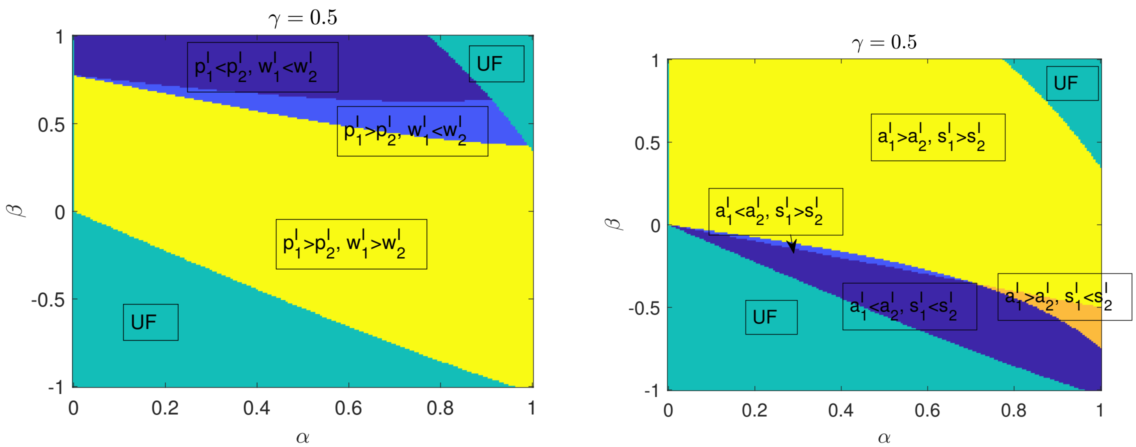

Figure 2: Case $\gamma = 0.5$. Scenario I: Comparison wholesale and retail prices (left) and advertising investments and supports (right) for the first and second periods.

3.3. Scenario V: The retailer advertises in each period and the manufacturer offers no advertising support

Scenario V is one of the corner solutions, where the retailer advertises without any cooperative advertising support from the manufacturer in each of the two periods. The following two propositions summarize the results of our analysis in this scenario.

Proposition 4. *When consumers are not sensitive to price changes over time ($\gamma = 0$), the strategies of the players in the two periods of Scenario V compare as follows:*
For any $\alpha \in (0, 1)$,

1. *If $\beta \in (0, 1)$, then $w_1^V < w_2^V$, $p_1^V < p_2^V$, $a_1^V > a_2^V$.*
2. *If $\beta = 0$, then $w_1^V = w_2^V$, $p_1^V = p_2^V$ $a_1^V = a_2^V$.*
3. *If $\beta \in (-1, 0)$, then $w_1^V > w_2^V$, $a_1^V < a_2^V$ and p_1^V can be greater or lower than p_2^V.*
4. *For any $\beta \in (-1, 1)$, $s_1^V = s_2^V = 0$.*

Proposition 5. *When consumers are sensitive to price changes over time ($\gamma > 0$), particularly when $\gamma = 0.5$ or $\gamma = 0.9$, the strategies of the players in the two periods of Scenario V compare as follows:*
For any $\alpha \in (0, 1)$,

1. *If $\beta \in (0,1)$, then $a_1^V > a_2^V$ and w_1^V and p_1^V can be greater or lower than w_2^V and p_2^V, respectively.*
2. *If $\beta = 0$, then $p_1^V > p_2^V$ $a_1^V > a_2^V$ and w_1^V can be greater or lower than w_2^V.*
3. *If $\beta \in (-1,0)$, then $w_1^V > w_2^V$, $p_1^V > p_2^V$ and a_1^V can be greater or lower than a_2^V.*
4. *For any $\beta \in (-1,1)$, $s_1^V = s_2^V = 0$.*

Again, the findings of the above two propositions critically depend on the nature of the long-term effects of retailer advertising (β). Particularly, where consumers are not sensitive to price changes over time as in Proposition 4, the two players adopt the same pricing strategies in the two periods. The retailer also advertises identically in the two periods when advertising does not impact long-term sales ($\beta = 0$). As in Scenario I, the games in the two periods are independent, meaning the strategies of the first-period game have no impact on second-period sales. When first-period advertising expands second period sales ($\beta > 0$), the retailer invests more in advertising in the first period to benefit from advertising carryover effects, which increase demand in the second period and provide room for price increases. Conversely, in the context where first-period advertising damages second-period demand ($\beta < 0$), the retailer invests more in advertising in the second period to mitigate the negative effect of first-period advertising. The manufacturer charges a higher first-period wholesale, while the retail price may increase or decrease over time depending on the values of the parameters α and β, as illustrated in Figure 3.

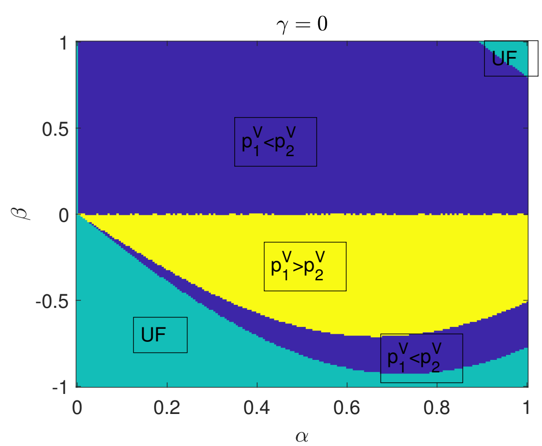

Figure 3: Case $\gamma = 0$. Scenario V: Comparison retail prices for the first and second periods.

The findings in Proposition 5, where consumers are sensitive to price changes over time ($\gamma > 0$), do not change qualitatively whether $\gamma = 0.5$ or $\gamma = 0.9$. The only way first-period decisions can affect second-period sales when first-period advertising has no long-term effects is through pricing. In such a context, it pays for the retailer to adopt skimming pricing strategy regardless whether the manufacturer adopts a lower or higher wholesale price in the first period. Accordingly, the retailer should also invest more in advertising in the first period than in the second period to justify high price and further expand first-period demand. As a result, the second-period discounted retail price helps to expand second-period baseline demand. When advertising positively impacts on long-term sales ($\beta > 0$), the retailer invests more in advertising in the first period to benefit from the carryover effects. Whether channel member prices fall or rise over time depends on the values of the model parameters, as shown in Figure 4 (left) for $\gamma = 0.5$. On the reverse, when advertising negatively impacts on long-term sales ($\beta < 0$), second-period discounted prices are more desirable to cope with lower second-period baseline demand. Whether or not the retailer should invest more in advertising in a given period depends on the values of the model parameters, as illustrated in Figure 4 (right) for $\gamma = 0.5$.

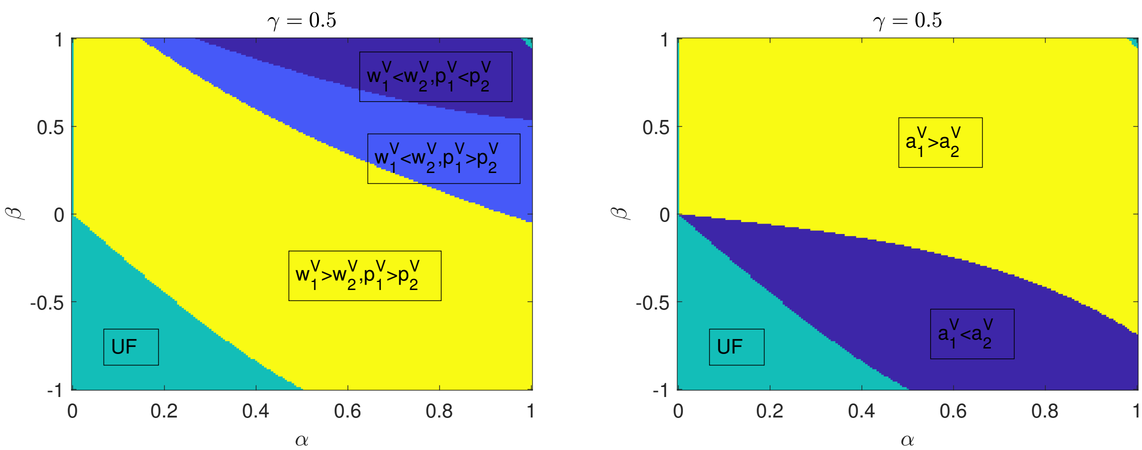

Figure 4: Case $\gamma = 0.5$. Scenario V: Comparison wholesale and retail prices (left) and advertising investments (right) for the first and second periods.

In summary, the main lesson to be learned from our analysis of the players' strategies above is that, both the manufacturer and retailer change their strategies depending, among others, on whether or not consumers are sensitive to price variations over time.

