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Mención en Física Matemática

KMOC Formalism with Spin: one-loop observables in gravity

Autor:

Selvin Roberto Vásquez

Tutores:

Andrés Luna Godoy
José María Muñoz Castañeda

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ABSTRACT

In this work the modern formalism, KMOC, is studied for calculating observables in a gauge and in gravity theories. In particular by applying this formalism we have calculated the linear momentum and the change in spin. These observables in classical gravity were obtained at one-loop and in the classical limit they are found to be in agreement with those obtained through traditional methods.

RESUMEN

En este trabajo se estudia el nuevo formalismo, KMOC, para el cálculo de observables en una teoría gauge y en gravedad. Con este formalismo hemos calculado el impulso lineal y el cambio en el espín. Estos observables en gravedad clásica se obtuvieron a un-lazo y en el límite clásico coinciden con los obtenidos a través de métodos tradicionales.

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1 INTRODUCTION

At macroscopic scales, the gravitational interaction is studied using Einstein's theory of general relativity (GR), which describes gravity as a geometric property of the space-time. There is no doubt that GR has made it possible to accurately understand and describe a huge variety of classical phenomena in our Universe (Misner *et al.*, 2017).
When general relativity was born a little over 100 years ago, experimental confirmation was almost a secondary issue. Today, in contrast, experimental gravitation is an important component of the field with aims to test or disprove the predictions of theory. In fact, a new window into the universe was oppened with the direct confirmation of Gravitational Waves (GWs) by the LIGO (LIGO Scientific Collaboration *et al.*, 2015) and VIRGO (Acernese *et al.*, 2015) collaborations. After this important achievement, the revolutionary era of GWs astronomy was inaugurated. Also new experimental efforts have come together such as KAGRA in Japan (Akutsu *et al.*, 2021), LISA (Amaro-Seoane *et al.*, 2017) while a new ground-based detector, the Einstein telescope (Punturo *et al.*, 2010), is also being developed by ESA.
The possibility to observe the universe through the spectrum of gravitational waves, which covers more than 20 orders of magnitude, requires different detectors for the frequency range of interest. In order to take this great step for humanity we need to improve the description of the gravitational interaction of two compact objects, to generate the theoretical waveforms necessary for the detection of events, as well as the extraction of parameters from observed mergers.

1.1 Gravity observables

We can think of GWs as small, local disturbances in spacetime that propagate through the universe at the speed of light. Currently, the main sources of GWs are binary systems of compact objects (Spurio, 2019). In fact the fist GWs' signal, GW150914 (Abbott *et al.*, 2016), came from the inward spiral and merger of a binary black hole (BH) system (see figure 1). GWs carry information from their sources such as angular and linear momentum, as well as their energy, which in turn can be measured by terrestrial detectors. Theoreticaly we can predict this quantities from the description of the binary dynamics of compact objetcs, but this is still a challenge of gravitational physics. In this sense is pressingly relevant to improve the description for the two-body problem in the different stages of a coalescing binary.

Historically the *Newtonian* approach was the first to be used to deal with the two-body problem. This approximation holds for low velocities, but at relativistic velocities, the *Newtonian* framework used to derive relations between quantities no longer applies. A more precise description of the dynamic of two-body systems is given by GR. The main discrepancies between the *Newtonian* gravitation and GR relies on the relativistic notion of trajectory, which in GR we call *worldline* (Misner *et al.*, 2017; Carroll, 2019). The geometric nature of gravity in GR has deep observational consequences, including the existence of GW radiation.
The perturbative arsenal of classical tools to attack the two-body problem is contained in three main methods : *Post Newtonian* (PN) aproximation, *Post Minkowskian* (PM) approximation and *Self Force* (SF). Every approximation scheme only can deal with the problem in a certain regime. For this description consider two bound compact objects with masses m_1, m_2; speeds u_1, u_2 and separated by a distance b:

- The PN approximation relies on a weak-field, slow-motion. Here the equation of motion are expanded in terms of the compactness GM/b and v^2. The weak-field, and slow-motion regime is particulary adequate for bound, virialized binary systems, and is valid during the spiralling phase, see Fig.1. This approximation naturally fails during the last cycles of the binary, when the velocities of the bodies become comparable to the speed of light.

- the *post-Minkowskian* (PM) aproach assumes weak fields, but we can assume any velocity regime, and perturb in powers of the Newton's constant G.

The regime of slow velocities and weak field is particularly adequate for bound systems and it is valid in the inspiral stage. The PM approximation results in a versatile method which is possible to apply for unbound systems, i.e scattering regime (Kälin and Porto, 2020b), see figure 3.

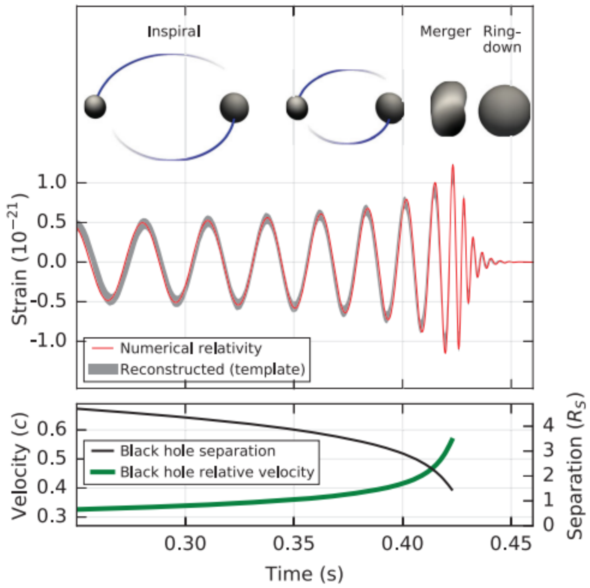


Figure 1: GW150914 signal interpretation as seen at Hanford observatory. There are tree stages in a GW event, the lower plot shows the velocity of the components as function of their spatial separation. Figure from (Abbott *et al.*, 2016).

The study of a weak gravitational scattering encounter system began at 1PM considering the bodies in the dispersion as two spinless particles with mass and with spherical symmetry (Portilla, 1979; Westpfahl, 1985; Damour, 2016; Ledvinka *et al.*, 2008). While in (Vines, 2018) the net changes in the momentum and spins of two spinning BHs were computed to 1PM. Before going beyond classical methods to compute observables, we discuss these significant results. Consider two spinning BHs with arbitrary mass, interacting through a weak gravitational scattering process. This classical system is pictorically represented in fig.2. This dynamic process takes place in an asymptotically flat space, where we can work it out as a Minkowski spacetime. Under this consideration the incoming and outgoing states can be characterized by constant linear momentum p^μ and the tensor field $J^{\mu\nu}$ which contain the total angular momentum about the point x, that is $J^{\mu\nu}(x') = J^{\mu\nu}(x) + 2p^{[\mu}(x' - x)^{\nu]}$.

Defining the body's proper CM worline to be the set of points z such that $J^{\mu\nu}(z)p_\nu = 0$ and we can then write:

$$J^{\mu\nu}(x) = 2p^{[\mu}(x' - x)^{\nu]} + S^{\mu\nu}, \tag{1.1}$$

here $S^{\mu\nu} = J^{\mu\nu}(z)$ is the intrinsic angular momentum tensor, satisfying the algebraic constraint

$$S^{\mu\nu}p_\nu = 0. \tag{1.2}$$

this constrain is called a spin supplementary condition (this is also known as the Tulczyjew-Dixon SSC), and imposes the vanishing of the body's mass dipole vector in the frame of p^μ. The kinematics of an object like a BH can be specified in terms of these worldline fields. So the momentum p^μ and the spin pseudo-vector allow us to determine the components of $S^{\mu\nu}$,

$$s_\mu = \frac{1}{2m}\varepsilon_{\mu\nu\rho\sigma}p^\nu S^{\rho\sigma} = \frac{1}{2m}\varepsilon_{\mu\nu\rho\sigma}p^\nu J^{\rho\sigma}(x), \tag{1.3}$$

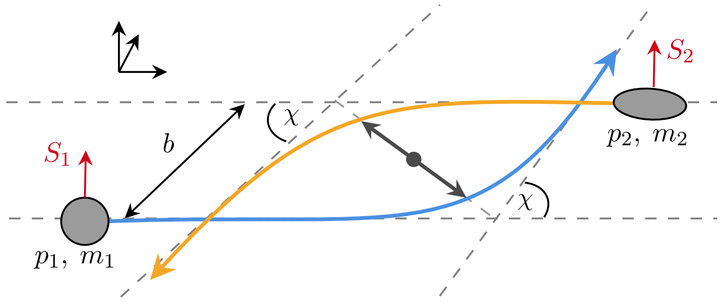


Figure 2: Scattering map of two spinning compact objects in the CM (gray-doted line). The incoming objects with momentum $p_{i=1,2}$ have masses $m_{i=1,2}$ and the spin is labeled with $s_{i=i,2}$, in the figure b represents the impact parameter and χ is the scattering angle. Figure adapted from (Antonelli *et al.*, 2020).

Consider two spinning BHs both approaching from the past infinity: BH 1 with incoming momenta and spin vector given by $p_1^\mu = m_1 u_1^\mu, s_1^\mu = m_1 a_1^\mu$ respectively. BH 2 with incoming momenta $p_2^\mu = m_2 u_2^\mu$, and incoming spin vector $s_2^\mu = m_2 a_2^\mu$. After approaching up to a distance b^μ interact gravitationally and move away to the future infinite, this scattering is represented in the figure 2.

According to the no-hair theorem, a Kerr BH is characterized uniquely by the mass and the intrinsic angular momentum (spin) (Misner *et al.*, 2017), then a natural consequence is that the net changes (Δ) in the linear and angular momentum depend only on their incoming linear and angular momentum. In order to determine these gravity observables we can take the equation of motions and integrate them over entire history of BH 1's zeroth-order state which corresponds to leading-order in the Newton's constant (G) expansion

$$\Delta p_{1\mu} = \int d\tau_1 \frac{\mathcal{L}_{\text{int}}}{\partial z_1^\mu}, \tag{1.4}$$

$$\Delta s_1^\mu = \int d\tau_1 \left(-\varepsilon^{\mu\nu}{}_{\alpha\beta} u_1^\alpha a_1^\beta \frac{\mathcal{L}_{\text{int}}}{\partial a_1^\nu} + u_1^\mu a_1^\nu \frac{\mathcal{L}_{\text{int}}}{\partial z_1^\nu} \right) \tag{1.5}$$

Here $\mathcal{L}_{\text{int}}$ is the interacting Lagrangian and we must to integrate over the entire *world-line* z_1. After some manipulations the results derived in Vines (2018) can be expressed as

$$\Delta p_1^\mu = \text{Re}\left\{\mathcal{Z}^\mu\right\} + \mathcal{O}(G^2) \tag{1.6}$$

$$\Delta s_1^\mu = -u_1^\mu a_1^\nu \text{Re}\left\{\mathcal{Z}_\nu\right\} - \varepsilon^{\mu\nu\alpha\beta} u_{1_\alpha} a_{1_\beta} \text{Im}\left\{\mathcal{Z}_\nu\right\} + \mathcal{O}(G^2) \tag{1.7}$$

where

$$\mathcal{Z}_\nu = \frac{2Gm_1m_2}{\sqrt{\gamma^2-1}} \left[(2\gamma^2-1)\eta_{\mu\nu} - 2i\gamma\varepsilon_{\mu\nu\alpha\beta} u_1^\alpha u_1^\beta \right] \frac{b^\nu + i\Pi^\nu_\rho (a_1+a_2)^\rho}{\left[b + i\Pi(a_1+a_1)\right]^2} \tag{1.8}$$

The gamma factor is $\gamma = u_1 \cdot u_2$ and also

$$\begin{aligned} \Pi^\mu_\nu &= \varepsilon^{\mu\rho\alpha\beta} \varepsilon_{\nu\rho\gamma\delta} \frac{u_{1_\alpha} u_{2_\beta} u_1^\gamma u_2^\delta}{\gamma^2-1} \\ &= \delta^\mu_\nu + \frac{1}{\gamma^2-1}\left[u_1^\mu (u_{1_\nu} - \gamma u_{2_\nu}) + u_2^\mu (u_{2\nu} - \gamma u_{1_\nu}) \right] \end{aligned} \tag{1.9}$$

being the projector into the plane orthogonal to both incoming velocities. Now expanding the previous expresions in the rescaled spin a_1^μ, and setting $a_2^\mu \to 0$ the linear impulse is

$$\begin{aligned} \Delta p_1^\mu = \frac{2Gm_1m_2}{\sqrt{\gamma^2-1}} \Bigg\{ & (2\gamma^2-1)\frac{b^\mu}{b^2} + \frac{2\gamma}{b^4}\left(2b^\mu b^\nu - b^2\Pi^{\mu\nu}\right) \varepsilon_{\nu\rho\alpha\beta\delta} u_1^\alpha u_2^\beta a_1^\rho \\ & - \frac{2\gamma^2-1}{b^6}\left(4b^\mu b^\nu b^\rho - 3b^2 b^{(\mu}\Pi^{\nu\rho)}\right) a_{1\nu} a_{1\rho} + \mathcal{O}(a^3) \Bigg\} + \mathcal{O}(G^2). \end{aligned} \tag{1.10}$$

Following the KMOC literature from now on we will call the net change in the spin (pseudo-)vector angular impulse and this quantity is given by

$$\Delta s_1^{\mu,(0)} = -u_1^{\mu} a_{1\nu} \Delta p_1^{\mu} - \frac{2Gm_1m_2}{\sqrt{\gamma^2-1}} \left\{ 2\gamma \varepsilon^{\mu\nu\rho\sigma} u_{1\rho} \varepsilon_{\sigma\alpha\beta\gamma} u_1^{\beta} u_2^{\gamma} \frac{b^{\alpha}}{b^2} a_{1\nu} \right.$$
$$\left. - \frac{2\gamma^2-1}{b^4} \varepsilon^{\mu\nu\kappa\lambda} u_{1\kappa} \left(2b_\nu b_\rho - b^2 \Pi_{\nu\rho}\right) a_{1\lambda} a_1^{\rho} + \mathcal{O}(a^3) \right\} + \mathcal{O}(G^2). \quad (1.11)$$

1.2 Beyond the classical methods

On the other hand, we have the scattering-amplitudes based treatment of gravitational scattering of compact objects. This astounding relation between amplitudes in quantum field theory (QFT) and classical physics is driven by tools developed for particle colliders, such as unitarity methods (Bern *et al.*, 1994, 1995), the double copy (Kawai *et al.*, 1986; Bern *et al.*, 2008), the spinor helicity formalism (Arkani-Hamed *et al.*, 2021), and differential equations (Kotikov, 1991; Bern *et al.*, 1993; Gehrmann and Remiddi, 2000; Henn, 2013, 2015) for loop integration. Computations in gravity are greatly simplified by harnessing the broad machinery of scattering amplitudes, and although these methods are valid only in scattering scenarios, the bound scenarios are partially understood through analytic continuation of scattering results. (Kälin and Porto, 2020a; Kälin and Porto, 2020; Saketh *et al.*, 2021).

Recently, a new approach has gained attention in this context, the observables-based formalism developed by Kosower, Maybee and O'Connell (KMOC) (Kosower *et al.*, 2019), which establishes a precise connection between scattering amplitudes and observables in classical physics. KMOC allows us to compute classical observables, such as the impulse of the particles (Herrmann *et al.*, 2021b; Maybee *et al.*, 2019) and the radiated momentum during a collision (Manu *et al.*, 2021; Herrmann *et al.*, 2021a), directly from on-shell scattering amplitudes at any order in electrodynamics and gravity. With the introduction of the KMOC formalism, a variety of classical problems in gauge theories and gravity can now be approached from a pure QFT perspective (Bern *et al.*, 2022; de la Cruz *et al.*, 2022).

The scattering of two spinning black holes in post-Minkowskian gravity, to all orders in spin was computed in (Vines, 2018) and to quadratic order in spin to 2PM order in (Kosmopoulos and Luna, 2021; Liu *et al.*, 2021). On the other hand, using KMOC, the results to 1PM in Vines (2018) and the linear momentum to 2PM order in (Cordero *et al.*, 2022) have been reproduced. In this work we focus on deriving the results of the impulse and spin kick of two spinning BHs to 2PM order due a gravitational interaction in a scattering process. The machinery offered by the KMOC formalism is applied to obtain the desired results.

This work is organized as follows: In section 2, we provide a description of the new techniques based on scattering amplitudes for the two-body problem. This new arsenal of tools will provide an overview of the fruitful results derived with these methods. Also we present a detailed description of the KMOC machinery. This will provide us with a robust framework for computing classical observables in gravity directly from the classical limit of QFT amplitudes. The heart of this frontier physics formalism lies in the on-shell amplitudes and in section 3, we present the necessary gravity amplitudes at tree and one-loop level. In section 4, we present the one-loop observables derived by applying the KMOC formalism. We finalize this work with the conclusions in section 5.

2 GRAVITY OBSERVABLES FROM SCATTERING AMPLITUDES AND THE KMOC FORMALISM

As we move to study nature at microscopic scales we invoke QFT, which has become a very precise calculation tool in this regime. Using it we can describe physical processes in a range of energies that goes from the few million electrovolts typical of nuclear physics to the billions of electrovolts of the Large Hadron Collider (LHC). The connection between QFT and experimental physics occurs through the computation of scattering amplitudes which describe the collisions between particles. In this section we provide a description of the scattering framework between two particles and a description in a nutshell about deriving classical observables from the physics of collisions.

2.1 Two spinning particles scattering in quantum field theory

In figure 3, an usual collision scheme is shown. The pragmatic idea is that two bunches of particles which are initially sufficiently distant from each other (so that the idealization that they do not interact is physically reasonable) are accelerated to relativistic velocities and made to collide. Here they produce some complicated interacting quantum state. After the collision this state evolves into several outgoing particles (the products) moving away in various directions until they are sufficiently well separated that the approximation of non-interaction is again reasonable.

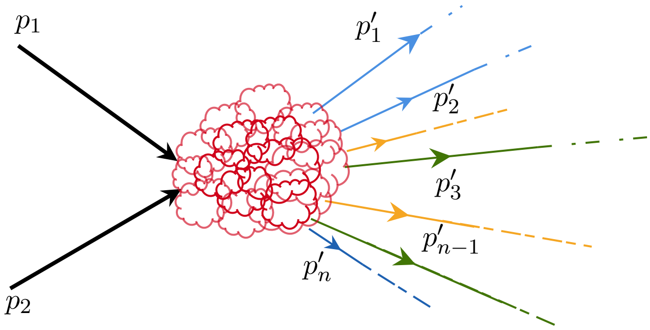


Figure 3: Usual setup for scattering experiments. The incoming particles are labeled with with momenta p_1, p_2 and the outgoing particles with p'_n.

In experiments such as LHC the outgoing particles are measured and recorded, but the collision process is not entirely deterministic. QFT is probabilistic, so if our goal is to find the transition probability between an initial and a final state in a collision process, how do we do it?

In collider experiments one measures scattering cross sections, but this is not straightforward. Our goal is to find the transition probability between an initial state and a final state in a collision process. So we need to prepare the initial and final states with definite momenta p_1, p_2 and $p'_1, p'_2, ..., p'_n$ respectively, at some initial time t_i before the collision and final time t_f after the collision

$$|\text{in}\rangle \sim |p_1, p_2\rangle\,, \quad \langle\text{out}| \sim \langle p'_1, p'_2, ..., p'_n|\,. \tag{2.1}$$

The probability amplitude that the $|\text{in}\rangle$ evolves to $|\text{out}\rangle$ is given by

$$\langle\text{out}|\text{in}\rangle = \langle\text{out}|\, e^{-iH(t_f-t_i)}\, |\text{in}\rangle\,. \tag{2.2}$$

We call the S *matrix* to the operator $e^{-iH(t_f-t_i)}$ and in the limit $(t_f - t_i) \to \infty$, where H is the Hamiltonian of the theory. The *scattering amplitude* is given by:

$$\langle\text{out}|\, S\, |\text{in}\rangle = \lim_{(t_f-t_i)\to\infty} \langle\text{out}|\, e^{-iH(t_f-t_i)}\, |\text{in}\rangle\,, \tag{2.3}$$

If $\langle\text{in}|\text{in}\rangle = 1$ and $|n\rangle$ is a complete basis of states $\sum_n |n\rangle\langle n| = 1$, so

$$1 = \sum_n |\langle n|\, S\, |\text{in}\rangle|^2 = \sum_n \langle\text{in}|\, S^\dagger\, |n\rangle\langle n|\, S\, |\text{in}\rangle = \langle\text{in}|\, S^\dagger S\, |\text{in}\rangle\,, \tag{2.4}$$

then we can deduce that $S^{\dagger}S = 1$, which means S is an unitary operator and this unitarity property of S express the conservation of probability.
The cross section σ can be related with the square-module of 2.3. For initial and final states with definite momenta, the S *matrix* contains a delta function $\delta^{(4)}(p_1 + ... + p_m - p'_1 - ... - p'_n)$ to conserve momentum. Inserting the unitarity relation we can express S in a convenient way:

$$S \equiv 1 + iT, \tag{2.5}$$

$$S^{\dagger}S = 1. \tag{2.6}$$

And the following relation can be established:

$$-i(T - T^{\dagger}) = T^{\dagger}T = 2\text{Im}(T) \tag{2.7}$$

An important quantity emerges if we rewrite (2.3) in terms of (2.5):

$$\langle p'_1, ..., p'_n | T | p_1, ..., p_m \rangle = \mathcal{A}(p_1, ..., p_m \to p'_1, ..., p'_n)(2\pi)^4 \delta^{(4)}(p_1 + ... + p_m - p'_1 - ... - p'_n). \tag{2.8}$$

In (2.8), $\mathcal{A}$ represent the matrix element between the initial and final states, in QFT this important quantity is usual called the amplitude. In order to calculate an amplitude, normally we use Feynman diagrams. Amplitudes in collider physics allows us to compute quantities such as the cross section, decay rate and also we can compute observables in terms of amplitudes. Scattering amplitudes are at the heart of high energy physics, but currently new techniques have been developed that allow to approach purely classical problems from the amplitudes in QFT. We comment further on this in the remainder of this section.

