Research Article

Eduardo Cuesta, Mokhtar Kirane*, Ahmed Alsaedi, and Bashir Ahmad

On the sub–diffusion fractional initial value problem with time variable order

Abstract: We consider a fractional derivative with order varying in time. Then, we derive for it a Leibniz' inequality and an integration by parts formula. We also study an initial value problem with our time variable order fractional derivative and present a regularity result for it, and a study on the asymptotic behavior.

1 Introduction

As early as 1972, G. Scarpi [21], based on an earlier work of Smit and de Vries [22] concerning rheological models with fractional derivatives, proposed an evolutive rheological model with fractional derivative with order varying in time as an ultime generalization. His definition seems to us appealing; we adopt it here. More than forty years later, in the contributions to Round Table Discussion "Fractional Calculus: Quo Vadimus? (Where are we going?) held at ICFDA 2014 Catania (Italy), 23-25 June 2014, M. Fabrizio commented on fractional derivatives of variable order in the following words: *I believe that a promising field of application of the fractional calculus is to study problems where the α-coefficient (order) of the fractional derivative is varying with time. In spite of several theoretical and numerical papers on this topic, it seems to me that applications are not yet all investigated. In fact, if in continuum mechanics we consider a visco-elastic materials described by a fractional derivative, its α-order is constant and assigns the constitutive law of viscoelasticity. In many problems we observe a change in the nature of the material due to the deformation or simply to the time. Such variation leads to a change of the α-exponent, that can especially be considered as a new variable of the problem. This can be an important step and a qualitative leap for the applications that the fractional derivatives can help to resolve. Phenomena with α-order variable are evident in fatigue, in the plasticity but also for magnetic hysteresis in electromagnetism and in some problems of phase transitions. So I think that considering fractional derivatives of variable order may be a promising way to apply fractional calculus to the above complex phenomena.*
Let us mention that in the literature one may find several definitions of time variable order fractional derivative [4], [5], [23], [25], [12], [13], [14], and even the order of derivation varying in time and the unknown of the system [20] to cite but a few. However, it seems to us that the one that should be adopted is the definition

Eduardo Cuesta, Department Matematica Aplicada, E.T.S.I. de Telecomunicacion, Campus Miguel Delibes, University of Valladolid, Paseo Belen 15, 47011 Valladolid, Spain, E-mail: eduardo@mat.uva.es
*Corresponding Author: Mokhtar Kirane, Department of Mathematics, College of Art and Sciences, Khalifa University of Science and Technology, Abu Dhabi, United Arab Emirates, E-mail: mokhtar.kirane@ku.ac.ae
Ahmed Alsaedi, Nonlinear Analysis and Applied Mathematics (NAAM)-Research Group, Department of Mathematics, Faculty of Science, King Abdulaziz University, P.O. Box 80257, Jeddah 21589, Saudi Arabia, E-mail: aalsaedi@kau.edu.sa
Bashir Ahmad, Nonlinear Analysis and Applied Mathematics (NAAM)-Research Group, Department of Mathematics, Faculty of Science, King Abdulaziz University, P.O. Box 80257, Jeddah 21589, Saudi Arabia, E-mail: bashirahmad_qau@yahoo.com

based on the Laplace transform.

Here after recalling the definition proposed by Scarpi, we address the important Leibniz rule from which we derive an important inequality that is useful to obtaining estimates for fractional differential equations as it was demonstrated for the case of a constant fractional order in [1].

Before stating and proving our results, we recall some definitions and preliminany results.

2 Background

Let us first recall the definition we consider along the paper; we will denote by $\mathcal{L}$ and $\mathcal{L}^{-1}$ the Laplace transform and the inverse Laplace transform operators, respectively. Let $\alpha : [0, +\infty) \to (0, 1)$ be a function that admits the Laplace transform denoted by $\tilde{\alpha}(z) := \mathcal{L}(\alpha)(z)$, for z belonging to certain complex domain $\mathcal{D}$ containing $\mathbb{C}_a^+ := \{z \in \mathbb{C} : \mathrm{Re}(z) \geq a\}$ (assume $a > 0$). For instance, a variety of piecewise continuous functions of exponential growth. Moreover, assume now and hereafter that $t > 0$ means to say *a.e.* $t > 0$.

Definition 1. *Given $f \in L^1([0, +\infty))$, we define*

$$\partial^{-\alpha(t)} f(t) := \int_0^t k(t - s) f(s) \, \mathrm{d}s, \qquad t > 0, \tag{2.1}$$

where $k : [0, +\infty) \to \mathbb{R}$ is given in terms of the Laplace transform as

$$k(t) = \mathcal{L}^{-1}(K)(t) = \frac{1}{2\pi \mathrm{i}} \int_\Gamma \mathrm{e}^{tz} K(z) \, \mathrm{d}z, \quad \text{where} \quad K(z) = \frac{1}{z^{z\tilde{\alpha}(z)}} \quad z \in D', \tag{2.2}$$

where $\mathcal{D}'$ is a certain complex domain such that $\mathbb{C}_a^+ \subset \mathcal{D}'$, Γ is a complex path according to the Bronwich formula for the inversion of the Laplace transform whose precise choice is discussed below, and $k(t)$ vanishes for $t < 0$. Note that in the case of $\alpha(t) = \alpha = constant$, the definition (2.1) matches the very well known definition of fractional integral of order α in the sense of Riemann–Liouville, this is why this definition is often known as the fractional integral of variable order in the sense of Riemann–Liouville.

Now we are in a position to define the fractional derivative of variable order $\alpha(t)$ (FDVO). To this end there are typically two different ways, both in terms of the definition of the fractional integral (2.1)–(2.2).