4. Equilibrium selection

Given the 9 feasible equilibrium solutions of Proposition 1, we evaluate from the point of view of the manufacturer and the retailer which equilibrium solution or scenario generates the greatest profits. The comparisons of players' profits are carried out considering the same type of grid as described in Section 3. The results of the comparisons of players' profits among scenarios are summarized in Figures 5 and 6, where for some selected values of parameter γ, $\gamma \in \{0, 0.5, 0.9\}$, parameters $\alpha \in (0, 1)$ and $\beta \in (-1, 1)$ vary. In Figures 5 and 6, Region i indicates that the manufacturer and the retailer, respectively, get the greatest profits under the corresponding Scenario i, $i \in \{I, II, III, IV, V, VI, VII, VIII, IX\}$.

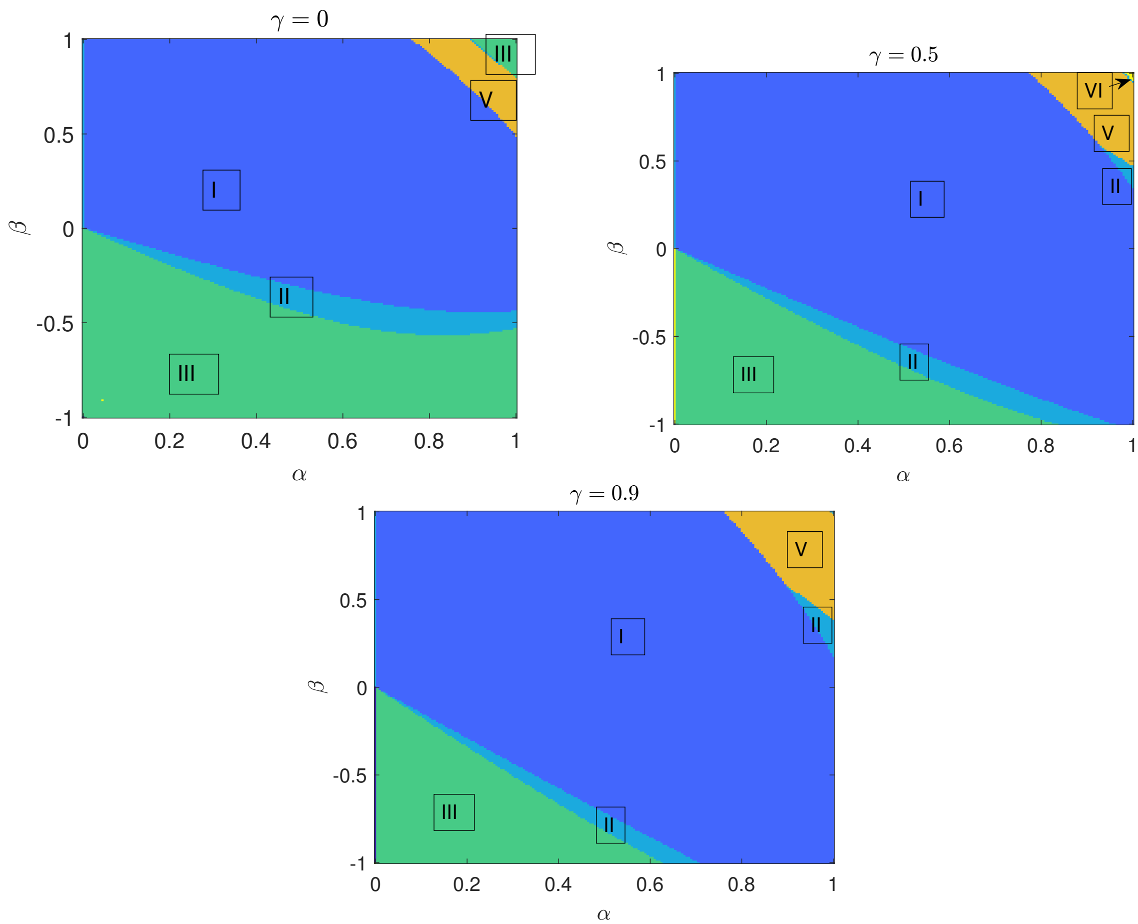

Figure 5: Comparison Manufacturer's profits across scenarios for different values of γ.

Two major observations can be made from Figure 5. First, given the feasibility conditions, Scenarios IV, VI, VII, VIII, and IX do not appear in Figure 5, which means they are never the most profitable for the manufacturer. This is expected from Scenarios VI, VIII, and IX where the retailer does not invest in advertising in the second period and from Scenario VI where the retailer does not invest in advertising in the first period. The most intriguing case is Scenario IV where the retailer advertises in both periods, and the manufacturer chooses to offer a cooperative advertising support only in the first period. This scenario is, however, dominated by Scenario I, which is the manufacturer's preferred scenario in the areas of the parameters space where they are both feasible.

Second, the profitability of the remaining scenarios for the manufacturer depends on the values of the parameters α, β, and γ. For instance, when first-period advertising does not impact second-period demand ($\beta = 0$), regardless of the values of the short-term effect of retailer advertising (α) and the reference price effect (γ), the manufacturer always obtains the greatest profits from Scenario I, which is also always feasible. When first-period advertising negatively affects second-period demand ($\beta < 0$), the manufacturer may get the greatest profits from Scenarios I, II, and III. Qualitatively, this result does not change with the reference price effect (γ), although the areas where each of these scenarios is preferred do change. On the contrary, when first-period advertising positively affects second-period demand ($\beta > 0$), the manufacturer's scenario preferences change qualitatively with the reference price effect. Particularly, if there is no reference price effect ($\gamma = 0$), depending on the values of the other parameters, the manufacturer may get more profits from Scenarios I, III, and V. As the reference price effect increases, Scenario III may become either unfeasible or less profitable for the manufacturer and other alternatives such as Scenarios II and VI can become more attractive.

A take away from this analysis is that it matters whether or not consumers are sensitive to price changes over time in the context where the manufacturer has the possibility to influence the retailer's pricing and advertising strategies. Providing cooperative advertising support to the retailer in the two periods is not always a feasible and most profitable alternative to the manufacturer. In some cases, the manufacturer is better off offering a cooperative advertising program only in the second period (Scenario II) and in others, no cooperative advertising should be offered at all (Scenario V).

The analysis of the profitability of the different scenarios for the retailer in Figure 6 provides a different perspective.

Similar to the case of the manufacturer above, Scenarios I, II, III, VII, and IX do not appear in Figure 6, suggesting that they are never the best choice for the retailer. The absence of Scenario I, which is the interior solution, among the scenarios favored by the retailer is rather surprising. A close examination, however, shows that from the retailer's perspective, Scenario IV is always preferred to Scenario I. In Scenario IV, where the retailer advertises in the two periods and receives only cooperative advertising support in the first period, second-period cooperative advertising induces the retailer to advertise more than is necessary to maximize profits, given the carryover effects of first-period advertising.

The profitability of the remaining scenarios for the retailer depends on the values of the parameters α, β, and γ. For instance, when first-period advertising does not affect second-

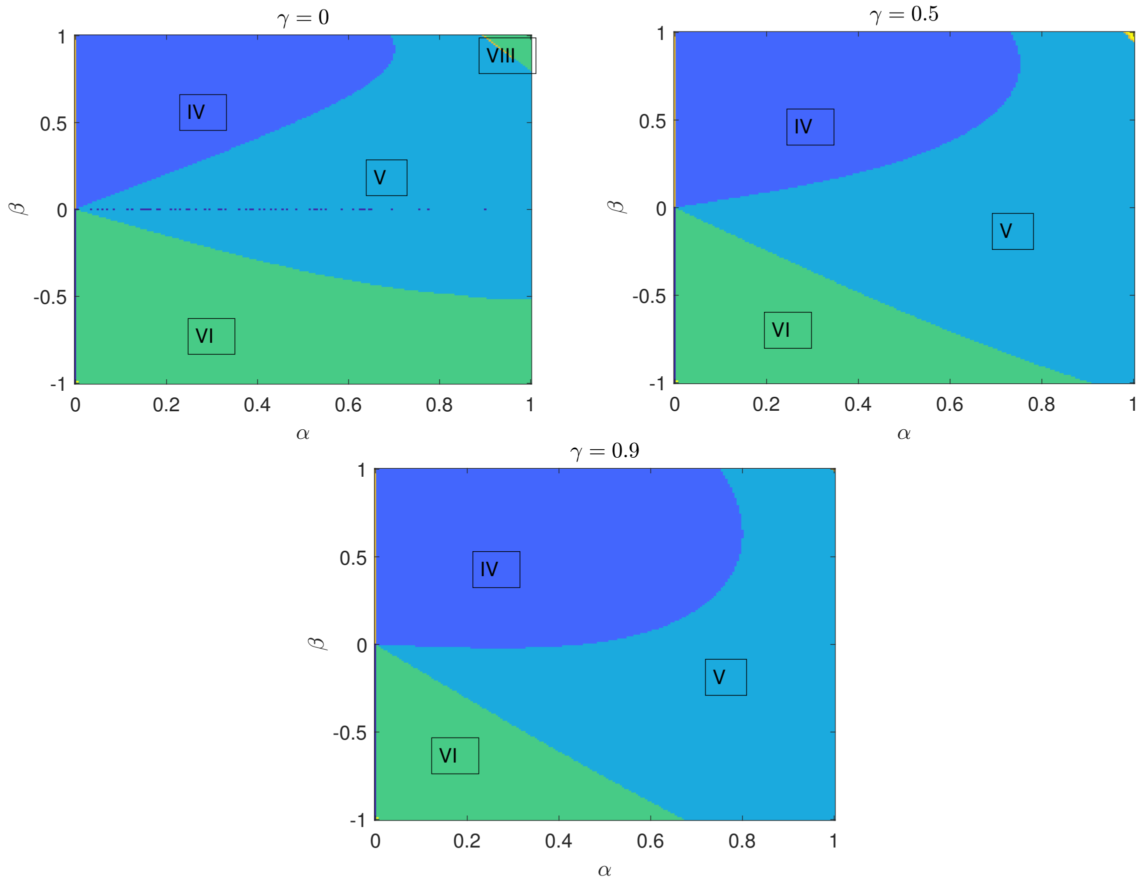

Figure 6: Comparison Retailer's profits across scenarios for different values of γ.

