Error analysis of projection methods for non inf-sup stable mixed finite elements. The Navier-Stokes equations.

Javier de Frutos* Bosco García-Archilla† Julia Novo‡

Abstract

We obtain error bounds for a modified Chorin-Teman (Euler non-incremental) method for non inf-sup stable mixed finite elements applied to the evolutionary Navier-Stokes equations. The analysis of the classical Euler non-incremental method is obtained as a particular case. We prove that the modified Euler non-incremental scheme has an inherent stabilization that allows the use of non inf-sup stable mixed finite elements without any kind of extra added stabilization. We show that it is also true in the case of the classical Chorin-Temam method. The relation of the methods with the so called pressure stabilized Petrov Galerkin method (PSPG) is established. We do not assume non-local compatibility conditions for the solution.

Keywords Projection methods, non inf-sup stable elements, Navier-Stokes equations, PSPG stabilization.

*Instituto de Investigación en Matemáticas (IMUVA), Universidad de Valladolid, Spain. Research supported under grants MTM2013-42538-P (MINECO, ES) and MTM2016-78995-P (AEI/FEDER, UE). frutos@mac.uva.es

†Departamento de Matemática Aplicada II, Universidad de Sevilla, Sevilla, Spain. Research under grant MTM2015-65608-P (MINECO, ES). bosco@esi.us.es

‡Departamento de Matemáticas, Universidad Autónoma de Madrid, Instituto de Ciencias Matemáticas CSIC-UAM-UC3M-UCM, Spain. Research supported under grants MTM2013-42538-P (MINECO, ES) and MTM2016-78995-P (AEI/FEDER, UE). julia.novo@uam.es

1 Introduction

We analyze a modified Chorin-Teman (Euler non-incremental) projection method for non inf-sup stable mixed finite elements with a pressure space $Q_h \subset H^1(\Omega)$. As a particular case we obtain the analysis of the classical Chorin-Temam method. We prove that both the modified and the standard methods have an inherent stabilization of PSPG type that allow the use of non inf-sup stable mixed finite elements without any kind of extra added stabilization. This result was known in the literature, see for example [11], but to our knowledge there were no available error bounds for the case of non inf-sup stable elements (see below for related results in [3]). In reference [9] we considered the case of the transient Stokes equations assuming enough regularity for the solution. In the present paper the analysis is applied to the evolutionary Navier-Stokes equations without assuming non-local compatibility conditions. We consider a explicit treatment of the nonlinear convection term since this is easier to implement in practice, although our analysis, if slightly modified, also covers the case of a fully implicit treatment of the nonlinear term. The analysis of the Chorin-Temam method holds under condition $\Delta t \geq Ch^2$ (and assuming also $\Delta t = O(h)$) which is in agreement with the error bounds in [3] where the authors prove error bounds for the Euler non-incremental scheme for LBB stable elements assuming also $\Delta t \geq Ch^2$. This result is also in agreement with the fact that had been observed in the literature that the standard Euler non-incremental scheme provides computed pressures that behave unstably for Δt small and fixed h if non inf-sup stable elements are used, see [7]. With our error analysis we clarify this question since we show that, when $\Delta t \to 0$, the inherent PSPG stabilization of the method disappears. On the other hand, for the modified Euler non-incremental method that we propose, the PSPG stabilization does not disappear when $\Delta t \to 0$, which allows to use Δt as small as desired in this modified method. Our results are also in agreement with the classical results for the continuous in space Euler non-incremental method (see for example [12]) since we prove that the rate of convergence in terms of Δt in the L^2 norm of the velocity is one and the rate of convergence in the H^1 norm of the velocity and the L^2 norm of the pressure is one half.

It is well-known (see e.g., [15, Corollary 2.1]) that the solution of the Navier-Stokes equations, no matter how smooth the initial velocity and the forcing term are, cannot be expected to have third spatial derivatives bounded up to $t = 0$, unless certain nonlocal compatibility conditions (which are difficult to check in practice and cannot be realistically assumed) are satisfied. For the pressure, the same can be said for second spatial derivatives. To cope with this fact in this paper we obtain error bounds that do not require the above-mentioned compatibility conditions to be satisfied.

Of course, the Chorin-Temam projection method is well known and this is not the first paper where the analysis of this method is considered. The analysis of the semidiscretization in time is carried out in [23], [24], [22], [21], [14]. In [7] the stability of the Chorin-Temam projection method is considered and, in case of non inf-sup stable mixed finite elements, some a priori bounds for the approximations to the velocity and pressure are obtained but no error bounds are proven for this method. In [3] the Chorin-Teman method is considered together with both non inf-sup stable and inf-sup stable mixed finite elements. In case of using non inf-sup stable mixed finite elements a local projection type stabilization is required in [3] to get the error bounds of the method. Both in the present paper and in [9], however, we get optimal error bounds without any extra stabilization for non inf-sup stable mixed finite elements.

For the Euler incremental scheme the analysis of the semidiscretization in time can be found in [21]. The Euler incremental scheme with a spatial discretization based on inf-sup stable mixed finite elements is analyzed in [13]. To our knowledge there is no error analysis for this method in case of using non-inf-sup stable elements other than the one in [9]. Some stability estimates can be found in [7] for the method with added stabilization terms more related to local projection stabilization than to the PSPG stabilization we consider in the present paper. A stabilized version of the incremental scheme is also proposed in [20] although no error bounds are proved. Finally, for an overview on projection methods we refer the reader to [12].

Being the Chorin-Temam projection method an old one, it has seen the appearance of many alternative methods during the years, many of which possess better convergence properties. The purpose of the present paper is not to discuss its advantages of disadvantages with respect to newer methods, but just to analyze the method when used in combination with non inf-sup stable elements, a task not fully carried out in the previous literature.

The outline of the paper is as follows. In the first section we introduce some notation. In Section 3 we state some results about a stabilized Stokes approximation that was introduced in [9]. In Section 4 we get the error analysis of the method for the transient Stokes equations. Finally, in the last section we prove the error bounds for the method for the Navier-Stokes equations. The analysis is based on a stability plus consistency arguments with stability restricted to h-dependent thresholds and is strongly based on the results for the transient Stokes equations obtained in Section 4.

2 Preliminaries and notation

Throughout the paper, standard notation is used for Sobolev spaces and corresponding norms. In particular, given a measurable set $\omega \subset \mathbb{R}^d$, $d = 2, 3$, its Lebesgue measure is denoted by $|\omega|$, the inner product in $L^2(\omega)$ or $L^2(\omega)^d$ is denoted by $(\cdot,\cdot)_\omega$ and the notation $(\cdot,\cdot)$ is used instead of $(\cdot,\cdot)_\Omega$. The semi norm in $W^{m,p}(\omega)$ will be denoted by $|\cdot|_{m,p,\omega}$ and, following [8], we define the norm $\|\cdot\|_{m,p,\omega}$ as

$$\|f\|^p_{m,p,\omega} = \sum_{j=0}^{m} |\omega|^{\frac{p(j-m)}{d}} |f|^p_{j,p,\omega} \,,$$

so that $\|f\|_{m,p,\omega} |\omega|^{\frac{m}{d}-\frac{1}{p}}$ is scale invariant. We will also use the conventions $\|\cdot\|_{m,\omega} = \|\cdot\|_{m,2,\omega}$ and $\|\cdot\|_m = \|\cdot\|_{m,2,\Omega}$. As it is usual we will use the special notation $H^s(\omega)$ to denote $W^{s,2}(\omega)$ and we will denote by $H^1_0(\Omega)$ the subspace of functions of $H^1(\Omega)$ satisfying homogeneous Dirichlet boundary conditions. Finally, $L^2_0(\Omega)$ will denote the subspace of function of $L^2(\Omega)$ with zero mean.

Let us denote by $\mathcal{T}_h$ a triangulation of the domain Ω, which, for simplicity, is assumed to have a Lipschitz polygonal boundary. On $\mathcal{T}_h$, we consider the finite element spaces $V_h \subset V = H^1_0(\Omega)^d$ and $Q_h \subset L^2_0(\Omega) \cap H^1(\Omega)$ based on local polynomials of degree k and l respectively. Equal degree polynomials for velocity and pressure are allowed. In the sequel it will be assumed that the family of meshes are regular.

Concerning the discretization, we shall assume that the family of meshes is quasi-uniform that is for a constant $\Lambda \geq 1$, the following inequality holds

$$h/h_K \leq \Lambda, \qquad \forall K \in \mathcal{T}_h, \tag{1}$$

where h_K is the diameter of the $K \in \mathcal{T}_h$ and $h = \max_{K \in \mathcal{T}_h} h_K$.

We shall also assume that the triangulations are regular enough so that for a constant $c_{\rm inv} > 0$ the following inequality holds for each $v_h \in V_h$, see e.g., [6, Theorem 3.2.6],

$$\|\mathbf{v}_h\|_{W^{m,p}(K)} \leq c_{\rm inv} h_K^{l-m-d\left(\frac{1}{q}-\frac{1}{p}\right)} \|\mathbf{v}_h\|_{W^{l,q}(K)}, \tag{2}$$

where $0 \le l \le m \le 1$, $1 \le q \le p \le \infty$, and h_K is the size (diameter) of the mesh cell $K \in \mathcal{T}_h$.

We will denote by $I_h\mathbf{u} \in V_h$ the Lagrange interpolant of a continuous function $\mathbf{u}$. The following bound can be found in [6, Theorem 3.1.6]

$$|\mathbf{u} - I_h\mathbf{u}|_{W^{m,p}(K)^d} \le c_{\rm int} h^{l'-m-d\left(\frac{1}{q}-\frac{1}{p}\right)} |\mathbf{u}|_{W^{l',q}(K)^d}, \quad 0 \le m \le 1 \le l', \tag{3}$$

where $l' > d/q$ when $1 < q \le \infty$ and $l' \ge d$ when $q = 1$.

Let λ be the smallest eigenvalue of $A = -\Delta$ subject to homogeneous Dirichlet boundary conditions, Δ being the Laplacian operator in Ω. Then, it is well-known that there exists a scale-invariant positive constant c_{-1} such that

$$\|\mathbf{v}\|_{-1} \le c_{-1}\lambda^{-1/2} \|\mathbf{v}\|_0, \qquad \mathbf{v} \in L^2(\Omega)^d, \tag{4}$$

and, also,

$$\|\mathbf{v}\|_0 \le \lambda^{-1/2} \|\nabla \mathbf{v}\|_0, \qquad \mathbf{v} \in H_0^1(\Omega)^d,$$

this last inequality also known as the Poincaré inequality. As a consequence of the above, there exist a scale-invariant constant $c_P > 0$ such that

$$\|\mathbf{v}\|_1 \le c_P \|\nabla \mathbf{v}\|_0, \qquad \mathbf{v} \in H_0^1(\Omega)^d, \tag{5}$$

We will use the following well-known inequalities:

i) Sobolev's inequality, [1]: For $s > 0$ there exist a scale-invariant constant $c_s >$ such that for $p \in [1, \infty)$ satisfying $\frac{1}{p} \ge \frac{1}{2} - \frac{s}{d}$, the following inequality holds

$$\|v\|_{L^p(\Omega)} \le c_s |\Omega|^{\frac{s}{d}-\frac{1}{2}+\frac{1}{p}} \|v\|_s, \qquad v \in H^s(\Omega). \tag{6}$$

For $p = \infty$, the relation is valid if $0 > \frac{1}{2} - \frac{s}{d}$.

ii) Agmon's inequality,

$$\|v\|_\infty \le c_{\rm A} \begin{cases} \|v\|_0^{1/2} \|v\|_2^{1/2}, & d = 2, \\ \|v\|_1^{1/2} \|v\|_2^{1/2}, & d = 3, \end{cases} \qquad v \in H^2(\Omega). \tag{7}$$

The case $d = 2$ is a direct consequence of [2, Theorem 3.9]. For $d = 3$, a proof for domains of class C^2 can be found in [8, Lemma 4.10], but thanks to the Calderón extension theorem (see e.g., [1, Theorem 4.32] the proof is valid for bounded Lipschitz domains.

iii) The following version of Hölder's inequality

$$\left|\int_\Omega v_1 v_2 v_3 \, dx\right| \le \|v_1\|_{L^{p_1}(\Omega)} \|v_2\|_{L^{p_2}(\Omega)} \|v_3\|_{L^{p_3}(\Omega)}, \qquad \frac{1}{p_1}+\frac{1}{p_2}+\frac{1}{p_3}=1. \tag{8}$$

We shall frequently apply this inequality with $p_1 = 2$, $p_2 = 2d/(d-1)$ and $p_3 = 2d$, or $p_1 = \infty$, and $p_2 = p_3 = 2$.

iv) The following inequality

$$\|v\|_{L^{\frac{2d}{d-1}}(\Omega)} \le c_1^{1/2} \|v\|_0^{1/2} \|\nabla v\|_0^{1/2}, \qquad v \in H^1(\Omega). \tag{9}$$

which is a consequence of Sobolev's inequality and the convexity inequality (see e. g., [10, § II.1]).