2.2 Gravity observables from scattering amplitude techniques

In section 1, we discuss the scattering of two black holes and how to calculate gravity observables from traditional methods. From a particle physicist's perspective, BHs are understood as point spinning particles. In this sense, the scattering of two BHs is not so different from the scattering of two quarks. For example both processes involve the interactions of tensorial particles, in non-linear theories. From the quantum amplitudes involving gluons, one can extract the corresponding information of the scattering process for two black holes, and derive observable quantities. Then a natural question at this point might be: but, how to switch from quantum to classical physics?
The correspondence principle states that classical physics emerges from the quantum theory in the limit of macroscopic conserved charges such as masses, electric charges, spins, orbital angular momenta. In the context of Black Hole Binary Dynamics and the Gravitational Scattering the transition has been discussed in (Bern *et al.*, 2019a; Vecchia *et al.*, 2021). For example, if we study the problem of two bodies in GR or in electrodynamics from the perspective of QFT, the classical limit is controlled by the Compton wavelength λ_c which is related to Plack's constant and the mass of the particles. In a scattering process of two black holes with a size of the Schwarzschil radius r_S, approached through a two-particle process $2 \to 2$ in QFT. The PM approximation requires the following condition $\lambda_c \ll r_S \ll b$ in order to extract the classical information.

However, the scattering processes in QFT are encoded in the amplitude and in the last decades an enormous machinery has been developed to push forward the comprehension and it's computations such as on-shell methods, double copy, unitarity methods, spinor helicity formalism, integration by parts identities, and advanced multiloop integration.
All these powerful tools from the modern amplitudes program have been brought to address problems that were the domain of classical methods. For example, in (Bern *et al.*, 2019b,a) the authors take advantage of these amplitude techniques such as double copy and generalized unitarity to derive the amplitude for classical scattering of gravitationally interacting massive scalars at 3PM order, also results

at 4PM have been derived in (Bern *et al.*, 2021b). Just like the classical methods, those approaches are valid and usually very powerful in different regions or stages in the binary dynamics. We can improve the accuracy that would be obtained with a single method by combining different approximations. Such "Tutti-Frutti" method which can combines several theoretical formalisms: PN, PM or effective field theory (EFT) among others, this method has been used to derive partial 5 and 6PN results in the conservative sector (Bini *et al.*, 2019, 2021). Again we emphasize that these methods are valid for unbound, scattering black holes, but they can also lead to results for bound orbits. Both scenarios are relevant for the detection of gravitational waves.

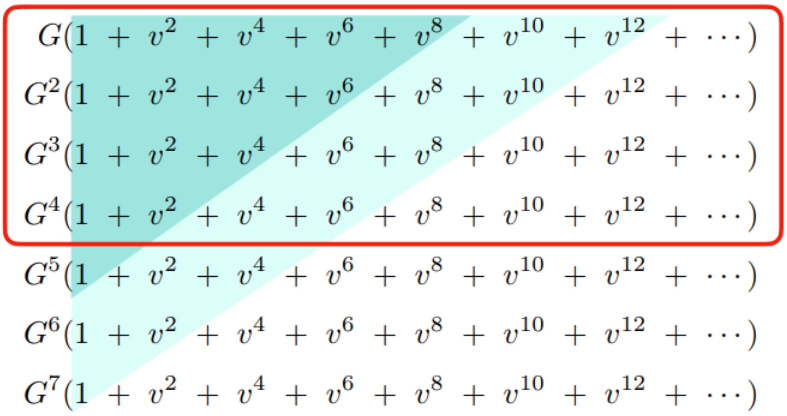


Figure 4: Corrections to the Newton's potential. In the red box the new results to $\mathcal{O}(G^4)$ using QFT tools are shown, the horizontal lines in the indicate the state of the art PM result. The vertical lines correspond then to the PN information currently available from PM approximation. These results overlap with the state-of-the-art from the PN approximation (dark triangle) and contributions required by future detectors (light triangle). Figure from (Buonanno *et al.*, 2022).

The spin and finite-size effects have been included in all those methods. And as described in (Larrouturou, 2021): for the traditional PN computations, the results with spins effects are gathered in (Blanchet, 2014). A double copy framework including spins has been derived, and applied up to 2PM to the spin-spin coupling (Bern *et al.*, 2021a). As for the EFT method, the conservative and radiative spin-spin effects at next-to-leading order have also been recently computed (Cho *et al.*, 2021), and were found to perfectly agree with previous results. Regarding finite-size effects, which enter at 5PN, they are known up to 2.5PN (Henry *et al.*, 2020b,a) and 2PM (Cheung and Solon, 2020; Kälin *et al.*, 2020) beyond the leading order, those results perfectly agreeing in the overlapping regions, as proven in (Henry *et al.*, 2020a).

In recent years we have seen how the methods that have allowed fruitful results in high energy physics have doubled efforts to deal with fundamental problems of classical physics, specifically those related to improving predictions of GW signals. We will now proceed to describe one of the frontier methods to extracting classical physics from amplitudes which focuses on determining physical observables which are well defined in both the classical and quantum theories.

2.3 The Kosower, Maybee and O'Connell Formalism (KMOC)

In this section we introduce the KMOC formalism with a detailed overview. This section provide us with the machinery offered from QFT amplitudes in order to obtain classical observables directly from on-shell scattering amplitudes. First we describe the KMOC formalism and how this robust frame allow us compute the linear impulse by rewriting this classical observable in terms of QFT scattering amplitudes.

In the KMOC formalism, the expectation value for the change of a quantum mechanical observable, $\langle\Delta\mathcal{O}\rangle$, due a scattering process is computed from scattering matrix, S, thought the formula:

$$\langle \Delta \mathcal{O}_j \rangle = {}_{\text{in}}\langle \Psi | \, S^{\dagger} \hat{\mathbb{O}}_j S \, | \Psi \rangle_{\text{in}} - {}_{\text{in}}\langle \Psi | \, \hat{\mathbb{O}}_j \, | \Psi \rangle_{\text{in}} \, . \tag{2.9}$$

Where the sub-index is used to labeled the observable and operator for particle j. In general the expression (2.9) corresponds to the difference between the in and out states expectation values, where we have relied on S as a time evolution operator determining the form of the asymptotic final state of the system $|\Psi\rangle_{\text{out}} = U(\infty, -\infty) \, |\Psi\rangle_{\text{in}} = S \, |\Psi\rangle_{\text{in}}$.

We emphasize that the KMOC formalism is based on amplitudes, so the connection of the quantum observable $\Delta\mathcal{O}$ to the scattering amplitude is made through the following procedure:

1. In (2.9) we proceed by writing the $S-$operator in terms of the transition matrix T via (2.5), now we use the unitarity condition of the $S-$operator expressed in Eq.(2.6), then this allow us to rewrite (2.9) in the form:
$$\langle \Delta \mathcal{O}_j \rangle = i \, {}_{\text{in}}\langle \Psi | \, [\hat{\mathbb{O}}_j, T] \, | \Psi \rangle_{\text{in}} + {}_{\text{in}}\langle \Psi | \, T^{\dagger} [\hat{\mathbb{O}}_j, T] \, | \Psi \rangle_{\text{in}} \, . \tag{2.10}$$
The first term is linear in a scattering amplitude and the latter will contain a product of scattering amplitudes. We can obtain all orders in perturbation theory

2. After deriving the KMOC formula Eq.(2.10), we proceed to insert a complete set of states between T and the operator $\mathbb{O}$ which is associated to the quantum observable
$$ {}_{\text{out}}\langle \Psi | \, \hat{\mathbb{O}}_j \, | \Psi \rangle_{\text{out}} = \sum_X \int \prod_i d\Phi(r_i) \mathcal{O}_j |\langle r_i X | \, S \, | \Psi \rangle|^2, \tag{2.11}$$
this complete set of states allow us to make explicit the amplitudes. In (2.11), r_i are the momentum of the intermediate massless particles, X and $d\Phi(r_i)$ represents the on-shell integrals (over Lorentz-invariant phase space) . The sum is over all states X with suitable quantum numbers as well as an integration over each statet's available phase space, and $\mathcal{O}_j$ is the value of the observable in that state. The expression (2.11) hints at the possibility of evaluating the momentum in terms of on-shell scattering amplitudes. The expression (2.11) requires the specification of the system's initial state.

As we have briefly described in the subsection (2.1) the n-particle asymptotic states in momentum space $|p_1, ..., p_n\rangle$ are the tensor products of the normalised single particle states $a_p^{\dagger} \, |0\rangle$, where $a_p^{\dagger}$ is the creation operator for the momentum p and $|0\rangle$ is the vacuum state which is annihilated by a_p operator. The conjugate states are labeled by $\langle p_1', ..., p_n'|$, then we define the amplitudes in four dimensions

$$\left\langle p_1', ..., p_n' \right| T \, |p_1, ..., p_m\rangle = \mathcal{A}(p_1, ..., p_m \to p_1', ..., p_n') \hat{\delta}^{(4)}(p_1 + ... + p_m - p_1' - ... - p_n'), \tag{2.12}$$

as it has been defined in (2.8), but from here so on we follow the notation $\hat{\delta}$ as in (Kosower *et al.*, 2019) to hide the 2π factor. Also if we denote by $[M]$ and $[L]$ the dimensions of mass and length, respectively. The dimensions in (2.12) are $[M]^{4-n}$, where $D - n$ is the dimension (D) for the n-point scattering amplitude. Now we must understand systematically how to extract the classical result using on-shell QFT scattering amplitudes in order to take full advantage of amplitude methods in the gravitational-wave problem, the ingredients we need are:

1. A parameter that allows us to explore the classical limit.

2. The adequate wave functions that describe the multi-particle initial state of the system and that it has the desired classical limit.

In fact, we control the classical limit with the constant $\hbar$, which as shown in (Kosower *et al.*, 2019) can be restored in an expression through dimensional analysis ($\hbar \neq 1$). This parameter appears in the coupling constants, which which are re-scaled in terms of $\hbar$, and also the wave numbers associated to massless momenta for the force carriers contains this parameter. In the subsection 2.3.2 the extraction of the classical limit is discussed in further detail. So, the classical piece of the observable, is then defined by:

$$\Delta \mathcal{O}_j = \lim_{\hbar \to 0} \hbar \left[{}_{\text{in}}i \langle \Psi | [\hat{\mathbb{O}}_j, T] | \Psi \rangle_{\text{in}} + {}_{\text{in}}\langle \Psi | T^\dagger [\hat{\mathbb{O}}_j, T] | \Psi \rangle_{\text{in}} \right]. \tag{2.13}$$

To denote the classical limit we remove the expectation value $\langle ... \rangle$ of the observable. The first term in (2.13) is linear in the amplitude and will be able to contribute at leading-order (LO), while the second one is quadratic and this will lead to the discontinuity of a scattering amplitude.
Note that we have dropped the "in" label and we will do so from now on.

The expression (2.13) is a formal definition to the observable at the quantum level, the factor of $\hbar$ ensures the classical scaling. As we can see this expectation value requires the appropriate initial state of the system. The particles are prepared in the far past so the appropriate states are incoming states $|\Psi\rangle_{\text{in}}$. The incoming particles are described by the relativistic wave-functions $\phi_i(p_i)$ which use to build a quantum state corresponding to a localized particle. So the initial state is defined as follows:

$$|\Psi\rangle_{\text{in}} = \int \prod_i [d\Phi(p_i)\phi_i(p_i)e^{ib\cdot p_1/\hbar}] \, |p_1 p_2\rangle. \tag{2.14}$$

Again Eq.(2.14) $d\Phi(p_i)$ represents the on-shell integrals (over Lorentz-invariant phase space). The short-hand notation for $d\Phi(p_i)$ will be convenient

$$d\Phi(p_i) \equiv \hat{d}^n p \hat{\delta}^{(+)}(p_i^2 - m_i^2), \tag{2.15}$$

to $\hat{d}^n p$ hide the factor of 2π throughout and is defined by

$$\hat{d}^n p \equiv \frac{d^n p}{(2\pi)^n}, \tag{2.16}$$

and finally the integrations are restricted to positive-energy solutions of the delta functions of $p_i^2 - m_i^2$, as indicated by the $(+)$ superscript in $\hat{\delta}^{(+)}$, as well as absorbing a factor of 2π just for $\hat{\delta}(p)$

$$\hat{\delta}^{(+)}(p_i^2 - m_i^2) \equiv \Theta(p^0)\hat{\delta}(p_i^2 - m_i^2), \tag{2.17}$$

where $\Theta(x)$ is the Heaviside step function and p^0 is the energy component of the four-vector.
For the dimensionless of the state we need $\phi_i(p_i)$ to have dimensions of inverse energy. This also allows us to normalize the state to unity, which we can accomplish by normalizing the wave-functions,

$$\int \prod_i d\Phi(p_i) |\phi_i(p_i)|^2 = 1. \tag{2.18}$$

2.3.1 Linear impulse in terms of amplitudes

Now consider a spinless scattering process between two stable quanta fields 1 and 2 with masses m_1 and m_2 respectively. Following the KMOC formalism we relate the observable (2.13) to the scattering amplitude using (2.12), together with the initial two-particle state (2.14). So the first observable written in terms of a scattering amplitude is the impulse of one of the particles, say particle 1. Let $\mathbb{P}_j^\mu$ be the momentum operator for quantum field j, the change in particle 1's momentum is then:

$$\langle \Delta p_1^\mu \rangle = i \langle \Psi | [\mathbb{P}_1^\mu, T] | \Psi \rangle + \langle \Psi | T^\dagger [\mathbb{P}_1^\mu, T] | \Psi \rangle. \tag{2.19}$$

This expression indicates at a quantum level the impulse is the difference between the expected outgoing and incoming momenta of such a particle. We can understand the link between amplitudes and the final-state value of the observable by inserting a complete set of states. For convenience we separate this on-shell observable in two terms:

$$I^{\mu}_{(1)} = i \langle \Psi | \left[\mathbb{P}^{\mu}_1, T \right] | \Psi \rangle , \tag{2.20}$$

$$I^{\mu}_{(2)} = \langle \Psi | T^{\dagger} [\mathbb{P}^{\mu}_1, T] | \Psi \rangle . \tag{2.21}$$

Replacing the initial state and its conjugate in (2.20), this contribution linear in the amplitude to the impulse turns into

$$I^{\mu}_{(1)} = \int d\Phi(p_1')d\Phi(p_2')d\Phi(p_1)d\Phi(p_2)\phi_1(p_1)\phi_1^*(p_1')\phi_2(p_2)\phi_2^*(p_2')e^{ib\cdot(p_1-p_1')/\hbar} i \langle p_1'p_2' | \mathbb{P}^{\mu}_1 T - T\mathbb{P}^{\mu}_1 | p_1 p_2 \rangle \tag{2.22}$$

$$= \int d\Phi(p_1')d\Phi(p_2')d\Phi(p_1)d\Phi(p_2)\phi_1(p_1)\phi_1^*(p_1')\phi_2(p_2)\phi_2^*(p_2') i(p_1'^{\mu} - p_1^{\mu})e^{ib\cdot(p_1-p_1')/\hbar} \langle p_1'p_2' | T | p_1 p_2 \rangle . \tag{2.23}$$

With the help of (2.12) it is straightforward to identify a familiar expression in (2.23) given by

$$\langle p_1'p_2' | T | p_1 p_2 \rangle = \mathcal{A}(p_1 p_2 \to p_1', p_2')\hat{\delta}^{(4)}(p_1' + p_2' - p_1 - p_2), \tag{2.24}$$

which is the four-point scattering amplitude of our two particles, the 4-fold delta function will allow us to perform $d-$integrals in $I^{\mu}_{(1)}$. Now in (2.23) we change the momentum of the outgoing particles in terms of the momentum transfer q and the initial momentum of the particles p_i, we do this by introducing the momentum shifts $q_i = p_i' - p_i$, then the amplitude becomes

$$\langle p_1'p_2' | T | p_1 p_2 \rangle = \mathcal{A}(p_1 p_2 \to p_1 + q_1, p_2 + q_2)\hat{\delta}^{(4)}(q_1 + q_2), \tag{2.25}$$

and changing the integration variables from $p_i' \to q_i$, and further using $\hat{\delta}^{(4)}(q_1 + q_2)$ to perform the integration over q_2, followed by the relabel $q_1 \to q$, the previous relation becomes

$$d\Phi(p_1 + q_1) = \hat{d}^4 q_1 \hat{\delta}^{(+)}[(p_1 + q_1)^2 - m_1^2]$$

$$= \hat{d}^4 q \Theta(p_1^0 + q^0)\hat{\delta}(2p_1 \cdot q + q^2)\Big|_{q_1 \to q} , \tag{2.26}$$

$$d\Phi(p_2 + q_2) = \hat{d}^4 q_2 \Theta(p_2^0 + q_2^0)\hat{\delta}(2p_2 \cdot q_2 + q_2^2) \tag{2.27}$$

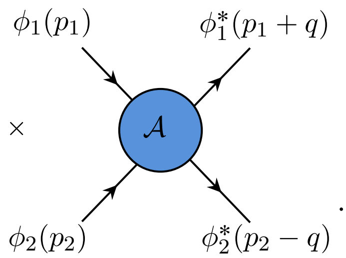
In (2.26) and (2.27) we apply the on-shell condition $p_i^2 - m_i^2 = 0$, but in (2.27) with the integration over q_2 the delta in (2.25) change q_2 by $-q$. Replacing all this together we finally get

$$I^{\mu}_{(1)} = \int d\Phi(p_1)d\Phi(p_2)\hat{d}^4 q \hat{\delta}(2p_1 \cdot q + q^2)\hat{\delta}(2p_2 \cdot q - q^2)\Theta(p_1^0 + q^0)\Theta(p_2^0 - q^0)$$
$$\times e^{-ib\cdot q/\hbar}\phi_1(p_1)\phi_1^*(p_1 + q)\phi_2(p_2)\phi_2^*(p_2 - q) i q^{\mu} \mathcal{A}(p_1, p_2 \to p_1 + q, p_2 - q). \tag{2.28}$$

The latter expression is the LO contribution for the linear impulse in terms of the amplitude $\mathcal{A}$, the i term assures us that the observable we will obtain will be a real quantity. In the section 2.3.2 we will proceed to obtain the classical limit of (2.28). From (2.24) we can deduce that the incoming and outgoing momenta in the amplitude correspond to the initial-state momenta. Diagrammatically this expression looks like:

$$I^{\mu}_{(1)} = \int d\Phi(p_1)d\Phi(p_2)\hat{d}^4 q \hat{\delta}(2p_1 \cdot q + q^2)\hat{\delta}(2p_2 \cdot q - q^2)\Theta(p_1^0 + q^0)\Theta(p_2^0 - q^0)$$

$$\times e^{-ib\cdot q/\hbar} i q^{\mu} \times \tag{2.29}$$



Now the quadratic term in the amplitude, $I^{\mu}_{(2)}$. Since there are two factors of T in this term, scattering amplitudes can be explicitly introduced by inserting a complete set of momentum eigenstates between the operators, in such way we can extract a momentum eigenvalue when the momentum operator hits the momentum eigenstates. In their first appearance this yields

$$I^{\mu}_{(2)} = \sum_X \int d\Phi(r_1) d\Phi(r_2) \langle \Psi | T^{\dagger} | r_1 r_2 X \rangle \langle r_1 r_2 X | [\mathbb{P}^{\mu}_1, T] | \Psi \rangle . \tag{2.30}$$

Expanding the wavefunctions and proceeding in an similar fashion as the LO contribution, that is, we need to replace the initial state and its conjugate in (2.30) and then make the amplitudes explicit from the complete base of states that have been introduced, the expression (2.12) is useful again to identify the dependence of the scattering amplitude (2.30). We label the states in the incoming wavefunction by

$$\begin{aligned} I^{\mu}_{(2)} = \sum_X \int \prod_{i=1,2} d\Phi(r_i) d\Phi(p_i) d\phi(p'_i) \phi_i(p_i) \phi^*_i(p'_i) e^{ib\cdot(p_1 - p'_1)/\hbar} (r^{\mu}_1 - p^{\mu}_1) \\ \times \hat{\delta}^{(4)}(p_1 + p_2 - r_1 - r_2 - r_X) \hat{\delta}^{(4)}(p'_1 + p'_2 - r_1 - r_2 - r_X) \\ \times \mathcal{A}(p_1, p_2 \to r_1, r_2, r_X) \mathcal{A}^*(p'_1, p'_2 \to r_1, r_2, r_X). \end{aligned} \tag{2.31}$$

In this expression, r_1, r_2 and r_X are the final momentum states and r_X is the total momentum carried by particles in state X. The product of amplitudes in the last expression gives us the impression of an amplitude weighted cut structure, in which the lowest order contribution will be a one-loop two particle cut. To simplify it we define again the momentum shifts $q_i = p'_i - p_i$, this allows us to change the integration variables $p'_i \to q_i$:

$$\begin{aligned} I^{\mu}_{(2)} = \sum_X \int \prod_{i=1,2} d\Phi(r_i) d\Phi(p_i) d\phi(q_i + p_i) \phi_i(p_i) \phi^*_i(p_i + q_i) e^{-ib\cdot q_1/\hbar} (r^{\mu}_1 - p^{\mu}_1) \\ \times \hat{\delta}^{(4)}(p_1 + p_2 - r_1 - r_2 - r_X) \hat{\delta}^{(4)}(q_1 + q_2) \\ \times \mathcal{A}(p_1, p_2 \to r_1, r_2, r_X) \mathcal{A}^*(p_1 + q_1, p_2 + q_2 \to r_1, r_2, r_X). \end{aligned} \tag{2.32}$$