period demand ($\beta = 0$), regardless of the values of the short-term effect of retailer advertising (α) and the reference price effect (γ), the retailer always obtains the greatest profits from Scenario V, in which he advertises in both periods without any cooperative advertising support from the manufacturer. When the reference price effect is zero ($\gamma = 0$), the retailer may obtain the greatest profits from Scenarios IV, V, VI, and VIII; while the retailer obtains the largest profits only with scenarios II, V and VI when the reference price effect takes on relatively higher values. Scenario VIII, in which the retailer exclusively advertises in the first period and receives no cooperative advertising support, is desirable for the retailer only when consumers are not sensitive to price changes over time and first-period advertising not only heavily affects first-period demand, but also significantly expands second-period demand. In such a context, the retailer adopts a kind of pulsing advertising strategy, which allows to benefit from advertising carryover effects and to keep advertising expenditures low by not advertising in the second period and to keep second-period retail price high.

By contrasting Figures 5 and 6, it appears that the manufacturer's and retailer's scenario preferences diverge almost everywhere in the parameter space, regardless of the value of the reference price effect (γ). Both players prefer the same scenario only in the situation where the manufacturer prefers scenario V, in which the retailer advertises in the two periods and receives no cooperative advertising support. As illustrated in Figure 5, this occurs only when both the short-term and long-term effects of retailer advertising are simultaneously high. This is because the retailer is intrinsically motivated to advertise to take advantage of the short and long-term effects of advertising that there is no need, for the manufacturer, to provide an additional cooperative advertising incentive. On the other hand, divergent scenario preferences between channel members can lead to conflicts between them. It is believed, however, that as the channel leader, the manufacturer can avoid such conflicts by taking exogenous measures to share the surplus of her preferred scenarios with the retailer or to design cooperative advertising programs that better align the interests of all channel members. An example of such measures is providing layouts and copies to the retailer for local advertising as part of a cooperative advertising program to further reduce the retailer's advertising costs and also to control the message communicated to consumers (Martín-Herrán and Sigué, 2017a).

5. Conclusion

This paper extends the two-period model that Martín-Herrán and Sigué (2017a) propose to study cooperative advertising decisions in the presence of the long-term effects of retailer advertising to incorporate reference price effects. In particular, it examines how members of a bilateral monopoly channel should set pricing and advertising decisions within a two-period planning horizon in a context where consumers take the price of the first period as their reference price of the second period. Nine out of the twelve possible combinations of channel members' advertising decisions over the two periods are feasible at the equilibrium, ranging from the two most expected where the retailer advertises and receives cooperative advertising support from the manufacturer in each period (Scenario I) and the retailer advertises in each period and receives no cooperative advertising support (Scenario V). The other remaining scenarios are essentially corner solutions where either the retailer does not advertise or the manufacturer does not support retailer advertising in a period (See Proposition 1). Among others, reference price effects affect how the manufacturer and retailer set their respective pricing and advertising strategies over time and their preferences with respect to the nine feasible advertising scenarios at the equilibrium.

The theoretical and managerial implications of our findings are discussed below.

From a theoretical perspective, the findings of this research complement those of Martín-Herrán and Sigué (2017a) and provide a most robust integrative framework for considering the dynamic nature of pricing decisions and its impact on consumer demand in the cooperative advertising literature. The particularity of our framework is that it can prescribe pricing and advertising strategies when consumers are sensitive to price variations over time, but also when they are not, as in Martín-Herrán and Sigué (2017a). It can also reproduce the results of static models in very specific situations where retailer advertising has no long-term effects and where consumers are not sensitive to price changes over time. Of course, our framework is more appropriate than static models in all other situations where, at least, one of these dynamic effects exists.

Compared to Zhang et al. (2013) where differential game modeling is used, we also found that the reference price effects affect channel strategies and profits. However, except in the specific case where retail advertising has no long-term effect and consumers are not sensitive to price changes over time, our findings qualitatively differ from theirs. Our framework prescribes evolving pricing and advertising strategies to better adapt to changes in consumer demand. For instance, in situations where retailer advertising has no long-term effects and consumers are sensitive to price variations over time, it is optimal for retailers to adopt skimming pricing strategies and heavily invest in first-period advertising, with or without cooperative advertising support, to build a strong brand image and benefit from higher profit margins from the start. In other circumstances, retailers may find it optimal to adopt penetration pricing strategies instead and to invest heavily in first-period advertising for its ability to expand second-period demand.

The existence of nine feasible solutions at equilibrium, including those where the retailer does not advertise or the manufacturer does not provide cooperative advertising support in a period, is further evidence that the proposed framework fits better to the complex reality of business practices. Continuous and constant advertising activities and advertising supports over time as often prescribed in differential game models remain a possibility, but other advertising arrangements may be worth considering as well (Mahajan and Muller,1986; Martín-Herrán and Sigué, 2017a&b; Mesak and Ellis, 2009).

From a management perspective, our results support the view that manufacturers and retailers should be aware of the importance of whether or not consumers use recent past prices as benchmarks to gauge the attractiveness of current prices in their industry. In any case, their pricing and advertising strategies must take this into account to maximize their profits. In particular, in a market segmented based on consumer sensitivity to price changes over time, the optimal approach would be to adopt differentiated advertising and pricing strategies for each market segment in order to increase their market performance.

Finally, our goal was to show that incorporating reference price effects into the cooperative advertising literature using two-period modeling can generate new insights. This objective achieved, our paper opens the doors to at least three possible extensions. First, we adopted a simplified conceptualization of the reference price. Other more elaborate conceptualizations available in the literature can be explored. Second, we have considered that perceived losses and perceived gains with respect to the reference price have symmetric effects on consumer demand. Asymmetric effects may be considered as well. Finally, considering other channel structures where competition exists will be another way of extending this work.

Appendix A. Proof of Proposition 1

The nine scenarios described in Table 2 in Proposition 1 can be defined mathematically as follows:

- Scenario I: $a_1 > 0$, $a_2 > 0$, $0 < s_1 < 1$, $0 < s_2 < 1$;
- Scenario II: $a_1 > 0$, $a_2 > 0$, $s_1 = 0$, $0 < s_2 < 1$;
- Scenario III: $a_1 = 0$, $a_2 > 0$, $0 \le s_1 \le 1$, $0 < s_2 < 1$;
- Scenario IV: $a_1 > 0$, $a_2 > 0$, $0 < s_1 < 1$, $s_2 = 0$;
- Scenario V: $a_1 > 0$, $a_2 > 0$, $s_1 = 0$, $s_2 = 0$;
- Scenario VI: $a_1 = 0$, $a_2 > 0$, $0 \le s_1 \le 1$, $s_2 = 0$;
- Scenario VII: $a_1 > 0$, $a_2 = 0$, $0 < s_1 < 1, 0 \le s_2 \le 1$;
- Scenario VIII: $a_1 > 0$, $a_2 = 0$, $s_1 = 0$, $0 \le s_2 \le 1$;
- Scenario IX: $a_1 = 0$, $a_2 = 0$, $0 \le s_1 \le 1$, $0 \le s_2 \le 1$.

In what follows we derive these nine scenarios or equilibria.

The two-period Stackelberg game between the manufacturer (leader) and the retailer (follower) is played in four stages. The subgame-perfect equilibrium solutions are obtained by solving backwards the game. The game is solved in four stages.