All previous inequalities are also valid for vector-valued functions.

3 A Stabilized Stokes approximation

Let us consider the Stokes problem

$$\begin{aligned} -\nu\Delta \mathbf{s} + \nabla z &= \hat{\boldsymbol{g}}, \quad \text{in} \quad \Omega \\ \nabla \cdot \mathbf{s} &= 0, \quad \text{in} \quad \Omega \\ \mathbf{s} &= \mathbf{0}, \quad \text{in} \quad \partial\Omega. \end{aligned} \tag{10}$$

As in [9] we define the stabilized Stokes approximation to (10) as the mixed finite element approximation $(\mathbf{s}_h, z_h) \in (V_h, Q_h)$ satisfying

$$\nu(\nabla \mathbf{s}_h, \nabla \boldsymbol{\chi}_h) + (\nabla z_h, \boldsymbol{\chi}_h) = (\hat{\boldsymbol{g}}, \boldsymbol{\chi}_h), \quad \forall \boldsymbol{\chi}_h \in V_h, \tag{11}$$

$$(\nabla \cdot \mathbf{s}_h, \psi_h) = -\delta(\nabla z_h, \nabla \psi_h), \quad \forall \psi_h \in Q_h, \tag{12}$$

where δ is a constant parameter. Observe that from (10) and (11) it follows that the errors $\mathbf{s}_h - \mathbf{s}$ and $z_h - z$ satisfy that

$$\nu(\nabla(\mathbf{s}_h - \mathbf{s}), \nabla \boldsymbol{\chi}_h) + (\nabla(z_h - z), \boldsymbol{\chi}_h) = 0, \qquad \forall \boldsymbol{\chi}_h \in V_h. \tag{13}$$

From now on we will use C to denote a generic non-dimensional constant.

We now state two lemmas that will be used in the sequel. The proof of the following lemma can be found in [4, Lemma 3], see also [17, Lemma 2.1].

Lemma 1 *For $\psi_h \in Q_h$ it holds*

$$\|\psi_h\|_0 \le Ch\|\nabla\psi_h\|_0 + C \sup_{\boldsymbol{\chi}_h\in V_h} \frac{(\psi_h, \nabla\cdot\boldsymbol{\chi}_h)}{\|\boldsymbol{\chi}_h\|_1}.$$

Lemma 2 *There exist a constant $C>0$ such that for any $\mathbf{v}\in H_0^1(\Omega)^d$ with* $\mathrm{div}(\mathbf{v})=0$, *$q\in\ L_0^2(\Omega)$, $\mathbf{v}_h\in V_h$ and $q_h\in Q_h$ satisfying*

$$\nu(\nabla(\mathbf{v}_h-\mathbf{v}),\nabla\boldsymbol{\chi}_h)+(\nabla(q_h-q),\boldsymbol{\chi}_h)=0, \qquad \forall\boldsymbol{\chi}_h\in V_h, \tag{14}$$

$$(\nabla\cdot(\mathbf{v}_h-\mathbf{v}),\psi_h)+\delta(\nabla q_h,\nabla\psi_h)=0, \qquad \forall\psi_h\in Q_h, \tag{15}$$

the following bounds hold:

$$\nu^{1/2}\left\|\nabla\mathbf{v}_h\right\|_0+\delta^{1/2}\left\|\nabla q_h\right\|_0 \le C\big(\nu^{1/2}\left\|\nabla\mathbf{v}\right\|_0+\nu^{-1/2}\left\|q\right\|_0\big), \tag{16}$$

$$\left\|\mathbf{v}_h-\mathbf{v}\right\|_0 \le C\left(h\left(\left\|\nabla(\mathbf{v}-\mathbf{v}_h)\right\|_0+\nu^{-1}\left\|q-q_h\right\|_0\right)+\delta\left\|\nabla q_h\right\|_0\right). \tag{17}$$

Proof Observe that since $\mathrm{div}(\mathbf{v})=0$, relation (15) can be written in the form $(\nabla\cdot\mathbf{v}_h,\psi_h)+\delta(\nabla q_h,\nabla\psi_h)=0$, for all $\psi_h\in Q_h$. Then taking $\psi_h=q_h$ in this relation, $\boldsymbol{\chi}_h=\mathbf{v}_h$ in (14) and summing both equations, the bound (16) easily follows. The proof of (17) can be found in [9, Lemma 2]. □

In the sequel we will assume

$$\frac{1}{\nu\rho_1^2}h^2 \le \delta, \tag{18}$$

for a positive constant ρ_1. The following bounds hold for the stabilized Stokes approximation solving (11)-(12) assuming condition (18) holds, see [9].

$$\begin{aligned}
\nu^{1/2}\|\nabla(\mathbf{s}-\mathbf{s}_h)\|_0+\delta^{1/2}\|\nabla(z-z_h)\|_0 & \\
\le C\frac{h}{\nu^{1/2}}(\nu\|\mathbf{s}\|_2+\|z\|_1)+C\delta^{1/2}\|z\|_1,& \\
\|z-z_h\|_0 \le Ch(\nu\|\mathbf{s}\|_2+\|z\|_1)+C(\nu\delta)^{1/2}\|z\|_1 & \\
\|\mathbf{s}-\mathbf{s}_h\|_0 \le C\frac{h^2}{\nu}(\nu\|\mathbf{s}\|_2+\|z\|_1)+C\delta\|z\|_1. &
\end{aligned} \tag{19}$$

3.1 A priori bounds for the stabilized Stokes approximation

We will get some a priori bounds for the stabilized Stokes approximation that will be needed in the sequel. They are a consequence of Lemma 2. In fact applying this result with $\mathbf{v}_h = \mathbf{s}_h$ and $q_h = z_h$, (16) implies that

$$\nu\|\nabla \mathbf{s}_h\|_0^2 + \delta\|\nabla z_h\|_0^2 \leq C\big(\nu\|\nabla \mathbf{s}\|_0^2 + \nu^{-1}\|z\|_0^2\big). \tag{20}$$

Similarly, from (17) it follows that

$$\begin{aligned}
\|\mathbf{s}_h - \mathbf{s}\|_0 &\leq C\left(h\left(\|\nabla(\mathbf{s}_h - \mathbf{s})\|_0 + \nu^{-1}\|z - z_h\|_0\right) + \delta\|\nabla z_h\|_0\right)\\
&\leq Ch\left(\|\nabla \mathbf{s}_h\|_0 + \|\nabla \mathbf{s}\|_0 + \nu^{-1}\left(\|z\|_0 + \|z_h\|_0\right)\right) + C\delta\|\nabla z_h\|_0\\
&\leq Ch\left(\|\nabla \mathbf{s}\|_0 + \nu^{-1}\left(\|z\|_0 + \|z_h\|_0\right)\right) + C\delta^{1/2}\left(\nu^{1/2}\|\nabla \mathbf{s}\|_0 + \nu^{-1/2}\|z\|_0\right),
\end{aligned} \tag{21}$$

where in the last inequality we have applied (20). Now observe that from Lemma 1 and (13) it follows that

$$\begin{aligned}
\|z_h\|_0 &\leq Ch\delta^{-1/2}\delta^{1/2}\|\nabla z_h\|_0 + \sup_{\boldsymbol{\chi}_h \in V_h} \frac{(z_h, \nabla\cdot\boldsymbol{\chi}_h)}{\|\boldsymbol{\chi}_h\|_1}\\
&\leq Ch\delta^{-1/2}\delta^{1/2}\|\nabla z_h\|_0 + \|z\|_0 + \nu\|\nabla(\mathbf{s} - \mathbf{s}_h)\|_0.
\end{aligned}$$

Recalling (18) and applying (20) we have

$$\|z_h\|_0 \leq C\big(\nu\|\nabla \mathbf{s}\|_0 + \|z\|_0\big). \tag{22}$$

To bound $\|\nabla z_h\|_0$ we add and subtract ∇z and apply (19) and (18) to obtain

$$\begin{aligned}
\|\nabla z_h\|_0 \leq \|\nabla(z_h - z)\|_0 + \|\nabla z\|_0 &\leq \delta^{-1/2}C\frac{h}{\nu^{1/2}}(\nu\|\mathbf{s}\|_2 + \|z\|_1) + \|z\|_1\\
&\leq C(\nu\|\mathbf{s}\|_2 + \|z\|_1).
\end{aligned} \tag{23}$$

From (21) and (22) we get

$$\|\mathbf{s}_h\|_0 \leq Ch\left(\|\nabla \mathbf{s}\|_0 + \nu^{-1}\|z\|_0\right) + C\|\mathbf{s}\|_0. \tag{24}$$

Finally, we will also use the following bound

$$\|\mathbf{s} - \mathbf{s}_h\|_0 \leq C(h + \nu^{1/2}\delta^{1/2})\left(\|\mathbf{s}\|_1 + \nu^{-1}\|z\|_0\right). \tag{25}$$

To prove (25) we first observe that taking into account (20) and (22) and adding and subtracting $\mathbf{s}$ and z respectively we get

$$\begin{aligned} \nu^{1/2}\|\nabla(\mathbf{s}-\mathbf{s}_h)\|_0 &\leq C(\nu^{1/2}\|\mathbf{s}\|_1+\nu^{-1/2}\|z\|_0), \\ \|z_h-z\|_0 &\leq C(\nu\|\mathbf{s}\|_1+\|z\|_0). \end{aligned} \tag{26}$$

Applying the bound (17) to $(\mathbf{v}_h, q_h) = (\mathbf{s}_h, z_h)$ together with (26) and (20) we reach (25).

4 Transient Stokes equations

We now consider the evolutionary Stokes equations

$$\begin{aligned} \mathbf{v}_t - \nu\Delta\mathbf{v} + \nabla q &= \mathbf{g}, &&\text{in } \Omega \\ \nabla\cdot\mathbf{v} &= 0, &&\text{in } \Omega \\ \mathbf{v} &= \mathbf{0}, &&\text{on } \partial\Omega, \\ \mathbf{v}(0,\mathbf{x}) &= \mathbf{v}_0(\mathbf{x}), &&\text{in } \Omega. \end{aligned} \tag{27}$$

We shall assume that there are positive constants M_1 and M_2 such that for $t \in [0, T]$,

$$\|\mathbf{v}(t)\|_1 + \nu^{-1}\|q(t)\|_0 \leq M_1, \quad \|\mathbf{v}(t)\|_2 + \nu^{-1}\left(\|q(t)\|_1 + \|\mathbf{v}_t(t)\|_0\right) \leq M_2, \tag{28}$$

and, following the analysis in [15], for $k \geq 2$ integer, we shall assume that the following quantities are finite

$$M_{k,1} = \max_{0\leq t\leq T}(t/T)^{k/2-1}\big(\|\mathbf{v}(t)\|_k + \nu^{-1}\|q(t)\|_{H^{k-1}/\mathbb{R}}\big), \tag{29}$$

$$M_{k,2} = \max_{0\leq t\leq T}(t/T)^{k/2-1}\big(\nu^{-1}\|\mathbf{v}_t(t)\|_{k-2} + \nu^{-2}\|q_t(t)\|_{H^{k-3}/\mathbb{R}}\big), \tag{30}$$

$$K_{k,2}^2 = \int_0^T \left(\frac{t}{T}\right)^{k-3}\big(\nu^{-2}\|\mathbf{v}_t(t)\|_{k-2}^2 + \nu^{-4}\|q_t(t)\|_{H^{k-3}/\mathbb{R}}^2\big)\,dt, \tag{31}$$

together with,

$$K_{4,3}^2 = \nu^{-4}\int_0^T \frac{t}{T}\|\mathbf{v}_{tt}\|_0^2\,dt, \tag{32}$$

and

$$\hat{K}_3^2 = \nu^{-2}\int_0^T \|\boldsymbol{g}_t\|_0^2\,dt. \tag{33}$$

We consider the modified Euler non-incremental scheme that has been introduced in [9]. We will denote by $(\mathbf{v}_h^n, \tilde{\mathbf{v}}_h^n, q_h^n)$, $n = 1, 2, \ldots$, $\tilde{\mathbf{v}}_h^n \in V_h$, $q_h^n \in Q_h$ and $\mathbf{v}_h^n \in V_h + \nabla Q_h$ the approximations to the velocity and pressure at time $t_n = n\Delta t$, $\Delta t = T/N$, $N > 0$ obtained with the modified Euler non-incremental scheme

$$
\begin{aligned}
&\left(\frac{\tilde{\mathbf{v}}_h^{n+1} - \mathbf{v}_h^n}{\Delta t}, \boldsymbol{\chi}_h\right) + \nu(\nabla \tilde{\mathbf{v}}_h^{n+1}, \nabla \boldsymbol{\chi}_h) = (\mathbf{g}^{n+1}, \boldsymbol{\chi}_h), \quad \forall \boldsymbol{\chi}_h \in V_h \\
&(\nabla \cdot \tilde{\mathbf{v}}_h^{n+1}, \psi_h) = -\delta(\nabla q_h^{n+1}, \nabla \psi_h), \quad \forall \psi_h \in Q_h, \\
&\mathbf{v}_h^{n+1} = \tilde{\mathbf{v}}_h^{n+1} - \delta \nabla q_h^{n+1}.
\end{aligned} \tag{34}
$$