Through (2.15) we can again perform the integral over q_2 using the four-fold delta function, and relabel $q_1 \to q$, these steps are are developed as follows:

$$\begin{aligned} I^{\mu}_{(2)} = \sum_X \int \prod_{i=1,2} & d\Phi(r_i) d\Phi(p_i) \hat{d}^4 q_1 \hat{d}^4 q_2 \hat{\delta}(2p_1 \cdot q_1 + q_1^2) \hat{\delta}(2p_2 \cdot q_2 + q_2^2) \Theta(p^0_1 + q^0_1) \Theta(p^0_2 + q^0_2) \\ & \times \phi_1(p_1) \phi^*_1(p_1 + q_1) \phi_2(p_2) \phi^*_2(p_2 + q_2) e^{-ib\cdot q_1/\hbar} (r^{\mu}_1 - p^{\mu}_1) \\ & \hat{\delta}^{(4)}(p_1 + p_2 - r_1 - r_2 - r_X) \hat{\delta}^{(4)}(q_1 + q_2) \\ & \times \mathcal{A}(p_1, p_2 \to r_1, r_2, r_X) \mathcal{A}^*(p_1 + q_1, p_2 + q_2 \to r_1, r_2, r_X), \end{aligned} \tag{2.33}$$

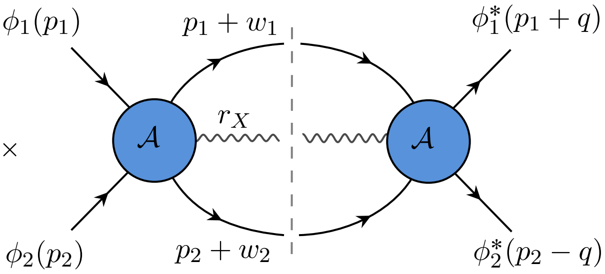
$$\begin{aligned} = \sum_X \int \prod_{i=1,2} & d\Phi(r_i) d\Phi(p_i) \hat{d}^4 q \hat{\delta}(2p_1 \cdot q + q^2) \hat{\delta}(2p_2 \cdot q - q^2) \Theta(p^0_1 + q^0) \Theta(p^0_2 - q^0) \\ & \times \phi_1(p_1) \phi^*_1(p_1 + q) \phi_2(p_2) \phi^*_2(p_2 - q) e^{-ib\cdot q/\hbar} (r^{\mu}_1 - p^{\mu}_1) \hat{\delta}^{(4)}(p_1 + p_2 - r_1 - r_2 - r_X) \\ & \times \mathcal{A}(p_1, p_2 \to r_1, r_2, r_X) \mathcal{A}^*(p_1 + q, p_2 - q \to r_1, r_2, r_X). \end{aligned} \tag{2.34}$$

Defining the momentum transfers $w_i = r_i - p_i$ we can change the integration variable r_i to w_i, using the momentum conserving delta function for each amplitude to perform the integration in w_2 and relabeling $w_1 \to w$ which leaves us with

$$\begin{aligned} I^{\mu}_{(2)} = \sum_X \int \prod_{i=1,2} & d\Phi(p_i) \hat{d}^4 w_i \hat{d}^4 q \hat{\delta}(2p_i \cdot w_i + w_i^2) \Theta(p^0_i + w^0_i) \\ & \times \hat{\delta}(2p_1 \cdot q + q^2) \hat{\delta}(2p_2 \cdot q - q^2) \Theta(p^0_1 + q^0) \Theta(p^0_2 - q^0) \\ & \times \phi_1(p_1) \phi^*_1(p_1 + q) \phi_2(p_2) \phi^*_2(p_2 - q) e^{-ib\cdot q/\hbar} w^{\mu}_1 \hat{\delta}^{(4)}(w_1 + w_2 + r_X) \\ & \times \mathcal{A}(p_1, p_2 \to p_1 + w_1, p_2 + w_2, r_X) \mathcal{A}^*(p_1 + q, p_2 - q \to p_1 + w_1, p_2 + w_2, r_X). \end{aligned} \tag{2.35}$$

We can give a pictorial representation of the NLO contribution to the impulse

$$
\begin{aligned}
I^{\mu}_{(2)} = \sum_X \int \prod_{i=1,2} & d\Phi(p_i)\hat{d}^4 w_i \hat{d}^4 q \hat{\delta}(2p_i \cdot w_i + w_i^2)\Theta(p_i^0 + w_i^0) \\
& \times \hat{\delta}(2p_1 \cdot q + q^2)\hat{\delta}(2p_2 \cdot q - q^2)\Theta(p_1^0 + q^0)\Theta(p_2^0 - q^0) \\
& \times e^{-ib\cdot q/\hbar} w_1^{\mu} \hat{\delta}^{(4)}(w_1 + w_2 + r_X)
\end{aligned}
\tag{2.36}
$$

$\times$  .

The gray dotted line in (2.36) represents a generalized unitary cut (it refers to a unitarity cut which has been generalized) it is possible thank to the unitarity property of the S-matrix. The cut tell us that the 1-loop amplitudes are given by products of tree level amplitudes. The quantum mechanical impulse particle 1 acquires during the scattering process, at NLO is simply given by the sum

$$
\left\langle \Delta p_1^{\mu,(1)} \right\rangle = I^{\mu}_{(1)} + I^{\mu}_{(2)}. \tag{2.37}
$$

2.3.2 Classical limit

The correspondence principle states classical physics should emerge from quantum physics in the limit of large quantum numbers, that is, in the limit of macroscopic conserved charges such as mass, electric charge, orbital angular momentum, spin angular momentum, etc. In the context of the two-body problem, the transition from quantum to classical physics has been extensively studied (Bern *et al.*, 2019a; Vecchia *et al.*, 2021).
In practice, taking the classical limit requires the following steps (Kosower *et al.*, 2022):

- Replace coupling g by $g/\sqrt{\hbar}$. In electrodynamics we replace e by $e/\sqrt{\hbar}$ while in gravity a factor of $\frac{1}{\sqrt{\hbar}}$ appears so that we replace $k = \sqrt{32\pi G}$ by $k = \sqrt{32\pi G/\hbar}$;
- Change all messenger momentum variables-mismatch, virtual, or real-emission to wavenumber variables, $\kappa = \hbar\bar{\kappa}$. The notation for wavenumber is $\bar{\kappa}$, which has dimensions of $[L]^{-1}$ and its associated momentum is κ;
- Approximate $\phi(p + \hbar\bar{q})$ by $\phi(p)$. In practice, this trivialises all massive-momentum integrals;
- Laurent-expand all integrands in $\hbar$. We must be careful with one-loop integrals, in the limit when $\hbar \to 0$ some might contain singular terms with inverse powers of $\hbar$ so to cancel them we must first sum all contributions and then evaluate the classical limit. For singular terms that survive they can be treated independently as well using a Laurent series expansion on $\hbar$;
- Replace all massive-particle momenta p_i by their classical values, $m_i u_i$.

To manipulate integrands we ensemble of these steps, that is, when we take the classical limit we will use the following notation,

$$
\left\langle\!\!\left\langle f(p_1, p_2, \ldots) \right\rangle\!\!\right\rangle \equiv \int d\Phi(p_1) d\Phi(p_2) |\phi_1(p_1)|^2 |\phi_2(p_2)|^2 f(p_1, p_2, \ldots). \tag{2.38}
$$

In this expression with the double angle brackets the approximation $\phi(p+q) \simeq \phi(p)$ has been used and $f(...)$ is a weighted integral over an amplitude or product of amplitudes.
We now proceed to calculate the classical limit of the linear impulse at LO

$$I^{\mu}_{(1),\text{cl}} = i\left\langle\!\!\left\langle \int \hat{d}^4 q \hat{\delta}(2p_1 \cdot q + q^2)\hat{\delta}(2p_2 \cdot q - q^2)\Theta(p_1^0 + q^0)\Theta(p_2^0 - q^0) \right.\right. \\ \left.\left. \times e^{-ib\cdot q/\hbar} q^{\mu} \mathcal{A}^{(0)}(p_1 p_2 \to p_1 + q, p_2 - q) \right\rangle\!\!\right\rangle. \tag{2.39}$$

Performing the coupling replacements and the change of variables detailed above, we obtain,

$$I^{\mu}_{(1),\text{cl}} = i\left\langle\!\!\left\langle \hbar^3 \int \hat{d}^4 \bar{q} \hat{\delta}(\bar{q} \cdot p_1)\hat{\delta}(\bar{q} \cdot p_2) \right.\right. \\ \left.\left. \times e^{-ib\cdot\bar{q}} \bar{q}^{\mu} \bar{\mathcal{A}}^{(0)}(p_1, p_2 \to p_1 + \hbar\bar{q}, p_2 - \hbar\bar{q}) \right\rangle\!\!\right\rangle. \tag{2.40}$$

Note inside the on-shell delta functions $\hat{\delta}(2p_i\cdot\bar{q}\pm\hbar\bar{q}^2)$ we have dropped the $\hbar\bar{q}^2$ factor, since non-singular terms in $\hbar$ appear in the amplitude. Now at NLO in the impulse we must consider both of the terms in (2.37)

$$I^{\mu}_{(2),\text{cl}} = i\left\langle\!\!\left\langle \int \hat{d}^4 q \hat{\delta}(2p_1 \cdot q + q^2)\hat{\delta}(2p_2 \cdot q - q^2)\Theta(p_1^0 + q^0)\Theta(p_2^0 - q^0) e^{-ib\cdot q/\hbar} \mathcal{I}^{\mu} \right\rangle\!\!\right\rangle, \tag{2.41}$$

The expression inside the double angle brackets $\mathcal{I}^{\mu}$ is the *impulse kernel* and and represents one of the important pieces to be able to calculate the observable at any order,

$$\mathcal{I}^{\mu} \equiv q^{\mu}\mathcal{A}(p_1, p_2 \to p_1 + q, p_2 - q) \\ -i\sum_X \int \prod_{i=1,2} \hat{d}^4 w_i \hat{\delta}(2p_i \cdot w_i + w_i^2)\Theta(p_i^0 + w_i^0) w_1^{\mu} \hat{\delta}^{(4)}(w_1 + w_2 + r_X) \\ \times\mathcal{A}(p_1, p_2 \to p_1 + w_1, p_2 + w_2, r_X)\mathcal{A}^*(p_1 + q, p_2 - q \to p_1 + w_1, p_2 + w_2, r_X). \tag{2.42}$$

At NLO the kernel contains the one-loop amplitude and the product of two amplitudes at the tree level that form the unitary cut, then $X = \varnothing$. As we will see in the section 4, when replacing the amplitudes to NLO, singular terms will appear that we must cancel with the help of the unitary cut and that is why in this case we still do not allow ourselves to neglect the terms $\hbar\bar{w}^2$ inside the on-shell deltas. Then the classical limit is given by

$$\Delta p_1^{\mu,(1)} = i\left\langle\!\!\left\langle \hbar^2 \int \hat{d}^4 \bar{q} \hat{\delta}(p_1 \cdot \bar{q})\hat{\delta}(p_2 \cdot \bar{q}) e^{-ib\cdot\bar{q}} \mathcal{I}^{\mu,(1)}_{\text{cl}} \right\rangle\!\!\right\rangle, \tag{2.43}$$

where,

$$\mathcal{I}^{\mu,(1)}_{\text{cl}} \equiv \hbar\bar{q}^{\mu}\bar{\mathcal{A}}^{(1)}(p_1, p_2 \to p_1 + \hbar\bar{q}, p_2 - \hbar\bar{q}) \\ -i\hbar^3 \int \hat{d}^4 \overline{w} \hat{\delta}(2p_1 \cdot \overline{w} + \hbar\overline{w}^2)\hat{\delta}(2p_2 \cdot \overline{w} - \hbar\overline{w}^2)\overline{w}^{\mu} \\ \times\mathcal{A}(p_1, p_2 \to p_1 + \hbar\overline{w}, p_2 - \hbar\overline{w})\mathcal{A}^*(p_1 + \hbar\bar{q}, p_2 - \hbar\bar{q} \to p_1 + \hbar\overline{w}, p_2 - \hbar\overline{w}). \tag{2.44}$$

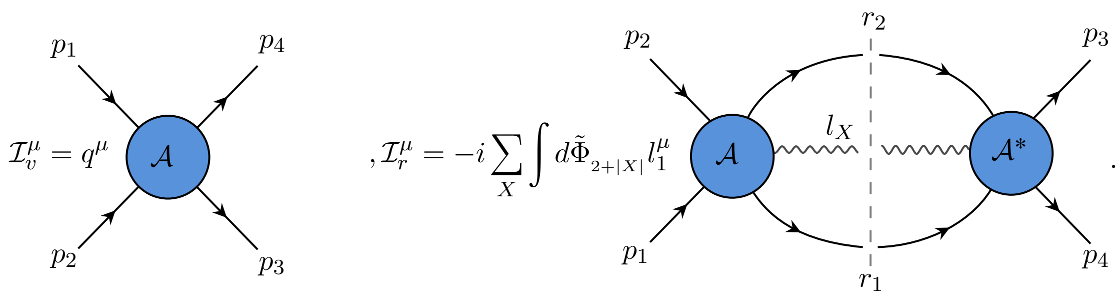
For future computations we neglect the sub-index cl in the kernel because it will be understood that we will work with the classical limit of this. We also will refer to the virtual ($\mathcal{I}^{\mu}_v$) and real part of the kernel ($\mathcal{I}^{\mu}_r$) which are defined as follow:

$$\mathcal{I}^{\mu,(1)}_{p_1} = \mathcal{I}^{\mu}_v + \mathcal{I}^{\mu}_r, \tag{2.45}$$

the observable can be obtained with the fourier transform as in (2.43) but now we include a dimensional regulator in a standard way $D = 4 - 2\epsilon$ in order to regulate potential divergences:

$$\Delta p_1^{\mu} = i\left\langle\!\!\left\langle \int \hat{d}^D q \hat{\delta}(-2p_1 \cdot q)\hat{\delta}(2p_2 \cdot q) e^{ib\cdot q/\hbar} \mathcal{I}^{\mu,(1)}_{p_1} \right\rangle\!\!\right\rangle. \tag{2.46}$$

The reason for this nomenclature (2.45) becomes apparent when one evaluates the expectation values. Diagrammatically (2.42) this notation can be represented as follows:



(2.47)

In this pictorial representation we labeled the outgoing particle momentum with p_3 and p_4. The sum is over a set of intermediate massless particles in the cut, X. The notation $d\tilde{\Phi}_{2+|X|}$ indicates the two-body phase-space measure and is understood to be computed over the legs crossing the dashed blue line, and includes the momentum-conserving delta function (Herrmann *et al.*, 2021b).

2.4 Adding spin to the KMOC formalism

This subsection describes the natural generalizations of the KMOC formalism to include the spin of particles. Classically this quantity is given by the spin pseudo-vector (1.3). The quantum mechanical version of this quantity is given by the expectation value of the Pauli-Lubanski (PL) operator

$$\mathbb{W}_\mu = \frac{1}{2m} \varepsilon_{\mu\nu\rho\sigma} \mathbb{P}^\nu \mathbb{J}^{\rho\sigma}. \tag{2.48}$$

The PL operator is defined in terms of the relativistic linear momentum operator $\mathbb{P}^\mu$ and $\mathbb{J}^{\rho\sigma}$ which is the relativistic tensor operator, both them are generators of the proper orthochronous Lorentz group. $\mathbb{W}_\mu$ is also used in quantum-relativistic description of the angular momentum, it describes the spin states of moving particles. The expectation value of the PL operator on a single particle state (which we need to define as soon as possible) is the quantum-mechanical generalisation of the classical spin pseudo-vector and is defined by

$$\langle s^\mu \rangle = \frac{1}{m} \langle \mathbb{W}^\mu \rangle = \frac{1}{2m} \varepsilon_{\mu\nu\rho\sigma} \langle \mathbb{P}^\nu \mathbb{J}^{\rho\sigma} \rangle. \tag{2.49}$$

Following the set-up established in the previous subsections, again we consider the scattering of two stable, massive particles which are quanta of different fields. With the the quantum-mechanical understanding of the spin vector, we move on to discuss the dynamics of the spin vector in a scattering process, we make this possible using (2.10) which yields to

$$\langle \Delta s_1^\mu \rangle = \frac{i}{m_1} \langle \Psi | [\mathbb{W}_1^\mu, T] | \Psi \rangle + \frac{1}{m_1} \langle \Psi | T^\dagger [\mathbb{W}_1^\mu, T] | \Psi \rangle. \tag{2.50}$$

Again, we will conveniently separate both contributions by representing the first term as $J^\mu_{(1)}$ and the term that, as we will see, emerges as a product of amplitudes, will be represented by $J^\mu_{(2)}$.

$$\langle \Delta s_1^\mu \rangle = J^\mu_{(1)} + J^\mu_{(2)} \tag{2.51}$$

In order to derive the quantum observable (2.50), an important issue here is to define the particle state. Then if we consider the scattering processes mediated by vector bosons and gravitons. So that, the incoming two-particle state is given by:

$$|\Psi\rangle = \sum_{a_k}^{j} \int \prod_{j=1} d\Phi(p_j) \phi_j(p_j) \xi_{a_j} e^{ib\cdot p_1/\hbar} |p_j; a_j\rangle. \tag{2.52}$$

Where b^μ is the impact parameter between the particles ($j = 1, 2$), also in the ket we have the state of a particle j with momentum p_j and spin a_j. Now we move to express the observable in terms of amplitudes, here we will take advantage of the notation we have used to separate the two observable terms and start with the term linear in the amplitude. By substituting the incoming state and the outgoing states in the expression for $J^\mu_{(1)}$, this leads to

$$J^\mu_{(1)} = \frac{i}{m_1}\sum_{a_1',a_1}\sum_{a_2',a_2}\int d\Phi(p_1')d\Phi(p_2')d\Phi(p_1)d\Phi(p_2)\phi_1(p_1)\phi_1^*(p_1')\phi_2(p_2)\phi_2^*(p_2') \times\xi^*_{1_{a_1'}}\xi^*_{2_{a_2'}}\xi_{1_{a_1}}\xi_{2_{a_2}}e^{ib\cdot(p_1-p_1')/\hbar}\left\langle p_1'p_2';a_1'a_2'\right|\mathbb{W}^\mu T - T\mathbb{W}^\mu\left|p_1p_2;a_1a_2\right\rangle. \tag{2.53}$$

Then we insert a complete set of states between the operators ($\mathbb{W}^\mu, T$), and with this, what we want again is to be explicit the amplitudes in this piece of the quantum observable. For the $\mathbb{W}^\mu T$ order we have

$$\left\langle p_1'p_2';a_1'a_2'\right|\mathbb{W}^\mu\mathbb{I}T\left|p_1p_2;a_1a_2\right\rangle = \sum_{b_1,b_2}\int d\Phi(r_1)d\Phi(r_2) \times\left\langle p_1'p_2';a_1'a_2'\right|\mathbb{W}^\mu\left|r_1r_2;b_1b_2\right\rangle\left\langle r_1r_2;b_1b_2\right|T\left|p_1p_2;a_1a_2\right\rangle. \tag{2.54}$$

We identify the following expressions

$$\left\langle p_1';a_1'\right|\mathbb{W}^\mu\left|r_1;b_1\right\rangle = m_1 s^\mu_{1_{a_1'b_1}}(r_1)\hat{\delta}_\Phi(r_1-p_1'), \tag{2.55}$$

$$\left\langle r_1p_2';a_2'b_1\right|T\left|p_1p_2;a_1a_2\right\rangle = \mathcal{A}_{b_1a_2'a_1a_2}(p_1,p_2\to r_1,p_2')\hat{\delta}^{(4)}(r_1+p_2'-p_1-p_2), \tag{2.56}$$

where (2.55) is the spin polarization vector of particle 1, and based in our definitions (2.55) is the scattering amplitude.
We will suppress the summation over repeated spin indices from now on. Replacing these identifications in (2.54) and integrating over the delta function $\hat{\delta}_\Phi(r_1)$, we obtain

$$\left\langle p_1'p_2';a_1'a_2'\right|\mathbb{W}^\mu\mathbb{I}T\left|p_1p_2;a_1a_2\right\rangle = m_1\sum_{b_1}s^\mu_{1_{a_1'b_1}}(p_1')\mathcal{A}_{b_1a_2'a_1a_2}(p_1,p_2\to p_1',p_2')\hat{\delta}^{(4)}(p_1'+p_2'-p_1-p_2). \tag{2.57}$$

The result for the other ordering of T and $\mathbb{W}^\mu$ is very similar:

$$\left\langle p_1'p_2';a_1'a_2'\right|T\mathbb{I}\mathbb{W}^\mu\left|p_1p_2;a_1a_2\right\rangle = m_1\sum_{b_1}\mathcal{A}_{a_1'a_2'b_1a_2}(p_1,p_2\to p_1',p_2')s^\mu_{1_{b_1a_1}}(p_1)\hat{\delta}^{(4)}(p_1'+p_2'-p_1-p_2). \tag{2.58}$$