Stage 4: At this stage of the game, the retailer looks at maximizing his second-period profits, and with this aim chooses the second-period price and advertising rate, p_2, and a_2, respectively. Therefore, the retailer's problem reads:

$$\max_{p_2,a_2} R_2 \tag{A.1}$$

where the retailer's second-period profits, R_2, and the second-period demand function, q_2, are given by

$$R_2 = (p_2 - w_2)q_2 - \frac{1}{2}(1 - s_2)a_2^2, \tag{A.2}$$

$$q_2 = g - p_2 + \beta a_1 + \alpha a_2 - \gamma(p_2 - p_1). \tag{A.3}$$

The retailer's second-period profits is a strictly concave function of his decision variables in this period, p_2, a_2 for any $\alpha, \gamma \in (0, 1)$ and $\beta \in (-1, 1)$. The first-order optimality conditions for an interior solution for problem (A.1) allow us to obtain the retailer's reaction functions. These functions express p_2 and a_2, as functions of the manufacturer's second-period decision variables, the wholesale price, w_2 and the cooperative advertising support offered by the manufacturer to the retailer, s_2, as well as of the retailer's first-period decision variables, retailer's price, p_1, and local advertising in the first period, a_1. The retailer's reaction functions in the case of an interior solution $a_2 > 0$ read:

$$p_2 = \frac{(1-s_2)(g+(1+\gamma)w_2+\gamma p_1+\beta a_1)-\alpha^2 w_2}{2(1+\gamma)(1-s_2)-\alpha^2}, \tag{A.4}$$

$$a_2 = \frac{(g-(1+\gamma)w_2+\beta a_1+\gamma p_1)\alpha}{2(1+\gamma)(1-s_2)-\alpha^2}. \tag{A.5}$$

We name this possibility as **First case** ($a_2 > 0$).

Alternatively, we can derive the retailer's reaction functions in the case of the corner solution $a_2 = 0$. In this case, the expression of p_2 is obtained from the optimality condition obtained from the maximization of the retailer's second-period profits with respect to p_2 taking into account that $a_2 = 0$. The retailer's reaction functions in this **Second case** ($a_2 = 0$) read:

$$p_2 = \frac{g+(1+\gamma)w_2+\gamma p_1+\beta a_1)}{2(1+\gamma)}, \tag{A.6}$$

$$a_2 = 0. \tag{A.7}$$

Stage 3: At this stage of the game the manufacturer looks at maximizing her second-period profits, and with this aim chooses the second-period wholesale price, w_2, and her cooperative advertising support offered to the retailer in the second period, s_2. Therefore, the manufacturer's problem reads:

$$\max_{w_2,s_2} M_2 \tag{A.8}$$

where the manufacturer's second-period profits, M_2, are given by

$$M_2 = w_2 q_2 - \frac{1}{2} s_2 a_2^2 \tag{A.9}$$

with q_2 the demand function in this period given in (A.3). At this stage of the game, the manufacturer as the Stackelberg leader knows the retailer's (follower's) reaction functions derived in Stage 4, and hence, takes into account these functions when making her optimal decisions on pricing and cooperative advertising in the second-period. Hence, the manufacturer substitutes either the reaction functions in (A.4) and (A.5) or alternatively in (A.6) and (A.7), in her objective function (A.8).

Solving this problem one can get the manufacturer's second-period decision variables, the wholesale price, w_2, and the cooperative advertising support offered to the retailer, s_2, as functions of the first-period retailer's price, p_1, and rate of local advertising, a_1.

In the following we consider the two different cases described in Stage 4.

First case: $a_2 > 0$. The retailer's reaction functions in (A.4) and (A.5) are considered.

The manufacturer's second-period profits is a strictly concave function of her decision variables in this period, w_2, s_2 for any $\alpha, \gamma \in (0,1)$ and $\beta \in (-1,1)$. The first-order conditions for problem (A.8) lead to two different pairs for w_2 and s_2. In the first pair $s_2 = 1$, which implies $a_2 = 0$, contradicting the hypothesis $a_2 > 0$ that applies in the first case. Hence, the other pair is the unique feasible case (**Subcase 1.1** ($a_2 > 0, 0 < s_2 < 1$)) and reads

$$w_2 = \frac{(g + \beta a_1 + \gamma p_1)(8(1+\gamma) - 3\alpha^2)}{(1+\gamma)(16(1+\gamma) - 9\alpha^2)}, \tag{A.10}$$

$$s_2 = \frac{1}{3}. \tag{A.11}$$

The second-period retail price and rate of local advertising as functions of the first-period price, p_1, and rate of local advertising, a_1, can be obtained substituting the expressions above into the retailer's reaction functions in (A.4) and (A.5):

$$p_2 = \frac{3(g + \beta a_1 + \gamma p_1)(4(1+\gamma) - \alpha^2)}{(1+\gamma)(16(1+\gamma) - 9\alpha^2)}, \tag{A.12}$$

$$a_2 = \frac{6\alpha(g + \beta a_1 + \gamma p_1)}{16(1+\gamma) - 9\alpha^2}. \tag{A.13}$$

Substituting the expressions (A.10), (A.11), (A.12) and (A.13) in (A.2) and (A.9), respectively, the second-period retailer's and manufacturer's optimal profits are obtained:

$$R_2 = \frac{4(g + \beta a_1 + \gamma p_1)^2(4(1+\gamma) - 3\alpha^2)}{(16(1+\gamma) - 9\alpha^2)^2}, \tag{A.14}$$

$$M_2 = \frac{2(g + \beta a_1 + \gamma p_1)^2}{16(1+\gamma) - 9\alpha^2}. \tag{A.15}$$

Alternatively, we can derive the second-period wholesale price when the manufacturer does not offer advertising support to the retailer in the second period, $s_2 = 0$. In this case, the expression of w_2 is obtained from the optimality condition obtained from the maximization of the manufacturer's second-period profits with respect to w_2 taking into account that $s_2 = 0$. Therefore, in this case (**Subcase 1.2** ($a_2 > 0, s_2 = 0$)),

$$w_2 = \frac{(2(\gamma+1) - \alpha^2)\,(a_1\beta + g + \gamma p_1)}{2(\gamma+1)\,(2(\gamma+1) - \alpha^2)}, \tag{A.16}$$

$$s_2 = 0. \tag{A.17}$$

The second-period retail price and rate of local advertising as functions of the first-period price, p_1, and rate of local advertising, a_1, can be obtained substituting w_2 by (A.16) and s_2 by zero into the retailer's reaction functions in (A.4) and (A.5):

$$p_2 = \frac{\left(3(\gamma+1)-\alpha^2\right)\left(a_1\beta+g+\gamma p_1\right)}{2(\gamma+1)\left(2(\gamma+1)-\alpha^2\right)}, \tag{A.18}$$

$$a_2 = \frac{\alpha(a_1\beta+g+\gamma p_1)}{2\left(2(\gamma+1)-\alpha^2\right)}. \tag{A.19}$$

Substituting the expressions (A.16), (A.17), (A.18) and (A.19) in (A.2) and (A.9), respectively, the second-period retailer's and manufacturer's optimal profits are obtained:

$$R_2 = \frac{(a_1\beta+g+\gamma p_1)^2}{8\left(2(\gamma+1)-\alpha^2\right)}, \tag{A.20}$$

$$M_2 = \frac{(a_1\beta+g+\gamma p_1)^2}{4\left(2(\gamma+1)-\alpha^2\right)}. \tag{A.21}$$

Second case: $a_2 = 0$. The retailer's reaction functions in (A.6) and (A.7) are considered.

In this case because the manufacturer's problem expressed in (A.8) does not depend on s_2, from the first-order optimality condition for this problem we get the following expression for w_2 and any value of s_2, $0 \leq s_2 \leq 1$:

$$w_2 = \frac{a_1\beta+g+\gamma p_1}{2(\gamma+1)}. \tag{A.22}$$

The second-period retail price and rate of local advertising as functions of the first-period price, p_1, and rate of local advertising, a_1, can be obtained substituting w_2 by (A.22) into the retailer's reaction functions in (A.6):

$$p_2 = \frac{3(g+\beta a_1+\gamma p_1)}{4(1+\gamma)}, \tag{A.23}$$

$$a_2 = 0. \tag{A.24}$$

The second-period retailer's and manufacturer's optimal profits in this case are given by

$$R_2 = \frac{(g+\beta a_1+\gamma p_1)^2}{16(1+\gamma)}, \tag{A.25}$$

$$M_2 = \frac{(g+\beta a_1+\gamma p_1)^2}{8(1+\gamma)}. \tag{A.26}$$

Stage 2: In the first period the retailer looks at maximizing his total profits over the two periods $R = R_1 + R_2$, and with this aim chooses the retail price, p_1, and his local advertising effort, a_1.

First case. Subcase 1.1: $a_2 > 0$ and $0 < s_2 < 1$.

The retailer's total profits read:

$$R = (p_1 - w_1)(g - p_1 + \alpha a_1) - \frac{1}{2}(1 - s_1)a_1^2 + \frac{4(g + \beta a_1 + \gamma p_1)^2(4(1 + \gamma) - 3\alpha^2)}{(16(1 + \gamma) - 9\alpha^2)^2},$$

where the second-period retailer's profits have been replaced by their expression in (A.14).