Let us observe that for $\delta = \Delta t$ in (34) we have the classical Chorin-Temam (Euler non-incremental) scheme [5], [25]. In case $\delta = \Delta t$ we can remove $\mathbf{v}_h^n$ from (34) inserting the expression of $\mathbf{v}_h^n$ from the last equation in (34) into the first equation to get

$$
\left(\frac{\tilde{\mathbf{v}}_h^{n+1} - \tilde{\mathbf{v}}_h^n}{\Delta t}, \boldsymbol{\chi}_h\right) + \nu(\nabla \tilde{\mathbf{v}}_h^{n+1}, \nabla \boldsymbol{\chi}_h) + (\nabla q_h^n, \boldsymbol{\chi}_h) = (\mathbf{g}^{n+1}, \boldsymbol{\chi}_h), \quad \forall \boldsymbol{\chi}_h \in V_h \tag{35}
$$

$$
(\nabla \cdot \tilde{\mathbf{v}}_h^{n+1}, \psi_h) = -\delta(\nabla q_h^{n+1}, \nabla \psi_h), \quad \forall \psi_h \in Q_h. \tag{36}
$$

The method we propose is (35) for δ, in general, different from Δt. We suggest to take δ satisfying (18). As a consequence of the error analysis of this section we will get the error bounds for the classical Euler non-incremental scheme assuming in that case $\delta = \Delta t$.

To get the error bounds of the method we compare the approximation $(\tilde{\mathbf{v}}_h^n, q_h^n)$ defined in (35)-(36) with the stabilized Stokes approximation defined in the previous section. More precisely, let us denote by $(\mathbf{s}_h^n, z_h^n) = (\mathbf{s}_h(t_n), z_h(t_n)) \in V_h \times Q_h$ the stabilized Stokes approximation of the solution $(\mathbf{v}, p)$ of (27) at time t_n satisfying

$$
\begin{aligned}
\nu(\nabla \mathbf{s}_h, \boldsymbol{\chi}_h) + (\nabla z_h, \boldsymbol{\chi}_h) &= (\hat{\mathbf{g}}, \boldsymbol{\chi}_h), \quad \boldsymbol{\chi}_h \in V_h, \\
(\nabla \cdot \mathbf{s}_h, \psi_h) &= -\delta(\nabla z_h, \nabla \psi_h), \quad \forall \psi_h \in Q_h,
\end{aligned} \tag{37}
$$

where $\hat{\mathbf{g}} = \mathbf{g} - \mathbf{v}_t$. Let us observe that the error bounds of Section 3.1 hold with $(\mathbf{s}, z) = (\mathbf{v}, q)$. Taking time derivatives in (20) and (24) we also reach

$$
\|(\mathbf{s}_h)_t\|_0 \le Ch\left(\|\nabla \mathbf{s}_t\|_0 + \nu^{-1}\|z_t\|_0\right) + C\|\mathbf{s}_t\|_0. \tag{38}
$$

In the sequel we will denote by

$$
\tilde{\mathbf{e}}_h^n = \tilde{\mathbf{v}}_h^n - \mathbf{s}_h^n, \quad r_h^n = q_h^n - z_h^n.
$$

From (35), (36) and (37) one obtains the following error equation for all $\boldsymbol{\chi}_h \in V_h$, $\psi_h \in Q_h$

$$\begin{aligned}\Big(\frac{\tilde{\mathbf{e}}_h^{n+1} - \tilde{\mathbf{e}}_h^n}{\Delta t}, \boldsymbol{\chi}_h\Big) + \nu(\nabla \tilde{\mathbf{e}}_h^{n+1}, \nabla \boldsymbol{\chi}_h) + (\nabla r_h^n, \boldsymbol{\chi}_h) = \\ (\tau_h^n, \boldsymbol{\chi}_h) - (\nabla(z_h^n - z_h^{n+1}), \boldsymbol{\chi}_h), \\ (\nabla \cdot \tilde{\mathbf{e}}_h^{n+1}, \psi_h) + \delta(\nabla r_h^{n+1}, \nabla \psi_h) = 0.\end{aligned} \tag{39}$$

where

$$\tau_h^n = \mathbf{v}_t^{n+1} - \frac{\mathbf{s}_h^{n+1} - \mathbf{s}_h^n}{\Delta t} = (\mathbf{v}_t^{n+1} - (\mathbf{s}_h)_t^{n+1}) + \left((\mathbf{s}_h)_t^{n+1} - \frac{\mathbf{s}_h^{n+1} - \mathbf{s}_h^n}{\Delta t}\right). \tag{40}$$

To estimate the errors $\tilde{\mathbf{e}}_h^n$ and r_h^n we will use the following stability result.

Lemma 3 *Let $(\mathbf{w}_h^n)_{n=0}^\infty$ and $(\mathbf{b}_h^n)_{n=0}^\infty$ sequences in V_h and $(y_h^n)_{n=0}^\infty$ and $(d_h^n)_{n=0}^\infty$ sequences in Q_h satisfying for all $\boldsymbol{\chi}_h \in V_h$ and $\psi_h \in Q_h$*

$$\begin{aligned}\left(\frac{\mathbf{w}_h^{n+1} - \mathbf{w}_h^n}{\Delta t}, \boldsymbol{\chi}_h\right) + \nu(\nabla \mathbf{w}_h^{n+1}, \nabla \boldsymbol{\chi}_h) + (\nabla y_h^n, \boldsymbol{\chi}_h) =& (\mathbf{b}_h^n + \nabla d_h^n, \boldsymbol{\chi}_h), \\ (\nabla \cdot \mathbf{w}_h^{n+1}, \psi_h) + \delta(\nabla y_h^{n+1}, \nabla \psi_h) =& 0.\end{aligned}$$

Assume condition

$$\Delta t \le \delta \tag{41}$$

holds. Then, for $0 \le n_0 \le n-1$ there exits a non-dimensional constant c_0 such that the following bounds hold

$$\begin{aligned}\|\mathbf{w}_h^n\|_0^2 + \sum_{j=n_0}^{n-1} \|\mathbf{w}_h^{j+1} - \mathbf{w}_h^j\|_0^2 + \Delta t \sum_{j=n_0}^{n-1} \big(\nu \|\nabla \mathbf{w}_h^{j+1}\|_0^2 + \delta \|\nabla y_h^{j+1}\|_0^2\big) \\ \le c_0 \Big(\|\mathbf{w}_h^{n_0}\|_0^2 + \Delta t \sum_{j=n_0}^{n-1} \big(\nu^{-1} \|\mathbf{b}_h^j\|_{-1}^2 + \delta \|\nabla d_h^j\|_0^2\big)\Big).\end{aligned} \tag{42}$$

$$\begin{aligned}t_n \|\mathbf{w}_h^n\|_0^2 + \sum_{j=n_0}^{n-1} t_{j+1} \|\mathbf{w}_h^{j+1} - \mathbf{w}_h^j\|_0^2 + \Delta t \sum_{j=n_0}^{n-1} t_{j+1} \big(\nu \|\nabla \mathbf{w}_h^{j+1}\|_0^2 + \delta \|\nabla y_h^{j+1}\|_0^2\big) \\ \le c_0 \Big(t_{n_0} \|\mathbf{w}_h^{n_0}\|_0^2 + \Delta t \sum_{j=n_0}^{n} \|\mathbf{w}_h^j\|_0^2 + \Delta t \sum_{j=n_0}^{n-1} t_{j+1} \big(t_{j+1} \|\mathbf{b}_h^j\|_0^2 + \delta \|\nabla d_h^j\|_0^2\big)\Big).\end{aligned} \tag{43}$$

$$\sum_{j=n_0}^{n-1} \Delta t \left\| \frac{\mathbf{w}_h^{j+1} - \mathbf{w}_h^j}{\Delta t} \right\|_0^2 + \nu \|\nabla \mathbf{w}_h^n\|_0^2 + \delta \|\nabla y_h^n\|_0^2 + \nu \sum_{j=n_0}^{n-1} \|\nabla(\mathbf{w}_h^{j+1} - \mathbf{w}_h^j)\|_0^2$$
$$\leq c_0 \Big(\nu \|\nabla \mathbf{w}_h^{n_0}\|_0^2 + \delta \|\nabla y_h^{n_0}\|_0^2 + \Delta t \sum_{j=n_0}^{n-1} \big(\|\mathbf{b}_h^j\|_0^2 + \|\nabla d_h^j\|_0^2 \big) \Big) \tag{44}$$

and

$$\sum_{j=n_0}^{n-1} t_{j+1} \Delta t \left\| \frac{\mathbf{w}_h^{j+1} - \mathbf{w}_h^j}{\Delta t} \right\|_0^2 + \nu t_n \|\nabla \mathbf{w}_h^n\|_0^2 + \delta t_n \|\nabla y_h^n\|_0^2$$
$$\leq c_0 \Big(\Delta t \sum_{j=n_0}^{n-1} t_{j+1} \big(\|b_h^j\|_0^2 + \|\nabla d_h^j\|_0^2 \big) + \Delta t \sum_{j=n_0}^{n-1} \big(\nu \|\nabla \mathbf{w}_h^j\|_0^2 + \delta \|\nabla y_h^j\|_0^2 \big) \Big). \tag{45}$$

Proof The proof of (42), (43) and (44) can be found in [9, Lemma 3]. The proof of (45) can be easily reached arguing as in the proof of (44). □

Remark 1 As commented in [9], it is possible to change condition (41) by $\Delta t < 2\delta$, but this requires a more elaborate proof than that presented in [9].

In the sequel, although it is not strictly necessary to prove our results, we will assume that

$$\delta \leq T. \tag{46}$$

to simplify some of the expressions below.

