Generalized multiple delay-dependent H_∞ functional observer design for non-linear system

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Abstract

Functional observers are the major alternative to many practical estimation problems where full-order observers cannot be used. This paper introduces a generalized approach to design H_∞ functional observers for a class of Lipschitz nonlinear systems with multiple time delays. Moreover, the considered system extends from previously published work in that it presents nonlinearity, multiple delay and external disturbance. Their main findings come from the development of a generalized augmented Lyapunov function that uses both the extended reciprocal convex combination and the wirtinger inequality. The stability of the observer is therefore guaranteed by an LMI optimization problem. Finally, the steps of the design procedure were condensed and proffered for the two numerical examples to test the recommended design approach.

Keywords: Functional observers, Nonlinear systems, Lyapunov Functions, Time delays, Disturbances, Linear Matrix Inequalities (LMIs)

1 Introduction

Time-delay phenomena are common in dynamical systems, caused by the transfer of matter, energy or information. These delays, also known as dead time or side effects, frequently induce instability, in particular in nonlinear systems. To deal with this, researchers are actively working on stability analysis and observer design for these systems. Recently, various works of research and studies have been conducted on nonlinear systems with constant or time delays (see, for example, [16, 34]).

On the other hand, the problem of observation is the main object of this work, which is of great interest in practical applications. The functional observer, first explored by Luenberger [20], is still a hot topic for both linear or nonlinear systems, e.g. [5, 7, 21–23, 26–28, 36, 37], due to its advantage in solving estimation problems and its inherent simplicity. Compared with the full-order or reduced-order Luenberger observers, the functional observers have lower orders, and the observability or detectability criteria are relaxed to less conservative conditions. In light of that, this observer is extensively applied in the design of observer-based controllers [17], fault detection and isolation [10, 18], power systems [2], multi-agent systems [14], etc. Although the publication of [8] produced a functional observer with minimal order by incorporating some additional linear estimation functions, the authors of [11] have recently shown some sufficient and necessary conditions for the existence of such an observer. In [5], a robust observer-based H_∞ controller has been developed to achieve stability of uncertain fractional-order systems with delays and perturbations. In [5], an observer-based robust H_∞ controller was evolved to achieve stability of uncertain fractional order systems with time delays and disturbances. The authors of [36] studied finite-time adaptive integral sliding mode control (ISMC) for an unknown nonlinear function systems using a functional observer. Based on a recent high-gain observer design approach, the authors developed a high-gain observer design for non-linear systems with time delays in [3]. In the research area of multiple time delays, the aim of the paper [30] was the design problem of functional observers for linear systems and in [32] was the H_∞ dynamic observer design for non-linear Lipschitz systems.

Furthermore, a major problem in the design of observers is to reduce the conservatism of current state estimation criteria: selecting the suitable Lyapunov-Krasovskii functional (LKF) and constructing a tighter upper bound on its derivative reduces conservatism while guaranteeing system performance. Based on this, modifications of the LKF are motivating many theoretical developments. For instance, the Jensen inequality [12] has been widely used to estimate upper bounds. However, this estimation may be improved by other methods, such as the Wirtinger-based integral inequality (WBII), introduced in [33] to establish a tighter upper bound for the single-integer quadratic terms. Based on the wirtinger inequality, a modified double-integral inequality is developed [24]. [19] discusses integral inequalities based on auxiliary functions, which are integral inequalities in infinite series form. Following the work of[39] and [40], the paper [38] introduces an extended reciprocal convex combination inequality (ERCC). When all these inequalities are considered, the wirtinger inequality and the Extended reciprocally convex combination strategy have the distinct advantage of decreasing the arbitrary conservativism.

Considering the above literature, the approach taken in this paper is based on the Wirtinger inequality [33] and the extended reciprocally convex matrix inequality [38], as well as a generalized augmented Lyapunov function, which extends the one proposed in [1] by including all the delay-dependent ones, resulting in less conservative hierarchical conditions. This solves the problem of asymptotic stability of a nonlinear system with multiple time delays and disturbances.

Nevertheless, the subject of functional observers for linear systems with multiple delays is well covered in many papers [26, 30]. In contrast, the problem of designing functional observers for nonlinear delayed systems in the presence of multiple delays and disturbances is completely unexplored.

Based on the above-mentioned shortcomings, the problem of functional observer is studied in this paper for lipschitz nonlinear systems with multiple delays and disturbances. The main contributions are threefold.

1. A generalized framework for the design of the functional observer for nonlinear lipschitz systems with disturbances and multiple delays is introduced, which allows further use of the proposed observer in practical and physical systems.
2. The generalized form of the Lyapunov function is obtained using the Reciprocally convex combination approach and Wirtinger's inequality, which calculated the analytical amount of improvement it provides in a less conservative and more convenient method than the associated Jensen inequality.
3. Comparisons with [37] prove that the approach presented in this paper is more advanced in the sense that much less conservative bounds are obtained while dealing with multiple delays. Not only the existence and stability conditions of the H_∞ functional observer are given, but also a design technique is proposed by using linear matrix inequalities (LMIs).

Ultimately, two examples are provided for illustrative purposes. These two examples show that the influence of the disturbance on the estimation error is significantly reduced and stability is attained. They also show that the results obtained in this paper are less conservative than those in the literature.

The article is organized as follows. Section 2 formulates the model description as well as the problem addressed in this paper. Section 3 summarizes the major results of the suggested technique for designing a H_∞ functional observer. Section 4 presents numerical simulation results that demonstrate the efficacy of the proposed approach. Section 5 draws this article to a close.

2 Problem Statement and Preliminaries

Consider the following nonlinear system with multiple delays and disturbances:

$$\begin{aligned} \dot{x}(t) &= A_x x(t) + \sum_{i=1}^{m} A_{xi} x(t-\tau_i(t)) + f(x) + B_u u(t) + B_w w(t) \\ y(t) &= C_x x(t) \\ x(t) &= \phi(t), \quad \forall t \in [-\tau_d, 0] \end{aligned} \tag{1}$$

where $x(t) \in \mathbb{R}^n$ is the state vector, $u(t) \in \mathbb{R}^r$ is the control input vector, $y(t) \in \mathbb{R}^p$ is the output vector, $f(x(t))$ is the nonlinear function, and $w(t) \in \mathbb{R}^q$ is the disturbance vector, we assume $w(t) \in L_2[0,\ \infty)$. $\tau_1(t), \ldots, \tau_m(t)$ are time delay functions. $\phi(t)$ is a continuous vector-valued initial function. The matrices $\{A_x, A_{xi}\} \in \mathbb{R}^{n\times n}, B_u \in \mathbb{R}^{n\times r}, C_x \in \mathbb{R}^{p\times n}$ and $B_w \in \mathbb{R}^{n\times q}$ are known.

Remark 1 In equation (1), we considered that the nonlinear system is characterized by multiple delays, which generalizes the results to different classes of systems [13, 42]. Furthermore, as it is well known in the literature [12], delays are in general a source of instability and poor performance.