Substituting into the full expression for (2.53) we find that the angular impulse at LO is

$$J^\mu_{(1)} = i\int d\Phi(p_1')d\Phi(p_2')d\Phi(p_1)d\Phi(p_2)\phi_1(p_1)\phi_1^*(p_1')\phi_2(p_2)\phi_2^*(p_2') \times\xi^*_{1_{a_1'}}\xi^*_{2_{a_2'}}\xi_{1_{a_1}}\xi_{2_{a_2}}e^{ib\cdot(p_1-p_1')/\hbar}\hat{\delta}^{(4)}(p_1'+p_2'-p_1-p_2) \times\left(s^\mu_{1_{a_1'b_1}}(p_1')\mathcal{A}_{b_1a_2'a_1a_2}(p_1,p_2\to p_1',p_2') - \mathcal{A}_{a_1'a_2'a_1b_1}(p_1,p_2\to p_1,p_2')s^\mu_{1_{b_1a_1}}(p_1)\right). \tag{2.59}$$

Now introducing the momentum shift $q_i = p_i' - p_i$, this allows us to change the integration variables $p_i' \to q_i$; performing the integral over q_2 using the delta function, and relabel $q_1 \to q$:

$$J^\mu_{(1)} = i\int d\Phi(p_1)d\Phi(p_2)\hat{d}^4q\hat{\delta}(2p_1\cdot q+q^2)\hat{\delta}(2p_2\cdot q-q^2) \times\phi_1(p_1)\phi_1^*(p_1+q)\phi_2(p_2)\phi_2^*(p_2-q)\xi^*_{1_{a_1'}}\xi^*_{2_{a_2'}}\xi_{1_{a_1}}\xi_{2_{a_2}}e^{-ib\cdot q/\hbar} \times\left(s^\mu_{1_{a_1'b_1}}(p_1+q)\mathcal{A}_{b_1a_2'a_1a_2}(p_1,p_2\to p_1+q,p_2-q)\right. \left. -\mathcal{A}_{a_1'a_2'a_1b_1}(p_1,p_2\to p_1+q,p_2-q)s^\mu_{1_{b_1a_1}}(p_1)\right). \tag{2.60}$$

Now to compute the second contribution to the angular impulse $J^{\mu}_{(2)}$ (see Eq.(2.51)) we follow similar steps, we start by inserting a complete set of states between the spin and the interaction operators and similar expressions such as those in (2.55) and (2.56) will allow us to arrive in a familiar expression with a product of two amplitudes, then by performing the integrals with the usual replacements $q_i = p'_i - p_i, l_i = r_i - p_i$ is straightforward arrive to

$$J^{\mu}_{(2)} = \int \prod_{i=1,2} d\Phi(p_i)\hat{d}^4q\hat{d}^4l\Theta(p_1^0+q^0)\Theta(p_2^0-q^0)\Theta(p_1^0+l^0)\Theta(p_2^0-l^0)$$
$$\hat{\delta}(2p_1\cdot q+q^2)\hat{\delta}(2p_2\cdot q-q^2)\hat{\delta}(2p_1\cdot l+l^2)\hat{\delta}(2p_2\cdot l-l^2)\phi_1(p_1)\phi_1^*(p_1+q)\phi_2(p_2)\phi_2^*(p_2-q)$$
$$\times\, \xi^*_{1_{a'_1}}\xi^*_{2_{a'_2}}\xi_{1_{a_1}}\xi_{2_{a_2}}e^{-b\cdot q/\hbar}\Bigg[s^{\mu}_{1_{b_1b'_1}}(p_1+l)\mathcal{A}_{b'_1b_2a_1a_2}(p_1,p_2\to p_1+l,p_2-l)-$$
$$\mathcal{A}_{b_1b_2b'_1a_2}(p_1,p_2\to p_1+l,p_2-l)s^{\mu}_{1_{b'_1a_1}}(p_1)\Bigg]\mathcal{A}^*_{a'_1a'_2b_1b_2}(p_1+l,p_2-l\to p_1+q,p_2-q). \tag{2.61}$$

2.4.1 Classical limit of the angular impulse

This is the quantum version of the angular impulse at LO. As a well-defined observable, we can extract the classical limit following the same procedure that is described in sec.2.3.2. First introducing a notation for the expectation values over the wavefunctions

$$\left\langle\!\!\left\langle f(p_1,p_2,...)\right\rangle\!\!\right\rangle \equiv \sum_{a'_1,a_1}\sum_{a'_2,a_2}\int d\Phi(p_1d\Phi(p_2|\phi_1(p_1)|^2|\phi_2(p_2)|^2$$
$$\times\xi^*_{1_{a'_1}}\xi^*_{1_{a'_2}}f^{a'_1a'_2a_1a_2}(p_1,p_2,...)\xi_{1_{a_1}}\xi_{1_{a_2}}, \tag{2.62}$$

so the angular impulse takes the form

$$\left\langle \Delta s_1^{\mu,(0)}\right\rangle = i\left\langle\!\!\left\langle\int \hat{d}^4q\hat{\delta}(2p_1\cdot q)\hat{\delta}(2p_2\cdot q)e^{-ib\cdot q/\hbar}\left(\, s^{\mu}(p_1+\hbar\bar{q})\mathcal{A}(p_1,p_2\to p_1+q,p_2-q)\right.\right.\right.$$
$$\left.\left.\left.-\mathcal{A}(p_1,p_2\to p_1+q,p_2-q)s_1^{\mu}(p_1)\right)\right\rangle\!\!\right\rangle. \tag{2.63}$$

Both the spin vector and the amplitude are matrices with spinor indices, some of which are contracted together (Kosower *et al.*, 2019). An important $\hbar$ shift remaining is that of the spin polarization vector $s^{\mu}_{1_{a'_1b_1}}(p_1+\hbar\bar{q})$. This object is a Lorentz boost of $s^{\mu}_{1_{a'_1b_1}}(p_1)$. The appropriate generator is

$$w^{\mu\nu} = -\frac{\hbar}{m_1^2}(p_1^{\mu}\bar{q}^{\nu}-\bar{q}^{\mu}p_1^{\nu}). \tag{2.64}$$

This result is valid for particles of any spin as it is purely kinematic, and therefore can be universally applied in our general formula for the angular impulse (Kosower *et al.*, 2019). In particular, since $w_{\mu\nu}$ is explicitly $\mathcal{O}(\hbar)$ the spin polarization vector transforms as

$$s^{\mu}_{1_{a'_1b_1}}(p_1+\hbar\bar{q}) = s^{\mu}_{1_{a'_1b_1}}(p_1) - \frac{\hbar}{m_1^2}p^{\mu}\bar{q}\cdot s_{ab}(p_1). \tag{2.65}$$

Then the final expression for the angular impulse is given by

$$\Delta s_1^{\mu,(0)} = i\left\langle\!\!\left\langle\int \hat{d}^4\bar{q}\hat{\delta}(2p_1\cdot\bar{q})\hat{\delta}(2p_2\cdot\bar{q})e^{-ib\cdot\bar{q}}\left(-\hbar^3\frac{p_1^{\mu}}{m_1^2}\bar{q}\cdot s_1(p_1)\mathcal{A}(p_1,p_2\to p_1+\hbar\bar{q},p_2-\hbar\bar{q})\right.\right.\right.$$
$$\left.\left.\left.-\hbar^2[s_1^{\mu}(p_1),\mathcal{A}(p_1,p_2\to p_1+\hbar\bar{q},p_2-\hbar\bar{q})]\right)\right\rangle\!\!\right\rangle. \tag{2.66}$$

In a similar fashion we derive the classical limit for the NLO in the angular impulse:

$$\Delta s_1^{\mu,(1)} = i\left\langle\!\!\left\langle \hbar^{D-2} \int \hat{d}^D \bar{q} \hat{\delta}(2p_1 \cdot \bar{q}) \hat{\delta}(-2p_2 \cdot \bar{q}) e^{-ib\cdot\bar{q}} \mathcal{I}_{s_1}^{\mu,(1)} \right\rangle\!\!\right\rangle, \tag{2.67}$$

where the kernel is given by

$$\begin{aligned}\mathcal{I}_{s_1}^{\mu,(1)} = &-\frac{\hbar}{m_1^2} p_1^\mu \bar{q} \cdot s_1(p_1) \mathcal{A}^{(1)}(p_1, p_2 \to p_1 + \hbar\bar{q}, p_2 - \hbar\bar{q}) \\ &+ \left[s_1^\mu(p_1), \mathcal{A}^{(1)}(p_1, p_2 \to p_1 + \hbar\bar{q}, p_2 - \hbar\bar{q}) \right] + i\hbar^2 \int \hat{d}^4 \bar{l} \hat{\delta}(2p_1 \cdot \bar{l} + \hbar\bar{l}^2) \hat{\delta}(2p_2 \cdot \bar{l} - \hbar\bar{l}^2) \times \\ &\left(\frac{\hbar}{m_1^2} p_1^\mu \bar{l} \cdot s_1(p_1) \mathcal{A}^{(0)}(p_1, p_2 \to p_1 + \hbar\bar{l}, p_2 - \hbar\bar{l}) \mathcal{A}^{(0)*}(p_1 + \hbar\bar{l}, p_2 - \hbar\bar{l} \to p_1 + \hbar\bar{q}, p_2 - \hbar\bar{q}) \right. \\ &\left. - \left[s_1^\mu(p_1), \mathcal{A}^{(0)}(p_1, p_2 \to p_1 + \hbar\bar{l}, p_2 - \hbar\bar{l}) \mathcal{A}^{(0)*}(p_1 + \hbar\bar{l}, p_2 - \hbar\bar{l} \to p_1 + \hbar\bar{q}, p_2 - \hbar\bar{q}) \right] \right). \end{aligned} \tag{2.68}$$

2.5 Simplification of KMOC

The traditional way to calculate the kernel tells us that the singular terms in the virtual part of the kernel will cancel by some contribution that come from the cut (Kosower *et al.*, 2019). So we need to be extremely careful with the computations at this level. We also know the integrals with a loop momentum in the numerator will appear, but we can solve this integrals rewriting l^μ in the momentum space basis as a combination of $\{q^\mu, \check{u}_1^\mu, \check{u}_2^\mu\}$. In the case of the spin pieces we also add the orthogonal product of those vectors $\varepsilon(\cdot, q^\mu, \check{u}_1^\mu, \check{u}_2^\mu)$ to the basis. In the spinless case this recipe and the flip symmetry of the cut results in the kernel setup in (Herrmann *et al.*, 2021b).

$$\mathcal{I}_{p_1}^{\mu,(1)} = q^\mu \mathcal{I}_\perp^{(1)} + \mathcal{I}_{u_i}^{\mu,(1)}, \tag{2.69}$$

$$= q^\mu \mathrm{Re}\left[\mathcal{M}^{(1)}\right] - iq^2 \left(\frac{\check{u}_1^\mu}{m_1} - \frac{\check{u}_2^\mu}{m_2} \right) \mathrm{Im}\left[\mathcal{M}^{(1)}\right]. \tag{2.70}$$

Here $\check{u}_i^\mu$ are dual 4-velocities defined in (Herrmann *et al.*, 2021b). We will work only at one-loop level so in this case we get

$$\check{u}_1^\mu = \frac{\gamma u_2^\mu - u_1^\mu}{\gamma^2 - 1} + \mathcal{O}(q), \quad \check{u}_2^\mu = \frac{\gamma u_1^\mu - u_2^\mu}{\gamma^2 - 1} + \mathcal{O}(q). \tag{2.71}$$

Then we can rewrite (2.70) as follow

$$\mathcal{I}_{p_1}^{\mu,(1)} = q^\mu \mathrm{Re}\left[\mathcal{M}^{(1)}\right] + iq^2 \left[\frac{(m_2 + \gamma m_1) u_1^\mu - (m_1 + \gamma m_2) u_2^\mu}{(\gamma^2 - 1) m_1 m_2} \right] \mathrm{Im}\left[\mathcal{M}^{(1)}\right]. \tag{2.72}$$

This setup will be useful when we compute observables in section 4

3 AMPLITUDES

In the previous section we presented the theoretical framework of the KMOC formalism. In a pragmatic way KMOC works like this: first we need the amplitudes at the desired order in perturbation theory, using these amplitudes we construct a kernel (2.45) and finally we compute the observable performing a Fourier transform-like integral of the kernel from momentum space to impact parameter space (2.46). And so the amplitude is the fundamental ingredient of the KMOC formalism. In this section we present our gravity amplitudes, first calculating the amplitudes at tree-level, while the one-loop amplitudes are extracted from (Cordero *et al.*, 2022).

3.1 Leading order

A practical way to compute this tree level amplitude in gravity makes use of a scalar Yang-Mills theory. This calculation is possible thanks to the color-kinematics relation established between these two theories by the double copy (see Section 2), more details within this context can be found in (Luna *et al.*, 2018). We will consider tree level scattering of a spin 1 particle off a scalar in Yang-Mills theory and gravity. Amplitudes in the former will be denoted by $\mathcal{A}_{n_1-0}$, while those for Einstein gravity are $\mathcal{M}_{n_1-0}$. For the amplitudes relevant to our on-shell observables only the t channel contributes.

The Yang-Mills with a minimally coupled gauge interaction given by the Higgs mechanism, in the fundamental representation of the gauge group our Lagrangian is given by:

$$\mathcal{L}_0 = -\frac{1}{4}F^a_{\mu\nu}F^{\mu\nu}_a - \Phi_i^\dagger(\Box + m_i^2)\Phi_i - igA^\mu[\Phi_i(\partial_\mu\Phi_i) - (\partial_\mu\Phi^\dagger)\Phi_i] + g^2A^2_\mu|\Phi_i|^2. \tag{3.1}$$

Where the *field strength* in the kinetic term of (3.1) is $F^a_{\mu\nu}(x) = 2\text{Tr}F_{\mu\nu}T^a$, as is usual we add the interactions via $D_\mu = \partial - igA_\mu(x)$ and Φ_i are massive scalar particles. After applying the Feynman rules this yields the tree-level amplitude:

$$i\mathcal{A}^{ij}_{1-0} = -\frac{i\bar{g}^2}{2q^2}\varepsilon_i^{\mu *}(p_1+q)\varepsilon_j^\nu(p_1)[\eta_{\mu\nu}(2p_{1\lambda}+q_\lambda) - \eta_{\nu\lambda}(p_{1\mu}-q_\mu) \\ - \eta_{\lambda\mu}(2q_\nu + p_{1\nu})](2p_2^\lambda - q^\lambda)\tilde{T}_1\cdot\tilde{T}_2. \tag{3.2}$$

Now we expand the product of polarization vectors to obtain the classical limit. The p_1+q momentum in the polarization vector ε_i^* is the outgoing momentum of the particle, so we can interpret ε_i^* as the outgoing polarization vector as being infinitesimally boosted.
Therefore expand this Lorentz boost two orders in the antisymmetric matrix $w_{\mu\nu}$, which come from an arbitrary infinitesimal transformation of the *Lorentz transformation matrix* $\Lambda^\mu_\nu = \delta^\mu_\nu + \omega^\mu_\nu$. In the *vectorial representation* of the Lorentz group $\Lambda^\mu_\nu = [e^{\{-\frac{i}{2}\omega_{\mu\nu}\Sigma^{\mu\nu}\}}]^\mu_\nu$, then this yields

$$\varepsilon_i^\mu(p) \to \Lambda^\mu_\nu\varepsilon_i^\nu(p) \simeq \left\{\delta^\mu_\nu - \frac{i}{2}w_{\mu\nu}(\Sigma^{\rho\sigma})^\mu_\nu - \frac{1}{8}\left[(w_{\rho\sigma}\Sigma^{\rho\sigma})^2\right]^\mu_\nu\right\}\varepsilon_i^\nu(p), \tag{3.3}$$

here $(\Sigma^{\rho\sigma})^\mu_\nu = i(\eta^{\rho\nu}\delta^\sigma_\nu - \eta^{\sigma\mu}\delta^\rho_\nu)$ is the covariant representation of the Lorentz generator for a spin 1 particle, this generators obey of course the Lorentz algebra. The appropriate infinitesimally parameter is the same to those used to derive (2.64), this allows us to obtain this new expression for the tensor product

$$\varepsilon_i^{*\mu}(p_1+\hbar\bar{q})\varepsilon_j^\nu(p_1) = \varepsilon_i^{*\mu}\varepsilon_j^\nu - \frac{\hbar}{m_1^2}(\bar{q}\cdot\varepsilon_i^*)p_1^\mu\varepsilon_j^\nu - \frac{\hbar^2}{2m_1^2}(\bar{q}\cdot\varepsilon_i^*)\bar{q}^\mu\varepsilon_j^\nu + \mathcal{O}(\hbar^3). \tag{3.4}$$

The scalar products $\varepsilon_i^*\cdot p_1 = \varepsilon_i\cdot p_1 = 0$ are satisfied in the classical limit which are a constrain of the *Lorenz gauge*. Without loss of generality we can replace(3.4) in the numerator of the amplitude and this leads us to

$$n_{ij} = 2(p_1\cdot p_2)(\varepsilon_i^*\cdot\varepsilon_j) - 2\hbar(p_2\cdot\varepsilon_i^*)(\bar{q}\cdot\varepsilon_j) + 2\hbar(p_2\cdot\varepsilon_j)(\bar{q}\cdot\varepsilon_i^*) \\ + \frac{1}{m_1^2}\hbar^2(p_1\cdot p_2)(\bar{q}\cdot\varepsilon_i^*)(\bar{q}\cdot\varepsilon_j) + \frac{\hbar^2}{2}\bar{q}^2(\varepsilon_i^*\cdot\varepsilon_j) + \mathcal{O}(\hbar^3). \tag{3.5}$$

We need to express explicitly the spin in the last expression (3.5), so we must invoke a Levi-Civita tensor identity

$$\varepsilon^{\delta\rho\sigma\nu}\varepsilon_{\delta\alpha\beta\gamma} = -2\left(\delta^\rho_\alpha\delta^\sigma_{[\beta}\delta^\nu_{\gamma]} + \delta^\rho_\beta\delta^\sigma_{[\gamma}\delta^\nu_{\alpha]} + \delta^\rho_\gamma\delta^\sigma_{[\alpha}\delta^\nu_{\beta]}\right). \tag{3.6}$$

Using (3.6) in the first line of (3.5) and truncate at $\mathcal{O}(\hbar)$ we obtain

$$\begin{aligned}\hbar(p_2\cdot\varepsilon_i^*)(\bar{q}\cdot\varepsilon_j)-\hbar(p_2\cdot\varepsilon_j)(\bar{q}\cdot\varepsilon_i^*) &= \frac{\hbar}{m_1^2}p_1^\rho\bar{q}^\sigma p_2^\lambda\varepsilon_{\delta\rho\sigma\lambda}\varepsilon^{\delta\alpha\beta\gamma}\varepsilon_{i_\alpha}^*\varepsilon_{j_\beta}p_{1_\gamma},\\ &\equiv -\frac{i}{m_1}p_1^\rho\bar{q}^\sigma p_2^\lambda\varepsilon_{\delta\rho\sigma\lambda}s_{1_{ij}}^\delta. \end{aligned} \tag{3.7}$$

where it is straightforward to identify the usual called spin polarization vector

$$s_{ij}^\mu(p)=\frac{i\hbar}{m}\varepsilon^{\mu\nu\rho\sigma}p_\nu\varepsilon_{i_\rho}^*(p)\varepsilon_{j_\sigma}(p), \tag{3.8}$$

which obeys the same algebra for the PL operator. We will continue using the expression (3.7) that allows us to make explicit the spin at $\mathcal{O}(s^2)$ in the one-loop amplitudes. Now we use a similar Levi-Civita identity but this time without repeating indices to make explicit spin to order $\mathcal{O}(s^2)$,

$$\varepsilon^{\mu\nu\rho\sigma}\varepsilon_{\alpha\beta\gamma\delta}=-4!\delta_\alpha^{[\mu}\delta_\beta^{\nu}\delta_\gamma^{\rho}\delta_\delta^{\sigma]}. \tag{3.9}$$

and after the expansion this piece of the spin at $\mathcal{O}(\hbar^2)$ we finally obtain the expression for spin polarization

$$\sum_k\left(\bar{q}\cdot s_1^{ik}\right)\left(\bar{q}\cdot s_1^{kj}\right)=-\hbar^2(\bar{q}\cdot\varepsilon_i^*)(\bar{q}\cdot\varepsilon_j^*)-\hbar^2\bar{q}^2\delta_{ij}+\mathcal{O}(\hbar^3). \tag{3.10}$$

Here eq.(3.10) depends on the sum over helicities $\sum_h\varepsilon_h^*\varepsilon_h^\nu=-\eta^{\mu\nu}+\frac{p_1^\mu p_1^\nu}{m_1^2}$ for massive vectors bosons, an additional consequence of which is that $\varepsilon_i^*\cdot\varepsilon_j=-\delta_{ij}$. Replacing these expressions the numerator in terms of spin vectors, the amplitude is

$$\begin{aligned}\hbar^3\mathcal{A}_{1-0}^{ij}=\frac{2g^2}{\bar{q}^2}\Bigg[&(p_1\cdot p_2)\delta^{ij}-\frac{i}{m_1}p_1^\rho\bar{q}^\sigma p_2^\lambda\varepsilon_{\rho\sigma\lambda\delta}s_1^{\delta\,ij}+\frac{1}{2m_1^2}(p_1\cdot p_2)(\bar{q}\cdot s_1^{ik})(\bar{q}\cdot s_1^{kj})\\ &-\frac{\hbar^2\bar{q}^2}{4m_1^2}\left(2(p_1\cdot p_2)+m_1^2\right)+\mathcal{O}(\hbar^3)\Bigg]\tilde{T}_1\cdot\tilde{T}_2. \end{aligned} \tag{3.11}$$