In spite of the regularity does not a matter here, to be precise the functions involved in the following results are required to belong to a convenient Sobolev space. In fact denote

$$W^{m,p}([0, +\infty)) := \left\{ f \in L^p([0, +\infty)) : \partial^n f \in L^p([0, +\infty)), 0 \leq n \leq m \right\}.$$

Now, given $f \in W^{1,1}([0, +\infty))$, and $\alpha : [0, +\infty) \to (0, 1)$:

1st.- *We define the FDVO as*

$$\partial^{\alpha(t)} f(t) := \frac{\mathrm{d}}{\mathrm{d}t} \left\{ \partial^{\alpha(t)-1} f(t) \right\}, \qquad t > 0. \tag{2.3}$$

Note that $\alpha(t) - 1 < 0$, therefore, inside the brackets, we apply definition (2.1)–(2.2) with order $1 - \alpha(t)$ instead of $\alpha(t)$, to have

$$\partial^{\alpha(t)} f(t) = k(t) f(0) + \int_0^t k(t - s) f'(s) \, \mathrm{d}s, \qquad t > 0, \tag{2.4}$$

where f' stands for the first time derivative of f, and $k(t)$ is given by

$$k(t) = \mathcal{L}^{-1}(K)(t) = \frac{1}{2\pi \mathrm{i}} \int_\Gamma \mathrm{e}^{tz} K(z) \, \mathrm{d}z, \quad \text{now with} \quad K(z) = \frac{1}{z^{1-z\tilde{\alpha}(z)}}, \tag{2.5}$$

and Γ is a suitable path in the complex plane. The choice of Γ is precisely discussed later. The equality (2.4) *has been already noticed for $\alpha(t) = \alpha = $ constant (e.g. formula (2.70), pp. 35 in [10], among many others references). In order to keep the notation as simple as possible and while not confusing, we denote again the Laplace transform of the convolution kernel by $K(z)$.*

2nd.- *Alternatively one may define the FDVO, for $0 < \alpha(t) < 1$ for f differentiable, as*

$$\partial^{\alpha(t)}f(t) := \partial^{\alpha(t)-1}\left\{\frac{\mathrm{d}}{\mathrm{d}t}f(t)\right\} = \partial^{\alpha(t)}f(t) = \int_0^t k(t-s)f'(s)\,\mathrm{d}s, \qquad t > 0, \tag{2.6}$$

where k is defined by (2.5).

Several comments related to the previous proposed definition for the FDVO can be made.

First of all note that, unlike what happens with the definition (2.3), with the definition (2.6) if $f(t) = $ constant, then the fractional derivative turns out to be zero. Moreover, if $f(0) = 0$, then both definitions coincide. Although both definitions show some differences, the treatment turns out to be quite similar for both, as we will show below.

Moreover, based on a fact that has already been observed in [6, 17, 18], we can assert that (2.2) and (2.5) are meaningful under very weak restriction on $\tilde{\alpha}$. In particular, let $K(z)$ be a complex−valued or operator−valued function, analytic outside of a complex sector of angle θ, $\theta < \pi/2$, denoted by

$$S_\theta := \{z \in \mathbb{C} : |\arg(-z)| < \theta\}, \tag{2.7}$$

and such that there exist $M > 0$ and $\beta \in \mathbb{R}$, satisfying

$$\|K(z)\| \le \frac{M}{|z|^\beta}, \quad \text{for} \quad z \notin S_\theta. \tag{2.8}$$

As explained in [6, 17, 18], $K(z)$ stands for the Laplace transform of a distribution $k(t)$ in the real line, so that $k(t) = 0$, for $t < 0$, whose singular support is empty, or merely concentrated at $t = 0$ (e.g. if $\beta < 0$), and which is analytic for $t > 0$. In that case the inverse (distributional) Laplace transform, i.e. $k(t)$, admits the integral representation

$$k(t) = \frac{1}{2\pi i}\int_\Gamma \mathrm{e}^{tz}K(z)\,\mathrm{d}z, \quad \text{for} \quad t > 0,$$

for a convenient path Γ in the complex plane running outside the sector of analyticity S_θ, e.g. one running parallel to the boundary of S_θ, and with increasing imaginary part. We will precisely state below one of these paths, according to our interest for the proofs.

In this context, let us recall the following lemma which has already been proven in [6], Lemma 2.1; it provides useful bounds for $k(t)$.

Lemma 1. *Let $K(z)$ be a analytic function outside a complex sector S_θ, $0 < \theta < \pi/2$, satisfying (2.8) for certain $M > 0$ and $\beta \in \mathbb{R}$.*

Then there exists $C > 0$ depending solely on M, β and θ (but not on t) such that the inverse Laplace transform of $K(z)$, $k(t)$, is bounded as follows

$$\|k(t)\| \le Ct^{\beta-1}, \qquad t > 0. \tag{2.9}$$

Observe that bound (2.9) suggests the lack of regularity of $k(t)$ at $t = 0$, if $\beta < 1$. Moreover, note that $k(t)$ is locally integrable is $\beta > 0$.

A more general case might be considered if instead of the sector S_θ of analyticity, one considers a shifted sector $a + S_\theta = \{a + z : z \in S_\theta\}$, for $a \ge 0$. However, since no noticeable additional difficulties arise in that case, and for the sake of the simplicity of the presentation of results, we merely consider the case $a = 0$, i.e. S_θ.

Before going ahead in the paper, we make some assumptions which will be required when studying the solutions of the sub–diffusion initial value problems of variable order $\alpha(t)$ in Section 4. These assumptions are related to sectoriality angle θ, which will also affect the linear operators considered there as it will be discussed in that section, and the fractional order $\alpha(t)$, and more particularly its Laplace transform $\tilde{\alpha}(z)$. In fact, assume the following:

[H0]. The Laplace transform of $\alpha(t)$, $\tilde{\alpha}(z) := \mathcal{L}(\alpha)(z)$, exists outside a sector S_θ, for $0 < \theta < \pi/2$.

[H1]. Denote

$$g(z) := z\tilde{\alpha}(z); \quad g_R(z) := \mathrm{Re}(g(z)); \quad g_I(z) = \mathrm{Im}(g(z)), \qquad z \notin S_\theta.$$

There exist $R > 0$ large enough, and $0 < m_1, m_2, C_1 < 1$, satisfying

$$1 - C_1 \le C_1, \ \ (\text{expectedly } C_1 \approx 1), \quad \text{and} \quad 1 - C_1 \le m_2,$$

such that $g_I(z)$ is bounded, for $|z| \le R$,

$$\left| g_I(z) \ln|z| \right| < C_1(\pi - \theta), \qquad |z| \ge R,$$

and

$$0 < m_1 \le g_R(z) \le m_2 < 1, \quad z \notin S_\theta.$$

[H2]. The analyticity angle θ satisfies

$$0 < \theta < \frac{\pi}{2}\left(2 - \frac{m_2}{1 - C_1}\right) \le \frac{\pi}{2}.$$

Note that, by hypothesis [H1], we have that $2 - m_2/(1 - C_1) \le 1$, so this hypothesis is meaningful.