The following functional observer is designed for system (1)

$$\begin{aligned}
\dot{\vartheta}(t) =& N\vartheta(t) + \sum_{i=1}^{m} M_i\vartheta(t-\tau_i(t)) + Dy(t) + \sum_{i=1}^{m} J_i y(t-\tau_i(t)) + f(\hat{x}) + Hu(t) \\
\hat{x}(t) =& R\vartheta(t) + Fy(t) \\
\vartheta(t) =& \vartheta_0, \quad \forall t \in [-\tau_d, 0]
\end{aligned} \tag{2}$$

where $\vartheta(t) \in \mathbb{R}^s$ is the state vector of functional observer, and $\hat{x}(t) \in \mathbb{R}^n$ is the estimate of $x(t)$. The matrices $N \in \mathbb{R}^{s\times s}$, $M_i \in \mathbb{R}^{s\times s}$ $(i = 1, \ldots, m)$, $D \in \mathbb{R}^{s\times p}$, $J_i \in \mathbb{R}^{s\times p}$ $(i = 1, \ldots, m)$, $H \in \mathbb{R}^{s\times r}$, $R \in \mathbb{R}^{n\times s}$, and $F \in \mathbb{R}^{n\times p}$ are to be selected such that $\hat{x}(t)$ converges asymptotically to $x(t)$ for $w(t) = 0$ and $\frac{\|e(t)\|_2}{\|w(t)\|_2} < \gamma$ for $w(t) \neq 0$, where the estimation error is $e(t) = x(t) - \hat{x}(t)$.

Remark 2 The systems treated here are more general than the single delay systems and the systems with consecutive delays that are more frequent in the literature. In fact, they generalize these systems: see, for example, [6], [28],[31] and [41]. Moreover, it has also been observed that when the analyses consider multiple time delays, the outcomes are significantly less conservative [9, 29, 30].

We now present some assumptions and lemmas, which play an important role in establishing the main results.

Assumption 1 The multiple delays $\tau_i(t)$, $i = 1, \ldots, m$ are bounded such that

$$0 \le \tau_i(t) \le \bar{\tau}_{d_i}, \qquad \tau_d = \underbrace{max}_{i}\, \bar{\tau}_{d_i} \tag{3}$$

where τ_d is positive integer.

Assumption 2 The derivative of the delays satisfies that

$$d_{1i} \le \dot{\tau}_i(t) \le d_{2i} < 1, \tag{4}$$

for $i = 1, \ldots, m$.

Assumption 3 The nonlinear function $f(x)$ is assumed to be Lipschitz, with a Lipschitz constant λ; that is, there exists a positive constant λ such that:

$$\|f(\hat{x})-f(x)\| \leqslant \lambda\|\hat{x}-x\|^2 \tag{5}$$

Lemma 1 (Wirtinger-Based inequality [33]) Given a matrix $S>0$, the following inequality holds for any continuous functions ω in $[a,b] \longrightarrow \mathbb{R}^n$:

$$\int_a^b \omega^T(\alpha)S\omega(\alpha)d\alpha \geqslant \frac{1}{b-a}\left(\int_a^b \omega(\alpha)d\alpha\right)^T S\left(\int_a^b \omega(\alpha)d\alpha\right)+\frac{3}{b-a}\Omega^T S\Omega,$$

where $\Omega=\int_a^b \omega(s)ds-\frac{2}{b-a}\int_a^b\int_a^s \omega(r)drds$.

Lemma 2 (Extended Reciprocally Convex Inequality [38]) For $\sigma \in \begin{bmatrix} 0 & 1 \end{bmatrix}$, the symmetric matrices $S_1>0$ and $S_2>0$, for any matrices R_1 and R_2, the following matrix inequality holds:

$$\begin{bmatrix} \frac{1}{\sigma}S_1 & 0 \\ 0 & \frac{1}{1-\sigma}S_2 \end{bmatrix} \geqslant \begin{bmatrix} S_1+(1-\sigma)T_1 & (1-\sigma)R_1+\sigma R_2 \\ * & S_2+\sigma T_2 \end{bmatrix},$$

where $T_1=S_1-R_2S_2^{-1}R_2^T$ and $T_2=S_2-R_1^TS_1^{-1}R_1$.

Lemma 3 (see [4]) For any $a \in \mathbb{R}^n$ and $b \in \mathbb{R}^n$ and any scalar $\nu>0$, the following inequality is obtained:

$$a^Tb+ba^T \leqslant \nu\|a\|^2+\frac{1}{\nu}a\|b\|^2. \tag{6}$$

The objectives of this paper are:

- Outline a general approach to designing the functional observer (2) of the nonlinear system (1) with multiple time delays and disturbance.
- Construct LMI conditions to design the functional observer (2) based on the advantages of the Wirtinger inequality and the extended reciprocally convex combination strategy which gives less conservative results.

3 Main Results

This section has three parts. First, the parameterization of the functional observer is described. Then, the process of finding the coefficients of the functional observer is given. Finally, the design approach is presented for the system (1).

3.1 H_∞ Functional observer parametrization

This subsection gives algebraic conditions to parametrize the proposed H_∞ functional observer (2).

Firstly, the estimation error for system (1) and observer (2) is

$$\bar{\varepsilon}(t)=\vartheta(t)-\bar{T}x(t) \tag{7}$$

where the matrix $\bar{T} \in \mathbb{R}^{s\times n}$ an arbitrary matrix.

Eq. (7) equivalently,

$$\begin{aligned}\dot{\bar{\varepsilon}}(t) =&\dot{\vartheta}(t) - \bar{T}\dot{x}(t)\\ =&N\bar{\varepsilon}(t) + \sum_{i=1}^{m} M_i\varepsilon(t-\tau_i(t)) + (N\bar{T} - \bar{T}A_x + DC_x)x(t)\\ &+\sum_{i=1}^{m}(M_i\bar{T} - \bar{T}A_{xi} + J_iC_x)x(t-\tau_i(t)) + (H\\ &-\bar{T}B_u)u(t) + \bar{T}\Delta f - \bar{T}B_w w(t)\end{aligned} \tag{8}$$

where $\Delta f = f(\hat{x}) - f(x(t))$.
It is calculated from (2) that

$$\hat{x}(t) = R\bar{\varepsilon}(t) + (R\bar{T} + FC_x)x(t) \tag{9}$$

The following result describes the criteria for the stability of the H_∞ functional observer (2).

Theorem 1 *System (2) is a H_∞ functional observer for the nonlinear system (1), and it ensures the performance index $\gamma > 0$ for any initial conditions, that is, $x(t)$, $\hat{x}(t)$ $\forall t \leq 0$, and $u(t)$, if*

1. *the estimation error dynamics*

$$\dot{\bar{\varepsilon}}(t) = N\bar{\varepsilon}(t) + \sum_{i=1}^{m} M_i\varepsilon(t-\tau_i(t)) + \bar{T}\Delta f - \bar{T}B_w w(t) \tag{10}$$

and the observation error is

$$e(t) = \hat{x}(t) - x(t) = R\bar{\varepsilon}(t) \tag{11}$$

$*$ *for $w(t) = 0$, are asymptotically stable;*
$*$ *for $w(t) \neq 0$, the influence of disturbances $w(t)$ on the estimation $\bar{\varepsilon}(t)$ provided by*

$$r(t) = \tfrac{\|e(t)\|_2}{\|w(t)\|_2} < \gamma,$$

that is equivalent to

$$J = \textstyle\int_0^\infty \left(e(t)^T e(t) - \gamma^2 w(t)^T w(t)\right) dt < 0.$$

2. $N\bar{T} - \bar{T}A_x + DC_x = 0$;
3. $M_i\bar{T} - \bar{T}A_{xi} + J_iC_x = 0 \qquad i = 1, \ldots, m$;
4. $H = \bar{T}B_u$;
5. $R\bar{T} + FC_x = I$.