Now we derive gravity amplitudes applying the double copy (Bern *et al.*, 2008). Also using the double copy ensures that the spin index structure passes to the gravity theory unchanged. We make possible the gravity amplitudes from a gauge theory with the following replacements, following the spirit of the double copy we must to replace the coupling constants between theories and the color-kinematic parts $T_i^a\to p_i$:

$$g\to\frac{k}{2};\quad \tilde{T}_1\cdot\tilde{T}_2=2T_1^a\cdot T_2^b\to 2p_1\cdot p_2 \tag{3.12}$$

after these double copy replacements in the numerator is straightforward to derive the LO gravity amplitude, then we get

$$\begin{aligned}\hbar^3\mathcal{M}^{ij}=-\left(\frac{k}{2}\right)^2\frac{4}{\bar{q}^2}\Bigg[&(p_1\cdot p_2)^2\delta^{ij}-\frac{i(p_1\cdot p_2)}{m_1}p_1^\rho\bar{q}^\sigma p_2^\lambda\varepsilon_{\rho\sigma\lambda\delta}s_1^{\delta\,ij}\\ &+\frac{1}{2m_1^2}(p_1\cdot p_2)^2(\bar{q}\cdot s_1^{ik})(\bar{q}\cdot s_1^{kj})+\mathcal{O}(\hbar^2)\Bigg]. \end{aligned} \tag{3.13}$$

The amplitude (3.13) is not a pure GR amplitude, here we have virtual dilatons and gravitons, but we can remove dilatons using a few methods such as (Luna *et al.*, 2018) or the advocated in (Luna *et al.*, 2017), we can examine the factorisation of the t channel to identify these contributions. Then pure GR classical scattering amplitude is given by

$$\hbar^3\mathcal{M}^{ij}_{1-0} = -\left(\frac{k}{2}\right)^2\frac{4}{\bar{q}^2}\Bigg[\left((p_1\cdot p_2)^2-\frac{1}{2}m_1^2m_2^2\right)\delta^{ij}-\frac{i}{m_1}(p_1\cdot p_2)p_1^\rho\bar{q}^\sigma p_2^\lambda\varepsilon_{\rho\sigma\lambda\delta}s_1^{\delta\, ij} + \frac{1}{2m_1^2}\left((p_1\cdot p_2)^2-\frac{1}{2}m_1^2m_2^2\right)(\bar{q}\cdot s_1^{ik})(\bar{q}\cdot s_1^{kj})+\mathcal{O}(\hbar^2)\Bigg]. \tag{3.14}$$

It will be useful to rewrite (3.14) in terms of the following tree-level factors (Cordero *et al.*, 2022):

$$M_1^{(0)} = 2m_1^2m_2^2(1-2\gamma^2), \quad M_2^{(0)} = 2m_2^2, \tag{3.15}$$

$$M_3^{(0)} = 2, \quad M_4^{(0)} = -4m_1m_2\gamma, \tag{3.16}$$

with $\sigma = \frac{p_1\cdot p_2}{m_1m_2} = u_1\cdot u_2 = \gamma$. So our tree-level aplitude with classical spin 1-spin 0 just be read as:

$$\hbar^3\mathcal{M}^{(0)} = \left(\frac{k}{2}\right)^2\frac{1}{\bar{q}^2}\Bigg\{M_1^{(0)}\delta^{ij}-\frac{iM_4^{(0)}}{m_1}p_1^\rho\bar{q}^\sigma p_2^\lambda\varepsilon_{\rho\sigma\lambda\delta}s_1^{\delta\, ij} + \frac{1}{2m_1^2}\Big[-M_1^{(0)}+2m_1^2M_2^{(0)}+2M_4^{(0)}(p_1\cdot p_2)\Big](\bar{q}\cdot s_1^{ik})(\bar{q}\cdot s_1^{kj})+\mathcal{O}(\hbar^2)\Bigg\}. \tag{3.17}$$

3.2 Next-to-leading order

The one-loop gravity amplitudes where computed in (Cordero *et al.*, 2022) employing the multi-loop numerical unitarity method, the NLO contributions are a combination of bubbles, triangles and boxes (see figure 5), the master integrals f_1, f_2 were solved employing integration by-parts (Chetyrkin and Tkachov, 1981)

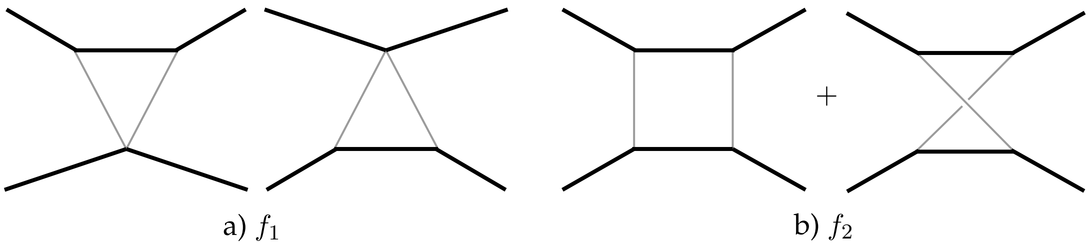


Figure 5: Contributions to the one-loop amplitude.

Then the NLO amplitude is given by

$$\mathcal{M}^{(1)} = \left(\frac{k}{2}\right)^4\frac{1}{q^2}\sum_{n=1}^{4}\sum_{k=0}^{2}M_n^{(1,k)}|q|^kT_n+\mathcal{O}(|q|). \tag{3.18}$$

Where $|q| = \sqrt{-q^2}$ and the tensor structure T_n in the amplitude is

$$T_n = \varepsilon_{1_\mu}T_n^{\mu\nu}\varepsilon^*_{4_\nu}, \tag{3.19}$$

here ε_1 is the polarization vector with a p_1 dependence, ε_4^* is the polarization vector in the frame of the outgoing particle p_1+q. The different tensor structures are given by

$$\left\{T_1^{\mu\nu},\ldots,T_4^{\mu\nu}\right\} = \left\{\eta^{\mu\nu}, q^\mu q^\nu, q^2\left(p_2^\mu-\frac{q^\mu}{2}\right)\left(p_2^\nu-\frac{q^\nu}{2}\right), \left(p_2^\mu-\frac{q^\mu}{2}\right)q^\nu-\left(p_2^\nu-\frac{q^\nu}{2}\right)q^\mu\right\}. \tag{3.20}$$

The superclassical contributions are formed by the superclassical parts of the box and the cut-box. At this order the superclassical pieces are contained in the imaginary part of the amplitude, in terms of the form factors $M_n^{(1,0)}$ are given by

$$M_1^{(1,0)} = \frac{-2m_1^3 m_2^3 f_2}{\sqrt{\sigma^2-1}} \left(\frac{1}{1-\varepsilon} - 2\sigma^2\right)^2, \tag{3.21}$$

$$M_2^{(1,0)} = -\frac{m_1 m_2^3 f_2}{(1-2\varepsilon)\sqrt{\sigma^2-1}} \left[\frac{2-3\varepsilon}{(1-\varepsilon)^2} - 4\sigma^2(1-\varepsilon)\right], \tag{3.22}$$

$$M_3^{(1,0)} = \frac{m_1 m_2 f_2}{\sqrt{\sigma^2-1}} \left[\frac{1-3\varepsilon}{2(\sigma^2-1)(1-\varepsilon)^2} + 4\sigma^2 - \frac{2\varepsilon}{(1-\varepsilon)(1-2\varepsilon)}\right], \tag{3.23}$$

$$M_4^{(1,0)} = \frac{4m_1^2 m_2^2 f_2}{\sqrt{\sigma^2-1}} \left[\frac{1}{1-\varepsilon} - 2\sigma^2\right]. \tag{3.24}$$

The classical contributions are the pure real part of the amplitude and these are formed by the triangles

$$M_1^{(1,1)} = \frac{m_1^2 m_2^2 (m_1+m_2) f_1}{4(\sigma^2-1)} \frac{1-2\varepsilon}{(1-\varepsilon)^2} \left[-3 + (3-4\varepsilon)\sigma^2 \left(6 - (5-4\varepsilon)\sigma^2\right)\right], \tag{3.25}$$

$$\begin{aligned} M_2^{(1,1)} = \frac{m_2^2 f_1}{32(\sigma^2-1)} \Bigg[& \frac{(5-4\varepsilon)(1-12\varepsilon+4\varepsilon^2)}{(1-\varepsilon)^2} m_1\sigma^4 - \frac{4(5-2\varepsilon)(1-2\varepsilon)}{(1-\varepsilon)^2} m_2 \\ & + \frac{4(1-\varepsilon)(5-2\varepsilon)(1-2\varepsilon)(3-4\varepsilon)}{(1-\varepsilon)^3} m_2\sigma^2 - \frac{27-\varepsilon\,(67-4\varepsilon(14-3\varepsilon))}{(1-\varepsilon)^3} m_1 \\ & + \frac{(78-2\varepsilon(147-2\varepsilon(113-2\varepsilon(35-6\varepsilon))))}{(1-\varepsilon)^3} m_1\sigma^2 \Bigg], \end{aligned} \tag{3.26}$$

$$\begin{aligned} M_3^{(1,1)} = \frac{f_1}{16(\sigma-1)^2} \Bigg[& \frac{2-4\varepsilon(3-4\varepsilon)}{(1-\varepsilon)^2} m_2 - \frac{8(1-2\varepsilon)(3-4\varepsilon)}{1-\varepsilon} m_2\sigma^2 \\ & + \frac{2(1-2\varepsilon)(5-4\varepsilon)(3-4\varepsilon)}{(1-\varepsilon)^2} m_2\sigma^4 + \frac{9-10\varepsilon+8\varepsilon^2}{(1-\varepsilon)^2} m_1 \\ & - \frac{2(33-2\varepsilon(45-4\varepsilon(11-4\varepsilon)))}{(1-\varepsilon)^2} m_1\sigma^2 + \frac{(5-4\varepsilon)(13-2\varepsilon(15-8\varepsilon))}{(1-\varepsilon)^2} m_1\sigma^4 \Bigg] \end{aligned} \tag{3.27}$$

$$, M_4^{(1,1)} = \frac{m_1 m_2 \sigma f_1}{4(\sigma^2-1)} \frac{1-2\varepsilon}{1-\varepsilon} \left[m_1(12-4(5-4\varepsilon)\sigma^2) + m_2\left(\frac{3(3-4\varepsilon)}{1-\varepsilon} - \frac{(5-4\varepsilon)(3-4\varepsilon)}{1-\varepsilon}\sigma^2\right)\right]. \tag{3.28}$$

In all these expressions we identify

$$f_1 \equiv \frac{1}{4\pi}\left(\frac{4\pi\mu^2}{q^2}\right)^{\varepsilon} \frac{\Gamma^2(\frac{1}{2}-\varepsilon)\Gamma(\frac{1}{2}+\varepsilon)}{2\sqrt{\pi}\Gamma(1-2\varepsilon)} \text{ and } f_2 \equiv \frac{1}{4\pi}\left(\frac{4\pi\mu^2}{q^2}\right)^{\varepsilon} \frac{i\Gamma^2(-\varepsilon)\Gamma(1+\varepsilon)}{4\Gamma(-2\varepsilon)} \tag{3.29}$$

are the results for the master integrals which corresponds to the topologies shown in the figure 5. The integrals were computed in a $D = 4 - 2\varepsilon$ dimension, here μ is a dimensional regularization scale.

3.2.1 Preparing the NLO amplitudes

We need transform the amplitude (3.18) into the classical version for the KMOC machinery. We start by replacing the gravitational coupling-constant i.e. $k = k/\sqrt{\hbar}$ and the momentum transfer $q = \hbar\bar{q}$. We will focus at first on the classical contributions $\mathcal{M}_{cl}^{(1)}$, expanding the sum in (3.18) the amplitude is as follow

$$\hbar^3\mathcal{M}_{cl}^{(1)} = \left(\frac{k}{2}\right)^4 \frac{|\bar{q}|}{\bar{q}^2}\Bigg[M_1^{(1,1)}\varepsilon_{1\mu}\eta^{\mu\nu}\varepsilon^*_{4\nu} + M_2^{(1,1)}\hbar^2\varepsilon_{1\mu}\bar{q}^\mu\bar{q}^\nu\varepsilon^*_{4\nu} + M_3^{(1,1)}\hbar^2\varepsilon_{1\mu}p_2^\mu p_2^\nu\varepsilon^*_{4\nu} + M_4^{(1,1)}\left(\hbar\varepsilon_{1\mu}(p_2^\mu\bar{q}^\nu - p_2^\nu\bar{q}^\mu)\varepsilon^*_{4\nu} - \frac{\hbar^2}{2}\varepsilon_{1\mu}(\bar{q}^\mu\bar{q}^\nu - \bar{q}^\nu\bar{q}^\mu)\varepsilon^*_{4\nu}\right)\Bigg]. \quad (3.30)$$

Now we need make the spin appear explicitly. So, we follow the similar steps described in "Spin-1 - spin-0 scattering" (Maybee *et al.*, 2019). Meaning we translate the polarization vectors into spin vectors, we must boost the polarization vector ξ_4^* into the frame of p_1.

$$\varepsilon_1^\mu(p_1)\varepsilon_4^{*\nu}(p_1+\hbar\bar{q}) = \varepsilon_1^\mu\varepsilon_4^{*\nu} - \frac{\hbar}{m_1^2}(\bar{q}\cdot\varepsilon_4^*)\varepsilon_1^\mu p_1^\nu - \frac{\hbar^2}{2m_1^2}(\bar{q}\cdot\varepsilon_4^*)\varepsilon_1^\mu\bar{q}^\nu + \mathcal{O}(\hbar^3). \quad (3.31)$$

The polarization vector $\varepsilon_i(p_1)$ is ortogonal with p_1, $\varepsilon_1\cdot p_1 = 0$, and the on-shell condition for the outgoing particle 1 requires $(p_1+q/2)\cdot q = 0$. With this set of replacements the numerator in the amplitude becomes

$$n_{14} = M_1^{(1,1)}(\varepsilon_1\cdot\varepsilon_4^*) + M_4^{(1,1)}\left[\hbar(p_2\cdot\varepsilon_1)(\bar{q}\cdot\varepsilon_4^*) - \hbar(p_2\cdot\varepsilon_4^*)(\bar{q}\cdot\varepsilon_1)\right] + \frac{1}{2m_1^2}\left[M_1^{(1,1)} - 2m_1^2M_2^{(1,1)} - 2M_4^{(1,1)}(p_1\cdot p_2)\right]\left[-\hbar^2(\bar{q}\cdot\varepsilon_4^*)(\bar{q}\cdot\varepsilon_1)\right] + M_3^{(1,1)}\hbar^2\bar{q}^2(p_2\cdot\varepsilon_1)(p_2\cdot\varepsilon_4^*) + \mathcal{O}(\hbar^3). \quad (3.32)$$

Now it's straightforward to convert the polarization relations in (3.32) to spin expressions. So, we use (3.7) and (3.10) and these sustitutions yields the classical one-loop amplitude:

$$\mathcal{M}_{cl}^{(1)} = -\left(\frac{k}{2}\right)^4\frac{1}{\sqrt{-q^2}}\Bigg\{M_1^{(1,1)}\delta^{ij} - \frac{iM_4^{(1,1)}}{m_1}p_1^\rho\bar{q}^\sigma p_2^\lambda\varepsilon_{\rho\sigma\lambda\delta}s_{1ij}^\delta + \frac{1}{2m_1^2}\left[-M_1^{(1,1)} + 2m_1^2M_2^{(1,1)} + 2M_4^{(1,1)}(p_1\cdot p_2)\right](\bar{q}\cdot s_1^{ik})(\bar{q}\cdot s_1^{kj}) + \mathcal{O}(\bar{q}^2)\Bigg\}. \quad (3.33)$$

In the same fashion we can obtain the superclassical $\mathcal{M}_{scl}^{(1)}$ and quantum contribution,$\mathcal{M}_{qc}^{(1)}$, with spin:

$$\mathcal{M}_{scl}^{(1)} = \left(\frac{k}{2}\right)^4\frac{1}{q^2}\Bigg\{M_1^{(1,0)}\delta^{ij} - \frac{iM_4^{(1,0)}}{m_1}p_1^\rho\bar{q}^\sigma p_2^\lambda\varepsilon_{\rho\sigma\lambda\delta}s_{1ij}^\delta + \frac{1}{2m_1^2}\left[-M_1^{(1,0)} + 2m_1^2M_2^{(1,0)} + 2M_4^{(1,0)}(p_1\cdot p_2)\right](\bar{q}\cdot s_1^{ik})(\bar{q}\cdot s_1^{kj}) + \mathcal{O}(\bar{q}^2)\Bigg\}, \quad (3.34)$$

$$\mathcal{M}_{qc}^{(1)} = -\left(\frac{k}{2}\right)^4\Bigg\{M_1^{(1,2)}\delta^{ij} - \frac{iM_4^{(1,2)}}{m_1}p_1^\rho\bar{q}^\sigma p_2^\lambda\varepsilon_{\rho\sigma\lambda\delta}s_{1ij}^\delta + \frac{1}{2m_1^2}\left[-M_1^{(1,2)} + 2m_1^2M_2^{(1,2)} + 2M_4^{(1,2)}(p_1\cdot p_2)\right](\bar{q}\cdot s_1^{ik})(\bar{q}\cdot s_1^{kj}) + \mathcal{O}(\bar{q}^2)\Bigg\}. \quad (3.35)$$

Although here we present quantum corrections, in the classical limit they vanish. The real part of the amplitude $\mathcal{M}^{(1)}$ will be formed only by the classical contributions even the term with the $\mathcal{O}(s)$ dipole is a real quantity. Here the contributions are given in terms of the form factors, and as we will see in the later section its expansion around $\varepsilon \to 0$ must be done very carefully including the distinction between the real and imaginary part of the amplitude. In order to prepare the amplitude that is necessary to compute the observables at NLO we expand each form factor and this leads to the contributions having the following form

$$\mathcal{M}^{(1)}_{scl} = \left(\frac{k}{2}\right)^4 \frac{1}{(q^2)^{1+\varepsilon}} \Bigg[- i\frac{m_1^3 m_2^3 (1-2\gamma^2)^2}{2\pi\sqrt{\gamma^2-1}} - \frac{m_1 m_2^2 \gamma (2\gamma^2-1)}{\pi\sqrt{\gamma^2-1}} p_1^\rho \bar{q}^\gamma p_2^\lambda \varepsilon_{\rho\gamma\lambda\delta} s_1^\delta - i\frac{m_1 m_2^3 (4(m_1^2-1)\gamma^4 + 2\gamma^2 - m_1^2)}{2\pi\sqrt{\gamma^2-1}} (\bar{q}\cdot s_1)(\bar{q}\cdot s_1) \Bigg] \tilde{f}_2, \quad (3.36)$$

$$\mathcal{M}^{(1)}_{cl} = \left(\frac{k}{2}\right)^4 \frac{1}{(-q^2)^{1/2+\varepsilon}} \Bigg[\frac{3}{16\pi} m_1^2 m_2^2 (m_1+m_2)(5\gamma^2-1) - i\frac{m_2(4m_1+3m_2)\gamma(5\gamma^2-3)}{16\pi(\gamma^2-1)} p_1^\rho \bar{q}^\gamma p_2^\lambda \varepsilon_{\rho\gamma\lambda\delta} s^\delta + \frac{m_2^2(60m_2(\gamma^2-1)\gamma^2 + 8m_2 + m_1(95\gamma^4 - 102\gamma^2 + 15))}{128\pi(\gamma^2-1)} (\bar{q}\cdot s_1)(\bar{q}\cdot s_1) \Bigg] \tilde{f}_1. \quad (3.37)$$

Where $\tilde{f}_1$ and $\tilde{f}_2$ are pieces which come from the triangle and box integrals

$$\tilde{f}_1 = \frac{\Gamma^2(\frac{1}{2}-\varepsilon)\Gamma(\frac{1}{2}-\varepsilon)}{2\sqrt{\pi}\Gamma(1-2\varepsilon)} \quad (3.38)$$

$$\tilde{f}_2 = i\frac{\Gamma^2(-\varepsilon)\Gamma(1+\varepsilon)}{4\Gamma(-2\varepsilon)} \quad (3.39)$$

4 OBSERVABLES TO 2PM WITH SPIN FROM KMOC FORMALISM

At this point it is also important to remember the state of the art of these observables both derived with classical and modern techniques: At 1 MP the linear impulse and the spin-kick have already been calculated using KMOC (Maybee *et al.*, 2019) and with classical methods in (Vines, 2018), also in (Cordero *et al.*, 2022) the authors computed the conservative two-body Hamiltonian of a compact binary system with a spinning black hole through $\mathcal{O}(G^3)$. By a generalization of the EFT approach to PM dynamics the authors in (Liu *et al.*, 2021) introduce a systematic procedure to compute the total change in momentum and spin in the gravitational scattering of compact objects and parallel to this last work in (Kosmopoulos and Luna, 2021) the authors obtain the quadratic-in-spin terms of the conservative Hamiltonian at $\mathcal{O}(G^2)$ from scattering amplitudes.
So far we have presented the framework of the KMOC formalism. In this section we use KMOC with spin to calculate linear and angular impulse in a two-BH scattering process. We start by computing the linear impulse to $\mathcal{O}(G^3)$, but first going through the calculation to LO ($\mathcal{O}(G^2)$), we will do the same for the angular impulse.

