Linear Algebra and its Applications 675 (2023) 338–343

Contents lists available at ScienceDirect

Linear Algebra and its Applications

journal homepage: www.elsevier.com/locate/laa

A note for the SNIEP in size 5 ☆

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A R T I C L E   I N F O

Article history:
Received 27 April 2023
Received in revised form 29 June 2023
Accepted 2 July 2023
Available online 5 July 2023
Submitted by R. Loewy

MSC:
15A18
15A29
15A42

Keywords:
SNIEP in size 5
Symmetric realizability

A B S T R A C T

The purpose of this note is to establish the current state of the knowledge about the SNIEP (symmetric nonnegative inverse eigenvalue problem) in size 5 with just one repeated eigenvalue.

The SNIEP (symmetric nonnegative inverse eigenvalue problem) is the problem of characterizing all possible real spectra of entrywise symmetric nonnegative matrices. A complete solution of this problem is known only for spectra of size $n \leq 4$. For these n's the most basic necessary conditions are also sufficient. That is, the Perron and the trace conditions characterize the SNIEP for $n \leq 4$. Spectra of size 5 are not characterized and this problem has proven to be a very challenging one.

☆ Partially supported by grant PGC2018-096446-B-C21 funded by MCIN/AEI/10.13039/501100011033, "ERDF A way of making Europe" and by the Spanish Research Council (Comisión Interministerial de Ciencia y Tecnología) under project PID2021-122501NB-I00.

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https://doi.org/10.1016/j.laa.2023.07.001
0024-3795/

The open case for size 5, $\sigma = \{\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \lambda_4 \geq \lambda_5\}$, is when there are 3 positive eigenvalues, the trace is positive, $\lambda_1 \geq |\lambda_5|$ and $\lambda_1 + \lambda_2 + \lambda_4 + \lambda_5 < 0$ (see [1]). Loewy in [4] studies this case. In fact, as he shows, when Loewy's result, see Theorem 1 below, is applied to the case of two repeated eigenvalues we have a wider area than the one excluded in [1, Theorem 1].

Theorem 1. ([4, Theorem 2.1]) *Let $\sigma = \{\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5\}$ be a list of real numbers, where the elements of σ are arranged in monotonically decreasing order and with $\lambda_3 - s_1(\sigma) \geq 0$. If σ is the spectrum of a nonnegative symmetric matrix, then $s_3(\sigma) \geq s_1(\sigma)^3 + 6\lambda_3 s_1(\sigma)(\lambda_3 - s_1(\sigma))$.*
($s_k(\sigma)$ is the kth moment of σ, i.e. $s_k(\sigma) = \sum_{i=1}^{5} \lambda_i^k$)

Before this result was published, another result appeared about symmetric realization of spectra with 5 eigenvalues.

Theorem 2. ([3, Theorem 4]) *Let $\sigma = \{\lambda_1, \ldots, \lambda_5\}$ be a list of monotonically decreasing real numbers such that $\sum_{i=1}^{5} \lambda_i \geq \frac{\lambda_1}{2}$. Necessary and sufficient conditions for σ to be the spectrum of a nonnegative symmetric matrix are:*

$$(1)\ \lambda_1 = \max_{\lambda \in \sigma} |\lambda|, \quad (2)\ \lambda_2 + \lambda_5 \leq \sum_{i=1}^{5} \lambda_i \quad \textit{and} \quad (3)\ \lambda_3 \leq \sum_{i=1}^{5} \lambda_i.$$

Previously, unresolved spectra with just one repeated eigenvalue are shown not to occur in [2]. The repetition could be either positive or negative, but the two situations are different.

Theorem 3. ([2, Theorem 3]) *Let $a, d_1 > 0$, $d_2 > d_1$ satisfy $a+d_2, d_1+d_2 < 1 < a+d_1+d_2$. If $(a + d_1)^3 + (a + d_2)^3 > 1 + a^3 + (a + d_1 + d_2 - 1)^3$, then $1, a, a, -(a + d_1), -(a + d_2)$ are not the eigenvalues of a 5-by-5 symmetric nonnegative matrix.*

Theorem 4. ([2, Theorem 4]) *The spectrum $1, a, a - r, -(a + d), -(a + d)$ with $d, r > 0$, $a > r$ and $a + d, r + 2d < 1 < a + 2d$ is not realizable by a symmetric nonnegative 5-by-5 matrix if*

$$2(a + d)^3 > 1 + a^3 + (a + 2d - 1)^3.$$

The purpose of this note is to establish the current state of the knowledge about the SNIEP in size 5 with just one repeated eigenvalue. The next theorems show that Loewy's result is strictly stronger than the results in [2] when it is particularized to one repeated eigenvalue.