3.2 Finding H_∞ functional observer matrices

As a consequence of Theorem 1, the design of the H_∞ functional observer is transformed into finding the functional observer matrices N, M_i $(i = 1,\dots,m)$, D, J_i $(i = 1,\dots,m)$, H, R, and F that satisfy conditions 1- 5. This is now discussed.

From conditions 2, 3, and 5 of Theorem 1, the following holds:

$$\begin{bmatrix} N & D \\ M_1 & J_1 \\ \vdots & \vdots \\ M_m & J_m \\ R & F \end{bmatrix} \begin{bmatrix} \bar{T} \\ C_x \end{bmatrix} = \begin{bmatrix} \bar{T}A_x \\ \bar{T}A_{x1} \\ \vdots \\ \bar{T}A_{xm} \\ I \end{bmatrix} \tag{12}$$

Further, the following rank condition is necessary for (12) to have a solution.

$$rank \begin{bmatrix} \bar{T} \\ C_x \end{bmatrix} = rank \begin{bmatrix} \bar{T} \\ C_x \\ \bar{T}A_x \\ \bar{T}A_{x1} \\ \vdots \\ \bar{T}A_{xm} \\ I \end{bmatrix} = n \tag{13}$$

Selecting a matrix $\bar{T}$ which satisfies condition (13), the solution to (12) is as follows:

$$\begin{bmatrix} N & D \\ M_1 & J_1 \\ \vdots & \vdots \\ M_m & J_m \\ R & F \end{bmatrix} = \begin{bmatrix} \bar{T}A_x \\ \bar{T}A_{x1} \\ \vdots \\ \bar{T}A_{xm} \\ I \end{bmatrix} \mathscr{M}^+ - Y(I - \mathscr{M}\mathscr{M}^+) \tag{14}$$

where $\mathscr{M} = \begin{bmatrix} \bar{T} \\ C \end{bmatrix}$, and $\mathscr{M}^+$ is any generalized inverse matrix of $\mathscr{M}$ (i.e., $\mathscr{M}\mathscr{M}^+\mathscr{M} = \mathscr{M}$) and Y is an arbitrary matrix of appropriate dimensions.

Left-multiplying (14) by $\begin{bmatrix} I & \overbrace{0 \ \dots \ 0}^{m} & 0 \end{bmatrix}$ and $\begin{bmatrix} 0 & \overbrace{I \ \dots \ I}^{m} & 0 \end{bmatrix}$ (14) yields,

$$\begin{aligned} \begin{bmatrix} N & D \end{bmatrix} &= \bar{T}A_x\mathscr{M}^+ - Y_0(I - \mathscr{M}\mathscr{M}^+), \\ \begin{bmatrix} M_i & J_i \end{bmatrix} &= \bar{T}A_{xi}\mathscr{M}^+ - Y_{1i}(I - \mathscr{M}\mathscr{M}^+). \end{aligned} \tag{15}$$

Subsequently, right-multiplying (15) by $\begin{bmatrix} I \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ I \end{bmatrix}$ respectively, we have

$$\begin{aligned} N &= \bar{T}A_x\Lambda_1 - Y_0\Gamma_1, & M_i &= \bar{T}A_{xi}\Lambda_1 - Y_{1i}\Gamma_1, \\ D &= \bar{T}A_x\Lambda_2 - Y_0\Gamma_2, & J_i &= \bar{T}A_{xi}\Lambda_2 - Y_{1i}\Gamma_2, \end{aligned} \tag{16}$$

where

$$\begin{aligned} \Lambda_1 &= \mathscr{M}^+ \begin{bmatrix} I \\ 0 \end{bmatrix}, & \Gamma_1 &= (I - \mathscr{M}\mathscr{M}^+) \begin{bmatrix} I \\ 0 \end{bmatrix}, \\ \Lambda_2 &= \mathscr{M}^+ \begin{bmatrix} 0 \\ I \end{bmatrix}, & \Gamma_2 &= (I - \mathscr{M}\mathscr{M}^+) \begin{bmatrix} 0 \\ I \end{bmatrix}, \end{aligned}$$

$$Y_0 = \begin{bmatrix} I & \overbrace{0 \ \dots \ 0}^{m} & 0 \end{bmatrix} Y, \tag{17}$$

$$Y_{1i} = \begin{bmatrix} \overbrace{0 \ \dots \ 0}^{i} & I & \overbrace{0 \ \dots \ 0}^{m-i-1} & 0 \end{bmatrix} Y.$$

for $i = 1, \dots, m$.

Similarly, R, and F are presented as follows:

$$R = \Lambda_1 - Y_3\Gamma_1, \qquad F = \Lambda_2 - Y_3\Gamma_2, \tag{18}$$

where

$$Y_3 = \begin{bmatrix} 0 & \overbrace{0 \ \dots \ 0}^{m+1} & I \end{bmatrix} Y.$$

Remark 3 It is clear from (11) that $e \longrightarrow 0$ when $\bar{\varepsilon}(t) \longrightarrow 0$, implying that the convergence of the error is not dependent R [15]. Then, we can set $Y_3 = 0$ to obtain that

$$R = \Lambda_1, \qquad F = \Lambda_2. \tag{19}$$

Using this result, the gain matrix Y can be used to determine all the H_∞ functional observer (2) parameters.

Then, the estimate error dynamics can be rewritten under condition (13) and using (16) as follows:

$$\begin{aligned} \dot{\bar{\varepsilon}}(t) =& (\bar{T}A_x\Lambda_1 - Y_0\Gamma_1)\bar{\varepsilon}(t) + \sum_{i=1}^{m} \bar{T}(A_{xi}\Lambda_1 - Y_{1i}\Gamma_1)\varepsilon(t - \tau_i(t)) \\ &+ \bar{T}\Delta f - \bar{T}B_w w(t) \end{aligned} \tag{20}$$

As a result, designing the H_∞ functional observer (2) is simplified to establishing a free matrix parameter Y that meets condition 1 in Theorem 1.

3.3 H_∞ Functional observer design

In this subsection, we propose a generalized condition using the augmented Lyapunov function, the Wirtinger-based Integral Inequality (WBII), and the Extended Reciprocal Convex Combination (ERCC), that makes it possible to design H_∞ functional observers for a nonlinear system (1) with multiple delays.