4.1 Linear impulse

4.1.1 Leading order

At leading order the contribution to the linear impulse is only *virtual* and we can compute the observable using (2.40):

$$\Delta p_1^{\mu,(0)} = i\left\langle\!\!\left\langle \int \hat{d}^4 q \hat{\delta}(2p_1\cdot q)\hat{\delta}(2p_2\cdot q) e^{-ib\cdot q/\hbar}\mathcal{I}_{p_1}^{\mu,(0)}\right\rangle\!\!\right\rangle, \tag{4.1}$$

where $\mathcal{I}_{p_1}^{\mu,(0)} = q^\mu \mathcal{M}^{(0)}$. Inserting the tree-level gravity amplitude in (4.1) and we reduce the double angle bracket to the single expectation value over the spin states by setting $p_i \to m_i u_i$ in the classical limit. Also we rescale $q \to \hbar\bar{q}$, all this unfolds as follows

$$\begin{aligned}
\Delta p_1^{\mu,(0)} &= i\left(\frac{k}{2}\right)^2 \frac{1}{4m_1m_2}\left\langle\!\!\left\langle \int \hat{d}^4\bar{q}\hat{\delta}(u_1\cdot\bar{q})\hat{\delta}(u_2\cdot\bar{q})e^{-ib\cdot\bar{q}}\frac{\bar{q}^\mu}{\bar{q}^2}\left\{M_1^{(0)} - \frac{iM_4^{(0)}}{m_1}p_1^\rho\bar{q}^\sigma p_2^\lambda\varepsilon_{\rho\sigma\lambda\delta}s_1^\delta\right.\right.\right.\\
&\left.\left.\left. +\frac{1}{2m_1^2}\left[-M_1^{(0)} + 2m_1^2M_2^{(0)} + 2M_4^{(0)}(p_1\cdot p_2)\right](\bar{q}\cdot s_1)(\bar{q}\cdot s_1)\right\}\right\rangle\!\!\right\rangle, \qquad (4.2)\\
&= i\left(\frac{k}{2}\right)^2 \frac{1}{4m_1m_2}\int \hat{d}^4\bar{q}\hat{\delta}(u_1\cdot\bar{q})\hat{\delta}(u_2\cdot\bar{q})e^{-ib\cdot\bar{q}}\frac{\bar{q}^\mu}{\bar{q}^2}\times\left\langle M_1^{(0)} - im_2M_4^{(0)}u_1^\rho\bar{q}^\sigma u_2^\lambda\varepsilon_{\rho\sigma\lambda\delta}s_1^\delta\right.\\
&\left. +\frac{1}{2m_1^2}\left[-M_1^{(0)} + 2m_1^2M_2^{(0)} + 2m_1m_2M_4^{(0)}(u_1\cdot u_2)\right](\bar{q}\cdot s_1)(\bar{q}\cdot s_1)\right\rangle, \qquad (4.3)\\
&= i\frac{2\pi G}{m_1m_2}\left\{M_1^{(0)}I_1^\mu - im_1m_2M_4^{(0)}u_1^\rho u_2^\lambda\varepsilon_{\rho\sigma\lambda\delta}\left\langle a_1^\delta\right\rangle I_1^{\mu\sigma}\right.\\
&\left. +\frac{1}{2}\left[-M_1^{(0)} + 2m_1^2M_2^{(0)} + 2m_1m_2M_4^{(0)}(u_1\cdot u_2)\right]\langle a_{1\sigma}a_{1\rho}\rangle I_1^{\mu\sigma\rho}\right\}. \qquad (4.4)
\end{aligned}$$

In (4.4) we rescale the spin $a^\mu = s^\mu/m$ and replace $\left(\frac{k}{2}\right)^2 = 8\pi G$. Here $I_1^\mu, I_1^{\mu\sigma}$ and $I_1^{\mu\sigma\rho}$ are integrals, we take their solution from Appx.A.1:

$$I_1^\mu = \int \hat{d}^4\bar{q}\hat{\delta}(u_1\cdot\bar{q})\hat{\delta}(u_2\cdot\bar{q})e^{-ib\cdot\bar{q}}\frac{\bar{q}^\mu}{\bar{q}^2} = \frac{i}{2\pi\sqrt{\gamma^2-1}}\frac{b^\mu}{b^2}, \tag{4.5}$$

$$I_1^{\mu\sigma} = \int \hat{d}^4\bar{q}\hat{\delta}(u_1\cdot\bar{q})\hat{\delta}(u_2\cdot\bar{q})e^{-ib\cdot\bar{q}}\frac{\bar{q}^\mu\bar{q}^\sigma}{\bar{q}^2} = \frac{1}{2\pi b^4\sqrt{\gamma^2-1}}\left(2b^\mu b^\sigma - b^2\Pi^{\mu\sigma}\right), \tag{4.6}$$

$$I_1^{\mu\sigma\rho} = \int \hat{d}^4\bar{q}\hat{\delta}(u_1\cdot\bar{q})\hat{\delta}(u_2\cdot\bar{q})e^{-ib\cdot\bar{q}}\frac{\bar{q}^\mu\bar{q}^\sigma\bar{q}^\rho}{\bar{q}^2} = -\frac{i}{\pi b^6\sqrt{\gamma^2-1}}\left(4b^\mu b^\sigma b^\rho - 3b^2 b^{(\mu}\Pi^{\sigma\rho)}\right). \tag{4.7}$$

After replacing the integral solutions, the factors $M_{i=1,2,3,4}^{(0)}$ and some algebra, we finally obtain the linear momentum at leading-order:

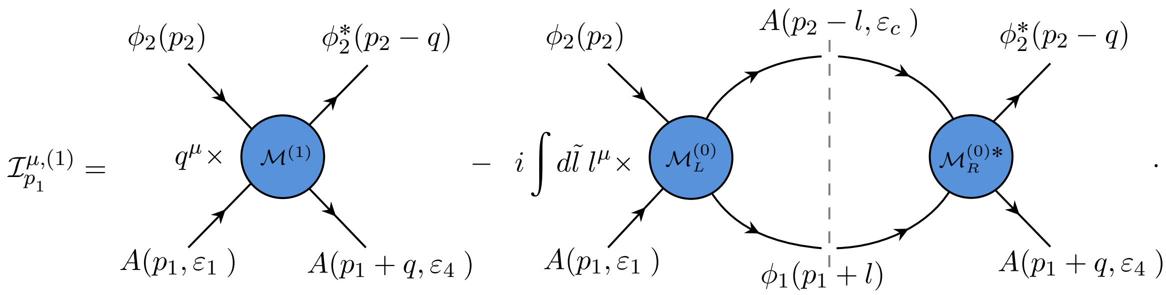
$$\begin{aligned}
\Delta p_1^{\mu,(0)} = \frac{2Gm_1m_2}{\sqrt{\gamma^2-1}}\Bigg((2\gamma^2-1)\frac{b^\mu}{b^2} &+ \frac{2\gamma}{b^4}\left(2b^\mu b^\alpha - b^2\Pi^{\mu\alpha}\right)\varepsilon_{\alpha\rho\sigma\delta}u_1^\rho u_2^\sigma\left\langle a_1^\delta\right\rangle\\
&- \frac{2\gamma^2-1}{b^6}\left(4b^\mu b^\nu b^\rho - 3b^2 b^{(\mu}\Pi^{\nu\rho)}\right)\langle a_{1\nu}a_{1\rho}\rangle\Bigg). \qquad (4.8)
\end{aligned}$$

In the latter expression (4.8) we can identify the spinless part which comes from the scalar part in the amplitude and the spinning part with two orders in the spin. It is straightforward to match this expression we have with the unperturbed result to 1PM in Liu *et al.* (2021), we just replace $\frac{b^\mu}{b^2} = -\frac{\hat{b}^\mu}{|b|}$. The appearance of (4.8) is the same for the linear momentum at leading orden in (Vines, 2018).

4.1.2 Next-to-leading order

Now we move to the NLO contribution in the momentum impulse, here the kernel have both virtual and real contributions. When we evaluate the NLO amplitudes in the virtual part of the kernel, divergent terms or terms with factors of $1/\varepsilon$ will appear which come from the superclassical contributions. The classical contributions lack this unfavorable detail and in the classical limit the quantum corrections will not contribute to the observable.
On the other hand, the cut in the real part contais a product of the left and right tree-level four-point amplitudes. And as it has been shown in Kosower *et al.* (2019) the superclassical contributions will be canceled by some terms that come from the unitary cut, however in the process integrals with l^μ in the numerator will appear and this represents the main difficulty of calculating and that is why we must clear the ground a bit before starting with the calculation of the observable.

Remembering that we are using scattering amplitudes for a $2 \to 2$ process involving a massive scalar ϕ and a massive spin-1 (vector) A_μ particles. In terms of the amplitudes the kernel $\mathcal{I}^{\mu,(1)}_{p_1}$ diagrammatically looks as follows



(4.9)

Where $\mathcal{M}_L^{(0)}$ and $\mathcal{M}_R^{(0)*}$ are the left and right amplitudes of the cut respectively. The virtual kernel is just $\mathcal{I}^\mu_v = q^\mu \mathcal{M}^{(1)}$ and $d\tilde{l}$ is the abbreviated notation for the on-shell integrals which explicitly are defined as

$$d\tilde{l} = \hat{d}l\hat{\delta}(2p_1 \cdot l + l^2)\hat{\delta}(2p_2 \cdot l - l^2). \tag{4.10}$$

As briefly mentioned in the subsection 2.5, unitarity and the cutting rules can be used to simplify the kernel by canceling the superclassical terms in the one-loop amplitude. We can perfectly accompany this simplification by reducing all the one-loop tensor integral to a linear combination of scalar integrals. This beautiful simplification start by decomposing the loop momentum l^μ in terms of the data $\{q^\mu, \check{u}_1^\mu, \check{u}_2^\mu, \varepsilon(\cdot, q, \check{u}_1, \check{u}_2)\}$

$$l^\mu = c_1 q^\mu + c_2 \check{u}_1^\mu + c_3 \check{u}_2^\mu + c_4 \varepsilon^\mu(q, \check{u}_1, \check{u}_2); \quad \varepsilon^\mu(q, \check{u}_1, \check{u}_2) = \varepsilon^{\mu\rho\alpha\beta}\check{u}_{1\rho}\check{u}_{2\alpha}q_\beta, \tag{4.11}$$

we can find the coefficients $c_{i=1,2,3,4}$ by contracting the 4-vectors and using their orthogonality properties $q \cdot u_i = 0, u_i \cdot \check{u}_j = \delta_{ij}$. Thus the appearance of l^μ in the new basis is

$$l^\mu = \frac{l \cdot q}{q^2} q^\mu + (l \cdot u_1)\check{u}_1^\mu + (l \cdot u_2)\check{u}_2^\mu + \frac{(\gamma^2 - 1)\varepsilon(l, q, \check{u}_1, \check{u}_2)}{2q^2}\varepsilon^\mu(q, \check{u}_1, \check{u}_2). \tag{4.12}$$

As a result, we must compute only the scalar integrals such as

$$I_q = \frac{1}{q^2}\int \hat{d}^4 l \frac{l \cdot q}{l^2(l-q)^2(l \cdot u_1)(l \cdot u_2)}. \tag{4.13}$$

From the latter expression one might average over over the two equivalent expressions after the linear change of variable $l \leftrightarrow q - l$

$$\frac{1}{2}\left[\frac{l \cdot q}{q^2} + \frac{(q-l)\cdot q}{q^2}\right] = \frac{1}{2}. \tag{4.14}$$

We can write the factors in eq. (4.12) by using $u_1 = (p_1 + \frac{q}{2})/m_1 + \mathcal{O}(q^2), u_2 = (p_2 - \frac{q}{2})/m_2 + \mathcal{O}(q^2)$, the on-shell conditions $(p_1 + l)^2 - m_1^2 = 0, (p_2 - l)^2 - m_2^2 = 0$

$$l \cdot u_1 = \frac{q^2}{4m_1}..., \quad l \cdot u_2 = -\frac{q^2}{4m_2}..., \tag{4.15}$$

we ignored terms such as quantum suppressed terms, then we get the final loop-momentum

$$l^\mu = \frac{1}{2}q^\mu + \frac{q^2}{4}\left(\frac{\check{u}_1^\mu}{m_1} - \frac{\check{u}_2^\mu}{m_2}\right) + \frac{(\gamma^2-1)\varepsilon(l,q,\check{u}_1,\check{u}_2)}{2q^2}\varepsilon^\mu(q,\check{u}_1,\check{u}_2). \tag{4.16}$$

Now we focus on the cut contribution and how it was established in Herrmann *et al.* (2021b): the S matrix descomposition $S = 1 + iT$ and the unitarity property $S^\dagger S = 1$ allows us to relate the imaginary part of amplitude $\langle p_1' p_2' | T | p_1 p_2 \rangle$ with the cut $\langle p_1' p_2' | T^\dagger T | p_1 p_2 \rangle$. This leads us to

$$2\text{Im}\left[\mathcal{M}^{(1)}\right] = \int d\tilde{l}\mathcal{M}_L^{(0)}(p_1, p_2 \to p_1 + l, p_2 - l)\mathcal{M}_R^{(0)*}(p_1 + q, p_2 - q \to p_1 + l, p_2 - l), \tag{4.17}$$

we identify that the first term in (4.16) is also another piece of the virtual kernel, the second expression of this decomposition is in the direction as u_i. After replacing the expression (4.17) in the kernel (4.9), the combination of the two pieces of the virtual kernel conduces to the real part of the amplitude $\mathcal{M}^{(1)} - i\text{Im}\left[\mathcal{M}^{(1)}\right] = \text{Re}\left[\mathcal{M}^{(1)}\right]$ and this simplification cancels the superclassical contributions. The remaining superclassical contribution in the second imaginary piece will have an order $1/\hbar^2$ just like the classical contributions, this is possible thanks to the fact that the factor q^2 will cancel the $1/q^2$ in the superclassical contributions. The resulting expression for the kernel is as follows

$$\begin{aligned}\mathcal{I}_{p_1}^{\mu,(1)} = q^\mu \text{Re}\left[\mathcal{M}^{(1)}\right] - i\frac{q^2}{2}\left(\frac{\check{u}_1}{m_1} - \frac{\check{u}_2}{m_2}\right)\text{Im}\left[\mathcal{M}^{(1)}\right] \\ - i\frac{(\gamma^2-1)}{2q^2}\varepsilon^\mu(q,\check{u}_1,\check{u}_2)\int d\tilde{l}\varepsilon(l,q,\check{u}_1,\check{u}_2)\mathcal{M}_L^{(0)}\mathcal{M}_R^{(0)*}.\end{aligned} \tag{4.18}$$

With the simplification of the KMOC formalism gathered in the kernel we can proceed to calculate the linear impulse. We start by construct the first line in the kernel and as is evident, it is essential to know the amplitudes. As we have mentioned for our interest, the one-loop amplitude is formed by the superclassical (3.36) and classical (3.37) contributions. The real part of the amplitude is formed by the classical contributions $\mathcal{M}_{cl}^{(1)}$. As mentioned, the superclassical contributions contained in the imaginary part of the amplitude contribute classically. After the insertion of this piece in the Fourier transform eq. (2.46) we will have to deal with integrals that look like

$$I^{\mu_1 \dots \mu_m} = \int \hat{d}^D q \hat{\delta}(u_1 \cdot q)\hat{\delta}(u_2 \cdot q)e^{-ib\cdot q}\frac{q^{\mu_1}\dots q^{\mu_m}}{(-q^2)^n}, \tag{4.19}$$

but whose solution we present in the Appx.A.1. The divergent terms or with factors of $1/\varepsilon$ will be healed by the gamma factor in the $D = 4 - 2\varepsilon$ integral of the kernel i.e. by the Fourier transform. Finally after a bit of algebra we get a part of the result for the linear impulse which comes from the virtual and real parts of the kernel

$$\begin{aligned}\Delta p_{1,a_1^0}^\mu = \frac{3\pi G^2 m_1 m_2 (m_1 + m_2)(5\gamma^2 - 1)}{4\sqrt{\gamma^2 - 1}}\frac{b^\mu}{|b|^3} \\ - \frac{G^2 m_1 m_2}{|b|^2}\frac{2(1-2\gamma^2)^2}{(\gamma^2-1)^2}\left[(m_2 + \gamma m_1)u_1^\mu - (m_1 + \gamma m_2)u_2^\mu\right],\end{aligned} \tag{4.20}$$

$$\Delta p^{\mu}_{1,_{a_1^1}} = \frac{G^2}{|b|^5} m_1 m_2 (4m_1 + 3m_2) \frac{\pi\gamma(5\gamma^2 - 3)}{4(\gamma^2 - 1)^{3/2}} (3b^\mu b^\sigma - b^2 \Pi^{\mu\sigma}) u_1^\rho u_2^\lambda \varepsilon_{\rho\sigma\lambda\delta} \left\langle a_1^\delta \right\rangle$$
$$+ \frac{G^2}{|b|^4} \frac{8m_1 m_2 \gamma(2\gamma^2 - 1)}{(\gamma^2 - 1)^2} \left[(m_2 + \gamma m_1) u_1^\mu - (m_1 + \gamma m_2) u_2^\mu\right] u_1^\rho u_2^\lambda b^\sigma \varepsilon_{\rho\sigma\lambda\delta} \left\langle a_1^\delta \right\rangle, \quad (4.21)$$

$$\Delta p^{\mu}_{1,_{a_1^2}} = \frac{4G^2}{|b|^7} m_1 m_2 3\pi \left[\frac{m_1(95\gamma^4 - 102\gamma^2 + 15)m_1 + 60 m_2(\gamma^2 - 1)\gamma^2 + 8m_2}{128(\gamma^2 - 1)^{3/2}}\right] \langle a_{1\sigma} a_{1\beta} \rangle$$
$$\times (5b^\mu b^\sigma b^\beta - 3b^2 b^{(\mu} \Pi^{\sigma\beta)})$$
$$+ \frac{4G^2}{|b|^6} \frac{m_1 m_2 \left(4(m_1^2 - 1)\gamma^4 + 2\gamma^2 - m_1^2\right)}{(\gamma^2 - 1)^2} \left[m_2 + \gamma m_1) u_1^\mu - (m_1 + \gamma m_2) u_2^\mu\right] \langle a_{1\sigma} a_{1\nu} \rangle$$
$$\times (4b^\sigma b^\nu - b^2 \Pi^{\sigma\nu}). \quad (4.22)$$

The latter piece in the kernel (4.18) contains a product orthogonal to q^μ and $\check{u}^\mu_{i=1,2}$, we will call this the cut kernel $\mathcal{I}^\mu_c$. By substituting the product of tree-level amplitudes, only up to linear order in the spin, the scalar part will vanish. This product looks like this

$$\mathcal{I}^\mu_{c_{a_1^1}} = -i \left(\frac{k}{2}\right)^4 \frac{(\gamma^2 - 1)}{2q^2} \frac{M_1^{(0)} M_4^{(0)}}{m_1} \varepsilon^\mu(q, \check{u}_1, \check{u}_2) \int \hat{d}^4 l \hat{\delta}(2p_1 \cdot l) \hat{\delta}(2p_2 \cdot l)$$
$$\times \frac{\varepsilon(l, q, \check{u}_1, \check{u}_2)}{l^2 (q - l)^2} \left[p_1^\rho l^\sigma p_2^\lambda \varepsilon_{\rho\sigma\lambda\delta} s_1^\delta + (p_1^\rho + l^\rho)(q_1^\sigma - l^\sigma)(p_2^\lambda - l^\lambda) \varepsilon_{\rho\sigma\lambda\delta} s_1^\delta\right]. \quad (4.23)$$

The term $\epsilon(l, q, \check{u}_1, \check{u}_2)$ in the integral contains a Levi-Civita which multiplies all those who come from the amplitudes, this will allow us to obtain products of scalars where we must use the orthogonal and normal conditions related to $\{s_1, u_{i=1,2}, q\}$ and after developing these products the kernel is reorganized as follows

$$\mathcal{I}^\mu_{c_{a_1^1}} = (8\pi G)^2 m_1^2 m_2^3 \gamma(2\gamma 2 - 1)(m_1 + \gamma m_2) \varepsilon^\mu(q, \check{u}_1, \check{u}_2) s_1 \cdot u_2 q^2 \int \hat{d}^4 l \frac{\hat{\delta}(u_1 \cdot l) \hat{\delta}(u_2 \cdot l)}{l^2 (q - l)^2}. \quad (4.24)$$

The familiar cut box integral in (4.24) and by performing this integral in a $D - 2$ dimensions we can derive some results from the cut Δp^μ_c

$$\Delta p^\mu_c = 4G^2 m_1 m_2 \frac{\gamma(2\gamma^2 - 1)(m_1 + \gamma m_2)}{\gamma^2 - 1} \langle a_1^\nu \rangle u_{2\nu} \varepsilon^\mu(q, \check{u}_1, \check{u}_2) \frac{b_\rho}{|a|^4} \quad (4.25)$$