This last expression is a concave function in the retailer's first-period decision variables, p_1, and a_1, if and only if the following condition applies:

$$\left(8\gamma^2\left(3\alpha^2-4(\gamma+1)\right)+2\left(9\alpha^2-16(\gamma+1)\right)^2\right)\left(8\beta^2\left(3\alpha^2-4(\gamma+1)\right)+(1-s_1)\left(9\alpha^2-16(\gamma+1)\right)^2\right)$$
$$-\left(8\beta\gamma\left(3\alpha^2-4(\gamma+1)\right)-\alpha\left(9\alpha^2-16(\gamma+1)\right)^2\right)^2 \geq 0. \tag{A.27}$$

The maximization of the retailer's total profits gives two pairs of first-period retailer's reaction functions. That is, two pairs of first-period retail price and advertising rate, (p_1, a_1), as functions of the manufacturer's first-period decision variables, the wholesale price, w_1, and the cooperative advertising support rate offered to the retailer, s_1.

The first pair that corresponds to an interior solution of a_1 (**Subcase 1.1.1** $(a_2 > 0, 0 < s_2 < 1, a_1 > 0)$) is given as follows:

$$p_1 = \frac{Nump_1}{Denp_1}, \tag{A.28}$$

$$a_1 = \frac{Numa_1}{Dena_1}, \tag{A.29}$$

where

$$Nump_1 = \left(8\gamma g(3\alpha^2-4(\gamma+1)) - (9\alpha^2-16(\gamma+1))^2(g+w_1)\right)\left(8\beta^2(3\alpha^2-4(\gamma+1))+(1-s_1)(9\alpha^2-16(\gamma+1))^2\right)$$
$$-\left(8\beta\gamma\left(3\alpha^2-4(1+\gamma)\right)-\alpha\left(9\alpha^2-16(\gamma+1)\right)^2\right)\left(8\beta g\left(3\alpha^2-4(\gamma+1)\right)+\alpha w_1\left(9\alpha^2-16(\gamma+1)\right)^2\right),$$

$$Denp_1 = -\left(8\gamma^2\left(3\alpha^2+4(\gamma+1)\right)+2\left(9\alpha^2-16(\gamma+1)\right)^2\right)\left(8\beta^2\left(3\alpha^2-4(\gamma+1)\right)+(1-s_1)(9\alpha^2-16(\gamma+1))^2\right)$$
$$+8\beta\gamma\left(3\alpha^2-4(\gamma+1)\right)-\alpha\left(9\alpha^2-16(\gamma+1)\right)^2\Big)^2,$$

$$Numa_1 = 81\alpha^5(w_1-g)+24\alpha^2(\alpha(13\gamma+12)g+\beta(\gamma+2)g+\alpha((\gamma-12)\gamma-12)w_1+\beta\gamma w_1)$$
$$-32(\gamma+1)(\alpha(9\gamma+8)g+\beta(\gamma+2)g+\alpha((\gamma-8)\gamma-8)w_1+\beta\gamma w_1),$$

$$Dena_1 = 24\alpha^2\left(-2\left(\beta^2+6\left(\alpha^2-2(1-s_1)\right)\right)-2\gamma(\alpha(6\alpha+\beta)-12(1-s_1))+\gamma^2(s_1-1)\right)$$
$$+32(\gamma+1)\left(2\left(4\alpha^2+\beta^2-8(1-s_1)\right)+2\gamma(\alpha(4\alpha+\beta)-8(1-s_1))+\gamma^2(1-s_1)\right)+81\alpha^4\left(\alpha^2-2(1-s_1)\right).$$

The second pair corresponds to the corner solution, $a_1 = 0$, (**Subcase 1.1.2** $(a_2 > 0, 0 < s_2 < 1, a_1 = 0)$):

$$a_1 = 0, \quad p_1 = \frac{\left(9\alpha^2-16(\gamma+1)\right)^2(g+w_1)-8\gamma g\left(3\alpha^2-4(\gamma+1)\right)}{8\gamma^2\left(3\alpha^2-4(\gamma+1)\right)+2\left(9\alpha^2-16(\gamma+1)\right)^2}. \tag{A.30}$$

First case. Subcase 1.2: $a_2 > 0$ and $s_2 = 0$.

The retailer's total profits reads:

$$R = (p_1 - w_1)(g - p_1 + \alpha a_1) - \frac{1}{2}(1 - s_1)a_1^2 + \frac{(a_1\beta + g + \gamma p_1)^2}{8\left(2(\gamma + 1) - \alpha^2\right)},$$

where the second-period retailer's profits in (A.20) have been replaced.

The retailer's total profits is a concave function in his first-period decision variables, p_1, and a_1, if and only if the following condition applies:

$$\left(\frac{\gamma^2}{8(\gamma+1) - 4\alpha^2} - 2\right)\left(\frac{\beta^2}{8(\gamma+1) - 4\alpha^2} - (1 - s_1)\right) - \left(\frac{\beta\gamma}{8(\gamma+1) - 4\alpha^2} + \alpha\right)^2 \geq 0. \quad \text{(A.31)}$$

The maximization of the retailer's total profits gives two pairs of first-period retailer's reaction functions. That is, two pairs of first-period retail price and advertising rate, (p_1, a_1), as functions of the manufacturer's first-period decision variables, the wholesale price, w_1, and the cooperative advertising support rate offered to the retailer, s_1.

The first pair corresponding to an interior solution of a_1, (**Subcase 1.2.1** $(a_2 > 0, s_2 = 0, a_1 > 0)$) reads:

$$p_1 = \frac{Nump_1}{Denp_1}, \quad \text{(A.32)}$$

$$a_1 = \frac{Numa_1}{Dena_1}, \quad \text{(A.33)}$$

where

$$\begin{aligned}
Nump_1 = {} & g\left(-4\alpha^2+\beta(\alpha-\beta)+9\gamma+s_1\left(4\alpha^2-9\gamma-8\right)+8\right) \\
& +w_1\left(\alpha\left(4\alpha\left(\alpha^2-3\right)-\gamma(8\alpha+\beta)\right)-\beta^2+8\gamma+4s_1\left(\alpha^2-2\gamma-2\right)+8\right), \\
Denp_1 = {} & -2\beta^2+4\left(\alpha^2-2\right)\left(\alpha^2-2(1-s_1)\right)-2\gamma(\alpha(4\alpha+\beta)-8(1-s_1))-\gamma^2(1-s_1), \\
Numa_1 = {} & g\left(-4\alpha^3+\alpha(9\gamma+8)+\beta(\gamma+2)\right)+w_1\left(4\alpha^3+\alpha((\gamma-8)\gamma-8)+\beta\gamma\right), \\
Dena_1 = {} & Denp_1.
\end{aligned}$$

The second pair corresponds to the corner solution, $a_1 = 0$, (**Subcase 1.2.2** $(a_2 > 0, s_2 = 0, a_1 = 0)$):

$$a_1 = 0, \quad p_1 = \frac{4\left(2-\alpha^2\right)(g+w_1)+\gamma(9g+8w_1)}{16-(\gamma-16)\gamma-8\alpha^2}. \quad \text{(A.34)}$$

Second case: $a_2 = 0$ and $s_2 \geq 0$.

The retailer's total profits reads:

$$R = (p_1 - w_1)(g - p_1 + \alpha a_1) - \frac{1}{2}(1 - s_1)a_1^2 + \frac{(g + \beta a_1 + \gamma p_1)^2}{16(1+\gamma)},$$

where the second-period retailer's profits given by (A.25) have been replaced.

The retailer's total profits is a concave function in the retailer's first-period decision variables, p_1, and a_1, if and only if the following condition applies:

$$\left(2 - \frac{\gamma^2}{8(\gamma+1)}\right)\left(1 - s_1 - \frac{\beta^2}{8(\gamma+1)}\right) - \left(\alpha + \frac{\beta\gamma}{8(\gamma+1)}\right)^2 \geq 0. \tag{A.35}$$

The maximization of the retailer's total profits gives two pairs of first-period retailer's reaction functions. That is, two pairs of first-period retail price and advertising rate, (p_1, a_1), as functions of the manufacturer's first-period decision variables, the wholesale price, w_1, and the cooperative advertising support rate offered to the retailer, s_1.

The first pair corresponding to an interior solution of a_1, (**Subcase 2.1** $(a_2 = 0, s_2 \geq 0, a_1 > 0)$) reads:

$$p_1 = \frac{Nump_1}{Denp_1}, \tag{A.36}$$

$$a_1 = \frac{Numa_1}{Dena_1}, \tag{A.37}$$

where

$$\begin{aligned}
Nump_1 &= g\left((9\gamma+8)(1-s_1) + \beta(\alpha-\beta)\right) - w_1\left(8(\gamma+1)(\alpha^2 - (1-s_1)) + \beta(\alpha\gamma+\beta)\right),\\
Denp_1 &= 8\alpha^2(\gamma+1) + 2\beta(\alpha\gamma+\beta) - (16 - (\gamma-16)\gamma)(1-s_1),\\
Numa_1 &= g(\alpha(9\gamma+8) + \beta(\gamma+2)) + w_1(\alpha((\gamma-8)\gamma-8) + \beta\gamma),\\
Dena_1 &= Denp_1.
\end{aligned}$$

The second pair corresponds to the corner solution, $a_1 = 0$, (**Subcase 2.2** $(a_2 = 0, s_2 \geq 0, a_1 = 0)$):

$$a_1 = 0, \quad p_1 = \frac{(9\gamma+8)g + 8(\gamma+1)w_1}{16(\gamma+1) - \gamma^2}. \tag{A.38}$$

Stage 1: At this stage of the game, the manufacturer looks at maximizing her total profits $M = M_1 + M_2$ and with this aim she chooses the first-period wholesale price, w_1, and the cooperative advertising support rate offered to the retailer, s_1.