Theorem 1 *Let* $(\mathbf{v}, q)$ *be the solution of (27) and let* $(\tilde{\mathbf{v}}_h^n, q_h^n)$, $n \geq 1$, *be the solution of (35)-(36). Assume* δ *satisfies condition (18) and (46), and that* Δt *satisfies condition (41). Then, the following bounds hold*

$$\begin{aligned} t_n \|\tilde{\mathbf{v}}_h^n - \mathbf{v}(t_n)\|_0^2 &\leq C t_n \left(\|\tilde{\mathbf{v}}_h^0 - \mathbf{v}(0)\|_0^2 + \Delta t^2 \left\|\nabla r_h^0\right\|_0^2 \right) \\ &\quad + C_1 t_n \Delta t^2 + C_2 t_n (h^4 + \delta^2 \nu^2), \end{aligned} \tag{47}$$

where C_1 *and* C_2 *are defined as*

$$C_1 = C\nu^2 \left(C_2 + \nu^2 K_{4,3}^2 T + \hat{K}_3^2 T + M_{2,2}^2 \right), \tag{48}$$
$$C_2 = C \left(\nu^2 K_{4,2}^2 T + \nu K_{3,2}^2 + M_{2,1}^2 \right), \tag{49}$$

Moreover, it also holds,

$$\Delta t \sum_{j=1}^{n} \big(\nu \|\nabla(\tilde{\mathbf{v}}_h^j - \mathbf{v}(t_j))\|_0^2 + \delta \|\nabla(q_h^j - q(t_j))\|_0^2\big) \\ \leq C\|\tilde{\mathbf{v}}_h^0 - \mathbf{v}(0)\|_0^2 + \tilde{C}_1 \Delta t + \tilde{C}_2(h^2 + \nu\delta). \tag{50}$$

where, assuming (46),

$$\tilde{C}_1 = C_1((\nu\lambda)^{-1} + T), \tag{51}$$

$$\tilde{C}_2 = C_2(\lambda^{-1} + \mathrm{diam}(\Omega)^2 + \nu T). \tag{52}$$

Proof In view of (39) we can apply (43) for $\mathbf{w}_h^n = \tilde{\mathbf{e}}_h^n$, $y_h^n = r_h^n$, $\mathbf{b}_h^n = \tau_h^n$ and $d_h^n = z_h^{n+1} - z_h^n$. It follows that

$$t_n \|\tilde{\mathbf{e}}_h^n\|_0^2 + \Delta t \sum_{j=1}^{n} t_j \big(\nu \|\nabla \tilde{\mathbf{e}}_h^j\|_0^2 + \delta \|\nabla r_h^j\|_0^2\big) \\ \leq c_0 \Big(\Delta t \sum_{j=0}^{n} \|\tilde{\mathbf{e}}_h^j\|_0^2 + \sum_{j=0}^{n-1} \Delta t \big(t_{j+1}^2 \|\tau_h^j\|_0^2 + \delta t_{j+1} \|\nabla(z_h^{j+1} - z_h^j)\|_0^2\big)\Big). \tag{53}$$

To bound the second term on the right-hand side of (53), we notice that $t_{j+1}/t_j \leq 2$ for $j = 1, \ldots, n-1$, so that we may write

$$\Delta t \sum_{j=0}^{n-1} t_{j+1}^2 \|\tau_h^j\|_0^2 \leq C t_n \Delta t \sum_{j=0}^{n-1} t_j' \|\tau_h^j\|_0^2,$$

where $t_j' = \max(\Delta t, t_j)$. From definition (40) we may write

$$\|\tau_h^j\|_0^2 \leq 2 \left\| \mathbf{v}_t^{j+1} - \frac{\mathbf{v}^{j+1} - \mathbf{v}^j}{\Delta t} \right\|_0^2 + \frac{2}{\Delta t^2} \left\| (\mathbf{v}^{j+1} - \mathbf{s}_h^{j+1}) - (\mathbf{v}^j - \mathbf{s}_h^j) \right\|_0^2. \tag{54}$$

To bound the first term on the right-hand side of (54), after taking Taylor expansion with integral reminder and applying Hölder's inequality we have

$$\Delta t \sum_{j=0}^{n-1} t_j' \left\| \mathbf{v}_t^{j+1} - \frac{\mathbf{v}^{j+1} - \mathbf{v}^j}{\Delta t} \right\|_0^2 \leq \sum_{j=0}^{n-1} t_j' \int_{t_j}^{t_{j+1}} (s - t_j)^2 \|\mathbf{v}_{ss}\|_0^2.$$

Now, for $j \geq 1$, we write $t_j'(s-t_j)^2 = t_j(s-t_j)^2 \leq t_j\Delta t^2 \leq s\Delta t^2$, and, for $j=0$, $t_0'(s-t_j)^2 = \Delta t(s)^2 \leq s\Delta t^2$, so that applying (32) we get

$$\Delta t \sum_{j=0}^{n-1} t_j' \left\| \mathbf{v}_t^{j+1} - \frac{\mathbf{v}^{j+1}-\mathbf{v}^j}{\Delta t} \right\|_0^2 \leq \Delta t^2 \int_{t_0}^{t_n} s\|\mathbf{v}_{ss}\|_0^2 \leq \Delta t^2 T \nu^4 K_{4,3}^2. \tag{55}$$

To bound the second term on the right-hand side of (54) we observe that

$$\begin{aligned}\left\|(\mathbf{v}^{j+1}-\mathbf{s}_h^{j+1})-(\mathbf{v}^j-\mathbf{s}_h^j)\right\|_0^2 &= \left\|\int_{t_j}^{t_{j+1}} (\mathbf{v}-\mathbf{s}_h)_s \ ds\right\|_0^2 \\ &\leq \Delta t \int_{t_j}^{t_{j+1}} \|(\mathbf{v}-\mathbf{s}_h)_s\|_0^2 \ ds,\end{aligned}$$

where, in the last inequality we have applied Hölder's inequality. Now, for $j \geq 1$ we write $t_j' = t_j \leq s$ and apply (19) to bound $\|(\mathbf{v}-\mathbf{s}_h)_s\|_0^2$, and, for $j=0$, $t_0' = \Delta t$ and apply (25), so that we have

$$\begin{aligned}&\frac{1}{\Delta t}\sum_{j=0}^{n-1} t_j' \left\|(\mathbf{v}^{j+1}-\mathbf{s}_h^{j+1})-(\mathbf{v}^j-\mathbf{s}_h^j)\right\|_0^2 \\ &\qquad \leq C\int_{t_1}^{t_n} (h^4+\delta^2\nu^2)s\left(\|\mathbf{v}_s\|_2^2+\nu^{-2}\|q_s\|_1^2\right) \ ds \\ &\qquad\quad + C\int_0^{t_1}\left(\nu\Delta t^2+\frac{h^4}{\nu}+\nu\delta^2\right)\left(\|\mathbf{v}_s\|_1^2+\nu^{-2}\|q_s\|_0^2\right) \ ds \\ &\qquad \leq C(h^4+\nu^2(\Delta t^2+\delta^2)(\nu^2K_{4,2}^2T+\nu K_{3,2}^2),\end{aligned} \tag{56}$$

where in the last inequality we have applied (31). Thus, from (54), (55) and (56) we finally reach

$$\Delta t \sum_{j=0}^{n-1} t_j' \|\tau_h^j\|_0^2 \leq C\left(\Delta t^2\nu^4 T K_{4,3}^2 + (h^4+\nu^2(\Delta t^2+\delta^2)(\nu^2K_{4,2}^2T+\nu K_{3,2}^2)\right), \tag{57}$$

so that for the second term on the right-hand side of (53) we write

$$\begin{aligned}\Delta t \sum_{j=0}^{n-1} t_{j+1}^2 \|\tau_h^j\|_0^2 \leq &Ct_n\Delta t^2\nu^4TK_{4,3}^2 \\ &+ Ct_n(h^4+\nu^2(\Delta t^2+\delta^2)(\nu^2K_{4,2}^2T+\nu K_{3,2}).\end{aligned} \tag{58}$$

Let us also observe that by writing $\Delta t^{-1}t'_j \geq 1$ and using (54), and repeating the arguments to prove (56), but using (25) instead of (19) for $j \geq 1$ we get

$$\Delta t \sum_{j=0}^{n-1} \|\tau_h^j\|_0^2 \leq C\left(\Delta t \nu^4 T(K_{4,2}^2 + K_{4,3}^3) + (h^2 + \nu\delta)\nu^2 K_{3,2}^2\right), \tag{59}$$

For the last term on the right-hand side of (53), applying Hölder's inequality and (16), we may write

$$\begin{aligned} \delta\|\nabla(z_h^{j+1} - z_h^j)\|_0^2 = \delta\Big\|\int_{t_j}^{t_{j+1}} (\nabla z_h)_s\Big\|_0^2 \leq \Delta t \int_{t_j}^{t_{j+1}} \delta\|(\nabla z_h)_s\|_0^2 \, ds \\ \leq C\Delta t \int_0^{t_1} (\nu\|\mathbf{v}_s\|_1^2 + \nu^{-1}\|q_s\|_0^2) \, ds. \end{aligned} \tag{60}$$

Thus,

$$\Delta t \sum_{j=0}^{n-1} \|\nabla(z_h^{j+1} - z_h^j)\|_0^2 \leq \frac{\Delta t^2}{\delta} \int_0^{t_n} (\nu\|\mathbf{v}_s\|_1^2 + \nu^{-1}\|q_s\|_0^2) \, ds, \tag{61}$$

and, consequently, for the last term on the right-hand side of (53), using also (31), we have

$$\Delta t\delta \sum_{j=0}^{n-1} t_{j+1}\|\nabla(z_h^{j+1} - z_h^j)\|_0^2 \leq C\nu^3\Delta t^2 K_{3,2}^2 t_{n+1}. \tag{62}$$

To conclude we need to bound the first term on the right-hand side of (53). For this purpose we denote

$$\mathbf{S}_h(t) = \int_0^t \mathbf{s}_h(s)\, ds,\ \widetilde{\mathbf{V}}_h^n = \Delta t \sum_{j=1}^n \tilde{\mathbf{v}}_h^j,\ P_h^n = \Delta t \sum_{j=1}^n q_h^j,\ \boldsymbol{G}(t) = \int_0^t \boldsymbol{g}(s)\, ds$$

and integrate (37) with respect to time taking into account that $\hat{\mathbf{g}} = \mathbf{g} - \mathbf{v}_t$. Thus,

$$(\mathbf{v}(t), \boldsymbol{\chi}_h) + \nu(\nabla\mathbf{S}_h, \nabla\boldsymbol{\chi}_h) + (\nabla \int_0^t z_h\, ds, \boldsymbol{\chi}_h) = (\boldsymbol{G} + \mathbf{v}(0), \boldsymbol{\chi}_h), \quad \forall \boldsymbol{\chi}_h \in V_h, \tag{63}$$

$$(\nabla \cdot \mathbf{S}_h, \psi_h) + \delta(\nabla \int_0^t z_h\, ds, \nabla\psi_h) = 0, \quad \forall \psi_h \in Q_h. \tag{64}$$

We also define

$$\tilde{\mathbf{E}}_h^n = \widetilde{\mathbf{V}}_h^n - \mathbf{S}_h(t_n), \qquad R_h^n = P_h^n - \int_0^{t_n} z_h\, ds,$$

and

$$\begin{aligned}\Upsilon_h^n = (\tilde{\mathbf{v}}_h^0 - \mathbf{v}(0)) + \Big(\Delta t \sum_{j=1}^{n+1} \boldsymbol{g}(t_j) - \boldsymbol{G}(t_{n+1})\Big) \\ + \Big(\frac{1}{\Delta t}\int_{t_n}^{t_{n+1}} \mathbf{s}_h(s)\ ds - \mathbf{v}(t_{n+1})\Big).\end{aligned} \tag{65}$$

We multiply (35)-(36) by Δt, sum from $j = 0$ to n, and subtract from (63)-(64) evaluated at $t = t_{n+1}$ to get

$$\begin{aligned}\left(\frac{\tilde{\mathbf{E}}_h^{n+1} - \tilde{\mathbf{E}}_h^n}{\Delta t}, \boldsymbol{\chi}_h\right) + \nu(\nabla \tilde{\mathbf{E}}_h^{n+1}, \nabla \boldsymbol{\chi}_h) + (\nabla R_h^n, \boldsymbol{\chi}_h) \\ = (\Upsilon_h^n, \boldsymbol{\chi}_h) - (\nabla D_h^n, \boldsymbol{\chi}_h), \quad \forall \boldsymbol{\chi}_h \in V_h, \\ (\nabla \cdot \tilde{\mathbf{E}}_h^{n+1}, \psi_h) + \delta(\nabla R_h^{n+1}, \nabla \psi_h) = 0, \quad \psi_h \in Q_h,\end{aligned}$$

where

$$D_h^n = \Delta t q_h^0 - \int_{t_n}^{t_{n+1}} z_h(s)\, ds = \Delta t r_h^0 + \Delta t z_h(0) - \int_{t_n}^{t_{n+1}} z_h(s)\, ds. \tag{66}$$