Theorem 2 *Given positive scalars τ_d, d_{1i}, d_{2i} and γ. If there are symmetric matrices $P>0$, $W_{1,1}>0,\dots,W_{1,m}>0$, W_2, $Z_1>0,\dots,Z_m>0$, an inversible matrix g_{est}, and any matrices X, X_i, $R_{1,i}$, $R_{2,i}$, such that the following inequalities hold for $i=1,2,\dots,m$,*

$$\begin{bmatrix} \Psi_j(\omega) & E_{4,i}^T g_{est}\bar{T} & \Xi_j \\ * & -\nu I & 0 \\ * & * & \breve{Z} \end{bmatrix} < 0, \quad j=1,2, \tag{21}$$

the dynamic error (10) is therefore asymptotically stable, where

$$\Psi_i(\omega) = \Sigma_i(\omega) + (1+\iota\kappa^2)e_1^T R^T R e_1 + 2E_{4,i}^T E_{3,i} - \gamma^2 e_{3m+6}^T e_{3m+6}, \tag{22}$$

$$\begin{aligned} \Sigma_i(\omega) =& \Pi_i(\omega) - \sum_{i=1}^{M}\left\{E_{2,i}^T \Upsilon_{2,i}(\omega) E_{2,i}\right\}, \\ & \breve{Z} = diag\{-\tilde{Z}_1,\dots,-\tilde{Z}_m\}, \end{aligned} \tag{23}$$

$$\begin{aligned} \Pi_i(\omega) =& E_{1,i}^T P E_{0,i}(\omega) + E_{0,i}^T(\omega) P E_{1,i} + diag\left\{\sum_{i=1}^{m} W_{1,i},\right. \\ & \left. + W_2 - \sum_{i=1}^{m}(1-\dot{d}_{2i}(t))W_{1,i}, -W_2, 0,\dots,0\right\} + e_{3m+5}^T \tau_d^2 (\sum_{i=1}^{m} Z_i) v_{3m+5}, \end{aligned} \tag{24}$$

$$\begin{aligned} \Upsilon_{2,i}(\omega) =& \begin{bmatrix} \breve{Z}_i + (1-\sigma)\breve{Z}_i & (1-\sigma)R_{1,i} + \sigma R_{2,i} \\ * & \breve{Z}_i + \sigma \breve{Z}_i \end{bmatrix}, \\ & \omega = 0, \tau_d, \quad \sigma = 0, 1, \end{aligned} \tag{25}$$

$$\Xi_1 = \left[\begin{pmatrix} e_1 - e_2 \\ e_1 + e_2 - 2e_4 \end{pmatrix}^T R_{1,1} \ \dots \ \begin{pmatrix} e_1 - e_{m+1} \\ e_1 + e_{m+1} - 2e_{2m+2} \end{pmatrix}^T R_{1,m} \right], \tag{26}$$

$$\Xi_2 = \left[\begin{pmatrix} e_2 - e_3 \\ e_2 + e_3 - 2e_5 \end{pmatrix}^T R_{2,1}^T \ \dots \ \begin{pmatrix} e_{m+1} - e_{m+2} \\ e_{m+1} + e_{m+2} - 2e_{2m+3} \end{pmatrix}^T R_{2,m}^T \right], \tag{27}$$

$$E_{0,i}(\omega) = \left[\, e_1^T \ \ \omega e_{i+3}^T \ \dots \ \omega e_{2m+3}^T \ \ (\tau_d - \omega) \ \times e_{2i+3}^T \ \dots \ (\tau_d - \omega) e_{2m+3+i}^T \,\right]^T, \tag{28}$$

$$E_{1,i} = \left[\, e_{2m+5+i}^T \ \ e_1^T - e_2^T \ \dots \ e_1^T - e_{m+1}^T \ \ e_2^T - e_3^T \ \dots \ e_{m+1}^T - e_{m+2}^T \,\right]^T, \tag{29}$$

$$E_{2,i} = \left[\, e_1^T - e_{i+1}^T \ \ e_1^T + e_{i+1}^T - e_{m+2+i}^T \ \ e_{i+1}^T - e_{m+2}^T \ \ e_{i+1}^T + e_{m+2}^T - e_{2m+3+i}^T \,\right]^T, \tag{30}$$

$$E_{3,i} = (g_{est} T A_x \Lambda_1 - Y_0 \Gamma_1) e_1 + \sum_{i=1}^{m} (g_{est} A_{xi} \Lambda_1 - Y_{1i}\Gamma_1) e_{1+i} - g_{est} T B_w e_{3m+6} - g_{est} v_{3m+5}, \tag{31}$$

$$E_{4,i} = e_1 + e_{3m+5} + \sum_{i=1}^{m} e_{1+i}, \qquad \tilde{Z}_i = \begin{bmatrix} Z_i & 0 \\ 0 & 3Z_i \end{bmatrix}, \tag{32}$$

$$e_l = \begin{bmatrix} 0_{n\times(l-1)n} & I_n & 0_{n\times(3m+5-l)n} \end{bmatrix},$$
$$l = 1,2,\ldots,3m+5, \tag{33}$$

The matrices parameter of the H_∞ functional observer are computed as follows:

$$Y_0 = g_{est}^{-1}X, \qquad Y_{1i} = g_{est}^{-1}X_i. \tag{34}$$

Proof Consider the following LKF:

$$V(\bar{\varepsilon}(t)) = V_1(\bar{\varepsilon}(t)) + V_2(\bar{\varepsilon}(t)) + V_3(\bar{\varepsilon}(t)) \tag{35}$$

where

$$V_1(\bar{\varepsilon}(t)) = \begin{bmatrix} \bar{\varepsilon}(t) \\ \int_{t-\tau_1(t)}^{t} \bar{\varepsilon}(s)ds \\ \vdots \\ \int_{t-\tau_m(t)}^{t} \bar{\varepsilon}(s)ds \\ \int_{t-\tau_d}^{t-\tau_1(t)} \bar{\varepsilon}(s)ds \\ \vdots \\ \int_{t-\tau_d}^{t-\tau_m(t)} \bar{\varepsilon}(s)ds \end{bmatrix}^T P \begin{bmatrix} \bar{\varepsilon}(t) \\ \int_{t-\tau_1(t)}^{t} \bar{\varepsilon}(s)ds \\ \vdots \\ \int_{t-\tau_m(t)}^{t} \bar{\varepsilon}(s)ds \\ \int_{t-\tau_d}^{t-\tau_1(t)} \bar{\varepsilon}(s)ds \\ \vdots \\ \int_{t-\tau_d}^{t-\tau_m(t)} \bar{\varepsilon}(s)ds \end{bmatrix} \tag{36}$$

$$V_2(\bar{\varepsilon}(t)) = \sum_{i=1}^{m} \int_{t-\tau_i(t)}^{t} \bar{\varepsilon}^T(s) W_{1,i} \bar{\varepsilon}(s)ds + \int_{t-\tau_d}^{t} \bar{\varepsilon}^T(s) W_2 \bar{\varepsilon}(s)ds \tag{37}$$

and

$$V_3(\bar{\varepsilon}(t)) = \tau_d \sum_{i=1}^{m} \int_{t-\tau_d}^{t} \int_{\theta}^{t} \dot{\bar{\varepsilon}}^T(s) Z_i \dot{\bar{\varepsilon}}(s)dsd\theta \tag{38}$$

Remark 4 The LKF (36) is chosen taking into account the nature of the delay. More precisely, the considered delay is noncommensurable, which means that each term is characterized by a derivative different from the other. Forthermore, the interesting aspect in constructing the LKF (36) is that Lemma 1 allows the LKF to consider the double integral term with vectors containing the state and its derivatives at the same time, which makes the LKF consider more information on the system with multiple delays.