On the other hand note that, under Hypotheses [H0]–[H2] we have for (2.5),

$$\|K(z)\| \le \frac{1}{|z|^{1-m_2}}, \quad z \notin S_\theta, \quad \text{and by Lemma 1} \quad \|k(t)\| \le Ct^{-m_2}, \quad t > 0. \tag{2.10}$$

Therefore these inequalities suggest a lack of regularity of $k(t)$ as $t \to 0^+$, and that the function $k(t)$ is locally integrable in $\mathbb{R}^+$. The same can be said for (2.2) related to the regularity and integrability since

$$\|K(z)\| \le \frac{1}{|z|^{m_1}}, \quad z \notin S_\theta, \quad \text{and by Lemma 1} \quad \|k(t)\| \le Ct^{m_1-1}, \quad t > 0. \tag{2.11}$$

Finally, before ending this section we provide some examples of functions $\alpha(t)$ which satisfy [H0]–[H1],

$$\alpha_1(t) = 0.5 + 0.4\sin(t) \quad t > 0, \quad \left(\tilde{\alpha}_1(z) = \frac{0.5}{z} + \frac{0.4}{1 + z^2}\right),$$

$$\alpha_2(t) = 0.5 + 0.4\cos(t) \quad t > 0, \quad \left(\tilde{\alpha}_2(z) = \frac{0.5}{z} + \frac{0.4z}{1 + z^2}\right),$$

$$\alpha_3(t) = 0.9\,e^{-t} \quad t > 0, \quad \left(\tilde{\alpha}_3(z) = \frac{0.9}{1 + z}\right).$$

3 Leibniz rule and integration–by–parts formula for fractional derivatives with time variable order

A Leibniz rule for fractional derivatives with constant order has already been provided in [1] as a generalization of the classical product rule for integer derivatives. It can be observed that this derivation rule (and others

of fractional type one can find in the literature) involve additional terms inspired by the non local character of the fractional derivatives, particularly in the case of FDVO. One of our main contributions in this paper is to go further in this generalization by extending such a property to the fractional derivatives, now with a variable order $\alpha(t)$.

On the one hand it is expected that the Leibniz rule we achieve will adopt different forms depending on the definition of FDVO one considers. On the other hand the definitions of FDVO we consider here may provide the simplest product rule among the ones provided by other definitions. An expected and desirable fact is that the Leibniz rule one achieves for fractional derivatives and particularly the ones achieved here for FDVO should be coherent with the classical product rule. That is the case for the equalities derived in this section.

The next theorem stands for the main result of this section.

Theorem 1. *Let f, g be two functions belonging to $W^{1,1}([0, +\infty))$. According to Definition (2.3), there holds*

$$\partial^{\alpha(t)}(fg)(t) = f(t)\partial^{\alpha(t)}g(t) + g(t)\partial^{\alpha(t)}f(t) + \partial^{\alpha(t)}(h(t, \cdot))(t) - k(t)f(t)g(t), \qquad t > 0, \tag{3.1}$$

and according to Definition (2.6)

$$\partial^{\alpha(t)}(fg)(t) = f(t)\partial^{\alpha(t)}g(t) + g(t)\partial^{\alpha(t)}f(t) + \partial^{\alpha(t)}(h(t, \cdot))(t), \qquad t > 0, \tag{3.2}$$

where

$$h(t, s) := (f(s) - f(t))(g(s) - g(t)), \qquad 0 < s < t.$$

Proof. Consider first the Definition (2.3). On the one hand,

$$\partial^{\alpha(t)}(fg)(t) = \frac{\mathrm{d}}{\mathrm{d}t}\left(\int_0^t k(t - s)f(s)g(s)\,\mathrm{d}s\right)$$

$$= k(t)f(0)g(0) - \int_0^t k(t - s)(f'(s)g(s) + f(s)g'(s))\,\mathrm{d}s;$$

on the other hand

$$f(t)\partial^{\alpha(t)}g(t) = f(t)\left\{k(t)g(0) - \int_0^t k(t - s)g'(s)\,\mathrm{d}s\right\},$$

$$g(t)\partial^{\alpha(t)}f(t) = g(t)\left\{k(t)f(0) - \int_0^t k(t - s)f'(s)\,\mathrm{d}s\right\}.$$

Therefore,

$$\partial^{\alpha(t)}(f(t)g(t)) - f(t)\partial^{\alpha(t)}g(t) - g(t)\partial^{\alpha(t)}f(t)$$

$$= k(t)\{f(0)g(0) - f(t)g(0) - f(0)g(t)\} - \int_0^t k(t - s)\{f'(s)g(s) + f(s)g'(s) - f(t)g'(s) - f'(s)g(t)\}\,\mathrm{d}s$$

$$= k(t)\{(f(0) - f(t))(g(0) - g(t)) - f(t)g(t)\} - \int_0^t k(t - s)\frac{\mathrm{d}}{\mathrm{d}s}\{(f(s) - f(t))(g(s) - g(t))\}\,\mathrm{d}s.$$

Moreover by (2.4) we have

$$\partial^{\alpha(t)}(f(t)g(t)) - f(t)\partial^{\alpha(t)}g(t) - g(t)\partial^{\alpha(t)}f(t)$$

$$= -k(t)f(t)g(t) - k(t)h(t,0) + \int_0^t k(t-s)\frac{\partial h}{\partial s}(t,s)\,ds$$

$$= -k(t)f(t)g(t) + \partial^{\alpha(t)}(h(t,\cdot))(t),$$

and the proof of the equality (2.3) ends.

For the Definition (2.6), we merely recall that

$$\partial^{\alpha(t)}(fg)(t) = \int_0^t k(t-s)\frac{d}{ds}\Big(f(s)g(s)\Big)\,ds = \int_0^t k(t-s)(f'(s)g(s) + f(s)g'(s))\,ds,$$

from which (3.2) straightforwardly follows, and the proof of the theorem ends. □

Here we have to highlight several facts.