We notice that

$$\frac{\tilde{\mathbf{E}}_h^{n+1} - \tilde{\mathbf{E}}_h^n}{\Delta t} = \tilde{\mathbf{e}}_h^{n+1} + \left(\mathbf{s}_h^{n+1} - \frac{1}{\Delta t}\int_{t_n}^{t_{n+1}} \mathbf{s}_h(s)\ ds\right), \qquad \tilde{\mathbf{E}}_h^0 = 0, \qquad R_h^0 = 0$$

so that applying (44) for $\mathbf{w}_h^n = \tilde{\mathbf{E}}_h^n$, $y_h^n = R_h^n$, $\mathbf{b}_h^n = \Upsilon_h^n$ and $d_h^n = D_h^n$ we get

$$\sum_{j=0}^{n-1} \Delta t \Big\| \frac{\tilde{\mathbf{E}}_h^{n+1} - \tilde{\mathbf{E}}_h^n}{\Delta t} \Big\|_0^2 \le c_0 \Delta t \sum_{j=0}^{n-1} (\|\Upsilon_h^j\|_0^2 + \|\nabla D_h^j\|_0^2)$$

and then

$$\sum_{j=1}^{n} \Delta t \|\tilde{\mathbf{e}}_h^n\|_0^2 \le C\Big(\Delta t \sum_{j=0}^{n-1} \big(\|\Upsilon_h^j\|_0^2 + \|\nabla D_h^j\|_0^2 + \big\|\mathbf{s}_h^{j+1} - \frac{1}{\Delta t}\int_{t_j}^{t_{j+1}} \mathbf{s}_h(s)\ ds\big\|_0^2\big)\Big). \tag{67}$$

We now bound the right-hand side of (67). We start with the second term of Υ_h^j in (65). We notice that

$$\int_{t_j}^{t_{j+1}} \boldsymbol{g}(t)\,dt - \Delta t \boldsymbol{g}(t_j) = \int_{t_j}^{t_{j+1}} (\boldsymbol{g}(t) - \boldsymbol{g}(t_j))\,dt$$

so that by successively applying Hölder's inequality and the mean value theorem we have

$$\begin{aligned}
\Big\|\int_{t_j}^{t_{j+1}} \boldsymbol{g}(t)\,dt - \Delta t \boldsymbol{g}(t_j)\Big\|_0^2 &\leq \Delta t \int_{t_j}^{t_{j+1}} \|\boldsymbol{g}(t) - \boldsymbol{g}(t_j)\|_0^2\,dt \\
&\leq \Delta t \int_{t_j}^{t_{j+1}} \Big\| \int_{t_j}^{t} \boldsymbol{g}_s(s) \Big\|_0^2\,dt \\
&\leq \Delta t \int_{t_j}^{t_{j+1}} (t - t_j) \int_{t_j}^{t} \|\boldsymbol{g}_s(s)\|_0^2 \,ds\,dt \\
&\leq \Delta t \int_{t_j}^{t_{j+1}} \|\boldsymbol{g}_s(s)\|_0^2 \,ds \int_{t_j}^{t_{j+1}} (t - t_j)\,dt \\
&\leq \frac{1}{2}\Delta t^3 \int_{t_j}^{t_{j+1}} \|\boldsymbol{g}_s(s)\|_0^2 \,ds.
\end{aligned}$$

Thus, applying Hölder's inequality we have

$$\begin{aligned}
\Big\|\Delta t \sum_{l=1}^{j+1} \boldsymbol{g}(t_l) - \boldsymbol{G}(t_{j+1})\Big\|_0^2 &= \Big\|\sum_{l=1}^{j+1} \Delta t \boldsymbol{g}(t_l) - \int_{t_{l-1}}^{t_l} \boldsymbol{g}(t)\,dt\Big\|_0^2 \\
&\leq (j+1) \sum_{l=1}^{j+1} \Big\|\Delta t \boldsymbol{g}(t_l) - \int_{t_{l-1}}^{t_l} \boldsymbol{g}(t)\,dt\Big\|_0^2 \\
&\leq t_{j+1} \frac{1}{2}\Delta t^2 \int_0^{t_{j+1}} \|\boldsymbol{g}_t(t)\|_0^2 \,dt.
\end{aligned}$$

And then

$$\sum_{j=0}^{n-1} \Delta t \Big\|\sum_{l=1}^{j+1} \Delta t \boldsymbol{g}(t_l) - \boldsymbol{G}(t_{j+1})\Big\|_0^2 \leq C t_n^2 \Delta t^2 \int_0^{t_n} \|\boldsymbol{g}_t(t)\|_0^2 \,dt \leq C t_n^2 \Delta t^2 \nu^2 \hat{K}_3^2, \tag{68}$$

where in the last inequality we have applied (33).

To bound the third term of Υ_h^j in (65) we observe that

$$\begin{aligned}
&\sum_{j=0}^{n-1} \Delta t \Big\| \frac{1}{\Delta t} \int_{t_n}^{t_{n+1}} \mathbf{s}_h(s)\ ds - \mathbf{v}(t_{n+1}) \Big\|_0^2 \\
&\qquad = \sum_{j=0}^{n-1} \frac{1}{\Delta t} \Big\| \int_{t_n}^{t_{n+1}} (\mathbf{s}_h(s) - \Delta t \mathbf{v}(t_{n+1}))\ ds \Big\|_0^2 \\
&\qquad \le \sum_{j=0}^{n-1} \frac{2}{\Delta t} \Big\| \int_{t_n}^{t_{n+1}} (\mathbf{s}_h(s) - \Delta t \mathbf{s}(t_{n+1}))\ ds \Big\|_0^2 \\
&\qquad\quad + \sum_{j=0}^{n-1} 2\Delta t \big\| \mathbf{s}_h(t_{n+1}) - \mathbf{v}(t_{n+1}) \big\|_0^2.
\end{aligned} \tag{69}$$

For the first term on the right-hand side of (69) arguing as in (68) and then applying (38), (30) and (31) we finally get

$$\begin{aligned}
\sum_{j=0}^{n-1} \frac{2}{\Delta t} \Big\| \int_{t_n}^{t_{n+1}} (\mathbf{s}_h(s) - \Delta t \mathbf{s}(t_{n+1}))\ ds \Big\|_0^2 &\le 2 \sum_{j=0}^{n-1} \Delta t^2 \int_{t_n}^{t_{n+1}} \|(\mathbf{s}_h)_s\|_0^2\ ds \\
&\le C \Delta t^2 \nu^2 \left(K_{3,2}^2 h^2 + M_{2,2}^2 t_n \right).
\end{aligned} \tag{70}$$

For the second term on the right-hand side of (69) applying (19) and (29) we get

$$\begin{aligned}
\sum_{j=0}^{n-1} 2\Delta t \, \|\mathbf{s}_h(t_{n+1}) - \mathbf{v}(t_{n+1})\|_0^2 &\le 2 t_n \max_{t_1 \le s \le t_n} \|\mathbf{s}_h(s) - \mathbf{v}(s)\|_0^2 \\
&\le C t_n (h^4 + \delta^2 \nu^2) M_{2,1}^2.
\end{aligned} \tag{71}$$

Inserting (70) and (71) in (69) we obtain

$$\begin{aligned}
\sum_{j=0}^{n-1} \Delta t \Big\| \frac{1}{\Delta t} \int_{t_n}^{t_{n+1}} \mathbf{s}_h(s)\ ds - \mathbf{v}(t_{n+1}) \Big\|_0^2 &\le C \Delta t^2 \nu^2 \left(K_{3,2}^2 h^2 + M_{2,2}^2 t_n \right) \\
&\quad + C t_n (h^4 + \delta^2 \nu^2) M_{2,1}^2.
\end{aligned} \tag{72}$$

Then, from (68) and (72) and taking into account the definition of Υ_h^j in (65) we finally reach

$$\begin{aligned}
\Delta t \sum_{j=0}^{n-1} \|\Upsilon_h^j\|_0^2 &\le C \left(t_n \|\tilde{\mathbf{v}}_h^0 - \mathbf{v}(0)\|_0^2 + t_n^2 \Delta t^2 \nu^2 \hat{K}_3^2 \right) \\
&\quad + C \left(\Delta t^2 \nu^2 \left(K_{3,2}^2 h^2 + M_{2,2}^2 t_n \right) + t_n (h^4 + \delta^2 \nu^2) M_{2,1}^2 \right).
\end{aligned} \tag{73}$$

To bound the second term on the right-hand side of (67) we notice that using definition (66) we get

$$\begin{aligned}\|\nabla D_h^j\|_0^2 &\le 4\Big\|\int_{t_j}^{t_{j+1}} \nabla z_h(s)\, ds\Big\|_0^2 + 4\Delta t^2 \,\|\nabla z_h(0)\|_0^2 + 2\Delta t^2 \left\|\nabla r_h^0\right\|_0^2 \\ &\le 4\Delta t \int_{t_j}^{t_{j+1}} \|\nabla z_h(s)\|_0^2 \; ds + 4\Delta t^2 \,\|\nabla z_h(0)\|_0^2 + 2\Delta t^2 \left\|\nabla r_h^0\right\|_0^2 .\end{aligned}$$

Thus, applying (23) and (29) we obtain

$$\begin{aligned}\Delta t \sum_{j=0}^{n-1} \|\nabla D_h^j\|_0^2 &\le C\Delta t^2 \left(\int_0^{t_n} \|\nabla z_h(s)\|_0^2 \; ds + t_n \,\|\nabla z_h(0)\|_0^2 + t_n \left\|\nabla r_h^0\right\|_0^2\right) \\ &\le C\Delta t^2 t_n \left(\max_{0\le t\le t_n} \|\nabla z_h(t)\|_0^2 + \left\|\nabla r_h^0\right\|_0^2\right) \\ &\le C\Delta t^2 t_n \left(\nu^2 M_{2,1}^2 + \left\|\nabla r_h^0\right\|_0^2\right). \qquad (74)\end{aligned}$$

On the other hand, for the last term in (67) we get

$$\begin{aligned}\big\|\mathbf{s}_h^{j+1} - \frac{1}{\Delta t}\int_{t_j}^{t_{j+1}} \mathbf{s}_h(s)\; ds\big\|_0^2 &= \big\|(\mathbf{S}_h)_t^{j+1} - \frac{\mathbf{S}_h^{j+1} - \mathbf{S}_h^j}{\Delta t}\big\|_0^2 \\ &= \big\|\frac{1}{\Delta t}\int_{t_j}^{t_{j+1}} (t_j - s)(\mathbf{S}_h)_{ss}\, ds\big\|_0^2 \\ &\le \Delta t \int_{t_j}^{t_{j+1}} \|(\mathbf{s}_h)_s\|_0^2\, ds \\ &\le C\Delta t \int_{t_j}^{t_{j+1}} h^2\big(\|\nabla \mathbf{v}_t\|_1^2 + \nu^{-2}\|q_t\|_0^2 \; ds + \|\mathbf{v}_t\|_0^2\,\big) ds,\end{aligned}$$

where in the last inequality we have applied (38). Consequently, using (30) and (31) we reach

$$\begin{aligned}&\Delta t \sum_{j=0}^{n-1} \Big\|\mathbf{s}_h^{j+1} - \frac{1}{\Delta t}\int_{t_j}^{t_{j+1}} \mathbf{s}_h(s)\; ds\Big\|_0^2 \\ &\qquad \le C\Delta t^2 \int_0^{t_n} \big(h^2\left(\|\nabla \mathbf{v}_t\|_1^2 + \nu^{-2}\|q_t\|_0^2 \; ds\right) + \|\mathbf{v}_t\|_0^2\,\big)\, ds \\ &\qquad \le C\Delta t^2 \nu^2 (K_{3,2}^2 h^2 + M_{2,2}^2 t_n). \qquad (75)\end{aligned}$$

Thus, in view of (67), (73), (74) and (75) we get

$$\sum_{j=1}^n \Delta t\|\tilde{\mathbf{e}}_h^j\|_0^2 \leq C\left(t_n\|\tilde{\mathbf{v}}_h^0-\mathbf{v}(0)\|_0^2+t_n\Delta t^2\left\|\nabla r_h^0\right\|_0^2+\nu^2 t_n^2\hat{K}_3^2\Delta t^2\right) \\ +C\left(\Delta t^2\nu^2\left(K_{3,2}^2h^2+t_n(M_{2,1}^2+M_{2,2}^2)\right)+t_n(h^4+\delta^2\nu^2)M_{2,1}^2\right). \tag{76}$$

Going back to (53) and inserting (58), (62) and (76) we finally reach

$$\begin{aligned} t_n\|\tilde{\mathbf{e}}_h^n\|_0^2 &+ \Delta t\sum_{j=1}^n t_j\left(\nu\|\nabla\tilde{\mathbf{e}}_h^j\|_0^2+\delta\|\nabla r_h^j\|_0^2\right) \\ &\leq C\left(t_n\|\tilde{\mathbf{v}}_h^0-\mathbf{v}(0)\|_0^2+\Delta t\|\tilde{\mathbf{e}}_h^0\|_0^2+t_n(\Delta t)^2\left\|\nabla r_h^0\right\|_0^2+t_n^2\Delta t^2\nu^2\hat{K}_3^2\right) \\ &\quad+C\left(t_n(h^4+\delta^2\nu^2)(\nu^2K_{4,2}^2T+\nu K_{3,2}^2+M_{2,1}^2)\right) \\ &\quad+C\Delta t^2\left(\nu^4T(K_{4,2}^2+K_{4,3}^2)t_n+t_{n+1}\nu^3K_{3,2}^2\right) \\ &\quad+C\Delta t^2\left(\nu^2\left(K_{3,2}^2h^2+t_n(M_{2,1}^2+M_{2,2}^2)\right)\right) \\ &\leq Ct_n\left(\|\tilde{\mathbf{v}}_h^0-\mathbf{v}(0)\|_0^2+\|\tilde{\mathbf{e}}_h^0\|_0^2+\Delta t^2\left\|\nabla r_h^0\right\|_0^2\right) \\ &\quad+C_1t_n\Delta t^2+C_2t_n(h^4+\delta^2\nu^2), \end{aligned} \tag{77}$$

where C_1 and C_2 are the constants in (48) and (49) and we have used the bounds $t_{n+1}\leq Ct_n$, $\Delta t\leq t_n$ and that $(\Delta t)^2\nu^2K_{3,2}^2h^2\leq t_n(\Delta t)\nu^2K_{3,2}^2h^2\leq t_n((\Delta t)^2\nu^3+h^4\nu)K_{3,2}^2$.