Finally we organize all these results as follow

$$\Delta p^{\mu}_{1,_{a_1^0}} = \frac{G^2 m_1 m_2}{(\gamma^2 - 1)} \left[\frac{3\pi(m_1 + m_2)(\gamma^2 - 1)(5\gamma^2 - 1)}{4\sqrt{\gamma^2 - 1}} \frac{b^\mu}{|b|^3}\right.$$
$$\left. - \frac{1}{|b|^2} \frac{2m_1 m_2 (1 - 2\gamma^2)^2}{(\gamma^2 - 1)} \Big((m_2 + \gamma m_1) u_1^\mu - (m_1 + \gamma m_2) u_2^\mu\Big)\right], \quad (4.26)$$

$$\Delta p^{\mu}_{1,_{a_1^1}} = \frac{G^2 m_1 m_2}{(\gamma^2 - 1)} \left[(E_1 \varepsilon_{\alpha\rho\beta\gamma} \langle a_1^\rho \rangle u_1^\beta u_2^\sigma \frac{1}{|b|^5} (3b^\mu b^\sigma - b^2 \Pi^{\mu\sigma})\right.$$
$$\left. + E_2 \varepsilon^{\mu\alpha\rho\beta} \langle a_{1\rho} \rangle u_{1_\beta} \frac{b_\alpha}{|b|^4} + \langle a_1^\rho \rangle u_1^\beta u_2^\sigma \frac{b^\alpha}{|b|^4} \varepsilon_{\alpha\rho\beta\sigma} (E_3 u_1^\mu + E_4 u_2^\mu)\right], \quad (4.27)$$

$$\Delta p^{\mu}_{1,_{a_1^2}} = \frac{G^2 m_1 m_2}{(\gamma^2-1)} E_5 \langle a_{1\sigma} a_{1\beta}\rangle \frac{1}{|b|^7}\Big(5b^\mu b^\sigma b^\beta - 3b^2 b^{(\mu}\Pi^{\sigma\beta)}\Big) + \frac{G^2 m_1 m_2}{\gamma^2-1}\big(E_7 u_1^\mu - E_8 u_2^\mu\big)\langle a_{1\sigma} a_{1\nu}\rangle \frac{1}{|b|^6}(4b^\sigma b^\nu - b^2\Pi^{\sigma\nu}). \tag{4.28}$$

4.2 Spin kick

4.2.1 Leading order

At LO we compute the angular impulse by inserting the LO amplitude (4.1) and performing the fourier transform eq. (2.66)

$$\Delta s_1^{\mu,(0)} = i\Big\langle\!\Big\langle \int \hat{d}^4\bar{q}\hat{\delta}(2p_1\cdot\bar{q})\hat{\delta}(-2p_2\cdot\bar{q})e^{-ib\cdot\bar{q}}\Big(-\hbar^3\frac{p_1^\mu}{m_1^2}\bar{q}\cdot s_1\mathcal{M}^{(0)}(p_1,p_2\to p_1+\hbar\bar{q},p_2-\hbar\bar{q}) -\hbar^2[s_1^\mu,\mathcal{M}^{(0)}(p_1,p_2\to p_1+\hbar\bar{q},p_2-\hbar\bar{q})]\Big)\Big\rangle\!\Big\rangle. \tag{4.29}$$

We conveniently separate the kernel part without and with the conmutator and label them as $J_1^{\mu,(0)}$ and $J_2^{\mu,(0)}$ respectively. We work first with $J_1^{\mu,(0)}$ and, as we have done, we rewrite the moments in a classical way and rescale the spin. The term that multiplies the amplitude introduces a new power in the spin, but we simply neglect it $\mathcal{O}(s^3)$. All these steps are summarized as follows

$$J_1^{\mu,(0)} = i\Big\langle\!\Big\langle \int \hat{d}^4\bar{q}\hat{\delta}(2p_1\cdot\bar{q})\hat{\delta}(-2p_2\cdot\bar{q})e^{-ib\cdot\bar{q}}\Big(-\hbar^3\frac{p_1^\mu}{m_1^2}\bar{q}\cdot s_1\mathcal{M}^{(0)}(p_1,p_2\to p_1+\hbar\bar{q},p_2-\hbar\bar{q})\Big)\Big\rangle\!\Big\rangle \tag{4.30}$$

$$= -i\left(\frac{k}{2}\right)^2\frac{p_1^\mu}{m_1^2}\Big\langle\!\Big\langle \int \hat{d}^4\bar{q}\hat{\delta}(2p_1\cdot\bar{q})\hat{\delta}(-2p_2\cdot\bar{q})e^{-ib\cdot\bar{q}}\bar{q}\cdot s_1\frac{1}{\bar{q}^2}\Big[M_1^{(0)} - \frac{iM_4^{(0)}}{m_1}p_1^\rho\bar{q}^\sigma p_2^\lambda\varepsilon_{\rho\sigma\lambda\delta}s_1^\delta + \frac{1}{2m_1^2}\Big(-M_1^{(0)}+2m_1^2M_2^{(0)}+2M_4^{(0)}(p_1\cdot p_2)\Big)(\bar{q}\cdot s_1)(\bar{q}\cdot s_1)\Big]\Big\rangle\!\Big\rangle, \tag{4.31}$$

$$= -i2\pi G\frac{u_1^\mu}{m_1m_2}\Big[M_1^{(0)}\langle a_{1\nu}\rangle I_2^\nu - im_1m_2M_4^{(0)}u_1^\rho u_2^\lambda\varepsilon_{\rho\sigma\lambda\delta}\left\langle a_{1\nu}a_1^\delta\right\rangle I_2^{\nu\sigma} + \mathcal{O}(a_1^3)\Big]. \tag{4.32}$$

In the last line the coupling constant $\left(\frac{k}{2}\right)^2 = 8\pi G$ has been replaced, the integrals here are already familiar to us and whose solution we invoke from the Appx.A.1 and after a bit of algebra this leads us to

$$J_1^{\mu,(0)} = -i2\pi G\frac{u_1^\mu}{4m_1m_2}\Big[2m_1^2m_2^2\langle a_{1\nu}\rangle\left(\frac{i}{2\pi\sqrt{\gamma^2-1}}\frac{b^\nu}{b^2}\right) + i4m_1^2m_2^2\gamma u_1^\rho u_2^\lambda\varepsilon_{\rho\sigma\lambda\delta}\left\langle a_{1\nu}a_1^\delta\right\rangle\left(\frac{1}{2\pi b^4\sqrt{\gamma^2-1}}\left(2b^\nu b^\sigma - b^2\Pi^{\nu\sigma}\right)\right)\Big], \tag{4.33}$$

$$= -\frac{2Gm_1m_2}{\sqrt{\gamma^2-1}}\Big[(2\gamma^2-1)u_1^\mu\frac{b^\nu}{b^2}\langle a_{1\nu}\rangle - 2\gamma u_1^\mu u_1^\rho u_2^\lambda\varepsilon_{\rho\sigma\lambda\delta}\left\langle a_{1\nu}a_1^\delta\right\rangle\frac{1}{b^4}\left(2b^\nu b^\sigma - b^2\Pi^{\nu\sigma}\right)\Big]. \tag{4.34}$$

Note that in the first line the form factors $M_1^{(0)}, M_4^{(0)}$ were substituted. Now the commutator term $[s_1^\mu, \mathcal{M}^{(0)}]$ requires more care in calculation details. For example note that the scalar part of the amplitud has diagonal spin indices δ^{ij}, so it's commutator vanishes, yielding to

$$\left[s^\mu, \mathcal{M}^{(0)}\right] = \frac{1}{\hbar^3}\left(\frac{k}{2}\right)^2 \frac{1}{\bar{q}^2}\Bigg(-\frac{iM_4^{(0)}}{m_1} p_1^\rho \bar{q}^\sigma p_2^\lambda \varepsilon_{\rho\sigma\lambda\delta}[s_1^\mu, s_1^\delta] + \frac{1}{2m_1^2}\Big(-M_1^{(0)} + 2m_1^2 M_2^{(0)} + 2M_4^{(0)}(p_1\cdot p_2)\Big)[s_1^\mu, (\bar{q}\cdot s_1)(\bar{q}\cdot s_1)]\Bigg). \tag{4.35}$$

The commutator that appear when introducing the amplitude at tree-level must of course be treated with the Lorentz algebra for the spin operator (see the Appx.B), specifically as follows

$$[s_1^\mu, s_1^\delta] = -i\frac{\hbar}{m_1}\varepsilon^{\mu\delta\alpha\beta} s_{1\alpha} p_{1\beta}, \tag{4.36}$$

$$[s_1^\mu, (\bar{q}\cdot s_1)(\bar{q}\cdot s_1)] = -2i\hbar \bar{q}\cdot s_1 \varepsilon^{\mu\alpha\beta\gamma}\bar{q}_\alpha s_{1\beta}\frac{p_{1\gamma}}{m_1} + \mathcal{O}(\hbar^2). \tag{4.37}$$

After setting the usual classical expressions and with the replacement of the equations (4.36) and (4.37) , we derive the new appearance for the commutator term in the kernel

$$J_2^{\mu,(0)} = \left(\frac{k}{2}\right)^2 \frac{1}{4m_1m_2}\int \hat{d}^4\bar{q}\hat{\delta}(u_1\cdot\bar{q})\hat{\delta}(u_2\cdot\bar{q})e^{-ib\cdot\bar{q}}\frac{1}{\bar{q}^2} \times\Bigg[-im_1m_2M_4^{(0)}u_1^\rho u_2^\lambda \varepsilon_{\rho\sigma\lambda\delta}\varepsilon^{\mu\delta\alpha\gamma}\langle a_{1\alpha}\rangle u_{1\gamma}\bar{q}^\sigma + \Big(-M_1^{(0)} + 2m_1^2M_2^{(0)} + 2m_1m_2M_4^{(0)}(u_1\cdot u_2)\Big)\varepsilon^{\mu\alpha\beta\gamma}\langle a_{1\sigma}a_{1\beta}\rangle \eta_{\alpha\nu}u_{1\gamma}\bar{q}^\sigma\bar{q}^\nu\Bigg]. \tag{4.38}$$

From here we can reduce the expression substituting the factors $M_{i=1,2,3,4}^{(0)}$

$$J_2^{\mu,(0)} = 2\pi G\frac{1}{m_1m_2}\Bigg[-4im_1^2m_2^2\gamma u_1^\rho u_2^\lambda\varepsilon_{\rho\sigma\lambda\delta}\varepsilon^{\mu\delta\alpha\gamma}\langle a_{1\alpha}\rangle u_{1\gamma}I_3^\sigma + 2m_1^2m_2^2(1-2\gamma^2)\varepsilon^{\mu\alpha\beta\gamma}\langle a_{1\sigma}a_{1\beta}\rangle\eta_{\alpha\nu}u_{1\gamma}I_3^{\sigma\nu}\Bigg]. \tag{4.39}$$

Again we replace the results of these familiar integrals in , arriving at

$$J_2^{\mu,(0)} = -\frac{2Gm_1m_2}{\sqrt{\gamma^2-1}}\Bigg[2\gamma\varepsilon^{\mu\delta\alpha\gamma}u_{1\gamma}\langle a_{1\alpha}\rangle\varepsilon_{\rho\sigma\lambda\delta}u_1^\rho u_2^\lambda\frac{b^\sigma}{b^2} + (2\gamma^2-1)\varepsilon^{\mu\alpha\beta\gamma}u_{1\gamma}\langle a_{1\sigma}a_{1\beta}\rangle\frac{1}{b^4}\Big(2b^\sigma b_\alpha - b^2\Pi_\alpha^\sigma\Big)\Bigg]. \tag{4.40}$$

Finally combining the results the angular impulse becomes

$$\Delta s_1^{\mu,(0)} = -\frac{2Gm_1m_2}{\sqrt{\gamma^2-1}}\Bigg[(2\gamma^2-1)u_1^\mu\frac{b^\nu}{b^2}\langle a_{1\nu}\rangle - 2\gamma u_1^\mu u_1^\rho u_2^\lambda\varepsilon_{\rho\sigma\lambda\delta}\left\langle a_{1\nu}a_1^\delta\right\rangle\frac{1}{b^4}\left(2b^\nu b^\sigma - b^2\Pi^{\nu\sigma}\right) + (2\gamma^2-1)\varepsilon^{\mu\alpha\beta\gamma}u_{1\gamma}\langle a_{1\sigma}a_{1\beta}\rangle\frac{1}{b^4}\Big(2b^\sigma b_\alpha - b^2\Pi_\alpha^\sigma\Big) + 2\gamma\varepsilon^{\mu\delta\alpha\gamma}u_{1\gamma}\langle a_{1\alpha}\rangle\varepsilon_{\rho\sigma\lambda\delta}u_1^\rho u_2^\lambda\frac{b^\sigma}{b^2}\Bigg]. \tag{4.41}$$

4.2.2 Next-to-leading order

Now we proceed to calculate the angular impulse, we already know the general scheme, first we invoke the kernel at NLO

$$\begin{aligned}\mathcal{I}_{s_1}^{\mu,(1)} = & -\frac{\hbar}{m_1^2} p_1^\mu \bar{q}\cdot s_1(p_1)\mathcal{M}^{(1)}(p_1,p_2\to p_1+\hbar\bar{q},p_2-\hbar\bar{q}) \\ & + \left[s_1^\mu(p_1),\mathcal{M}^{(1)}(p_1,p_2\to p_1+\hbar\bar{q},p_2-\hbar\bar{q})\right] + i\hbar^2\int \hat{d}^4\bar{l}\hat{\delta}(2p_1\cdot\bar{l}+\hbar\bar{l}^2)\hat{\delta}(2p_2\cdot\bar{l}-\hbar\bar{l}^2)\times \\ & \left(\frac{\hbar}{m_1^2}p_1^\mu\bar{l}\cdot s_1(p_1)\mathcal{M}_L^{(0)}(p_1,p_2\to p_1+\hbar\bar{l},p_2-\hbar\bar{l})\mathcal{M}_R^{(0)*}(p_1+\hbar\bar{l},p_2-\hbar\bar{l}\to p_1+\hbar\bar{q},p_2-\hbar\bar{q})\right. \\ & \left. - \left[s_1^\mu(p_1),\mathcal{M}_L^{(0)}(p_1,p_2\to p_1+\hbar\bar{l},p_2-\hbar\bar{l})\mathcal{M}_R^{(0)*}(p_1+\hbar\bar{l},p_2-\hbar\bar{l}\to p_1+\hbar\bar{q},p_2-\hbar\bar{q})\right]\right). \end{aligned} \tag{4.42}$$

As in the case of linear impulse, in order to calculate the angular impulse, some subtleties must be considered. Let's start by separating our kernel into its real and virtual part. The virtual part of the kernel is formed by the terms that have the amplitude a one-loop, while the product of a tree-level left and right amplitude is present only in the real part of the kernel, explicitly both pieces of the kernel are given by

$$\begin{aligned}\mathcal{I}_v^\mu = & -\frac{\hbar}{m_1^2}p_1^\mu\bar{q}\cdot s_1(p_1)\mathcal{M}^{(1)}(p_1,p_2\to p_1+\hbar\bar{q},p_2-\hbar\bar{q}) \\ & + \left[s_1^\mu(p_1),\mathcal{M}^{(1)}(p_1,p_2\to p_1+\hbar\bar{q},p_2-\hbar\bar{q})\right] \end{aligned} \tag{4.43}$$

$$\begin{aligned}\mathcal{I}_r^\mu = & \, i\hbar^2\int \hat{d}^4\bar{l}\hat{\delta}(2p_1\cdot\bar{l}+\hbar\bar{l}^2)\hat{\delta}(2p_2\cdot\bar{l}-\hbar\bar{l}^2)\times \\ & \left(\frac{\hbar}{m_1^2}p_1^\mu\bar{l}\cdot s_1(p_1)\mathcal{M}_L^{(0)}(p_1,p_2\to p_1+\hbar\bar{l},p_2-\hbar\bar{l})\mathcal{M}_R^{(0)*}(p_1+\hbar\bar{l},p_2-\hbar\bar{l}\to p_1+\hbar\bar{q},p_2-\hbar\bar{q})\right. \\ & \left. - \left[s_1^\mu(p_1),\mathcal{M}_L^{(0)}(p_1,p_2\to p_1+\hbar\bar{l},p_2-\hbar\bar{l})\mathcal{M}_R^{(0)*}(p_1+\hbar\bar{l},p_2-\hbar\bar{l}\to p_1+\hbar\bar{q},p_2-\hbar\bar{q})\right]\right). \end{aligned} \tag{4.44}$$

The superclassical contributions should be canceled with those from the cut leaving only classical contributions. By substituting the loop momentum (4.16) in the first term of the real kernel and again using the relationship between the imaginary part of the amplitude at NLO with the cut (4.17) it becomes more evident that the superclassical contributions encoded in this kernel term cancel those superclassical contributions from the first term in the virtual kernel. This simplifies our kernel (4.42) as follows

$$\begin{aligned}\mathcal{I}_{s_1}^{\mu,(1)} = & -p_1^\mu\frac{q\cdot s_1}{m_1^2}\text{Re}[\mathcal{M}^{(1)}] + \left[s_1^\mu,\mathcal{M}^{(1)}\right] + ip^\mu s_{1\alpha}\frac{q^2}{2m_1^2}\left(\frac{\check{u}_1}{m_1}-\frac{\check{u}_2}{m_2}\right)\text{Im}\left[\mathcal{M}^{(1)}\right] \\ & + i\frac{p_1^\mu}{m_1^2}\frac{(\gamma^2-1)}{2q^2}\varepsilon^\mu(q,\check{u}_1,\check{u}_2)\int d\tilde{l}\varepsilon(l,q,\check{u}_1,\check{u}_2)\mathcal{M}_L^{(0)}\mathcal{M}_R^{(0)*} - i\int d\tilde{l}\left[s_1^\mu,\mathcal{M}_L^{(0)}\mathcal{M}_R^{(0)*}\right] \end{aligned} \tag{4.45}$$

here we again are using the short hand notation for the integrals. The superclassical contributions that still remain in the term with the commutator cancel with those that come from the cut in the last term of (4.45). If we work with the virtual kernel, the first term up to $\mathcal{O}(s_1^3)$. As in the case of the linear impulse the superclassical and classical pieces of the one-loop amplitude are given by (3.36) and (3.37) respectively.

The expressions in (4.36) and (4.37) will continue to be useful for the commutator terms, and again the scalar part contains spin indices only in the diagonal, so its commutator vanishes. When we perform the Fourier transform of the imaginary part of (4.45) in a D dimension, the result of this integral contains a factor $\Gamma(D/2-\varepsilon)/\Gamma(\varepsilon)$ which will be multiplied by $\tilde{f}_2$ and by expanding in $\varepsilon \to 0$ the divergences that come from $\tilde{f}_2$ will be healed. Now we can replace the real part of the amplitude, the classical contribution and the imaginary part of the amplitude in the first line in (4.45) and we derive the following contributions to the angular impulse

$$\Delta s^{\mu}_{1,|\mathrm{Re}} = \pi G^2 m_1 m_2 u_1^{\mu}\Bigg[-\frac{3(m_1+m_2)(5\gamma^2-1)}{4(\gamma^2-1)^{1/2}}\langle a_{1\alpha}\rangle\frac{b^{\alpha}}{|b|^3} + \frac{(4m_1+3m_2)\gamma(5\gamma^2-3)}{4(\gamma^2-1)^{3/2}}u_1^{\rho}u_2^{\lambda}\varepsilon_{\rho\gamma\lambda\delta}\frac{\langle a_{1\alpha}a_1^{\delta}\rangle}{|b|^5}(3b^{\alpha}b^{\gamma}-b^2\Pi^{\alpha\gamma})\Bigg] \tag{4.46}$$

$$\Delta s^{\mu}_{1,|[s,\mathrm{cl}]} = \frac{\pi G^2 m_1 m_2}{4(\gamma^2-1)^{3/2}}\Bigg[\frac{(4m_1+3m_2)\gamma(5\gamma^2-3)}{4(\gamma^2-1)^{3/2}}u^{\rho}u_2^{\lambda}\varepsilon_{\rho\gamma\lambda\delta}\varepsilon^{\mu\delta\alpha\beta}\langle a_{1\alpha}\rangle u_{1\beta}\frac{b^{\gamma}}{|b|^3} - \frac{(60m_2(\gamma^2-1)\gamma^2+8m_2(95\gamma^4-102\gamma^2+15))}{4} \times \langle a_{1\nu}a_{1\beta}\rangle\varepsilon^{\mu\alpha\beta\gamma}u_{1\gamma}\frac{1}{|b|^5}(3b^{\nu}b^{\alpha}-b^2\Pi^{\nu\alpha})\Bigg] \tag{4.47}$$

$$\Delta s^{\mu}_{1,|\mathrm{Im}} = -2G^2\frac{m_1^2m_2^2}{(\gamma^2-1)^{1/2}}u_1^{\mu}\left(\frac{\check{u}_1^{\alpha}}{m_1}-\frac{\check{u}_2^{\alpha}}{m_2}\right)\Bigg[\frac{(1-2\gamma^2)^2}{(\gamma^2-1)^{1/2}}\langle a_{1\alpha}\rangle\frac{1}{|b|^2} - \frac{4\gamma(2\gamma^2-1)}{(\gamma^2-1)^{1/2}}u_1^{\rho}u_2^{\lambda}\varepsilon_{\rho\gamma\lambda\delta}\left\langle a_1^{\delta}a_{1\alpha}\right\rangle\frac{b^{\gamma}}{|b|^4}\Bigg] \tag{4.48}$$

With a bit of algebra we can rewrite the Levi-Civita product in the second term by using the equation (3.6) also we can substitute the expressions (2.71) for the 4-dual velocities, this will lead us to

$$\Delta s^{\mu}_{1,a_1^1} = \pi G^2 m_1 m_2\frac{(4m_1+3m_2)\gamma(5\gamma^2-3)}{2(\gamma^2-1)^{3/2}}\frac{1}{|b|^3}\Big[(u_2^{\mu}-\gamma u_1^{\mu})(\langle a_{1\gamma}\rangle b^{\gamma}) - b^{\mu}(u_2^{\alpha}\langle a_{1\alpha}\rangle)\Big] \tag{4.49}$$

$$\Delta s^{\mu}_{1,|\mathrm{Im}} = -2G^2\frac{m_1^2m_2^2}{(\gamma^2-1)^2}u_1^{\mu}\Big[(m_2+\gamma m_1)u_1^{\alpha}-(m_1+\gamma m_2)u_2^{\alpha}\Big]\Bigg[(1-2\gamma^2)^2\langle a_{1\alpha}\rangle\frac{1}{|b|^2} - 4\gamma(2\gamma^2-1)u_1^{\rho}u_2^{\lambda}\varepsilon_{\rho\gamma\lambda\delta}\left\langle a_1^{\delta}a_{1\alpha}\right\rangle\frac{b^{\gamma}}{|b|^4}\Bigg] \tag{4.50}$$