Theorem 5. *Let $\sigma = \{1, a, a, -(a+d_1), -(a+d_2)\}$ with $a, d_1 > 0$, $d_2 > d_1$ and $a+d_2, d_1 + d_2 < 1 < a + d_1 + d_2$. If $(a + d_1)^3 + (a + d_2)^3 - 1 - a^3 - (a + d_1 + d_2 - 1)^3 > 0$, then $s_1(\sigma)^3 + 6as_1(\sigma)(a - s_1(\sigma)) - s_3(\sigma) > 0$. The reverse is not true.*

Proof. First of all, note that the hypothesis $\lambda_3 > s_1(\sigma)$ of Theorem 1 applied to our list is the hypothesis $1 < a + d_1 + d_2$. It is straightforward to show that

$$\begin{aligned} & s_1(\sigma)^3 + 6as_1(\sigma)(a - s_1(\sigma)) - s_3(\sigma) \\ &= (1 - d_1 - d_2)^3 + 6a(1 - d_1 - d_2)(a + d_1 + d_2 - 1) - 1 - 2a^3 + (a + d_1)^3 + (a + d_2)^3 \\ &= (a + d_1)^3 + (a + d_2)^3 - 1 - a^3 - a^3 + (1 - d_1 - d_2)^3 + 6a(1 - d_1 - d_2)(a + d_1 + d_2 - 1) \\ &> (a + d_1 + d_2 - 1)^3 - a^3 + (1 - d_1 - d_2)^3 + 6a(1 - d_1 - d_2)(a + d_1 + d_2 - 1) \\ &= 3a(1 - d_1 - d_2)(a + d_1 + d_2 - 1) > 0, \end{aligned}$$

where both inequalities are from the hypothesis of the theorem.

For $a = \frac{1}{2}$, $d_1 = \frac{1}{4}$ and $d_2 = \frac{3}{8}$ we are under the hyphotesis of the theorem and for these values we have

$$s_1(\sigma)^3 + 6as_1(\sigma)(a - s_1(\sigma)) - s_3(\sigma) = \frac{9}{256} > 0$$

and

$$(a + d_1)^3 + (a + d_2)^3 - 1 - a^3 - (a + d_1 + d_2 - 1)^3 = -\frac{9}{256} < 0. \quad \square$$

Let a, d_1 and d_2 be under the hyphotesis of Theorem 5 and let define the functions

$$\begin{aligned} F(a, d_1, d_2) &= \left(\frac{4 + d_1 + d_2}{5}\right)^3 + 2\left(\frac{5a - 1 + d_1 + d_2}{5}\right)^3 \\ &\quad + \left(\frac{d_2 - 5a - 1 - 4d_1}{5}\right)^3 + \left(\frac{d_1 - 5a - 1 - 4d_2}{5}\right)^3 \\ G(a, d_1, d_2) &= (a + d_1)^3 + (a + d_2)^3 - 1 - a^3 - (a + d_1 + d_2 - 1)^3 \\ L(a, d_1, d_2) &= s_1(\sigma)^3 + 6as_1(\sigma)(a - s_1(\sigma)) - s_3(\sigma) \end{aligned}$$

and, for a fixed a, the curves $s \equiv F(a, d_1, d_2) = 0$, $t \equiv G(a, d_1, d_2) = 0$, $n \equiv L(a, d_1, d_2) = 0$ and $\ell = \{d_1 + d_2 = \frac{1}{2}\} \cap \{X = a\}$. See [2] for a more detailed explanation. For the spectra σ under the hyphotesis of Theorem 5 we have:

- If $a < \frac{4\sqrt{5}-5}{11}$, for some (d_1, d_2) the spectrum σ is symmetrically realizable with constant diagonal, those under or on curve s in Fig. 1. And for others, σ is not symmetrically realizable those above curve t in Fig. 1 (Theorem 3) and those above curve n in Fig. 1 (Theorem 1). The question mark in Fig. 1 means that the region between s and n (including n) is unresolved.
- If $\frac{4\sqrt{5}-5}{11} \le a \le \frac{1}{2}$, for some (d_1, d_2) the spectrum σ is not symmetrically realizable, those above curve t in Fig. 1 (Theorem 3) and those above curve n in Fig. 1 (Theorem 1). The question mark in Fig. 1 means that the region under or on n is unresolved.
- If $\frac{1}{2} < a < \frac{3}{4}$, for some (d_1, d_2) the spectrum σ is not symmetrically realizable, those above curve t (Theorem 3), or those above curve n (Theorem 1) in Fig. 1, and for others neither, those under or on line ℓ in Fig. 1 (Theorem 2). Since all the section is covered between both, σ is not symmetrically realizable.