To conclude (47) we apply (77) together with triangle inequality, (19) and (29).

Finally to prove (50) we apply (42) instead of (43). Then, using (4) and then applying (59), (61) and (31), we have that

$$\begin{aligned} \Delta t\sum_{j=1}^n&\left(\nu\|\nabla\tilde{\mathbf{e}}_h^h\|_0^2+\delta\|\nabla r_h\|_0^2\right)\leq C\left(\|\tilde{\mathbf{v}}_h^0-\mathbf{v}(0)\|_0^2+M_{2,1}^2(h^4+(\nu\delta)^2)\right) \\ &+\frac{C}{\nu\lambda}\left(\nu^4(K_{4,2}^2+K_{4,3}^2)T\Delta t+\nu^2K_{3,2}(h^2+\nu\delta+\nu^2\lambda\Delta t^2\right) \\ &\leq C(\|\tilde{\mathbf{v}}_h^0-\mathbf{v}(0)\|_0^2 \\ &\quad+C\big(C_1\Delta t\big((\nu\lambda)^{-1}+\Delta t\big)+C_2(\lambda^{-1}(h^2+\nu\delta)+h^4+(\nu\delta)^2\big) \\ &\leq C\|\tilde{\mathbf{v}}_h^0-\mathbf{v}(0)\|_0^2+\tilde{C}_1\Delta t+\tilde{C}_2(h^2+\nu\delta), \end{aligned} \tag{78}$$

where $\tilde{C}_1$ and $\tilde{C}_2$ are the constants in (51) and (52). Taking into account (19) the estimate (50) follows.

□

Remark 2 Let us observe that taking $\delta = \Delta t$ the analysis carried out applies to the standard Euler non-incremental scheme. However, since condition (18) implies

$$\frac{1}{\nu\rho_1^2}h^2 \leq \Delta t, \tag{79}$$

the analysis for the standard Euler non-incremental scheme holds under condition (79). This result is in agreement with the error bounds in [3] where the authors prove error bounds for the Euler non-incremental scheme for LBB stable elements assuming $\Delta t \geq Ch^2$. It is also in agreement with the classical results for the continuous in space Euler non-incremental method (see for example [12]) since for $\delta = \Delta t$ the rate of convergence in terms of Δt in the L^2 norm of the velocity is one and the rate of convergence in the H^1 norm of the velocity and the L^2 norm of the pressure is one half, see (47)-(50).

Remark 3 In view of (47) and (50) any initial approximation $\tilde{\mathbf{v}}_h^0$ based on linear elements such us the linear interpolant of the initial condition gives the optimal order for the term $\|\tilde{\mathbf{v}}_h^0 - \mathbf{v}(0)\|_0$. For the initial pressure any initial pressure giving $\|\nabla r_h^0\|_0 = O(1)$ keeps the optimal rate of convergence. In particular, choosing $q_h^0 = 0$ which means taking in (34) $\mathbf{v}_h^0 = \tilde{\mathbf{v}}_h^0$ gives $\|\nabla r_h^0\|_0 = \|\nabla z_h\|_0$ that is bounded thanks to (23).

Remark 4 As commented in [9], the restriction (41) for the modified Euler non-incremental scheme is not just a requirement of the proof but, as it can be easily checked in practice, the method becomes unstable if Δt is taken larger than 2δ.

We will now prove a bound for the pressure error.

Theorem 2 *Under the assumptions of Theorem 1 the following bound holds*

$$\begin{aligned}\Delta t \sum_{j=1}^{n} t_j \|q_h^j - q(t_j)\|_0^2 \leq & C(t_{n+1}\nu + \lambda^{-1})\|\tilde{\mathbf{v}}_h^0 - \mathbf{v}(0)\|_0^2 \\ & + Ct_{n+1}\nu\left(\|\tilde{\mathbf{e}}_h^0\|_0^2 + \Delta t^2\|\nabla r_h^0\|_0^2\right) \\ & + C_3\Delta t + C_4(h^2 + \nu\delta),\end{aligned} \tag{80}$$

where, (assuming (46)),

$$C_3 = C(\nu T + \lambda^{-1})\tilde{C}_1 \tag{81}$$

$$C_4 = C\big(\nu T + \lambda^{-1})\tilde{C}_2, \tag{82}$$

and where $\tilde{C}_1$ and $\tilde{C}_2$ are the constants in (51) and (52) respectively.

Proof Applying Lemma 1 and (39) it is easy to obtain

$$\begin{aligned}\Delta t \sum_{j=1}^n t_j \|r_h^j\|_0^2 \le{}& C\Delta t \sum_{j=1}^n t_j \nu\delta \|\nabla r_h^j\|_0^2 \\ &+ C\Delta t \sum_{j=1}^n t_j \left\| \frac{\tilde{\mathbf{e}}_h^{j+1} - \tilde{\mathbf{e}}_h^j}{\Delta t} \right\|_{-1}^2 + C\Delta t \sum_{j=1}^n t_j \|\tau_h^j\|_{-1}^2 \\ &+ C\Delta t \sum_{j=1}^n t_j \nu^2 \|\nabla \tilde{e}_h^{j+1}\|_0^2 + C\Delta t \sum_{j=1}^n t_j \|(z_h^j - z_h^{j+1})\|_0^2.\end{aligned} \tag{83}$$

We will bound all the terms on the right-hand side of (83). We first observe that the first and forth terms are already bounded in (77) and then

$$\begin{aligned}&C\Delta t \sum_{j=1}^n t_j \nu\delta \|\nabla r_h^j\|_0^2 + C\Delta t \sum_{j=1}^n t_j \nu^2 \|\nabla \tilde{e}_h^{j+1}\|_0^2 \\ &\quad \le C t_{n+1}\nu \left(\|\tilde{\mathbf{v}}_h^0 - \mathbf{v}(0)\|_0^2 + \|\tilde{\mathbf{e}}_h^0\|_0^2 + \Delta t^2 \|\nabla r_h^0\|_0^2 \right) \\ &\qquad + \nu t_{n+1}\big(C_1 \Delta t^2 + C_2 (h^4 + (\nu\delta)^2) \big).\end{aligned} \tag{84}$$

To bound the third term on the right-hand side of (83) we first apply (4) and then (57) to get

$$\begin{aligned}\Delta t \sum_{j=1}^n t_j \|\tau_h^j\|_{-1}^2 &\le \frac{C}{\lambda} \Delta t \sum_{j=1}^n t_j \|\tau_h^j\|_0^2 \\ &\le \frac{C}{\lambda} \left(\Delta t^2 \nu^4 T K_{4,3}^2 + (h^4 + \nu^2(\Delta t^2 + \delta^2))(\nu^2 K_{4,2}^2 T + \nu K_{3,2}^2) \right) \\ &\le C\lambda^{-1} \left(C_1 \Delta t^2 + C_2 (h^4 + (\nu\delta)^2) \right).\end{aligned} \tag{85}$$

For the last term on the right-hand side of (83) arguing as in (60) and applying (22) to the time derivative $\|(z_h)_t\|_0$, and (31) we get

$$\begin{aligned}
\Delta t\sum_{j=1}^n t_j\|(z_h^j-z_h^{j+1})\|_0^2 &\le \Delta t^2\sum_{j=1}^n t_j\int_{t_j}^{t_{j+1}}\|(z_h)_s\|_0^2\,ds\\
&\le \Delta t^2 C t_n\int_{t_1}^{t_{n+1}} s\left(\nu^2\|\mathbf{v}_s\|_1^2+\|q_s\|_0^2\right)\\
&\le C t_n\Delta t^2\nu^4 K_{3,2}^2\\
&\le C\nu T C_1\Delta t^2.
\end{aligned}\tag{86}$$

To conclude we will bound the second term on the right-hand side of (83). Since $t_j\le t_{j+1}$ and taking into account (4) we can write

$$\begin{aligned}
\Delta t\sum_{j=1}^n t_j\left\|\frac{\tilde{\mathbf{e}}_h^{j+1}-\tilde{\mathbf{e}}_h^j}{\Delta t}\right\|_{-1}^2 &\le \Delta t\sum_{j=1}^n t_{j+1}\left\|\frac{\tilde{\mathbf{e}}_h^{j+1}-\tilde{\mathbf{e}}_h^j}{\Delta t}\right\|_{-1}^2\\
&\le C\lambda^{-1}\Delta t\sum_{j=1}^n t_{j+1}\left\|\frac{\tilde{\mathbf{e}}_h^{j+1}-\tilde{\mathbf{e}}_h^j}{\Delta t}\right\|_{0}^2.
\end{aligned}$$

Applying (45) we get

$$\begin{aligned}
\lambda^{-1}\Delta t&\sum_{j=1}^n t_{j+1}\left\|\frac{\tilde{\mathbf{e}}_h^{j+1}-\tilde{\mathbf{e}}_h^j}{\Delta t}\right\|_{0}^2\\
&\le c_0\lambda^{-1}\Bigg(\Delta t\sum_{j=1}^n t_{j+1}\|\tau_h^j\|_0^2+\Delta t\sum_{j=1}^n t_{j+1}\|\nabla(z_h^{j+1}-z_h^j)\|_0^2\\
&\quad+\Delta t\nu\sum_{j=1}^n\|\nabla\tilde{\mathbf{e}}_h^j\|_0^2+\Delta t\delta\sum_{j=1}^n\|\nabla r_h^j\|_0^2\Bigg).
\end{aligned}\tag{87}$$

To conclude we will bound the four terms on the right-hand side of (87). For the first one recalling that $t_{j+1}/t_j\le 2$ for $j\ge 1$ and applying (57) we get

$$\begin{aligned}
\Delta t\sum_{j=1}^n t_{j+1}\|\tau_h^j\|_0^2 &\le C\left(\Delta t^2\nu^4 T K_{4,3}^2+(h^4+\nu^2(\Delta t^2+\delta^2))(\nu^2K_{4,2}^2T+\nu K_{3,2}^2)\right)\\
&\le C\lambda^{-1}\left(C_1\Delta t^2+C_2(h^4+(\nu\delta)^2)\right).
\end{aligned}\tag{88}$$

To bound the second term on the right-hand side of (87) arguing as usual and applying (61),(41) and (31) we get

$$\Delta t \sum_{j=1}^{n} t_{j+1} \|\nabla (z_h^{j+1} - z_h^j)\|_0^2 \leq C t_{n+1} \Delta t \nu^3 K_{3,2}^2 \leq C T C_1 \Delta t \leq C \tilde{C}_1 \Delta t. \quad (89)$$

To conclude we observe that the last two terms in (87) have been bounded in (78). Then

$$\Delta t \nu \sum_{j=1}^{n} \|\nabla \tilde{\mathbf{e}}_h^j\|_0^2 + \Delta t \delta \sum_{j=1}^{n} \|\nabla r_h^j\|_0^2 \leq C \|\tilde{\mathbf{v}}_h^0 - \mathbf{v}(0)\|_0^2 + \tilde{C}_1 \Delta t + \tilde{C}_2 (h^2 + \nu\delta). \quad (90)$$