Now for the commutator and cut terms the relevant of the pieces of the amplitude are given by

$$\mathcal{M}_L^{(0)}\times\mathcal{M}^{(0)*}_{R|\mathcal{O}(s^2)} = (8\pi G)^2\frac{1}{l^2(l-q)^2}\Bigg[4m_1^4m_1^4(1-2\gamma^2)^2 - i8m_1^2m_2^3\gamma(2\gamma^2-1)\Big(p_1^{\rho}l^{\sigma}p_2^{\lambda}\varepsilon_{\rho\gamma\lambda\delta}s_1^{\delta} + (p_1^{\alpha}+l^{\alpha})(q^{\beta}-l^{\beta})(p_2^{\theta}-l^{\theta})\varepsilon_{\alpha\beta\theta\nu}s_1^{\nu}\Big) + \mathcal{O}(s_1^2)\Bigg] \tag{4.51}$$

$$\mathcal{M}_L^{(0)} \times \mathcal{M}_{R|\mathcal{O}(s^3)}^{(0)*} = (8\pi G)^2 \frac{1}{l^2(l-q)^2}\Big[- i8m_1m_2^3\gamma(2\gamma^2-1)\Big(p_1^\rho l^\sigma p_2^\lambda \varepsilon_{\rho\gamma\lambda\delta}s_1^\delta + (p_1^\alpha + l^\alpha)(q^\beta - l^\beta)(p_2^\theta - l^\theta)\varepsilon_{\alpha\beta\theta\nu}s_1^\nu\Big) + 2m_1m_2^3(2\gamma^2-1)\Big((m_1(2m_2\gamma^2+m_2)-4\gamma^2)(l\cdot s_1)(l\cdot s_1) + (m_1(2m_2\gamma^2+m_2)-4\gamma(p_1\cdot p_2 - l^2))((q-l)\cdot s_1)((q-l)\cdot s_1)\Big) + \mathcal{O}(s^3)\Big] \quad (4.52)$$

As for the first term in the second line of (4.45) which we will continue to refer to as the kernel of the cut $\mathcal{I}_c^\mu$, we first proceed to develop the product of $\varepsilon(l, q, \check{u}_1, \check{u}_2)$ and the Levi-Civita that come from the product of amplitudes with linear spin (4.51) and this allows us to obtain

$$\mathcal{I}_c^\mu = -(8\pi G)^2 m_2^3(m_1+\gamma m_2)\gamma(2\gamma^2-1)\varepsilon(s_1, q, \check{u}_1, \check{u}_2)q^2 s_1\cdot u_2 I_1 \quad (4.53)$$

where I_1 is the familiar cut box integral and whose solution is

$$I_1 = \int \hat{d}^D l \frac{\delta(2p_1\cdot l)\delta(2p_2\cdot l)}{l^2(l-q)^2} = -\frac{1}{16\pi m_1 m_2(\gamma^2-1)^{1/2}}\frac{1}{(-q^2)^{1+\varepsilon}}\frac{\Gamma^2(-\varepsilon)\Gamma(1+\varepsilon)}{\Gamma(-2\varepsilon)} \quad (4.54)$$

We proceed to insert both (4.53) and (4.54) in the Fourier transform (2.67) and this leads us to the following result for the angular impulse

$$\Delta s_{1,c}^\mu = -4G^2 m_1 m_2(m_1+\gamma m_2)\frac{\gamma(2\gamma^2-1)}{\gamma^2-1}u_1^\mu \langle a_{1\nu}a_{1\sigma}\rangle \varepsilon^\nu(b, \check{u}_1, \check{u}_2)u_2^\sigma \frac{1}{|b|^4} \quad (4.55)$$

The results derived for the angular impulse are a combination of (4.46),(4.47), (4.50) and those derived form the cut. Finally, it is important to mention that, as for the linear impulse, these results for the angular impulse contain terms that are missing to fully derive the results of (Liu *et al.*, 2021), however these can be obtained by considering the classical limit of the amplitudes at a time later than the time we did it since in the process they have been lost those terms.

5 CONCLUSIONS

After the direct detection of GWs, a new era has been established for physics, astronomy and cosmology. Through GWs we can derive new knowledge: such as the measurement of properties of GWs sources, tests of general relativity including extreme strong-field conditions or tests of the fundamental no-hair theorem, new techniques to measure the Hubble constant and studies of galactic dynamics and evolution.
The ability to make high-precision predictions for GW signals detected in experiments represents a fundamental challenge for current physics. In the face of this vigorous era, describing the two-body problem in GR is of paramount importance since binary systems, and in particular of BHs, represent the main sources for GWs.
The study of the two-body problem in GR has been approached mainly from perturbative and NR methods. Recently, QFT methods based on scattering amplitudes have joined to develop new machinery for the most recent stage in predictions of GW signals.

Within this context, in this work we have presented a study of a new formalism based in on-shell amplitudes, dubbed KMOC after its authors (Kosower *et al.*, 2019; Maybee *et al.*, 2019) and through which

we can derive classical observables. The problem addressed here was the computation of two observables (namely the change in linear and angular impulse) in the scattering of two Kerr BHs up to 2PM approximation using KMOC.
We started by reviewing the KMOC formalism. As we have emphasized, this is based on Amplitudes, which encode the scattering processes at the quantum level and in the classical limit this leads to the emergence of classical structures. Then we proceeded to calculate gravity amplitudes with spin: at LO they were computed from a Yang-Mills theory using another powerful tool, the double copy. While at NLO the amplitudes were extracted from (Cordero *et al.*, 2022).
These pieces of the formalism are inserted into the kernel in order to compute observables. At LO this is straightforward. However, for the one-loop calculations it was necessary to simplify the real kernel and after this simplification we obtained an extra l^μ-dependent kernel in the numerator (4.18). We summarize our results as follows:

- We have implemented the KMOC formalism to compute the change in linear and angular impulse in a scattering process of two spinning BHs. In the classical limit $\hbar \to 0$, the amplitudes with which these observables were rewritten allow us to obtain results that coincide with those existing to 2PM derived in (Liu *et al.*, 2021), where the authors generalized the EFT approach to PM dynamics to include rotational degrees of freedom.

- Regarding the calculation of observables in general relativity, this work offers an example of the new and fascinating perspective brought by modern QFT methods driven by the amplitudes program.

Finally, we are living the nascent and exciting era of the GWs physics, with methods from collider physics joining forces with traditional methods to try to solve fundamental problems at the classical level, the recent incorporation of the GWs detector KAGRA in Japan, the construction of its own LIGO in India, the new kind of WG signal detected by NANOGrav. The latest upgrades to the LIGO and Virgo instruments will result in more sensitive detectors and detections, also ESA prepares LISA and the Newton Telescope which represent the next generation in GWs detectors. This is not the end of the road.

6 APPENDICES

A Computational details in the classical observables

A.1 Integrals

The general form of the integrals is defined by

$$I^{\mu_1 \cdots \mu_n} = \int \hat{d}^4\bar{q}\hat{\delta}(u_1 \cdot \bar{q})\hat{\delta}(u_2 \cdot \bar{q})\frac{e^{-ib\cdot\bar{q}}}{\bar{q}^2}\bar{q}^{\mu_1}...\bar{q}^{\mu_n} \tag{A.1}$$

In the lowest case

$$I^{\mu_1} = \int \hat{d}^4\bar{q}\hat{\delta}(u_1 \cdot \bar{q})\hat{\delta}(u_2 \cdot \bar{q})\frac{e^{-ib\cdot\bar{q}}}{\bar{q}^2}\bar{q}^{\mu_1} \tag{A.2}$$

To perform this integral we hace to work in the rest frame of particle 1, which implies $u_1 = (1,0,0,0)$. Also we can orientate the spatial coordinates in this frame so that particle 2 is moving along the z axis with proper velocity $u_2 = (\gamma, 0, 0, \gamma\beta)$. Now the standar Lorentz gamma factor is given by $\gamma = u_1 \cdot u_2$ and β satisfy $\gamma^2(1-\beta^2) = 1$. Resuming these steps

$$\bar{q} = (\bar{q}_0, \bar{q}_1, \bar{q}_2, \bar{q}_3) \tag{A.3}$$

$$u_1 = (1,0,0,0); \quad u_2 = (\gamma, 0, 0, \gamma\beta) \tag{A.4}$$

$$\gamma = u_1 \cdot u_2; \quad \gamma^2(1-\beta^2) = 1 \tag{A.5}$$

$$\hat{\delta}(u_1 \cdot \bar{q}) = \hat{\delta}(\bar{q}^0) \tag{A.6}$$

$$\hat{\delta}(u_2 \cdot \bar{q}) = \hat{\delta}(\gamma\bar{q}^0 - \gamma\beta\bar{q}^3) \tag{A.7}$$

With these replacings I^{μ_1} becomes

$$I^{\mu_1} = \int \hat{d}^4\bar{q}\hat{\delta}(\bar{q}^0)\hat{\delta}(\gamma\bar{q}^0 - \gamma\beta\bar{q}^3)\frac{e^{-ib\cdot\bar{q}}}{\bar{q}^2}q^{\mu_1} \tag{A.8}$$

A very quickly simplification of the integrals comes from the deltas, with

$$\hat{\delta}(\bar{q}^0)\hat{\delta}(\gamma\bar{q}^0 - \gamma\beta\bar{q}^3) = \frac{\hat{\delta}(\bar{q}^0)\hat{\delta}(\bar{q}^0 - \beta\bar{q}^3)}{\gamma} \tag{A.9}$$

$$= \frac{1}{\gamma\beta} \tag{A.10}$$

The last line is possible because $\bar{q}^0 = \bar{q}^3 = 0$. Now the non-vanishing components of $\bar{q}^\mu$ in the xy plane of our coordinate system are $\bar{\boldsymbol{q}}_\perp$. After the two delta-integrals in (A.9), we perform the 4-integral (A.8) using $\hat{d}^2\bar{q} = \frac{d^2\bar{q}}{(2\pi)^2}$, where the last step represents the integral over $\bar{\boldsymbol{q}}_\perp$. So, our integral becomes

$$I^{\mu_1} = \frac{1}{(2\pi)^2\gamma\beta}\int d^2\bar{q}e^{i\boldsymbol{b}\cdot\bar{\boldsymbol{q}}_\perp}\frac{1}{\bar{\boldsymbol{q}}_\perp^2}q^{\mu_1} \tag{A.11}$$

Now we use polar coordinates to perform this last integral. Let the magnitude of $\bar{\boldsymbol{q}}_\perp$ be χ and orient the x and y axes so that $\boldsymbol{b} \cdot \bar{\boldsymbol{q}}_\perp = |\boldsymbol{b}|\chi\cos\theta$. Then the integral becomes

$$I^{\mu_1} = \frac{1}{(2\pi)^2\gamma\beta}\int_0^\infty d\chi\chi\int_{-\pi}^{\pi} d\theta e^{i|\boldsymbol{b}|\chi\cos\theta}\frac{1}{\chi^2}(0, \chi\cos\theta, \chi\sin\theta, 0) \tag{A.12}$$

$$= \frac{1}{(2\pi)^2\gamma\beta}\int_0^\infty d\chi\int_{-\pi}^{\pi} d\theta e^{i|\boldsymbol{b}|\chi\cos\theta}(0, \cos\theta, \sin\theta, 0) \tag{A.13}$$

We use Mathematica for this hardcore integral and we obtain

$$I^{\mu_1} = -\frac{2\pi i}{(2\pi)^2\gamma\beta}\int_0^\infty d\chi J_1(|\boldsymbol{b}|\chi)(0,1,0,0) \tag{A.14}$$

$$= -\frac{i}{2\pi\gamma\beta}\frac{(0,1,0,0)}{|\boldsymbol{b}|} \tag{A.15}$$

$$= -\frac{i}{2\pi\sqrt{\gamma^2-1}}\frac{\hat{\boldsymbol{b}}}{|\boldsymbol{b}|} \tag{A.16}$$

The impact parameter is always transverse ,$\boldsymbol{b}_\perp = -b^2 \equiv |b|^2$. And to restore the Lorentz invariant note that (Kosower *et al.*, 2019):

$$\frac{1}{|\beta|} = \frac{\gamma}{\sqrt{\gamma^2-1}}, \quad \frac{\hat{\boldsymbol{b}}}{|\boldsymbol{b}|} = -\frac{b^\mu}{b^2} \tag{A.17}$$

Now to solve the higher rank integrals, we have that the results must lie in the plane orthogonal to the four velocities. This plane is spanned by the impact parameter b^μ, and the projector Π^μ_ν defined in (citar ecuación). So we could have

$$I^{\mu\nu} = \alpha_2 b^\mu b^\nu + \beta_2 \Pi^{\mu\nu} \tag{A.18}$$

The left hand side in (citar ecuación) is traceless and $\beta_2 = -\alpha_2 b^2/2$. Then contracting both sides with b_ν, we derive

$$\alpha_2 b^2 b^\mu = 2\int \hat{d}\bar{q}\hat{\delta}(u_1\cdot\bar{q})\hat{\delta}(u_2\cdot\bar{q})\frac{e^{-ib\cdot\bar{q}}}{\bar{q}^2}q^\mu(b\cdot\bar{q}) = \frac{i}{\pi\sqrt{\gamma^2-1}}\frac{b^\mu}{b^2} \tag{A.19}$$

Finally we find

$$I^{\mu\nu} = \frac{1}{\pi b^4\sqrt{\gamma^2-1}}\left(b^\mu b^\nu - \frac{1}{2}b^2\Pi^{\mu\nu}\right) \tag{A.20}$$

And with the same steps we find

$$I^{\mu\nu\rho} = -\frac{4i}{\pi b^6\sqrt{\gamma^2-1}}\left(b^\mu b^\nu b^\rho - \frac{3}{4}b^2 b^{(\mu}\Pi^{\nu\rho)}\right). \tag{A.21}$$

A.2 *D* Integrals

The previous results can be generalized as follows

$$\int\frac{\hat{d}^Dq\hat{\delta}(q\cdot u_1)\hat{\delta}(q\cdot u_2)e^{iq\cdot b}q^{\mu_1}\cdots q^{\mu_m}}{(-q^2)^n} = \left(-i\frac{\partial}{\partial b_{\mu_1}}\right)\ldots\left(-i\frac{\partial}{\partial b_{\mu_m}}\right)\int\frac{\hat{d}^Dq\hat{\delta}(q\cdot u_1)\hat{\delta}(q\cdot u_2)e^{iq\cdot b}}{(-q^2)^n} \tag{A.22}$$

and as in (Liu *et al.*, 2021) the projector is given by

$$\frac{\partial b^\nu}{\partial b_\mu} = \Pi^{\mu\nu} = \eta^{\mu\nu} + \frac{u_1^\mu(u_1^\nu - \gamma u_2^\nu) + u_2^\mu(u_2^\nu - \gamma u_1^\nu)}{\gamma^2-1} \tag{A.23}$$

and without lost of generality we can and without loss of generality we can still continue working in the rest frame of particle 1. The master integral in (A.22) is computed in a $D-2$ dimensions

$$\int\frac{\hat{d}^Dq\hat{\delta}(q\cdot u_1)\hat{\delta}(q\cdot u_2)e^{iq\cdot b}}{(-q^2)^n} = \frac{2^{-2n}\pi^{(2-D)/2}}{\sqrt{\gamma^2-1}|b|^{D-2-2n}}\frac{\Gamma(\frac{D-2}{2}-n)}{\Gamma(n)} \tag{A.24}$$

it is easy to obtain the result of tensor integrals of different orders by partial derivative with respect to b in the latter expression

$$\int \frac{\hat{d}^D q \hat{\delta}(q\cdot u_1)\hat{\delta}(q\cdot u_2)e^{iq\cdot b}q^{\mu_1}q^{\mu_2}}{(-q)^n} = -i\frac{2^{1-2n}\pi^{(2-D)/2}}{\sqrt{\gamma^2-1}}\frac{b^{\mu_1}}{|b|^{D-2n}}\frac{\Gamma(\frac{D}{2}-n)}{\Gamma(n)} \tag{A.25}$$

$$\int \frac{\hat{d}^D q \hat{\delta}(q\cdot u_1)\hat{\delta}(q\cdot u_2)e^{iq\cdot b}q^{\mu_1}q^{\mu_2}}{(-q^2)^n} = -\frac{2^{1-2n}\pi^{(2-D)/2}}{\sqrt{\gamma^2-1}}\frac{1}{|b|^{D+2-2n}}\Big[(D-2n)b^{\mu_1}b^{\mu_2} - b^2\Pi^{\mu_1\mu_2}\Big] \tag{A.26}$$

$$\times\frac{\Gamma(\frac{D}{2}-n)}{\Gamma(n)} \tag{A.27}$$

$$\int \frac{\hat{d}^D q \hat{\delta}(q\cdot u_1)\hat{\delta}(q\cdot u_2)e^{iq\cdot b}q^{\mu_1}q^{\mu_2}q^{\mu_3}}{(-q^2)^n} = i\frac{2^{1-2n}\pi^{(2-D)/2}}{\sqrt{\gamma^2-1}}\frac{1}{|b|^{D+4-2n}} \tag{A.28}$$

$$\times\Big[(D+2-2n)b^{\mu_1}b^{\mu_2}b^{\mu_3} - 3b^{2\,(\mu_1}\Pi^{\mu_2\mu_3)}\Big]\frac{\Gamma(\frac{D}{2}-n)}{\Gamma(n)} \tag{A.29}$$

B Lorentz algebra for the Pauli-Lubanski operator

The Pauli-Lubanski operator is a basic quantity in the classification of free particle states, although it receives less attention in introductory accounts of quantum field theory than it should. With the help of the Lorentz algebra

$$[\mathbb{J}^{\mu\nu},\mathbb{P}^{\rho}] = i\hbar\left(\eta^{\mu\rho}\mathbb{P}^{\nu}-\eta^{\nu\rho}\mathbb{P}^{\mu}\right), \tag{B.1}$$

$$[\mathbb{J}^{\mu\nu},\mathbb{J}^{\rho\sigma}] = i\hbar\left(\eta^{\mu\rho}\mathbb{J}^{\mu\sigma}-\eta^{\mu\rho}\mathbb{J}^{\nu\rho}-\eta^{\nu\sigma}\mathbb{J}^{\mu\rho}+\eta^{\mu\sigma}\mathbb{J}^{\mu\sigma}\right), \tag{B.2}$$

The Pauli-Lubanski operator commutes with the momentum:

$$[\mathbb{P}^{\mu},\mathbb{W}^{\mu}] = 0. \tag{B.3}$$

$\mathbb{W}^{\mu}$ is a vector operator, it satisfies

$$[\mathbb{J}^{\mu\nu},\mathbb{P}^{\rho}] = i\hbar\left(\eta^{\mu\rho}\mathbb{W}^{\nu}-\eta^{\nu\rho}\mathbb{W}^{\mu}\right). \tag{B.4}$$

And the conmutationrelations of $\mathbb{W}$ with itself are

$$[\mathbb{W}^{\mu},\mathbb{W}^{\nu}] = i\hbar\varepsilon^{\mu\nu}\mathbb{W}_{\rho}\mathbb{P}_{\rho}. \tag{B.5}$$

On single particle states this last commutation relation takes a particularly instructive form. Working in the rest frame of our massive particle state, evidently $W^0 = 0$. Then remaining generators satisfy

$$[\mathbb{W}^{i},\mathbb{W}^{j}] = i\hbar\varepsilon^{ijk}\mathbb{W}^{k}, \tag{B.6}$$

so that the Pauli-Lubanski operators are nothing but the generators of the little group. Not only is this the basis for their importance, but also you can see these commutation relations are directly useful in the computation of the change in a particles spin during scattering.

C Linear and angular impulse coefficients

$$E_1 = \frac{\pi\gamma(5\gamma^2 - 3)}{4(\gamma^2 - 1)^{1/2}}(4m_1 + 3m_2) \tag{C.1}$$

$$E_2 = (8\gamma^3 + 4\gamma^2 - 4\gamma - 1) + (8\gamma^3 - 4\gamma^2 - 4\gamma + 1)\delta(m_1 + m_2) \tag{C.2}$$

$$E_3 = \frac{\gamma(m_1 + m_2)}{\gamma^2 - 1}[(2\gamma + 1)(8\gamma^2 + 2\gamma - 5) + (2\gamma - 1)(8\gamma^2 - 2\gamma - 5)\delta] \tag{C.3}$$

$$E_4 = \frac{m_1 + m_2}{\gamma^2 - 1}[(-8\gamma^4 - 16\gamma^3 + 8\gamma + 1) + (8\gamma^4 - 16\gamma^3 + 8\gamma - 1)\delta] \tag{C.4}$$

$$E_5 = 12\pi\left[\frac{m_1(95\gamma^4 - 102\gamma^2 + 15)m_1 + 60m_2(\gamma^2 - 1)\gamma^2 + 8m_2}{128(\gamma^2 - 1)^{3/2}}\right] \tag{C.5}$$

$$E_7 = \frac{4\left(4(m_1^2 - 1)\gamma^4 + 2\gamma^2 - m_1^2\right)}{(\gamma^2 - 1)}(m_2 + \gamma m_1) \tag{C.6}$$

$$E_8 = \frac{4\left(4(m_1^2 - 1)\gamma^4 + 2\gamma^2 - m_1^2\right)}{(\gamma^2 - 1)}(m_1 + \gamma m_2) \tag{C.7}$$

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