- The terms $\partial^{\alpha(t)}(h(t,\cdot))(t) - k(t)f(t)g(t)$ in (3.1) and $\partial^{\alpha(t)}(h(t,\cdot))(t)$ in (3.2) stand for the difference (in the form of an additional term) with respect to the classical product rule, i.e. the case $\alpha(t) \equiv 1$. Moreover, the formulation (3.1) and (3.2) of the fractional product rules according to the definitions (2.3) and (2.6), respectively, may represent the simplest representations if compared to the ones achieved by considering other definitions of FDVO.
- It is straightforward to check that equality (3.1) (analogously (3.2)) is satisfied in the particular case of $\alpha(t) \equiv \alpha$, $0 < \alpha < 1$. It can be easily illustrated by taking for example $f(t) = t^p$, $g(t) = t^q$, $p, q \in \mathbb{Z}^+$, and any $0 < \alpha < 1$. In particular, in the constant case $\alpha(t) \equiv \alpha$, $0 < \alpha < 1$, the product rule achieved here perfectly matches with the one derived in Eq. 3.12, [2],

$$\partial^\alpha(fg)(t) = f(t)\partial^\alpha g(t) + g(t)\partial^\alpha f(t) - \frac{\alpha}{\Gamma(1-\alpha)}\int_0^t \frac{(f(s)-f(t))(g(s)-g(t))}{(t-s)^{\alpha+1}}\,ds - \frac{t^{-\alpha}f(t)g(t)}{\Gamma(1-\alpha)}, \tag{3.3}$$

(∂^α stands for the fractional derivative in the sense of Riemann–Liouville of order $\alpha > 0$).

- Another relevant fact is that both equalities (3.1) and (3.2)) are coherent with the classical product rule, namely the classical product rule turns out to be the limit case as $\alpha(t)$ tends to the constant function $\alpha(t) \equiv 1$. In that case the corresponding limit of the convolution kernel in the sense of distributions, and always according to definition (2.5), turns out to be $k(t) = \delta_0(t)$ where $\delta_0(t)$ stands for Dirac's delta distribution with density concentrated at $t = 0$. In that manner, since $k(t) = 0$, for $t < 0$, then $k(t)f(t)g(t) = 0$, for $t < 0$. Moreover, according to the definition (2.3)

$$\partial^{\alpha(t)}(h(t,\cdot))(t) = \frac{d}{dt}\left\{\partial^{\alpha(t)-1}h(t,\cdot)(t)\right\},$$

and in the limit

$$\partial^{\alpha(t)-1}h(t,\cdot)(t) = \int_0^t k(t-s)h(t,s)\,ds = \int_0^t \delta_0(t-s)h(t,s)\,ds = h(t,t) = 0, \qquad t > 0.$$

Therefore it comes $\partial^{\alpha(t)}(h(t,\cdot))(t) = 0$. Similarly, according to the definition (2.6)

$$\partial^{\alpha(t)}(h(t,\cdot))(t) = \partial^{\alpha(t)-1}\frac{d}{ds}h(t,\cdot)(t),$$

and since in the limit

$$\partial^{\alpha(t)}(h(t,\cdot))(t) = \int_0^t \delta_0(t-s)\frac{d}{ds}h(t,s)\,ds = \frac{d}{ds}h(t,s)|_{s=t} = 0, \qquad t > 0,$$

both equalities (3.1) and (3.2) boil into the usual Leibniz rule

$$(f(t)g(t))' = f(t)g'(t) + f'(t)g(t), \qquad t > 0,$$

as conjectured above.

– Finally, in spite of the inequality

$$\partial^\alpha \left(f^2(t) \right) \le 2f(t)\partial^\alpha \left(f(t) \right), \qquad t > 0, \qquad \text{(which implies } f(t)\partial^\alpha f(t) \ge 0)$$

has been proven in [2] from the equality (3.3) for the case of fractional derivatives of constant order $\alpha > 0$, in the case of the derivatives with time variable order considered here, the inequality

$$\partial^{\alpha(t)} \left(f^2(t) \right) \le 2f(t)\partial^{\alpha(t)} \left(f(t) \right), \tag{3.4}$$

is not satisfied in general. Only if we can guarantee that $\partial^{\alpha(t)}(h(t, \cdot))(t) - k(t)f(t)g(t) \le 0$, (or simply $\partial^{\alpha(t)}(h(t, \cdot))(t) = 0$ depending on the definition one adopts) for $t > 0$, the inequality (3.4) could become valid, although this is not obvious at all for a so general class of functions $\alpha(t)$ we are here considering, i.e., $\alpha(t)$ under the hypotheses made in Section 2.

From Theorem 1 several integration–by–parts formulas for FDVO can be straightforwardly derived. We show two of such formulas in the following corollaries.

Corollary 2. *Let f, g be two functions belonging to $W^{1,1}([0, +\infty))$. According to the Definition (2.3) we have the following integration–by–parts formula for FDVO*

$$\partial^{-\alpha(t)} \left(\partial^{\alpha(t)}f(t) \cdot g(t) \right) = f(t)g(t) - \partial^{-\alpha(t)} \left(f(t) \cdot \partial^{\alpha(t)}g(t) \right) + \partial^{-\alpha(t)} \left(k(t)f(t)g(t) \right), \quad t > 0, \tag{3.5}$$

and according to the Definition (2.6)

$$\partial^{-\alpha(t)} \left(\partial^{\alpha(t)}f(t) \cdot g(t) \right) = f(t)g(t) - f(0)g(0) - \partial^{-\alpha(t)} \left(f(t) \cdot \partial^{\alpha(t)}g(t) \right) + h(t, 0), \quad t > 0, \tag{3.6}$$

where $k(t)$ is in both cases the kernel defined in (2.5).

Proof. Let φ be a function belonging to $L^1([0, +\infty))$.

First of all, we show an equality that will be useful for the proof. In fact if we consider the Definition (2.3), then applying the Laplace transform we straightforwardly have that

$$\partial^{-\alpha(t)} \left(\partial^{\alpha(t)}\varphi(t) \right) = \partial^{-\alpha(t)} \left(\frac{\mathrm{d}}{\mathrm{d}t} \partial^{\alpha(t)-1}\varphi(t) \right) = \frac{1}{2\pi\mathrm{i}} \int_\Gamma \mathrm{e}^{tz} \mathcal{L}(\varphi)(z)\,\mathrm{d}z = \varphi(t), \qquad t \ge 0. \tag{3.7}$$

Note that this equality means that $\partial^{-\alpha(t)}$ is the inverse operator of $\partial^{\alpha(t)}$. Moreover if $\alpha(t) \equiv 1$ and $\varphi(0) = 0$, then this equality coincides with the classical one.