The derivative of $V_1(\bar{\varepsilon}(t))$ is as follows:

$$\dot{V}_1(\bar{\varepsilon}(t)) = 2\delta(t) \begin{bmatrix} e_1 \\ \omega e_{i+3} \\ \vdots \\ \omega e_{2m+3} \\ (\tau_d-\omega)e_{2i+3} \\ \vdots \\ (\tau_d-\omega)e_{2m+3+i} \end{bmatrix}^T P \begin{bmatrix} e_{2m+5+i} \\ e_1 - e_2 \\ \vdots \\ e_1 - e_{m+1} \\ e_2 - e_3 \\ \vdots \\ e_{m+1} - e_{m+2} \end{bmatrix} \delta(t), \tag{39}$$

where

$$\delta(t)=\big[\ \bar{\varepsilon}^T(t)\ \ \bar{\varepsilon}^T(t-\tau_1(t))\ \ \ldots\ \ \bar{\varepsilon}^T(t-\tau_i(t))\ \ \bar{\varepsilon}^T(t-\tau_d)$$
$$\tfrac{1}{\tau_1(t)}\textstyle\int_{t-\tau_1(t)}^{t}\bar{\varepsilon}^T(s)ds\ \ \ldots\ \ \tfrac{1}{\tau_i(t)}\int_{t-\tau_i(t)}^{t}\bar{\varepsilon}^T(s)ds$$
$$\tfrac{1}{\tau_d-\tau_1(t)}\textstyle\int_{t-\tau_d}^{t-\tau_1(t)}\bar{\varepsilon}^T(s)ds\ \ \ldots\ \ \tfrac{1}{\tau_d-\tau_i(t)}\int_{t-\tau_d}^{t-\tau_i(t)}\bar{\varepsilon}^T(s)ds$$
$$\dot{\bar{\varepsilon}}^T(t)\ \ w^T(t)\ \big]^T.$$

Using (4), we have that

$$\begin{aligned}\dot{V}_2(\bar{\varepsilon}(t))\leqslant\sum_{i=1}^{m}\Big(&\bar{\varepsilon}^T(t)W_{1,i}\bar{\varepsilon}(t)-(1-\dot{\tau}_i(t))\bar{\varepsilon}^T(t\\&-\tau_i(t))W_{1,i}\varepsilon(t-\tau_i(t))\Big)+\bar{\varepsilon}^T(t)W_2\bar{\varepsilon}(t)\\&-\bar{\varepsilon}^T(t-\tau_d)W_2\varepsilon(t-\tau_d).\end{aligned}\tag{40}$$

Now, taking the derivative of $V_3(\bar{\varepsilon}(t))$, we obtain that

$$\dot{V}_3(\bar{\varepsilon}(t))=\sum_{i=1}^{m}\left(\tau_d^2\dot{\bar{\varepsilon}}^T(t)Z_i\dot{\bar{\varepsilon}}(t)-\tau_d\int_{t-\tau_d}^{t}\dot{\bar{\varepsilon}}^T(s)Z_i\dot{\bar{\varepsilon}}(s)ds\right).\tag{41}$$

Assuming assumptions 1, 2, the expression of $\Pi_i(\tau_i(t))$ in (24), $E_{0,i}(\tau_i(t))$, and $E_{1,i}$ in (28)-(29), the derivative of $V(\bar{\varepsilon}(t))$ is stated as

$$\dot{V}(\bar{\varepsilon}(t))\ \le\ \delta^T(t)\Pi_i(\tau_i(t))\delta(t)-\tau_d\textstyle\int_{t-\tau_d}^{t}\dot{\bar{\varepsilon}}^T(s)Z_i\dot{\bar{\varepsilon}}(s)ds\tag{42}$$

By defining $\sigma=\tau_i(t)/\tau_d$, and applying Lemma 1 to the second integral after introducing $\tau_i(t)$ through the Chasles relation yields that

$$\begin{aligned}-\tau_d\int_{t-\tau_d}^{t}\dot{\bar{\varepsilon}}^T(s)Z_i\dot{\bar{\varepsilon}}(s)ds\leqslant-&\begin{bmatrix}\bar{\varepsilon}(t)-\bar{\varepsilon}(t-\tau_i(t))\\ \bar{\varepsilon}(t)+\bar{\varepsilon}(t-\tau_i(t))-\frac{2}{\tau_i(t)}\int_{\tau_i(t)}^{t}\bar{\varepsilon}(s)ds\\ \bar{\varepsilon}(t-\tau_i(t))-\bar{\varepsilon}(t-\tau_d)\\ \bar{\varepsilon}(t-\tau_i(t))-\bar{\varepsilon}(t-\tau_d)-\frac{2}{\tau_d-\tau_i(t)}\int_{t-\tau_d}^{t-\tau_i(t)}\bar{\varepsilon}(s)ds\end{bmatrix}^T\\&\times\begin{bmatrix}\frac{1}{\sigma}\breve{Z}_i & 0\\ 0 & \frac{1}{1-\sigma}\breve{Z}_i\end{bmatrix}\\&\times\begin{bmatrix}\bar{\varepsilon}(t)-\bar{\varepsilon}(t-\tau_i(t))\\ \bar{\varepsilon}(t)+\bar{\varepsilon}(t-\tau_i(t))-\frac{2}{\tau_i(t)}\int_{t-\tau_i(t)}^{t}\bar{\varepsilon}(s)ds\\ \bar{\varepsilon}(t-\tau_i(t))-\bar{\varepsilon}(t-\tau_d)\\ \bar{\varepsilon}(t-\tau_i(t))-\bar{\varepsilon}(t-\tau_d)-\frac{2}{\tau_d-\tau_i(t)}\int_{t-\tau_d}^{t-\tau_i(t)}\bar{\varepsilon}(s)ds\end{bmatrix}.\end{aligned}\tag{43}$$

$$\tag{44}$$

Substituting (44) into (42) yields that

$$\dot{V}(\bar{\varepsilon}(t))\ \le\ \delta^T(t)\left(\Pi_i(\tau_i(t))-E_{2,i}^T\Upsilon(\tau_i(t))E_{2,i}\right)\delta(t),\tag{45}$$

where $E_{2,i}$ is given in (30) and

$$\Upsilon(\tau_i(t))=\begin{bmatrix}\frac{1}{\sigma}\breve{Z}_i & 0\\ 0 & \frac{1}{1-\sigma}\breve{Z}_i\end{bmatrix},\tag{46}$$

Then, by using Lemma 2, it follows that for any matrices R_1 and R_2, the following holds:

$$\Upsilon(\tau_i(t))\geqslant\alpha\Upsilon_1(0)+(1-\sigma)\Upsilon_1(\tau_d),\tag{47}$$

with

$$\Upsilon_1(\tau_i(t)) = \begin{bmatrix} \breve{Z}_i + (1-\sigma)(\breve{Z}_i - R_{2,i}\breve{Z}_i^{-1}R_{2,i}^T) & (1-\sigma)R_{1,i} + \alpha R_{2,i} \\ * & \breve{Z}_i + \alpha(\breve{Z}_i - R_{1,i}^T\breve{Z}_i^{-1}R_{1,i}) \end{bmatrix}.$$

Note that $\Pi_i(\tau_i(t))$ is $\tau_i(t)$ dependent, and that $\Pi_i(\tau_i(t)) = \sigma\Pi(0) + (1-\sigma)\Pi(\tau_d)$ so it holds that

$$\dot{V}(\bar{\varepsilon}(t)) \leq \delta^T(t)\left(\sigma\Sigma_i(0) + (1-\sigma)\Sigma_i(\tau_d)\right)\delta(t) \tag{48}$$

with $\Sigma_i(.)$ given in (23). Let $E_{3,i}$ be the matrix given in (31), and notice that for any invertible matrix g_{est}, that from $g_{est}^{-1}E_{3,i}\delta(t) + \bar{T}\Delta f(x(t)) = 0$, the following equation holds:

$$2\delta^T(t)E_{4,i}^TE_{3,i}\delta(t) + 2\delta^T(t)E_{4,i}^Tg_{est}\bar{T}\Delta f(x(t)) = 0, \tag{49}$$