<u>Subcase 1.1.1 $(a_2 > 0, 0 < s_2 < 1, a_1 > 0)$</u>

The manufacturer's total profits reads

$$M = w_1(g - p_1 + \alpha a_1) - \frac{1}{2}s_1 a_1^2 + \frac{2(g + \beta a_1 + \gamma p_1)^2}{16(1+\gamma) - 9\alpha^2},$$

where the second-period manufacturer's profits given by (A.15) have been taken into account.

Substituting the retailer's reaction functions given by (A.28) and (A.29) in the expression above, one obtains the manufacturer's total profits to be maximized.

The first-order optimality conditions for the maximization of the manufacturer's total profits with respect to w_1 and s_1 lead to two pairs of solutions. However, one of the possibilities implies $a_2 = 0$, and therefore, it is unfeasible in this scenario.

The unique feasible interior case (**Subcase 1.1.1.1**, $a_2 > 0, 0 < s_2 < 1, a_1 > 0, 0 < s_1 < 1$, **Scenario I** in Proposition 1) corresponds to the following pair of solutions:

$$w_1 = g\frac{Numw_1}{Denw_1}, \tag{A.39}$$

$$s_1 = \frac{Nums_1}{Dens_1}, \tag{A.40}$$

where

$$\begin{aligned} Numw_1 = {} & 19683\alpha^{10} - 1944\alpha^8(70\gamma + 99) - 3888\alpha^7\beta(7\gamma - 4) - 144\alpha^6\left(270\beta^2 - 22\gamma^3 - 2415\gamma^2 - 7716\gamma - 5184\right) \\ & -192\alpha^4\left(3\beta^2\left(7\gamma^2 - 342\gamma - 348\right) + 2\left(34\gamma^4 + 1019\gamma^3 + 5951\gamma^2 + 8710\gamma + 3744\right)\right) \\ & -576\alpha^5\beta\left(7\gamma^3 - 235\gamma^2 - 99\gamma + 141\right) + 1536\alpha^3\beta\left(7\gamma^4 - 140\gamma^3 - 207\gamma^2 + 32\gamma + 92\right) \\ & +512\alpha^2(\gamma+1)\left(3\beta^2\left(7\gamma^2 - 218\gamma - 224\right) + 30\gamma^4 + 270\gamma^3 + 3488\gamma^2 + 5936\gamma + 2688\right) \\ & -1024(\gamma+1)^2\left(\beta^2\left(7\gamma^2 - 186\gamma - 192\right) + 4\left(\gamma^4 - 7\gamma^3 + 136\gamma^2 + 272\gamma + 128\right)\right) \\ & -1024\alpha\beta(\gamma+1)^2\left(7\gamma^3 - 123\gamma^2 - 48\gamma + 80\right), \end{aligned}$$

$$Denw_1 = 59049\alpha^{10} - 3888\alpha^8\left(2\gamma^2+108\gamma+135\right) - 85536\alpha^7\beta\gamma + 576\alpha^5\beta\gamma\left(2\gamma^2+771(1+\gamma)\right)$$
$$-144\alpha^6\left(540\beta^2-4\gamma^4-300\gamma^3-7887\gamma^2-20736\gamma-12960\right)-3072\alpha^3\beta\gamma\left(\gamma^3+251\gamma^2+500\gamma+250\right)$$
$$+192\alpha^4\left(3\beta^2\left(\gamma^2+696\gamma+696\right)-8\left(\gamma^5+53\gamma^4+878\gamma^3+3850\gamma^2+5184\gamma+2160\right)\right)$$
$$+512\alpha^2(\gamma+1)\left(2\left(\gamma^5+49\gamma^4+444\gamma^3+3852\gamma^2+6336\gamma+2880\right)-3\beta^2\left(\gamma^2+448(\gamma+1)\right)\right)$$
$$+2048\alpha\beta\gamma(\gamma+1)^2\left(\gamma^2+216(\gamma+1)\right)+1024(\gamma+1)^2\left(\beta^2\left(\gamma^2+384(\gamma+1)\right)+128\left(\gamma^3-7\gamma^2-16\gamma-8\right)\right),$$
$$Nums_1 = -243\alpha^9\left(4\gamma^2-12\gamma-27\right)-972\alpha^8\beta(5\gamma-3)-36\alpha^7\left(108\beta^2-80\gamma^3+489\gamma^2+2097\gamma+1296\right)$$
$$+288\alpha^6\beta\left(10\gamma^3+91\gamma^2-3\gamma-117\right)+96\alpha^5\left(3\beta^2\left(20\gamma^2+89\gamma+75\right)+2\gamma^4+570\gamma^3+2887\gamma^2+3615\gamma+1296\right)$$
$$+96\alpha^4\beta\left(6\beta^2(5\gamma+4)-80\gamma^4-542\gamma^3-49\gamma^2+1697\gamma+1284\right)$$
$$-512\alpha^3(\gamma+1)\left(\beta^2\left(30\gamma^2+99\gamma+78\right)+11\gamma^4+149\gamma^3+702\gamma^2+852\gamma+288\right)$$
$$-512\alpha^2\beta(\gamma+1)\left(3\beta^2(5\gamma+4)-10\gamma^4-53\gamma^3+131\gamma^2+534\gamma+360\right)$$
$$+1024\alpha(\gamma+1)^2\left(\beta^2\left(10\gamma^2+31\gamma+24\right)+3\gamma^4+35\gamma^3+168\gamma^2+200\gamma+64\right)$$
$$+1024\beta(\gamma+1)^2\left(\beta^2(5\gamma+4)+3\gamma^3+59\gamma^2+152\gamma+96\right),$$
$$Dens_1 = 19683\alpha^9-972\alpha^7\left(5\gamma^2+161\gamma+144\right)-3888\alpha^6\beta(5\gamma+8)+288\alpha^5\left(2\gamma^4+98\gamma^3+1695\gamma^2+2895\gamma+1296\right)$$
$$+288\alpha^4\beta\left(2\gamma^3+359\gamma^2+921\gamma+564\right)-1536\alpha^3(\gamma+1)^2\left(\gamma^3+34\gamma^2+388\gamma+288\right)$$
$$-1536\alpha^2\beta(\gamma+1)^2\left(\gamma^2+118\gamma+184\right)+1024\alpha(\gamma+1)^3\left(\gamma^3+32\gamma^2+280\gamma+192\right)$$
$$+1024\beta(\gamma+1)^3\left(\gamma^2+104\gamma+160\right).$$

We compute the first (h_{11}) and second ($h_{11}h_{22}-h_{12}h_{21}$) minors of the Hessian matrix of function M with respect to the manufacturer's decisions variables in the first period, w_1 and s_1. h_{11} depends on s_1, while $h_{11}h_{22}-h_{12}h_{21}$ depends on s_1 and w_1. The expressions of the entries of the Hessian matrix, h_{ij}, are cumbersome and they do not provide any insight. For brevity the expressions of the first and second minor are omitted. The sign of these minors once the expressions in (A.39) and (A.40) have been replaced is not defined. However, when Scenario I ($a_1>0, a_2>0, 0<s_1<1, 0<s_2<1$) described in Proposition 1 has been considered, in all the numerical simulations we have checked that the quadratic form associated with the Hessian matrix is negative semidefined, ensuring that M is a concave function, and that the interior solution given by (A.39) and (A.40) is a maximum.