In the same manner if one takes the definition (2.6) there is a subtle difference, in fact applying again the Laplace transform we have

$$
\begin{aligned}
\partial^{-\alpha(t)} \left(\partial^{\alpha(t)}\varphi(t) \right) &= \partial^{-\alpha(t)} \left(\partial^{\alpha(t)-1}\varphi'(t) \right) \\
&= \frac{1}{2\pi\mathrm{i}} \int_\Gamma \mathrm{e}^{tz} \mathcal{L}(\varphi')(z)\,\mathrm{d}z \\
&= \frac{1}{2\pi\mathrm{i}} \int_\Gamma \mathrm{e}^{tz} \left(\mathcal{L}(\varphi)(z) - \frac{h(0)}{z} \right) \mathrm{d}z \\
&= \varphi(t) - \varphi(0), \qquad t \ge 0.
\end{aligned}
\tag{3.8}
$$

This formula perfectly matches with the classical one.

Now consider Definition (2.3). Thanks to Theorem 1 and equality (3.7), we have

$$
\begin{aligned}
\partial^{-\alpha(t)} \left(\partial^{\alpha(t)}f(t) \cdot g(t) \right) &= \partial^{-\alpha(t)} \left(\partial^{\alpha(t)} \left(f(t)g(t) \right) \right) - \partial^{-\alpha(t)} \left(f(t) \cdot \partial^{\alpha(t)}g(t) \right) \\
&\quad - \partial^{-\alpha(t)} \left(\partial^{\alpha(t)}(h(t, \cdot))(t) \right) + \partial^{-\alpha(t)} \left(k(t)f(t)g(t) \right) \\
&= f(t)g(t) - h(t, t) - \partial^{-\alpha(t)} \left(f(t) \cdot \partial^{\alpha(t)}g(t) \right) + \partial^{-\alpha(t)} \left(k(t)f(t)g(t) \right),
\end{aligned}
$$

and the equality (3.5) follows (recall that $h(t, t) = 0$, for $t \ge 0$).

In the same fashion (3.6) is obtained. □

Notice that, by the same arguments as in Theorem 1, as $\alpha(t)$ tends to the constant function $\alpha(t) \equiv 1$, the kernel $k(t)$ tends in the distributional sense to Dirac's delta distribution $\delta_0(t)$, and therefore the term $\partial^{-\alpha(t)}\left(k(t)f(t)g(t)\right)$ tends to $\delta_0(t)f(0)g(0) = 0$ therefore the equality stated in Corollary 2 reads

$$\int_0^t f(s)g'(s)\,\mathrm{d}s = f(t)g(t) - \int_0^t f'(t)g(s)\,\mathrm{d}s, \qquad t > 0. \tag{3.9}$$

Therefore, if $f(0)g(0) = 0$ (in particular if $f(0) = g(0) = 0$), then (3.9) is coherent with the classical integration−by−parts formula.

However according to the Definition (2.6) although $f(0) = g(0) = 0$, the formulation is not coherent with classical case since $h(t, 0) \neq 0$.

An alternative form of the integration−by−parts formula for FDVO in given in the following corollary. This time the corollary is stated only for the Definition (2.3) since we did not find the related formula for the Definition (2.6) coherent with the classical formula i.e. for $\alpha(t) \equiv 1$.

Corollary 3. *Let f, g be two functions belonging to $W^{1,1}([0, +\infty))$. According to the Definition (2.3) we have the following integration−by−parts formula for FDVO*

$$\int_0^T \partial^{\alpha(t)}f(t) \cdot g(t)\,\mathrm{d}t = -\int_0^T f(t) \cdot \partial^{\alpha(t)}g(t) + \int_0^T \left(k(T-t) - k(t)\right)f(t)g(t)\,\mathrm{d}t + \int_0^T k(T-t)h(t, T)\,\mathrm{d}t, \tag{3.10}$$

for $t > 0$, where $k(t)$ is the kernel defined in (2.5).

Proof. By Theorem 1, one has

$$\int_0^T \partial^{\alpha(t)}f(t) \cdot g(t)\,\mathrm{d}t = \int_0^T \partial^{\alpha(t)}(fg)(t)\,\mathrm{d}t - \int_0^T f(t) \cdot \partial^{\alpha(t)}g(t)\,\mathrm{d}t + \int_0^T \partial^{\alpha(t)}(h(t, \cdot))(t)\,\mathrm{d}t - \int_0^T k(t)f(t)g(t).$$

If one considers the Definition (2.3), then

$$\int_0^T \partial^{\alpha(t)}(fg)(t)\,\mathrm{d}t = \int_0^T \frac{\mathrm{d}}{\mathrm{d}t}\left(\partial^{\alpha(t)-1}(fg)(t)\right)\,\mathrm{d}t = \partial^{\alpha(t)-1}(fg)(t)\Big|_0^T$$

$$= \partial^{\alpha(t)-1}(fg)(T) = \int_0^T k(T-t)f(t)g(t)\,\mathrm{d}t,$$

and since

$$\int_0^T \partial^{\alpha(t)}(h(t, \cdot))(t)\,\mathrm{d}t = \int_0^T \frac{\mathrm{d}}{\mathrm{d}t}\partial^{\alpha(t)-1}(h(t, \cdot))(t)\,\mathrm{d}t = \partial^{\alpha(t)-1}h(t, T) = \int_0^T k(T-t)h(t, T)\,\mathrm{d}t,$$

the equality (3.10) straightforwardly follows. □

The equality (3.10) turns out to be coherent with the classical integration−by−parts formula as $\alpha(t)$ tends to the function 1 since as above $k(t)$ tends in the distributional sense to Dirac's delta distribution $\delta_0(t)$ and therefore,

$$\int_0^T \left(k(T-t) - k(t)\right)f(t)g(t)\,\mathrm{d}t \qquad \text{tends to} \qquad f(T)g(T) - f(0)g(0),$$

and

$$\int_0^T k(T-t)h(t, T)\,\mathrm{d}t \qquad \text{tends to} \qquad h(T, T) = 0.$$

In that case, that is if $\alpha(t)$ tends to the constant function $\alpha(t) \equiv 1$, then we have again the classical integration−by−parts formula

$$\int\limits_0^T f'(t) \cdot g(t)\, \mathrm{d}t = f(T)g(T) - f(0)g(0) - \int\limits_0^T f(t) \cdot g'(t).$$

4 On the initial data for fractional ODE's

4.1 Formulation

In this section, we formulate precisely the sub−diffusion initial value problem with time dependent order fractional derivatives with particular attention to the choice of the initial data. In fact, we here discuss how the initial data of the value problem has to be chosen in coherence with the regularity of the solution and the variable order of the fractional derivatives in the equation.