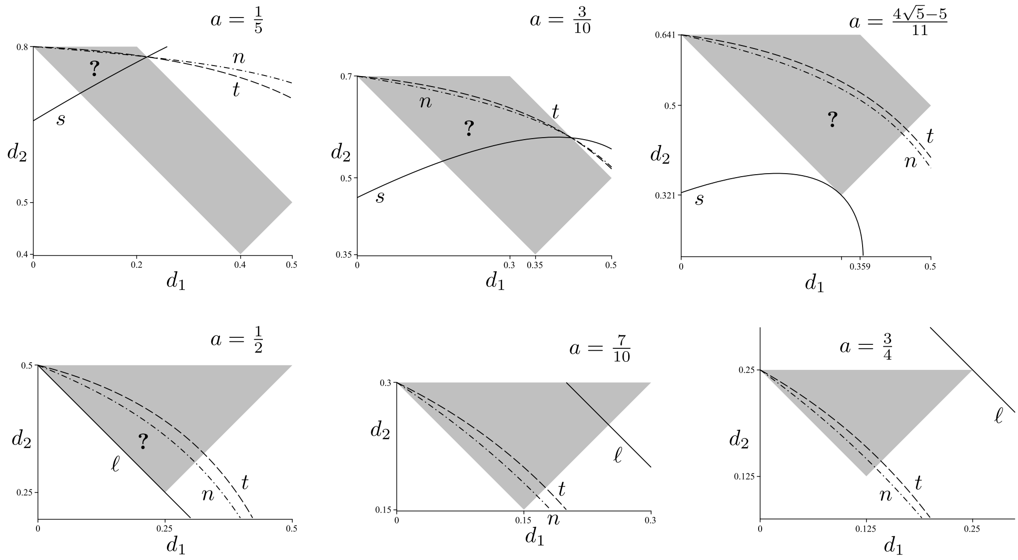

Fig. 1. $d_1 d_2$-sections of the domain of Theorem 5 with curve $s \equiv F(a, d_1, d_2) = 0$, curve $t \equiv G(a, d_1, d_2) = 0$, curve $n \equiv L(a, d_1, d_2) = 0$ and curve $\ell = \{d_1 + d_2 = \frac{1}{2}\} \cap \{X = a\}$ for $a \in \left\{\frac{1}{5}, \frac{3}{10}, \frac{4\sqrt{5}-5}{11}, \frac{1}{2}, \frac{7}{10}, \frac{3}{4}\right\}$.

- If $a \geq \frac{3}{4}$, the spectrum σ is not symmetrically realizable by Theorem 2, see Fig. 1.

The new area that was unresolved is the one between curves n and t for $a \leq \frac{1}{2}$.

Theorem 6. *Let $\sigma = \{1, a, a-r, -(a+d), -(a+d)\}$ with $d, r > 0$, $a > r$ and $a+d, r+2d < 1 < a+2d$. If $2(a+d)^3 - 1 - a^3 - (a+2d-1)^3 > 0$, then $s_1(\sigma)^3 + 6(a-r)s_1(\sigma)(a-r-s_1(\sigma)) - s_3(\sigma) > 0$. The reverse is not true.*

Proof. The hypothesis $\lambda_3 > s_1(\sigma)$ of Theorem 1 applied to our list is the hypothesis $1 < a + 2d$. It is straightforward to show that

$$\begin{aligned} & s_1(\sigma)^3 + 6(a-r)s_1(\sigma)(a-r-s_1(\sigma)) - s_3(\sigma) \\ &= (1-r-2d)^3 + 6(a-r)(1-r-2d)(a+2d-1) - 1 - a^3 - (a-r)^3 + 2(a+d)^3 \\ &= 2(a+d)^3 - 1 - a^3 + (1-r-2d)^3 + 6(a-r)(1-r-2d)(a+2d-1) - (a-r)^3 \\ &> (a+2d-1)^3 + (1-r-2d)^3 + 6(a-r)(1-r-2d)(a+2d-1) - (a-r)^3 \\ &= 3(a-r)(1-r-2d)(a+2d-1) > 0, \end{aligned}$$

where both inequalities are from the hypothesis of the theorem.