The second possibility corresponds to the following corner option (**Subcase 1.1.1.2** ($a_2>0, 0<s_2<1, a_1>0, s_1=0$), **Scenario II** in Proposition 1):

$$w_1 = \frac{g}{2}\frac{Numw_1}{Denw_1}, \qquad \text{(A.41)}$$
$$s_1 = 0, \qquad \text{(A.42)}$$

where

$$
\begin{aligned}
Numw_1 =\, & 243\alpha^{10}(4\gamma+27) - 972\alpha^9\beta(\gamma-1) - 54\alpha^8\left(18\beta^2+130\gamma^2+1030\gamma+1107\right) \\
& +72\alpha^7\beta\left(100\gamma^2-108\gamma-111\right) - 12\alpha^6\left(24\beta^2\left(4\gamma^2-29\gamma+6\right)-1576\gamma^3-14189\gamma^2-31210\gamma-18144\right) \\
& -192\alpha^5\beta\left(6\beta^2(2\gamma-1)+89\gamma^3-115\gamma^2-325\gamma-127\right) \\
& -32\alpha^4\left(36\beta^4-6\beta^2\left(16\gamma^3-89\gamma^2+32\gamma+134\right)+706\gamma^4+6957\gamma^3+25029\gamma^2+31162\gamma+12384\right) \\
& +512\alpha^3\beta(\gamma+1)\left(6\beta^2(2\gamma-1)+29\gamma^3-97\gamma^2-184\gamma-64\right) \\
& +512\alpha^2(\gamma+1)\left(6\beta^4-\beta^2\left(4\gamma^3-29\gamma^2+74\gamma+104\right)+21\gamma^4+177\gamma^3+1084\gamma^2+1632\gamma+704\right) \\
& -1024\alpha\beta(\gamma+1)^2\left(\beta^2(4\gamma-2)+3\gamma^3-31\gamma^2-48\gamma-16\right) \\
& -1024(\gamma+1)^2\left(2\beta^4+\beta^2\left(3\gamma^2-30\gamma-32\right)+\gamma^4-7\gamma^3+136\gamma^2+272\gamma+128\right), \\
Denw_1 =\, & 243\alpha^{10}\left(2\gamma^2+27\right)+972\alpha^9\beta\gamma+54\alpha^8\left(9\beta^2-32\gamma^3-50\gamma^2-864\gamma-1107\right)-36\alpha^7\beta\gamma\left(32\gamma^2+96\gamma+339\right) \\
& -6\alpha^6\left(144\beta^2\left(4\gamma^2+2\gamma+11\right)-160\gamma^4-1088\gamma^3-21001\gamma^2-57024\gamma-36288\right) \\
& -384\alpha^5\beta\gamma\left(9\beta^2-8\gamma^3-13\gamma^2-124\gamma-116\right) \\
& -192\alpha^4\left(6\beta^4-\beta^2\left(48\gamma^3+53\gamma^2+214\gamma+206\right)-8\gamma^5+8\gamma^4+710\gamma^3+3526\gamma^2+4896\gamma+2064\right) \\
& +1024\alpha^3\beta\gamma(\gamma+1)\left(9\beta^2-2\gamma^3+\gamma^2-63\gamma-63\right) \\
& +512\alpha^2(\gamma+1)\left(6\beta^4-3\beta^2\left(4\gamma^3+3\gamma^2+40\gamma+40\right)-2\gamma^5-2\gamma^4+71\gamma^3+903\gamma^2+1536\gamma+704\right) \\
& -2048\alpha\beta\gamma(\gamma+1)^2\left(3\beta^2+\gamma^2-16\gamma-16\right)-1024(\gamma+1)^2\left(2\beta^4+\beta^2\left(\gamma^2-32\gamma-32\right)-16\left(\gamma^3-7\gamma^2-16\gamma-8\right)\right)
\end{aligned}
$$

Subcase 1.1.2 ($a_2 > 0, 0 < s_2 < 1, a_1 = 0$)

The manufacturer's total profits read:

$$M = w_1(g - p_1 + \alpha a_1) - \frac{1}{2}s_1 a_1^2 + \frac{2(g + \beta a_1 + \gamma p_1)^2}{16(1+\gamma) - 9\alpha^2},$$

where the second-period manufacturer's profits given by (A.15) have been replaced.

Substituting the retailer's reaction functions given by (A.30) in the expression above, one obtains the manufacturer's total profits to be maximized. In this case, because the manufacturer's problem does not depend on s_1, from the optimality conditions we get the expression for w_1 and any value of s_1, $0 \leq s_1 \leq 1$:

$$w_1 = g\frac{Numw_1}{Denw_1}, \tag{A.43}$$

where

$$
\begin{aligned}
Numw_1 =\, & 6561\alpha^8+486\alpha^6\left(3\gamma^2-98\gamma-96\right)+144\alpha^4\left(2\gamma^4-43\gamma^3+861\gamma^2+1770\gamma+864\right) \\
& -768\alpha^2(\gamma+1)^2\left(\gamma^3-12\gamma^2+208\gamma+192\right)+512(\gamma+1)^3\left(\gamma^3-8\gamma^2+144\gamma+128\right), \\
Denw_1 =\, & 2\left(9\alpha^2-16(\gamma+1)\right)^2\left(81\alpha^4+3\alpha^2\left(7\gamma^2-96\gamma-96\right)-32\left(\gamma^3-7\gamma^2-16\gamma-8\right)\right).
\end{aligned}
$$

This case corresponds to **Scenario III** ($a_2 > 0, 0 < s_2 < 1, a_1 = 0, 0 \leq s_1 \leq 1$) in Proposition 1. For any $\alpha, \gamma \in (0,1)$ and $\beta \in (-1,1)$, the manufacturer's total profits is a strictly concave function of her decision variable, w_1.

<u>Subcase 1.2.1 ($a_2 > 0, s_2 = 0, a_1 > 0$)</u>

The manufacturer's total profits read:

$$M = w_1(g - p_1 + \alpha a_1) - \frac{1}{2}s_1 a_1^2 + \frac{(a_1\beta + g + \gamma p_1)^2}{4\left(2(\gamma+1) - \alpha^2\right)},$$

where the second-period manufacturer's profits given by (A.21) have been taken into account.

Substituting the retailer's reaction functions given by (A.32) and (A.33) in the expression above, one obtains the manufacturer's total profits to be maximized. The maximization of the manufacturer's total profits with respect to w_1 and s_1 gives two pairs of solutions. However, one of the pairs implies $a_1 = 0$, and therefore, it is unfeasible in this scenario.

The unique feasible interior case (**Subcase 1.2.1.1**, $a_2 > 0, s_2 = 0, a_1 > 0, 0 < s_1 < 1$) **Scenario IV** in Proposition 1) corresponds to the following pair of solutions:

$$w_1 = g\frac{Numw_1}{Denw_1}, \tag{A.44}$$

$$s_1 = g\frac{Nums_1}{Dens_1}, \tag{A.45}$$

where

$$\begin{aligned}
Numw_1 =\ & 48\alpha^6 - 4\alpha^4\left(\gamma^2+47\gamma+80\right) - 8\alpha^3\beta(8\gamma-5) + \alpha^2\left(-96\beta^2+9\gamma^3+176\gamma^2+920\gamma+704\right)\\
&+\alpha\beta\left(-7\gamma^3+123\gamma^2+48\gamma-80\right)+\beta^2\left(-7\gamma^2+186\gamma+192\right)-4\left(\gamma^4-7\gamma^3+136\gamma^2+272\gamma+128\right),\\
Denw_1 =\ & 144\alpha^6 - 8\alpha^4\left(3\gamma^2+72\gamma+104\right) - 216\alpha^3\beta\gamma + \alpha^2\left(-192\beta^2+\gamma^4+48\gamma^3+560\gamma^2+2176\gamma+1600\right)\\
&+2\alpha\beta\gamma\left(\gamma^2+216\gamma+216\right)+\beta^2\left(\gamma^2+384\gamma+384\right)+128\left(\gamma^3-7\gamma^2-16\gamma-8\right),\\
Nums_1 =\ & 4\alpha^5(3\gamma+4) - 12\alpha^4\beta(\gamma-1) - \alpha^3\left(12\beta^2+\gamma^3+40\gamma^2+124\gamma+64\right) + \alpha^2\beta\left(5\gamma^3+26\gamma^2-28\gamma-72\right)\\
&+\alpha\left(\beta^2\left(10\gamma^2+31\gamma+24\right)+3\gamma^4+35\gamma^3+168\gamma^2+200\gamma+64\right)+\beta\left(\beta^2(5\gamma+4)+3\gamma^3+59\gamma^2+152\gamma+96\right),\\
Dens_1 =\ & 48\alpha^5 - 4\alpha^3\left(4\gamma^2+59\gamma+48\right) - 4\alpha^2\beta(13\gamma+20) + \alpha\left(\gamma^4+33\gamma^3+312\gamma^2+472\gamma+192\right)\\
&+\beta\left(\gamma^3+105\gamma^2+264\gamma+160\right).
\end{aligned}$$

We compute the first and second minors of the Hessian matrix of function M with respect to the manufacturer's decision variables in the first period, w_1 and s_1. These minors are long expressions and they are omitted here. These expressions can take positive or negative values. In the numerical simulations when Scenario IV ($a_1 > 0, a_2 > 0, 0 < s_1 < 1, s_2 = 0$) is considered, we check that the quadratic form associated with the Hessian matrix is negative semidefined, and hence, M is a concave function, and the interior solution given by (A.44) and (A.45) is a maximum.