Another issue is the time regularity of the solution of the fractional initial value problems with non constant order. In the case of constant order $\alpha(t) = \alpha$ the regularity has been already studied [6] in the context of the numerical solutions to such a kind of problems. Several properties of the fractional diffusion equation with time dependent order, including the time regularity, have been already studied in [8] if $1 < \alpha(t) < 2$. Here, we show the precise form of the non regular part of the solution as $t \to 0^+$ in the case of $0 < \alpha(t) < 1$. This is a crucial subject for instance when discretizing in time such a kind of equations since this lack of regularity restricts the order of convergence of the numerical schemes.

We conclude this section with the study of the asymptotic behavior of the solution of such a initial value problems.

For the shortness of the presentation and since no relevant differences arise on the proofs, in this section we solely focus on the FDVO given by (2.3). Therefore consider the abstract time varying order fractional differential equation

$$\partial^{\alpha(t)} u(t) = Au(t) + f(t), \qquad t > 0, \tag{4.1}$$

where A is a linear operator, f stands for a suitable source term, and where we adopt the definition (2.3) for the FDVO.

First of all by the definition (2.3)−(2.5) we have that

$$\partial^{\alpha(t)} u(t) = \frac{\mathrm{d}}{\mathrm{d}t}\left(\partial^{\alpha(t)-1} u(t)\right) = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2\pi \mathrm{i}} \int\limits_\Gamma \mathrm{e}^{tz} K(z) U(z)\, \mathrm{d}z\right), \tag{4.2}$$

where $U(z)$ represents the Laplace transform of $u(t)$, and $K(z)$ is defined by (2.5). Therefore, the Laplace transform in both (4.2) leads to

$$\mathcal{L}(\partial^{\alpha(t)} u(\cdot))(z) = zK(z)U(z) - \left.\partial^{\alpha(t)-1} u(t)\right|_{t=0}.$$

In the same manner the Laplace transform in both sides of (4.1) leads to

$$zK(z)U(z) - \left.\partial^{\alpha(t)-1} u(t)\right|_{t=0} = AU(z) + \tilde{f}(z), \qquad z \in \mathcal{D}, \tag{4.3}$$

where $\tilde{f}(z)$ stands for the Laplace transform of $f(t)$, and $\mathcal{D}$ for a complex domain in $\mathbb{C}/S_\theta$ according to the choice of θ below. Though that the resolvent $(zK(z)-A)^{-1}$ of the operator A exists (as it will happen according to the functional setting stated below) we have

$$U(z) = (zK(z) - A)^{-1}\left(\left.\partial^{\alpha(t)-1} u(t)\right|_{t=0} + \tilde{f}(z)\right). \tag{4.4}$$

Expressions (4.3) or (4.4) actually suggest that the initial data for (4.1) must be

$$\partial^{\alpha(t)-1} u(t)\Big|_{t=0} = u_0, \tag{4.5}$$

which will be consistent with the regularity of $u(t)$ stated in Theorem 4 below, and advances the lack of regularity of $u(t)$ at $t = 0$.

Now, we are in a position to formulate precisely the fractional sub–diffusion initial value problem with non constant order we study in this section. Here, we opt for a very general/abstract framework which is the one of the linear sectorial operators. In fact, consider

$$\begin{cases} \partial^{\alpha(t)} u(t) &= Au(t) + f(t), \qquad t > 0, \\ \partial^{\alpha(t)-1} u(t)\Big|_{t=0} &= u_0, \qquad \text{(the initial data)}, \end{cases} \tag{4.6}$$

where $f \in L^1([0, +\infty), X)$, X stands for a complex Banach space, and the linear operator $A : \mathcal{D}(A) \subset X \to X$ is a θ–sectorial operator, for $0 < \theta < \pi/2$ in X. Let us recall that a linear and closed operator A is θ–sectorial if its resolvent $R(z) = (zI - A)^{-1}$ (where I is the identity operator in X) is analytic outside the sector S_θ in the sense of (2.8) with $\beta = 1$, i.e. there exists $M > 0$, such that $R(z)$ is analytic outside the sector

$$S_\theta := \{z \in \mathbb{C} : |\arg(-z)| < \theta\}, \qquad \text{and satisfies} \qquad \|(zI - A)^{-1}\|_{X \to X} \le \frac{M}{|z|}, \quad z \notin S_\theta.$$

For the sake of the simplicity of the presentation, and without loss of generality, as we have discussed in Section 2 we consider a non shifted sector S_θ (according to the notation in Section 2 $a = 0$). Moreover if not confusion, the we denote by simplicity $\| \cdot \|$ instead of $\| \cdot \|_{X \to X}$.

Note that a lot of linear operators fit in that general framework, e.g. complex scalars, finite dimensional operators like matrices (e.g. the ones coming out from most of the spatial discretization of elliptic operators), or infinite dimensional operators (like Laplacian Δ, or fractional powers of the Laplacian $-(-\Delta)^\beta$ with $\beta > 0$), and so on.

Moreover, the assumptions [H0]–[H2] arising from the definition of FDVO we have opted for, force us to make additional assumptions now for A. In fact assume

[H3]. A is a θ–sectorial operator such that the sectoriality angle θ obeys the Hypothesis [H2].

4.2 Regularity

This section is devoted to study the regularity of the solution of (4.6) under the hypotheses [H0]–[H3] stated in previous sections, in particular the statement of the theorem presented in this section shows how far goes the lack of regularity at $t = 0$ of the mild solution of (4.6) or in other words how is the structure of the singularity of the solution as $t \to 0^+$.

Under the Hypotheses [H0]–[H3], the resolvent $(zI - A)^{-1}$ exists and the equality (4.4) is now meaningful, for $z \notin S_\theta$. But even more, the same hypotheses unable us to apply the Bromwich formula for the inverse Laplace transform to obtain a closed form of the mild solution of (4.6)

$$u(t) = \frac{1}{2\pi i} \int_\Gamma e^{zt} (zK(z) - A)^{-1} (u_0 + \tilde{f}(z)) \, dz = \frac{1}{2\pi i} \int_\Gamma e^{zt} (z^{z\tilde{\alpha}(z)} - A)^{-1} (u_0 + \tilde{f}(z)) \, dz, \qquad t > 0, \tag{4.7}$$

where Γ is a suitable complex path connecting $-i\infty$ and $+i\infty$ with increasing imaginary part. We should not get confused with the complex path Γ considered in (2.3)-(2.5) and the one in (4.7) in spite of the underlaying ideas to define one of these paths are the same in both cases. Let us define for (4.7) one of these complex paths that is suitable for us, in particular according to Hypotheses [H0]–[H3] one may define Γ lying outside the sector S_θ as $\Gamma = \Gamma_1 \cup \Gamma_2$ where

$$\Gamma_1 : \gamma_1(\rho) \quad := \quad \rho\, e^{\pm i\Psi}, \ \rho \ge R, \quad \text{for} \quad R = \frac{1}{t}, \ \text{(for each } t > 0) \ (\pm \text{ means upper and lower branches)}. \tag{4.8}$$

$$\Gamma_2 : \gamma_2(\varphi) \quad := \quad R\, e^{i\varphi}, \quad -\Phi \le \varphi \le \Phi, \quad \text{with} \quad \Phi := \frac{(1 - C_1)(\pi - \theta)}{m_2}. \tag{4.9}$$