Thus, inserting (88), (89) and (90) into (87) we have

$$\begin{aligned} \lambda^{-1} \Delta t \sum_{j=1}^{n} t_{j+1} \left\| \frac{\tilde{\mathbf{e}}_h^{j+1} - \tilde{\mathbf{e}}_h^j}{\Delta t} \right\|_0^2 \leq & C \lambda^{-1} \big(\|\tilde{\mathbf{v}}_h^0 - \mathbf{v}(0)\|_0^2 + \tilde{C}_1 \Delta t \big) \\ & + C \lambda^{-1} \big((h^2 + \nu\delta) \tilde{C}_2 + C_1 \Delta t^2 \big) \\ & + \lambda^{-1} C_2 (h^4 + (\nu\delta)^2) \\ \leq & C \lambda^{-1} \big(\|\tilde{\mathbf{v}}_h^0 - \mathbf{v}(0)\|_0^2 + \tilde{C}_1 \Delta t + \tilde{C}_2 (h^2 + \nu\delta) \big). \end{aligned} \quad (91)$$

Finally, inserting (84), (85), (86) and (91) in (83) we obtain for the modified Euler non-incremental method

$$\begin{aligned} \Delta t \sum_{j=1}^{n} t_j \|r_h^j\|_0^2 \leq & C (t_{n+1} \nu + \lambda^{-1}) \|\tilde{\mathbf{v}}_h^0 - \mathbf{v}(0)\|_0^2 + C t_{n+1} \nu \left(\|\tilde{\mathbf{e}}_h^0\|_0^2 + \Delta t^2 \|\nabla r_h^0\|_0^2 \right) \\ & + C \lambda^{-1} \big(\tilde{C}_1 \Delta t + \tilde{C}_2 (h^2 + \nu\delta) \big) \\ & + C (\nu t_{n+1} + \lambda^{-1}) \big(C_1 \Delta t^2 + C_2 (h^4 + (\nu\delta)^2) \big) \\ \leq & \, C (\nu T + \lambda^{-1}) \big(\|\tilde{\mathbf{v}}_h^0 - \mathbf{v}(0)\|_0^2 + \|\tilde{\mathbf{e}}_h^0\|_0^2 + \Delta t^2 \|\nabla r_h^0\|_0^2 \big) \\ & + C (\nu T + \lambda^{-1}) \big(\tilde{C}_1 \Delta t + \tilde{C}_2 (h^2 + \nu\delta) \big) \end{aligned} \quad (92)$$

Now observing that due to (19) we have

$$\|z_h^j - q(t_j)\|_0^2 \leq \nu^2 M_{2,1}^2 (h^2 + \nu\delta) \leq \nu^2 C_2 (h^2 + \nu\delta) \leq (\nu/T) \tilde{C}_2 (h^2 + \nu\delta),$$

for $j = 0, 1, \ldots, N$, applying the triangle inequality in (92), we finally reach (80).
□

Remark 5 Let us observe that any initial approximation for the velocity such that $\|\tilde{\mathbf{v}}_h^0 - \mathbf{v}(0)\|_0 = O(h^2)$ and any initial approximation for the pressure satisfying $\|\nabla r_h^0\|_0 = O(1)$ (which includes the choice $q_h^0 = 0$) keep the optimal rate of convergence for the pressure $O(h^2 + \Delta t)$ for the Euler non-incremental method or $O(h^2 + \delta + \Delta t)$ for the modified Euler non-incremental method.

We also observe that the error bounds (47) and (50) for the Euler non-incremental method hold under assumption (79), $h^2/(\nu\rho_1^2) \leq \Delta t$. This means that for the Euler non-incremental method $\Delta t = O(h)$ could be a possible choice.

On the other hand, the bounds (47) and (50) and (80) for the modified Euler non-incremental method hold only under assumption $\Delta t \leq \delta$. Then, for the modified Euler non-incremental one can choose Δt as small as possible, and, in particular, one can make $\Delta t \to 0$.

5 Navier-Stokes equations

We now consider the following initial value problem associated with the Navier-Stokes equations.

$$\begin{aligned} \partial_t \mathbf{u} - \nu \Delta \mathbf{u} + (\mathbf{u} \cdot \nabla)\mathbf{u} + \nabla p = \boldsymbol{f} \qquad & \text{in } (0,T) \times \Omega, \\ \nabla \cdot \mathbf{u} = 0 \qquad & \text{in } (0,T) \times \Omega, \\ \mathbf{u}(0,\cdot) = \mathbf{u}_0(\cdot) \qquad & \text{in } \Omega, \end{aligned} \tag{93}$$

and its discretization by the modified semi implicit Euler non-incremental method,

$$\begin{aligned} \left(\frac{\tilde{\mathbf{u}}_h^{n+1} - \tilde{\mathbf{u}}_h^n}{\Delta t}, \boldsymbol{\chi}_h\right) + \nu(\nabla \tilde{\mathbf{u}}_h^{n+1}, \nabla \boldsymbol{\chi}_h) + (B(\tilde{\mathbf{u}}_h^n, \tilde{\mathbf{u}}_h^n), \boldsymbol{\chi}_h) + (\nabla p_h^n, \boldsymbol{\chi}_h) \\ = (\mathbf{g}^{n+1}, \boldsymbol{\chi}_h), \qquad \forall \boldsymbol{\chi}_h \in V_h \\ (\nabla \cdot \tilde{\mathbf{u}}_h^{n+1}, \psi_h) + \delta(\nabla p_h^{n+1}, \nabla \psi_h) = 0, \qquad \forall \psi_h \in Q_h, \end{aligned} \tag{94}$$

together with the initial condition to be specified later. In (94) and in the sequel, $B(\cdot,\cdot)$ denotes the following bilinear form

$$B(\mathbf{v}, \mathbf{w}) = \mathbf{u} \cdot \nabla \mathbf{w} + \frac{1}{2}(\nabla \cdot \mathbf{v})\mathbf{w}, \qquad \mathbf{v}, \mathbf{w} \in V.$$

Notice the well-known skew-symmetric property,

$$(B(\mathbf{v},\mathbf{w}),\mathbf{y}) = -(B(\mathbf{v},\mathbf{y}),\mathbf{w}), \qquad \mathbf{v},\mathbf{w},\mathbf{y} \in V, \tag{95}$$

so that in particular, $(B(\mathbf{v},\mathbf{w}),\mathbf{w}) = 0$.

The numerical approximation $(\tilde{\mathbf{u}}_h^n, p_h^n)$ of (94) will be compared with the solution $(\tilde{\mathbf{v}}_h^n, q_h^n)$ of (35)-(36) for

$$\boldsymbol{g} = \mathbf{f} - B(\mathbf{u},\mathbf{u}). \tag{96}$$

On the other hand, along this section we apply to $(\tilde{\mathbf{v}}_h^n, q_h^n)$ the error bounds obtained in the previous section where $(\tilde{\mathbf{v}}_h^n, q_h^n)$ is compared with the stabilized Stokes approximation $(\mathbf{s}_h^n, z_h^n)$ defined in (11)-(12) for

$$\hat{\boldsymbol{g}} = \boldsymbol{g} - \mathbf{u}_t. \tag{97}$$

Whenever $\|\boldsymbol{g}_t\|^2$ is integrable in $(0,T]$, i.e. the constant $\hat{K}_3^2$ in (33) is finite, this approximation will satisfy the error bounds (47), (50) and (80).

In the rest of this section we shall assume that $\mathbf{f}, \mathbf{f}_t, \mathbf{f}_{tt} \in L^2(0,T]$ and that $\mathbf{v} = \mathbf{u}$, $q = p$ satisfies the bounds (28–32). Since we now prove $\hat{K}_3^2$ in (33) is finite all appearances of the constants in (28–33) will be for $\mathbf{v} = \mathbf{u}$ and $q = p$.

To prove $\hat{K}_3^2$ is finite we first observe that $\boldsymbol{g}_t = \mathbf{f}_t - B(\mathbf{u},\mathbf{u})_t \in L^2(0,T)$ if $\mathbf{u}\cdot\nabla\mathbf{u}_t \in L^2(0,T)$ and $\mathbf{u}_t\nabla\mathbf{u} \in L^2(0,T)$. We now show that this is so for the more difficult case $d = 3$. Applying (7) we have

$$\|\mathbf{u}\cdot\nabla\mathbf{u}_t\|_0 \le \|\mathbf{u}\|_\infty \|\nabla\mathbf{u}_t\|_0 \le c_{\mathrm{A}} \|\mathbf{u}\|_1^{1/2} \|\mathbf{u}\|_2^{1/2} \|\nabla\mathbf{u}_t\|_0 \le c_{\mathrm{A}} M_1^{1/2} M_2^{1/2} \|\nabla\mathbf{u}_t\|_0.$$

Similarly, applying Hölder's inequality with $p = d$ and $q = d/(d-1)$, and then (6) and (9), we have

$$\begin{aligned}\|\mathbf{u}_t\cdot\nabla\mathbf{u}\|_0 \le \|\mathbf{u}_t\|_{L^6} \|\nabla\mathbf{u}\|_{L^3} &\le c_1^{3/2} \|\nabla\mathbf{u}_t\|_0 \|\nabla\mathbf{u}\|_0^{1/2} \|\mathbf{u}\|_2^{1/2} \\ &\le c_1^{3/2} M_1^{1/2} M_2^{1/2} \|\nabla\mathbf{u}_t\|_0,\end{aligned}$$

so that $\|\mathbf{u}\cdot\nabla\mathbf{u}_t\|_{L^2(0,T)} + \|\mathbf{u}_t\cdot\nabla\mathbf{u}\|_{L^2(0,T)} \le C\nu(M_1M_2)^{1/2}K_{3,2}$ and consequently $\hat{K}_3^2$ in (33) is finite. We can state the following result.

Theorem 3 *Let $(\mathbf{u},p)$ be the solution of (93) and let $(\tilde{\mathbf{v}}_h^n, q_h^n)$ be the solution of (35)-(36) with $\boldsymbol{g}$ defined in (96) and $(\tilde{\mathbf{v}}_h^0, q_h^0) = (\mathbf{s}_h^0, z_h^0)$. Under the assumptions of Theorem 1 the following bounds hold*

$$\max_{0\le t_n\le T} \|\tilde{\mathbf{v}}_h^n - \mathbf{u}(t_n)\|_0^2 \le C_1(\Delta t)^2 + C_2(h^4 + (\nu\delta)^2), \tag{98}$$

where C_1 and C_2 are the constants in (48) and (49), and, assuming for simplicity that (46) holds,

$$\Delta t \sum_{j=1}^{n} \big(\nu\|\nabla(\tilde{\mathbf{v}}_h^j - \mathbf{u}(t_j))\|_0^2 + \delta\|q_h^j - p(t_j)\|_0^2\big) \le \tilde{C}_1 \Delta t + \tilde{C}_2(h^2 + \nu\delta), \quad (99)$$

$$\Delta t \sum_{j=1}^{n} t_j\big(\nu\|\nabla(\tilde{\mathbf{v}}_h^j - \mathbf{u}(t_j))\|_0^2 + \delta\|q_h^j - p(t_j)\|_0^2\big) \le C_1 t_n \Delta t^2 + \tilde{C}_2 t_n(h^2 + \nu\delta). \quad (100)$$

where $\tilde{C}_1$ and $\tilde{C}_2$ are the constants in (51) and (52).

Proof The error bounds (98), (99) follow from (47) and (50) respectively taking into account that applying (19) and (29) $\|\tilde{\mathbf{v}}_h^0 - \mathbf{u}(0)\|_0^2 \le CM_{2,1}^2(h^4 + \delta^2\nu^2)$ and $\nu\|\nabla(\tilde{\mathbf{v}}_h^0 - \mathbf{u}(0))\|_0^2 \le C\nu M_{2,1}^2(h^2 + \delta\nu)$ and that $\|\nabla r_h^0\|_0 = 0$. The error bound (100) follows from (77), the decomposition $\tilde{\mathbf{v}}_h^j - \mathbf{u}(t_j) = \tilde{\mathbf{e}}_h^j + \mathbf{s}_h(t_j) - \mathbf{u}(t_j)$ and $q_h^j - p(t_j)) = r_h^j + z_h(t_j) - p(t_j)$ and the error bound (19). □

The error bounds for the discretization (94) will be obtained as a consequence of several previous results that we now state. The first one is a discrete Gronwall lemma whose proof can be easily obtained by induction (see e.g., [16]).