The second possibility corresponds to the following corner option (**Subcase 1.2.1.2** ($a_2 > 0, s_2 = 0, a_1 > 0, s_1 = 0$), **Scenario V** in Proposition 1):

$$w_1 = \frac{g}{2}\frac{Numw_1}{Denw_1}, \tag{A.46}$$
$$s_1 = 0, \tag{A.47}$$

where

$$\begin{aligned}Numw_1 =& \left(\alpha^2(\gamma-4)+\alpha\beta(1-\gamma)-\beta^2-\gamma^2+7\gamma+8\right)\left(-4\alpha^4+8\alpha^2(\gamma+2)+2\alpha\beta\gamma+2\beta^2+\gamma^2-16(\gamma+1)\right)\\ &+8\left(\alpha^2-2(\gamma+1)\right)\left(\alpha^2-\alpha\beta-\gamma-2\right)\left(\alpha^2\gamma+\alpha\beta-\gamma\right),\\ Denw_1 =& \left(4\alpha^4-8\alpha^2(\gamma+2)-2\alpha\beta\gamma-2\beta^2-\gamma^2+16\gamma+16\right)\left(\alpha^2\left(\gamma^2+4\right)+2\alpha\beta\gamma+\beta^2-8(\gamma+1)\right)\\ &-4\left(\alpha^2-2(\gamma+1)\right)\left(\alpha^2\gamma+\alpha\beta-\gamma\right)^2.\end{aligned}$$

Subcase 1.2.2 ($a_2 > 0,\ s_2 = 0,\ a_1 = 0$)

The manufacturer's total profits read follows:

$$M = w_1(g-p_1+\alpha a_1) - \frac{1}{2}s_1 a_1^2 + \frac{(a_1\beta+g+\gamma p_1)^2}{4\left(2(\gamma+1)-\alpha^2\right)},$$

where the second-period manufacturer's profits given by (A.21) have been replaced.

Substituting the retailer's reaction functions given by (A.34) in the expression above, one obtains the manufacturer's total profits to be maximized. In this case, because the manufacturer's problem does not depend on s_1, from the optimality conditions we get the expression for w_1 and for any value of s_1, $0 \le s_1 \le 1$:

$$w_1 = g\frac{Numw_1}{Denw_1}, \tag{A.48}$$

where

$$\begin{aligned}Numw_1 &= 4\left(\alpha^2+34\right)\gamma^2 - 136\left(\alpha^2-2\right)\gamma + 32\left(\alpha^2-2\right)^2 + \gamma^4 - 7\gamma^3,\\ Denw_1 &= 16\left(\left(\alpha^2+14\right)\gamma^2 + 16\left(2-\alpha^2\right)\gamma + 4\left(\alpha^2-2\right)^2 - 2\gamma^3\right).\end{aligned}$$

This case corresponds to **Scenario VI** ($a_2 > 0, s_2 = 0, a_1 = 0, 0 \le s_1 \le 1$) in Proposition 1. For any $\alpha, \gamma \in (0,1)$ and $\beta \in (-1,1)$, the manufacturer's total profits is a strictly concave function of her decision variable, w_1.

Subcase 2.1 ($a_2 = 0$, any value of s_2, $0 \le s_2 \le 1$, $a_1 > 0$)

The manufacturer's total profits read:

$$M = w_1(g - p_1 + \alpha a_1) - \frac{1}{2}s_1 a_1^2 + \frac{(a_1\beta + g + \gamma p_1)^2}{8(\gamma+1)},$$

where the second-period manufacturer's profits given by (A.26) have been taken into account.

Substituting the retailer's reaction functions given by (A.36) and (A.37) in the expression above, one obtains the manufacturer's total profits to be maximized.

The maximization of the manufacturer's total profits with respect to w_1 and s_1 gives two pairs of solutions. However, one of the pairs implies $a_1 = 0$, and therefore, it is unfeasible in this scenario.

The unique feasible interior case (**Subcase 2.1.1**, $a_2 = 0$, any value of $s_2, 0 \leq s_2 \leq 1, a_1 > 0, 0 < s_1 < 1$ **Scenario VII** in Proposition 1) corresponds to the following pair of solutions:

$$w_1 = g\frac{Numw_1}{Denw_1}, \tag{A.49}$$

$$s_1 = g\frac{Nums_1}{Dens_1}, \tag{A.50}$$

where

$$\begin{aligned} Numw_1 =& \alpha^2\left(9\gamma^3+192\gamma^2+376\gamma+192\right) - \alpha\beta\left(7\gamma^3-123\gamma^2-48\gamma+80\right) - \beta^2\left(7\gamma^2-186\gamma-192\right) \\ & -4\left(\gamma^4-7\gamma^3+136\gamma^2+272\gamma+128\right), \\ Denw_1 =& \alpha^2\left(\gamma^2+24(\gamma+1)\right)^2 + 2\alpha\beta\gamma\left(\gamma^2+216(\gamma+1)\right) + \beta^2\left(\gamma^2+384(\gamma+1)\right) + 128\left(\gamma^3-7\gamma^2-16\gamma-8\right), \\ Nums_1 =& -\alpha^3\gamma\left(\gamma^2+24\gamma+24\right) + \alpha\left(\beta^2\left(10\gamma^2+31\gamma+24\right)+3\gamma^4+35\gamma^3+168\gamma^2+200\gamma+64\right) \\ & +\alpha^2\beta\left(5\gamma^3+26\gamma^2-24\right) + \beta\left(\beta^2(5\gamma+4)+3\gamma^3+59\gamma^2+152\gamma+96\right), \\ Dens_1 =& (\gamma+1)\left(\alpha\left(\gamma^3+32\gamma^2+280\gamma+192\right)+\beta\left(\gamma^2+104\gamma+160\right)\right). \end{aligned}$$

We compute the first and second minors of the Hessian matrix of function M with respect to the manufacturer's decision variables in the first period, w_1 and s_1. These minors are long expressions and they are omitted here. These expressions can take positive or negative values. In the numerical simulations when Scenario VII ($a_2 = 0$, any value of $s_2, 0 \leq s_2 \leq 1, a_1 > 0, 0 < s_1 < 1$) is considered, we check that the quadratic form associated with the Hessian matrix is negative semidefined, and hence M is a concave function, and the interior solution given by (A.49) and (A.50) is a maximum.

The second possibility corresponds to the following corner option (**Subcase 2.1.2**, $a_2 = 0$, any value of s_2, $0 \leq s_2 \leq 1$, $a_1 > 0$, $s_1 = 0$ **Scenario VIII** in Proposition 1):

$$w_1 = \frac{g}{2}\frac{Numw_1}{Denw_1}, \tag{A.51}$$

$$s_1 = 0, \tag{A.52}$$

where

$$
\begin{aligned}
Numw_1 =& 8\alpha^4\gamma(\gamma+1)+2\alpha^3\beta\left(4-5\gamma^2\right)+\alpha^2\left(2\beta^2\left(\gamma^2-6\gamma-4\right)-9\gamma^3-96\gamma^2-152\gamma-64\right)\\
&+\alpha\beta\left(\beta^2(4\gamma-2)+3\gamma^3-31\gamma^2-48\gamma-16\right)+2\beta^4+\beta^2\left(3\gamma^2-30\gamma-32\right)+\gamma^4-7\gamma^3+136\gamma^2+272\gamma+128,\\
Denw_1 =& 2\alpha^3\beta\gamma^3+\alpha^2\left(\left(6\beta^2-64\right)\gamma^2+\gamma^4-128\gamma-64\right)+2\alpha\beta\gamma\left(3\beta^2+\gamma^2-16(\gamma+1)\right)+2\beta^4\\
&+\beta^2\left(\gamma^2-32(\gamma+1)\right)-16\left(\gamma^3-7\gamma^2-16\gamma-8\right),
\end{aligned}
$$

For any $\alpha, \gamma \in (0,1)$ and $\beta \in (-1,1)$, the manufacturer's total profits is a strictly concave function of her decision variable, w_1.

Subcase 2.2 ($a_2 = 0$, any value of s_2, $0 \le s_2 \le 1$, $a_1 = 0$)

The manufacturer's total profits read:

$$M = w_1(g - p_1 + \alpha a_1) - \frac{1}{2}s_1 a_1^2 + \frac{(a_1\beta + g + \gamma p_1)^2}{8(\gamma+1)}.$$

where the second-period manufacturer's profits given by (A.26) have been replaced.

Substituting the retailer's reaction functions given by (A.38) in the expression above, one obtains the manufacturer's total profits to be maximized. In this case, because the manufacturer's problem does not depend on s_1, from the optimality conditions we get the expression for w_1 and any value of s_1, $0 \le s_1 \le 1$

$$w_1 = \frac{(\gamma^3 - 8\gamma^2 + 144\gamma + 128)\,g}{32\left(8(\gamma+1) - \gamma^2\right)}. \tag{A.53}$$

This case corresponds to **Scenario IX** ($a_2 = 0$, any value of s_2, $0 \le s_2 \le 1$, a_1, any value of s_1, $0 \le s_1 \le 1$) in Proposition 1. For any $\alpha, \gamma \in (0,1)$ and $\beta \in (-1,1)$, the manufacturer's total profits is a concave function of her decision variables w_1 and s_1.

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