It is straightforward to see that the path Γ defined by (4.8)–(4.9) satisfies the following properties: $zK(z) \notin S_\theta$ for $z \in \Gamma$, Γ keeps outside of S_θ, and the real part of z is negative for all $z \in \Gamma_1$.

The representation (4.7) is now completely determined once determined the complex path (4.8)–(4.9), and the mild solution of (4.6) can be written in terms of the corresponding evolution operator, i.e. we can write

$$u(t) = E(t)u_0 + \int_0^t E(t - s)f(s)\,ds, \qquad \text{where} \qquad E(t) := \frac{1}{2\pi i} \int_\Gamma e^{tz}(z^{z\tilde{\alpha}(z)} - A)^{-1}\,dz, \quad t > 0, \tag{4.10}$$

where Γ is defined by (4.8)–(4.9). Note that it serves as a proof of the well posedness of (4.6).

Next theorem shows the regularity of the solution of (4.6) depending on the regularity of the initial data u_0, and source term $f(t)$.

Theorem 4. *Under the hypotheses [H0]–[H3] for A and $\alpha(t)$, if $u_0 \in \mathcal{D}(A^p)$ where $p := \max\{r \in \mathbb{Z}^+ : rm_1 - 1 < 0\}$, and $f \in C^2([0, T], X)$, then the mild solution of (4.6) can be written as*

$$u(t) = \sum_{(r,s) \in J} k_{r,s}(t) + v(t), \quad t > 0, \qquad J = \{(r, s) : r \in \mathbb{Z}^+, s \in \mathbb{Z}^+ \cup \{0\}, rm_1 - 1 + s < 0\}, \tag{4.11}$$

where, for each $(r, s) \in J$, there exists $C > 0$ (a generic constant which may change in each case) independent on t, such that

$$\|k_{r,s}(t)\| \le Ct^{rm_1 - 1 + s}, \quad t \to 0^+, \tag{4.12}$$

and $v \in C([0, T], X)$.

Proof. The proof makes use of the superposition principle with the cases $f \equiv 0$ first, then with $u_0 = 0$.

We first consider $f \equiv 0$, and define, for $(r, 0) \in J$,

$$K_{r,0}(z) = \frac{1}{2\pi i} \int_\Gamma e^{tz} \frac{1}{z^{rz\tilde{\alpha}(z)}} A^{r-1} u_0,$$

where Γ is the complex path (4.8)–(4.9). We now prove that $k_{r,0}(t) = \mathcal{L}^{-1}(K_{r,0})(t)$, for $t > 0$, are the functions provided by the statement of the theorem, and in particular that

$$u(t) - \sum_{(r,0) \in J} k_{r,0}(t),$$

is regular, for $t \ge 0$. To prove it let's we go back to the Laplace domain where we have

$$U(z) - \sum_{(r,0) \in J} K_{r,0}(z) \quad = \quad \frac{1}{2\pi i} \int_\Gamma e^{tz} \left\{ (z^{z\tilde{\alpha}(z)} - A)^{-1} u_0 - \sum_{(r,0) \in J} \frac{1}{z^{rz\tilde{\alpha}(z)}} A^{r-1} u_0 \right\} dz$$

$$= \quad \frac{1}{2\pi i} \int_\Gamma e^{tz} \frac{1}{z^{pz\tilde{\alpha}(z)}} (z^{z\tilde{\alpha}(z)} - A)^{-1} A^p u_0 \, dz.$$

By the sectoriality of A,

$$\left\| \frac{1}{z^{pz\tilde{\alpha}(z)}} (z^{z\tilde{\alpha}(z)} - A)^{-1} \right\| \le \frac{M}{\left| z^{(p+1)z\tilde{\alpha}(z)} \right|}$$

and defining the function $v(t)$ as follows,

$$v(t) := u(t) - \sum_{(r,0) \in J} k_{r,0}(t), \qquad t > 0,$$

we have by Lemma 2.9 that there satisfies,

$$\|v(t)\| = \left\| u(t) - \sum_{(r,0)\in J} k_{r,0}(t) \right\| = O\big(t^{(p+1)m_1 - 1}\big), \quad t \to 0^+.$$

Since by the definition of p, $(p+1)m_1 - 1 \geq 0$, the regularity of $v(t)$ is proven as said in the statement of the theorem. The bounds for each $k_{r,0}(t)$, for $(r,0) \in J$, straightforwardly follows from Lemma 2.9.

Moreover the contribution of remainder terms $k_{r,s}(t)$ ($s \neq 0$), i.e. the case $u_0 \equiv 0$ and $f \neq 0$, follows similar steps merely by having into account the regularity of $f(t)$, and consequently that $f(t)$ can be written as

$$f(t) = f(0) + f^{'}(0)t + R_f(t), \quad t > 0,$$

where $R_f(t)$ stands for the Taylor residual of f around $t = 0$.

Finally, let us show the coherence of (4.11) with the choice of the initial data. In fact, if we take the fractional integral of order $1 - \alpha(t)$ of $u(t)$ we have

$$\partial^{-(1-\alpha(t))} u(t) = \sum_{(r,s)\in J} \partial^{-(1-\alpha(t))} k_{r,s}(t) + \partial^{-(1\alpha(t))} v(t), \quad t > 0,$$

which, in the Laplace domain and according to the definition of $k_{r,s}(t)$, reads

$$\frac{U(z)}{z^{1-z\tilde{\alpha}(z)}} = \frac{1}{z^{1-z\tilde{\alpha}(z)}} \frac{u_0}{z^{z\tilde{\alpha}(z)}} + \sum_{(r,s)\in J-\{(1,0)\}} \frac{1}{z^{1-z\tilde{\alpha}(z)}} K_{r,s}(z) + \frac{V(z)}{z^{1-z\tilde{\alpha}(z)}},$$

whose inverse Laplace transform turn out to be zero at $t = 0$ for all terms in the left hand side, excepting the first one, which is exactly u_0 as expected. This concludes the proof. □

Note that the spatial regularity of u_0 is too demanding for practical cases, in fact if m_1 is close to 0 we may require a very high spatial regularity for u_0 which might be unacceptable in practical cases. However, a case particularly interesting is the case $m_1 = 1/2$ (see e.g. [9]) where the required regularity is $u_0 \in \mathcal{D}(A)$ which is commonly acceptable in practice. In the scalar case (i.e. A being merely a constant or even a matrix) this discussion is meaningless.