$E_{4,i}$ is defined in (32). Summing (49) with (48) and apply Lemma 3, leads to,

$$\begin{aligned}\dot{V}(\bar{\varepsilon}(t)) \leq &\delta^T(t)\left(\sigma\Sigma_i(0) + (1-\sigma)\Sigma_i(\tau_d)\right)\delta(t) \\ &+ 2\delta^T(t)E_{4,i}^TE_{3,i}\delta(t) + \frac{1}{\nu}\delta^T(t)E_{4,i}^T \\ &\times g_{est}\bar{T}\bar{T}^Tg_{est}^TE_{4,i}\delta(t) + \nu\Delta f(x(t))^T\Delta f(x(t)),\end{aligned} \tag{50}$$

with $\nu > 0$ and by using $\Delta f(x(t))^T\Delta f(x(t)) \leq \lambda^2 e^T(t)e(t)$, with $e(t) = C\bar{\varepsilon}(t) = Ce_1\delta(t)$, Eq.(50) becomes

$$\begin{aligned}\dot{V}(\bar{\varepsilon}(t)) \leq &\delta^T(t)\Big(\sigma\Sigma_i(0) + (1-\sigma)\Sigma_i(\tau_d) + 2E_{4,i}^TE_{3,i} \\ &+ \frac{1}{\nu}E_{4,i}^Tg_{est}\bar{T}\bar{T}^Tg_{est}^TE_{4,i} + \nu\lambda^2e_1^TC^TCe_1\Big)\delta(t) \\ \leq &\delta^T(t)\Phi_1\delta(t)\end{aligned} \tag{51}$$

It follows from the convexity property over $\tau_i(t)$ that if $\Phi_1(0) < 0$ and $\Phi_1(\tau_d) < 0$ are verified, then, $\Phi_1 < 0$ and thus, the negativity of the Lyapunov derivative.

Additionally, the following inequality must be satisfied to minimize the influence of disturbances, with a performance level γ,

$$\dot{V}(\bar{\varepsilon}(t)) + e^T(t)e(t) - \gamma^2w^T(t)w(t) < 0, \tag{52}$$

by integrating (52), it follows that,

$$\int_0^\infty \dot{V}(\bar{\varepsilon}(t)) + \int_0^\infty e^T(t)e(t) - \int_0^\infty \gamma^2w^T(t)w(t) < 0,$$

Alternatively,$V(\infty) - V(0) + \parallel e(t) \parallel_2^2 - \gamma^2 \parallel w(t) \parallel_2^2 < 0$, under zero conditions $V(0) = 0$; since $V(infty)geqsant0$, then

$$V(\infty) + \parallel e(t) \parallel_2^2 - \gamma^2 \parallel w(t) \parallel_2^2 < 0, \tag{53}$$

resulting in $\parallel e(t) \parallel_2^2 < \gamma^2 \parallel w(t) \parallel_2^2$.

Now, from inequalities (51) and (52), we obtain the following inequality:

$$\dot{V}(\bar{\varepsilon}(t)) \leq \delta^T(t)\Phi_2\delta(t) \tag{54}$$

where

$$\Phi_2 = \sigma\Sigma_i(0) + (1-\sigma)\Sigma_i(\tau_d) + 2E_{4,i}^TE_{3,i} + \frac{1}{\nu}E_{4,i}^Tg_{est}T \times \bar{T}^Tg_{est}^TE_{4,i}$$

$$+\nu\lambda^2 e_1^T C^T C e_1 + e_1^T C^T C e_1 - \gamma^2 e_{3i+6}^T e_{3i+6} \tag{55}$$

On the other hand, if the LMIs (21) are met, then, using Schur complement and allowing for the change of variables $X_0 = g_{est}Y_0,\ \ X_i = g_{est}Y_{1i}$, it is guaranteed that $\Phi_2(0) < 0$ and $\Phi_2(\tau_d) < 0$. Therefore, we can assume that the system (14) is asymptotically stable for any delay varying $\tau_i(t)$. This completes the proof.

□

Remark 5 Since finding the existence conditions and designing a functional observer of nonlinear lipschitz systems with multiple delays and disturbances are the main objectives, this work considers the generalized form of the quadratic Lyapunov function to construct the stability condition. However, this quadratic Lyapunov function developed here can be explored to find the parameters of the observer to increase the solution domain by reducing the conservative condition.

Remark 6 There are two parts to the design of the H_∞ functional observer (2) for a nonlinear system (1) with multiple delays and perturbations. Theorem 1 is used in the first section to solve the design parameters of the observer. The observer matrices are then determined in the second section based on the LMIs of Theorem 2, for example using the Matlab LMI toolbox.

Remark 7 The problem of finding the matrices of the observer (2) that guarantee asymptotic stability can be approached by solving a feasibility problem of the set of inequalities of Theorem 2 combined with the constraints of Theorem 1. Therefore, these conditions can be tested very easily in terms of feasibility in the numerical examples presented in the next section.

Algorithm 1 in Fig. 1 is then proposed to determine the parameters of the H_∞ functional observer (2).

4 NUMERICAL EXAMPLES

Two examples are developed in this section to demonstrate the efficacy of the produced findings. The YALMIP toolbox and the SDP solver were used to code the LMIs conditions in MATLAB.

4.1 Example

Consider the nonlinear system (1) with $m = 1$ and its matrices provided by:

$$A_x = \begin{bmatrix} -0.5 & 0 \\ 0 & 0.5 \end{bmatrix},\ A_{x1} = \begin{bmatrix} -1 & 0 \\ 0 & -0.5 \end{bmatrix},\ B_u = \begin{bmatrix} 0 \\ 0.5 \end{bmatrix},\ B_w = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$
$$C_x = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix},\ \ \bar{T} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix},\ \ f(x,t) = \begin{bmatrix} 0.01 + 0.05\, sin(x_1(t)) \\ 0 \end{bmatrix}.$$

In this example, the time delays is $\tau_1(t) = 0.7\, sin(t - \frac{\pi}{2})$, $d_{21} = 0.7$, and $\tau_d = 0.9$, and the Lipschitz constant is $\lambda = 0.8$.

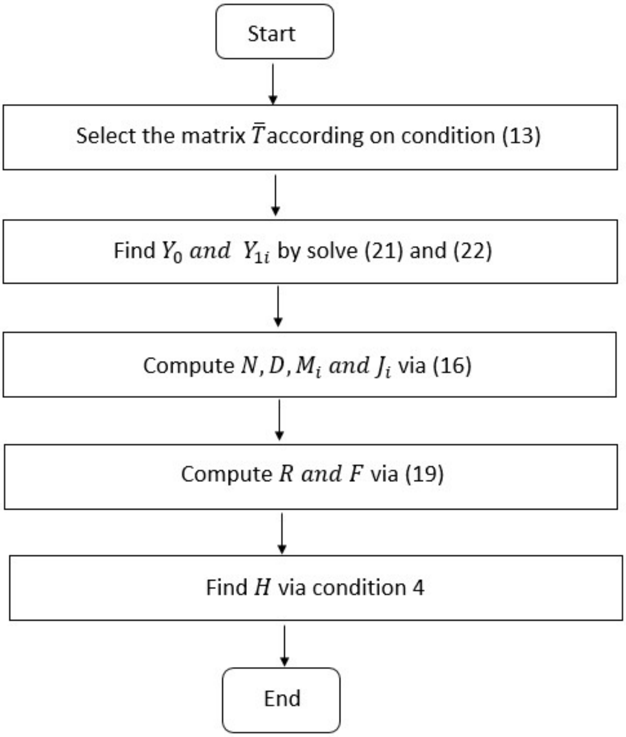

Fig. 1 Algorithm 1: Functional Observer Design

Table 1 presents the H_∞ performance bounds determined using the proposed approach, compared with the technique proposed in [37]. The results produced by the suggested technique are less conservative than those of [37], as shown in Table 1.