For $a = \frac{1}{2}$, $d = \frac{8}{25}$ and $r = \frac{1}{10}$ we are under the hyphotesis of the theorem and for these values we have

$$s_1(\sigma)^3 + 6(a-r)s_1(\sigma)(a-r-s_1(\sigma)) - s_3(\sigma) = \frac{1167}{62500} > 0$$

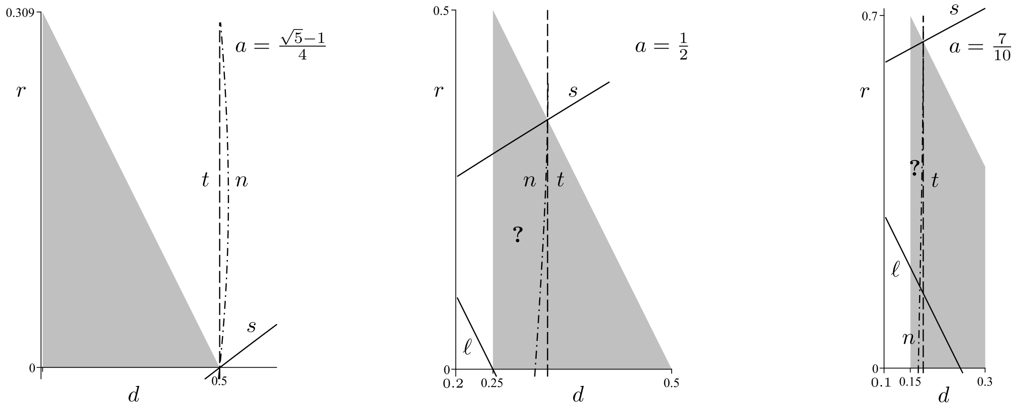

Fig. 2. dr-sections of the domain of Theorem 6 with curve $s \equiv H(a,d,r) = 0$, curve $t \equiv J(a,d,r) = 0$, curve $n \equiv L(a,d,r) = 0$ and curve $\ell = \{r + 2d = \frac{1}{2}\} \cap \{X = a\}$ for $a \in \left\{\frac{\sqrt{5}-1}{4}, \frac{1}{2}, \frac{7}{10}\right\}$.

and

$$2(a+d)^3 - 1 - a^3 - (a+2d-1)^3 = -\frac{1563}{62500} < 0. \quad \square$$

Let a, d and r be under the hyphotesis of Theorem 6 and let define the functions

$$\begin{aligned} H(a,d,r) &= \left(\frac{4+r+2d}{5}\right)^3 + \left(\frac{5a-1+r+2d}{5}\right)^3 \\ &\quad + \left(\frac{5a-4r-1+2d}{5}\right)^3 + 2\left(\frac{r-5a-3d-1}{5}\right)^3 \\ J(a,d,r) &= 2(a+d)^3 - 1 - a^3 - (a+2d-1)^3 \\ L(a,d,r) &= s_1(\sigma)^3 + 6(a-r)s_1(\sigma)(a-r-s_1(\sigma)) - s_3(\sigma) \end{aligned}$$

and, for a fixed a, the curves $s \equiv H(a,d,r) = 0$, $t \equiv J(a,d,r) = 0$, $n \equiv L(a,d,r) = 0$ and $\ell = \{r + 2d = \frac{1}{2}\} \cap \{X = a\}$. See [2] for a more detailed explanation. For the spectra σ under the hyphoteses of Theorem 6 we have:

- If $a \leq \frac{\sqrt{5}-1}{4}$, the spectrum σ is always symmetrically realizable with constant diagonal.
- If $\frac{\sqrt{5}-1}{4} < a \leq \frac{1}{2}$, for some (d,r) the spectrum σ is symmetrically realizable with constant diagonal, those above or on curve s in Fig. 2. And for others σ is not symmetrically realizable, those on the right hand side of curve t (Theorem 4) and those on the right hand side of curve n (Theorem 1) in Fig. 2. The question mark in Fig. 2 means that the region under s and on the left hand side of n (including n) is unresolved.
- If $a > \frac{1}{2}$, for some (d,r) the spectrum σ is symmetrically realizable with constant diagonal, those above or on curve s, for others σ is not symmetrically realizable, those on the right hand side of curve t (Theorem 4), those on the right hand side of curve n

(Theorem 1), and those under or on line ℓ (Theorem 2) in Fig. 2. The question mark in Fig. 2 means that the region among ℓ, n and s (including only n) is unresolved.

The new area that was unresolved is the one between curves n and t for $a > \frac{\sqrt{5}-1}{4}$.

Declaration of competing interest

None declared.

Data availability

No data was used for the research described in the article.

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