4.3 Asymptotic behavior

We conclude this section with the study of the asymptotic behavior of the solution of (4.6). First of all note that, the asymptotic behavior is determined merely by the evolution operator (4.10) of the problem, in other words it does not depend on u_0, therefore we state the following theorem in terms of the evolution operator $E(t)$.

Theorem 5. *Under the hypotheses [H0]–[H3] for A and $\alpha(t)$, there exists a constant $C > 0$, such that*

$$\|E(t)\|_{X\to X} \leq Ct^{m_2}, \quad as \quad t \to +\infty,$$

where $E(t)$ is the evolution operator given in (4.6), and m_2 is the constant stated in the Hypothesis [H1].

Proof. The proof consists of two parts: The behavior of $E(t)$ over Γ_1 and Γ_2 according to (4.8)–(4.9) respectively. Therefore, the evolution operator (4.10) reads

$$E(t) = I_1(t) + I_2(t) \quad \text{where} \quad I_j(t) := \frac{1}{2\pi i} \int_{\Gamma_j} e^{zt}(z^{z\tilde{\alpha}(z)} - A)^{-1} \, dz, \quad j = 1, 2$$

Assume that $t > 0$ is large enough since the asymptotic behavior as $t \to +\infty$ is the matter.

First case. If $z \in \Gamma_1$, then $z = \rho\, e^{\pm i\Phi}$, for $\rho \geq R = 1/t$. From the sectoriality property of A it follows

$$\|I_1(t)\| \leq \frac{1}{2\pi} \int_{\Gamma_1} |e^{zt}| \, \|(z^{z\tilde{\alpha}(z)} - A)^{-1}\| \, |\,dz| \leq \frac{M}{2\pi} \int_{\Gamma_1} \frac{|e^{zt}|}{|z^{z\tilde{\alpha}(z)}|} |\,dz|.$$

On the other hand by Hypothesis [H1] there exist $c_m, C_m > 0$ such that $c_m \leq \exp(-\arg(z)g_I(z)) \leq C_m$, for $z \in \Gamma_1$. Henceforth

$$|z^{z\tilde{\alpha}(z)}| = \exp(g_R(z)\ln|z| - \arg(z)g_I(z)) \geq \exp(m_2 \ln(1/t))c_m = \frac{c_m}{t^{m_2}}, \quad t > 0,$$

and

$$\|I_1(t)\| \leq \frac{Mt^{m_2}}{2\pi c_m} \int_{\Gamma_1} |e^{zt}| \, |\,dz| = \frac{CMt^{m_2}}{2\pi c_m}, \tag{4.13}$$

where, by the definition of Γ_1, $C = \int_{\Gamma_1} |e^{zt}| \, |\,dz| > 0$ is independent on t.

Second case. If $z \in \Gamma_2$, then $z = (1/t)\, e^{i\varphi}$, for $-\Phi \leq \varphi \leq \Phi$. With the same arguments as in the first case it is straightforward that

$$\|I_2(t)\| \leq \frac{Mt^{m_2}}{2\pi c_m} \int_{\Gamma_2} |e^{zt}| \, |\,dz| = \frac{CMt^{m_2}}{2\pi c_m}, \tag{4.14}$$

where, now by the definition of Γ_2, $C = \int_{\Gamma_2} |e^{zt}| \, |\,dz| > 0$ is again independent on t.

Bounds (4.13) and (4.14) end the proof. □

Here we have to highlight two relevant facts. The asymptotic behavior provided by the Theorem 5 for the solution of (4.6) might seem unexpected, instead a bounded behavior (or even exponential decay) might be the expected one, however it is the best one can get. The reason why is twofold: On the one hand the integrand of the evolution operator (4.10) cannot be analytically extended to $(-\infty, 0]$, even if it were possible to shift the sector S_θ to the left-hand complex plane, that is taking $-a + S_\theta$, for $a > 0$, instead of S_θ. If this were the case, then a slightly improved bound might be achieved but anymore. This fact has been already noticed in [7] in the case of fractional equations with constant order; And the second issue is the initial data in (4.6). In this respect one may thing of the integral counterpart of (4.6) as

$$u(t) = u_0 + \partial^{-\alpha(t)} A u(t), \quad t > 0, \tag{4.15}$$

(homogeneous by simplicity). In this case, and following the ideas in [8] (there with $1 < \alpha(t) < 2$), it can be proven that there exists a constant $C > 0$ such that

$$\|E(t)\|_{X \to X} \leq C, \quad t > 0,$$

where $E(t)$ stands for the evolution operator associated to (4.15). However (4.15) does not correspond to the integral equivalent formulation to (4.6) since both initial data do not correspond to each other. The true integral equivalent formulation to (4.6) actually reads as

$$u(t) = \partial^{1-\alpha(t)} u_0 + \partial^{-\alpha(t)} A u(t), \quad t > 0. \tag{4.16}$$

It is straightforward to show applying the fractional derivative $\partial^{\alpha(t)}$ in both sides of (4.16) that equation in (4.6) satisfies, and applying the fractional integral $\partial^{\alpha(t)-1}$ also in both sides of (4.16), at $t = 0$, the initial condition in (4.6) satisfies as well. The evolution operator $E(t)$ associated to (4.15) now applies to $\partial^{1-\alpha(t)} u_0$ by mean of the variation of constants formula, and having in mind that this evolution operator is in fact bounded the statement of Theorem 5 is straightforwardly confirmed.

Secondly, another issue to be emphasized is that the solution of (4.6) with $f \equiv 0$ is given by $u(t) = E(t)u_0$, $E(t)$ defined in (4.10), and it follows by Theorem 5 that

$$\|u(t)\| \leq Ct^{m_2}, \quad t \to +\infty,$$

whatever the initial data u_0 is merely belonging to X, i.e. without any regularity requirement.

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