Table 1 Comparison of H_∞ performance with [37] for Example 1.

	[37]	Theorem 2
γ_{min}	2.3819	0.7935

Figure 2 presents the normalized ratio $r(t) = \frac{\|e(t)\|_2}{\|w(t)\|_2}$ in with $w(t) = \begin{bmatrix} 0.7\,sin(2t) & 0 \end{bmatrix}^T$. It can be seen that this ratio tends to be 0.7935, as shown in Figure 2.

4.2 Example

Take a look at the truck trailer system [35] provided by:

$$\dot{x}_1(t) = -a\frac{v\bar{t}}{Lt_0}x_1(t) - (1-a)\frac{v\bar{t}}{Lt_0}x_1(t-\tau_1(t)) + \frac{v\bar{t}}{lt_0}u(t) + 0.5w(t),$$

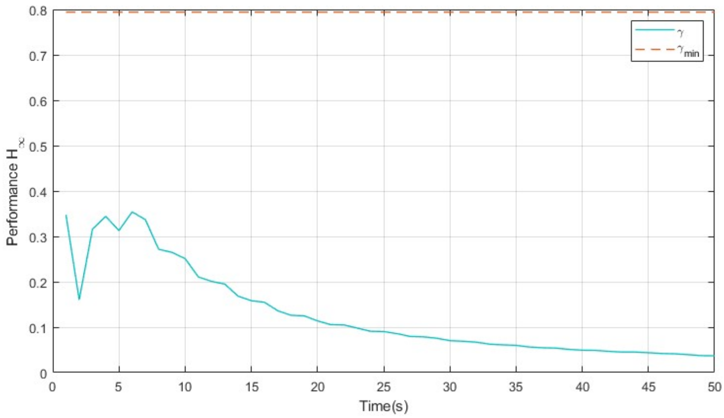

Fig. 2 The ratio $r(t)$ and the set value γ_{min}.

$$
\begin{aligned}
\dot{x}_2(t) =& \frac{\upsilon\bar{t}}{Lt_0}x_1(t) - (1-a)\frac{\upsilon\bar{t}}{Lt_0}x_1(t-\tau_1(t)) + 0.5w(t),\\
\dot{x}_3(t) =& \frac{\upsilon\bar{t}}{t_0}sin\left[x_2(t) - \frac{\upsilon\bar{t}}{2L}x_1(t) + (1-a)\frac{\upsilon\bar{t}}{2L}x_1(t-\tau_1(t))\right] + 0.5w(t).
\end{aligned}
\tag{56}
$$

where $l = 2.8$, $L = 5.5$, $\upsilon = -1.0$, $\bar{t} = 2.0$, $t_0 = 0.5$, $a = 0.7$ and $\tau_1(t)$ is the time delay satisfying Assumptions 1 and 2. The reader is referred to [35] for more details on the linearization of the nonlinear system (56).

By the linearisation, the following matrices are obtained:

$$
A_x = \begin{bmatrix} -a\frac{\upsilon\bar{t}}{Lt_0} & 0 & 0 \\ a\frac{\upsilon\bar{t}}{Lt_0} & 0 & 0 \\ -a\frac{\upsilon^2\bar{t}^2}{2Lt_0} & \frac{\upsilon\bar{t}}{t_0} & 0 \end{bmatrix}, \qquad A_{x1} = \begin{bmatrix} -(1-a)\frac{\upsilon\bar{t}}{Lt_0} & 0 & 0 \\ (1-a)\frac{\upsilon\bar{t}}{Lt_0} & 0 & 0 \\ (1-a)\frac{\upsilon^2\bar{t}^2}{2Lt_0} & 0 & 0 \end{bmatrix},
$$

$$
B_u = \begin{bmatrix} \frac{\upsilon\bar{t}}{lt_0} \\ 0 \\ 0 \end{bmatrix}, \qquad B_w = \begin{bmatrix} 0.5 \\ 0.5 \\ 0.5 \end{bmatrix}.
$$

The rest of the system (1) is chosen to be:

$$
C_x = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \; \bar{T} = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 2 \end{bmatrix}, \; f(x,t) = \begin{bmatrix} 0.01 + 0.05\,sin(x_1(t)) \\ 0 \\ 0 \end{bmatrix}.
$$

In this example, the time varying delays are $\tau_1(t) = 0.5\,sin(t - \frac{\pi}{2})$, $d_{21} = 0.5$, $\tau_d = 0.8$, andthe Lipschitz constant is $\lambda = 0.9$.

In order to obtain the parameters of the H_∞ functional observer, Algorithm 1 is applied, obtaining the following results:

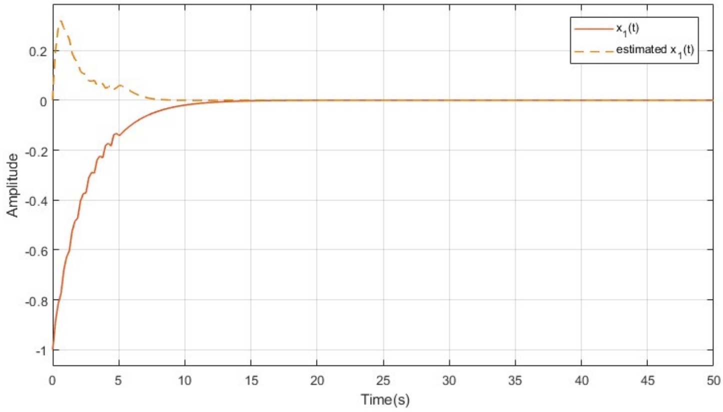

Fig. 3 The vector $x_1(t)$ and its estimation.

$$\begin{aligned}
\dot{\vartheta}(t) &= \begin{bmatrix} -2.5346 & -0.0038 & 0.0444 \\ -0.0038 & -2.5308 & -0.0047 \\ -0.0047 & 0.0491 & -0.4081 \end{bmatrix} \vartheta(t) + \begin{bmatrix} -0.3781 & 0.0050 & 0.4556 \\ 0.0050 & -0.3831 & -0.0047 \\ 0.0491 & 0.0047 & -0.2021 \end{bmatrix} \vartheta(t-\tau_1(t)) \\
&+ \begin{bmatrix} -0.0038 & 0.0321 \\ -2.5308 & 0.0050 \\ 0.0030 & -0.3831 \end{bmatrix} y(t) + \begin{bmatrix} 0.0050 & 0.2546 \\ -0.3831 & 0.0.5432 \\ 0.3612 & -0.2963 \end{bmatrix} y(t-\tau_1(t)) \\
&+ \begin{bmatrix} 0.01+0.05\,sin(x_1(t)) \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0.5 \\ 0.5 \\ 0.5 \end{bmatrix} u(t) \\
\hat{x}(t) &= \begin{bmatrix} 0.5 & -0.5 & 0 \\ 0 & 0.5 & 0 \\ 0 & 0 & 0.5 \end{bmatrix} \vartheta(t) + \begin{bmatrix} -0.5 & 0 \\ 0.5 & 0.5 \\ -0.5 & 0 \end{bmatrix} y(t)
\end{aligned} \tag{57}$$

For $\gamma = 0.3851$, a viable solution of the LMI requirements of Theorem 2 is achieved, and the effect of disturbances on the system is reduced.

MATLAB/Simulink was used to simulate the designed observers. The findings are shown in Figures 3 - 8. Figures 3 - 5 show the temporal behavior of the states $x_1(t)$, $x_2(t)$, $x_3(t)$, as well as their estimates. Figures 6 - 8 show the estimation errors $e(t)$, with the simulation started from $x(0) = \begin{bmatrix} -1 & 1 & 1 \end{bmatrix}^T$, $\hat{x}(0) = 1$, and the disturbance $w(t)$ is chosen to be:

$$w(t) = \begin{cases} 0 & t > 5 \\ 0.5\,sin(t) & else \end{cases}$$

According to these figures, the estimate $\hat{x}(t)$ tracks the real state $x(t)$ with a minor error, and the estimate error $e(t)$ converges to zero over time. This is consistent with the minimal value of the disturbance attenuation criteria, $\gamma = 0.3851$.

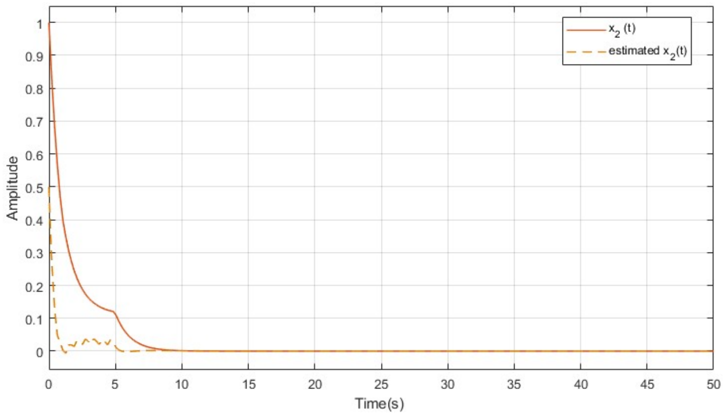

Fig. 4 The vector $x_2(t)$ and its estimation.

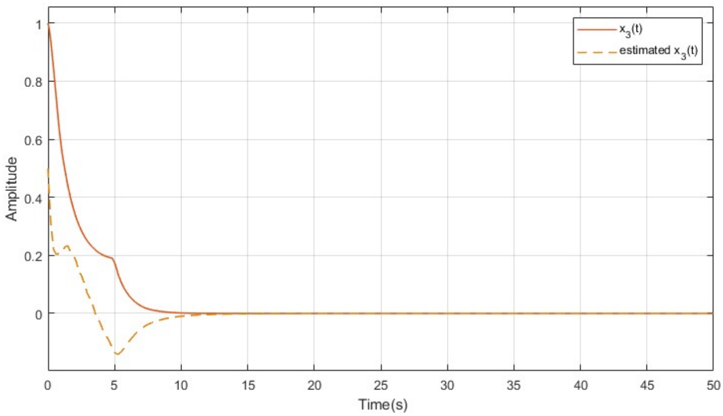

Fig. 5 The vector $x_3(t)$ and its estimation.

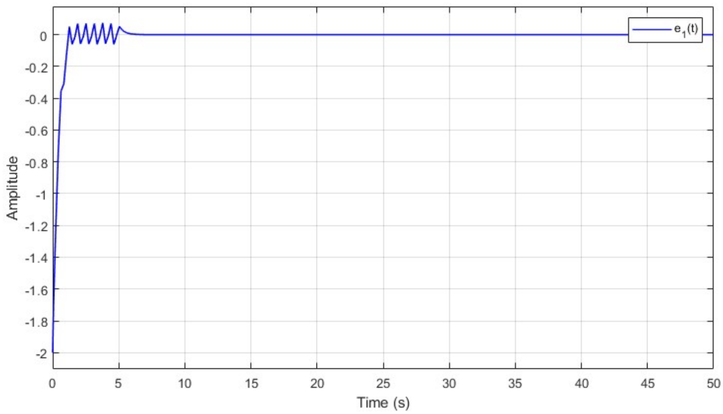

Fig. 6 The error e_1.

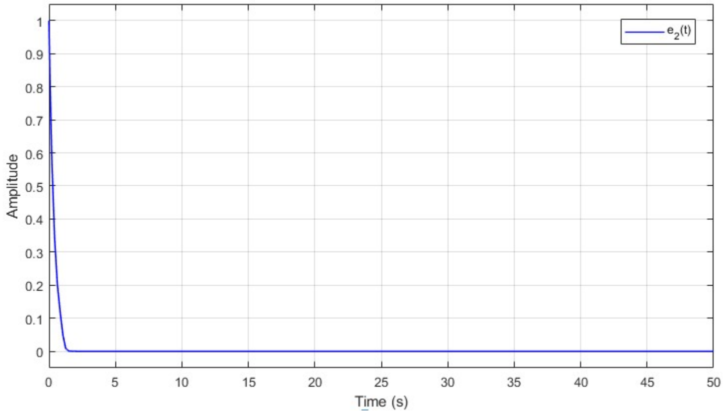

Fig. 7 The error e_2.

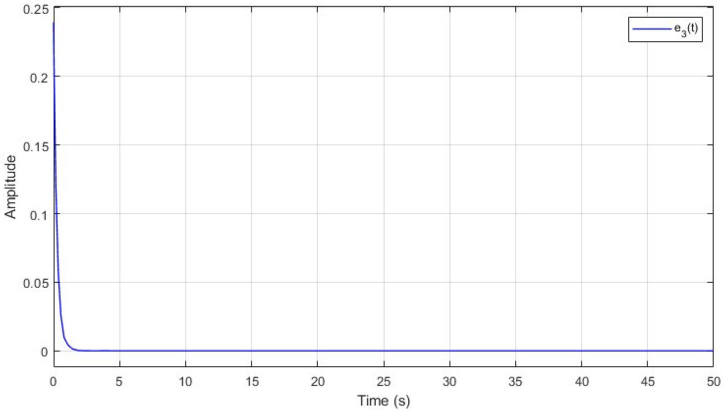

Fig. 8 The error e_3.

5 Conclusion

This research has proposed an approach to design H_∞ functional observers for lipschitz nonlinear systems with disturbances and multiple delays. The main advantage of this scheme is that our methodology incorporates nonlinearity, multiple delays, and external disturbances, which more realistically describe a physical system. In the proposed method, first, the observer design problem was simplifed to determine a parameter matrix based on the parameterisation of algebraic constraints acquired from the estimation error analysis. Then, by introducing a generalized augmented Lyapunov functions, less conservative conditions have been achieved, which allow the optimization of the stabilization issue with LMI constraints; moreover, this makes possible to further exploit the advantages of the wirtinger inequality and an extended reciprocal convex combination. An algorithm to determine the parameters of the observer has been proposed. The influence of the perturbation on the estimation error is greatly reduced, and stability is obtained, as shown in the last two examples, which also show the efficiency of the results.

Data Availability The datasets generated during and/or analysed during the current study are available from the corresponding author on reasonable request.

Conflict of interest The authors declared that they have no conflicts of interest to this work.

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