Lemma 4 *Let k, B, and a_n, b_n, c_n, γ_n be nonnegative numbers such that*

$$a_n + k\sum_{j=0}^{n} b_j \le k\sum_{j=0}^{n-1} \gamma_j a_j + k\sum_{j=0}^{n} c_j + B, \quad n \ge 0.$$

Then, the following bound holds

$$a_n + k\sum_{j=0}^{n} b_j \le \exp\bigg(k\sum_{j=0}^{n-1} \gamma_j\bigg)\bigg(k\sum_{j=0}^{n} c_j + B\bigg), \quad n \ge 0.$$

Remark 6 The statement of Lemma 4 above is very similar to Lemma 5.1 in [16], where the sum involving the terms $\gamma_j a_j$ includes also the term $\gamma_n a_n$. In order to extend the analysis in the present paper to the fully implicit backward Euler method, Lemma 4 must be replaced by [16, Lemma 5.1].

Lemma 5 *For* $\mathbf{v}, \mathbf{w}, \boldsymbol{\phi} \in V$ *the following bounds hold*

$$\begin{aligned}\|B(\mathbf{v},\mathbf{v}) - B(\mathbf{w},\mathbf{w})\|_0 \leq & \big(\|\nabla\mathbf{v}\|_{L^{2d/(d-1)}} + \|\nabla\mathbf{w}\|_{L^{2d/(d-1)}}\big)\, \|\mathbf{e}\|_{L^{2d}} \\ & + \big(\|\mathbf{v}\|_\infty + \|\mathbf{w}\|_\infty\big)\, \|\nabla\mathbf{e}\|_0 \,. \end{aligned} \tag{101}$$

$$\begin{aligned}|(B(\mathbf{v},\mathbf{v}) - B(\mathbf{w},\mathbf{w}), \boldsymbol{\phi})| \leq \|\mathbf{e}\|_0 \Big(& \big(\|\nabla\mathbf{v}\|_{L^{2d/(d-1)}} + \|\nabla\mathbf{w}\|_{L^{2d/(d-1)}}\big)\, \|\boldsymbol{\phi}\|_{L^{2d}} \\ & + \big(\|\mathbf{v}\|_\infty + \|\mathbf{w}\|_\infty\big)\, \|\nabla\boldsymbol{\phi}\|_0\Big), \end{aligned} \tag{102}$$

where $\boldsymbol{e} = \mathbf{v} - \mathbf{w}$.

Proof From the identity

$$B(\mathbf{v},\mathbf{v}) - B(\mathbf{w},\mathbf{w}) = B(\mathbf{e},\mathbf{v}) + B(\mathbf{w},\mathbf{e}), \tag{103}$$

and applying Hölder inequality we have

$$\begin{aligned}|B(\mathbf{e},\mathbf{v})| &\leq \|\mathbf{e}\|_{L^{2d}}\, \|\nabla\mathbf{v}\|_{L^{2d/(d-1)}} + \frac{1}{2}\, \|\nabla\cdot\mathbf{e}\|_0\, \|\mathbf{v}\|_\infty \,, \\ |B(\mathbf{w},\mathbf{e})| &\leq \|\mathbf{w}\|_\infty\, \|\nabla\mathbf{e}\|_0 + \frac{1}{2}\, \|\nabla\cdot\mathbf{w}\|_{L^{2d/(d-1)}}\, \|\mathbf{e}\|_{L^{2d}} \,, \end{aligned}$$

and the bound (101) follows. To prove (102), we multiply (103) by $\boldsymbol{\phi} \in H_0^1$ and integrate in Ω, integrating by parts adequately and using the skew-symmetry property (95) we have

$$\begin{aligned}(B(\mathbf{v},\mathbf{v}) - B(\mathbf{w},\mathbf{w}), \boldsymbol{\phi}) = & \frac{1}{2}\big((\mathbf{e}\cdot\nabla\mathbf{v}, \boldsymbol{\phi}) - (\mathbf{e}\cdot\nabla\boldsymbol{\phi}, \mathbf{v})\big) \\ & + (B(\mathbf{w},\boldsymbol{\phi}), \mathbf{e}). \end{aligned}$$

and the bound follows by applying Hölder inequality (8). □

Lemma 6 *Let* $(\tilde{\mathbf{v}}_h^n, q_h^n)$ *be the solution of (35)-(36) with* $\boldsymbol{g}$ *defined in (96) and* $(\tilde{\mathbf{v}}_h^0, q_h^0) = (\mathbf{s}_h^0, z_h^0)$*. Under the assumptions of Theorem 1, and assuming also*

$$\nu\delta \leq c_M \mathrm{diam}(\Omega) h, \tag{104}$$

for a scale-invariant $c_M > 0$*, there exists a scale invariant constant* $c_{\mathrm{r}} > 0$ *depending on the constants* c_{inv}*,* c_P *and* c_A *in (2), (5) and (7), respectively, and constant* $C_{\mathrm{th}} > 0$ *depending also on the constants in (99),* ν^{-1}*,* T*, the* *constants M_1 and M_2 in (28), the constant $c_{\rm int}$ in (3), and the constant Λ in (1) (and also on $\max_{0\le t\le T}\|\mathbf{u}(t)\|_0$ in the case $d=2$) such that the following bounds hold for all sequences $(\mathbf{w}_h^n)_{n=0}^{N=T/\Delta t}$ in V_h,*

$$\Delta t\sum_{j=0}^n \|\nabla \mathbf{w}_h^j\|^2_{L^{2d/(d-1)}} \le \frac{c_{\rm r}\Lambda}{h}\Delta t\sum_{j=0}^n \|\nabla(\mathbf{w}_h^j-\tilde{\mathbf{v}}_h^j)\|_0^2 + C_{\rm th},$$

$$\Delta t\sum_{j=0}^n \|\mathbf{w}_h^j\|^2_{\infty} \le \left(\frac{c_{\rm r}\Lambda}{h}\Delta t\sum_{j=0}^n \|\nabla(\mathbf{w}_h^j-\tilde{\mathbf{v}}_h^j)\|_0^2 + C_{\rm th}\right)|\Omega|^{(3-d)/(2d)},$$

and $n=0,1,\ldots,N=T/\Delta t$.

Proof We start with the first bound. We write

$$\mathbf{w}_h^j = (\mathbf{w}_h^j-\tilde{\mathbf{v}}_h^j) + (\tilde{\mathbf{v}}_h^j - I_h(\mathbf{u}(t_j))) + (I_h(\mathbf{u}(t_j))) - \mathbf{u}(t_j)) + \mathbf{u}(t_j) \qquad (105)$$

We notice that for $p=2d/(d-1)$ and $q=2$ we have

$$d\left(\frac{1}{q}-\frac{1}{p}\right) = d\left(\frac{1}{2}-\frac{d-1}{2d}\right) = \frac{1}{2},$$

so that applying (2) with $m=1$, $p=2d/(d-1)$, $q=2$ and $l=1$, and using (1) and (105) we get

$$\begin{aligned}\left\|\nabla \mathbf{w}_h^j\right\|_{L^{2d/(d-1)}} \le & c_{\rm inv}\frac{\|\mathbf{w}_h^j-\tilde{\mathbf{v}}_h^j\|_1 + \|\tilde{\mathbf{v}}_h^j - I_h(\mathbf{u}(t_j))\|_1}{(h/\Lambda)^{1/2}} \\ & + \|\nabla(I_h(\mathbf{u}(t_j)) - \mathbf{u}(t_j))\|_{L^{2d/(d-1)}} + \|\nabla \mathbf{u}(t_j)\|_{L^{2d/(d-1)}}.\end{aligned}$$

We notice that due to the interpolation bound (3) we have

$$\|\nabla(I_h(\mathbf{u}(t_j)) - \mathbf{u}(t_j))\|_{L^{2d/(d-1)}} \le c_{\rm int}h^{1/2}\left\|\mathbf{u}(t_j)\right\|_2 \le c_{\rm int}h^{1/2}M_2,$$

and due to (9), $\|\nabla \mathbf{u}(t_j)\|_{L^{2d/(d-1)}} \le (c_1\left\|\nabla\mathbf{u}(t_j)\right\|_0\left\|\mathbf{u}(t_j)\right\|_2)^{1/2} \le (c_1M_1M_2)^{1/2}$. The proof is finished by writing $\tilde{\mathbf{v}}_h^j - I_h(\mathbf{u}(t_j)) = (\tilde{\mathbf{v}}_h^j - \mathbf{u}(t_j)) + (\mathbf{u}(t_j) - I_h(\mathbf{u}(t_j)))$ and applying (3) and (99).

For the second bound we observe that from [15, Lemma 4.4] it follows $\|\mathbf{w}_h\|_\infty \le Ch^{-1/2}\|\nabla\mathbf{w}_h\|_1\,|\Omega|^{(3-d)/(2d)}$, where C depends on $c_{\rm inv}$ and c_A, and then we argue as before. □

In the sequel, for sequences $(\mathbf{w}_h^n)_{n=0}^N$ of $N+1$ terms in V_h we denote

$$|||(\mathbf{w}_h^n)_{n=0}^N|||_{\delta,\Delta t} = \max_{0\le n\le N}\Big(\|\mathbf{w}_h^n\|_0^2 + \Delta t\sum_{j=0}^{n-1}\big(\nu\|\nabla\mathbf{w}_h^{j+1}\|_0^2 + \delta\|\nabla Z_h\mathbf{w}_h^{j+1}\|_0^2\big)\Big)^{1/2},$$

where the mapping $Z_h : V_h \to Q_h$ is defined for every $\mathbf{w}_h \in V_h$ as the solution of

$$\delta(\nabla Z_h\mathbf{w}_h, \nabla\psi_h) = -(\nabla\cdot\mathbf{w}_h, \psi_h), \qquad \forall\psi_h \in Q_h.$$

The following result establishes the stability of discretization (94) restricted to h-dependent thresholds, a concept due to López-Marcos and Sanz-Serna [18] (see also [19]).

Lemma 7 *Fix $\Gamma_1 > 0$, $\rho_1 > 0$ and $\Lambda \ge 1$, and let $(\tilde{\mathbf{v}}_h^n, q_h^n)$ be the solution of (35)-(36) with $\boldsymbol{g}$ defined in (96) and initial condition $(\mathbf{v}_h^0, q_h^0) = (\mathbf{s}_h^0, z_h^0)$. Then, under the assumptions of Lemma 6, there exist positive constants h_0 and S (the stability constant given by (110) below) such that for any $h \le h_0$, and any two sequences $(\mathbf{w}_{1,h}^n)_{n=1}^N$ and $(\mathbf{w}_{2,h}^n)_{n=0}^N$ in V_h satisfying the threshold condition*

$$\Big(\Delta t\sum_{j=0}^{N}\nu\|\nabla(\mathbf{w}_{i,h}^j - \tilde{\mathbf{v}}_h^j)\|_0^2\Big)^{1/2} \le \Gamma_1 h^{1/2}, \quad i=1,2, \quad \Delta t = T/N. \qquad (106)$$

the following bound holds

$$|||(\mathbf{w}_h^n)_{n=0}^N|||_{\delta,\Delta t} \le S\Big(\|\mathbf{w}_h^0\|_0^2 + \Delta t\sum_{j=1}^{N}\frac{1}{\nu}\|\boldsymbol{\tau}_h^j\|_{-1}^2\Big)^{1/2}, \qquad (107)$$

where, for $n = 0,1,\ldots,N$, $\mathbf{w}_h^n = \mathbf{w}_{h,1}^n - \mathbf{w}_{h,2}^n$, and $\boldsymbol{\tau}_h^n$ is defined by

$$\begin{aligned}(\boldsymbol{\tau}_h^n, \boldsymbol{\chi}_h) =& \Big(\frac{\mathbf{w}_h^n - \mathbf{w}_h^{n-1}}{\Delta t}, \boldsymbol{\chi}_h\Big) + \nu(\nabla\mathbf{w}_h^n, \nabla\boldsymbol{\chi}_h) + (\nabla Z_h\mathbf{w}_h^{n-1}, \boldsymbol{\chi}_h)\\ &+ \big(B(\mathbf{w}_{1,h}^{n-1}, \mathbf{w}_{1,h}^{n-1}) - B(\mathbf{w}_{2,h}^{n-1}, \mathbf{w}_{2,h}^{n-1}), \boldsymbol{\chi}_h\big), \qquad \boldsymbol{\chi}_h \in V_h.\end{aligned}$$

Proof Applying (42) with $d_h^n = 0$ and

$$\mathbf{b}_h^n = \boldsymbol{\tau}_h^{n+1} - \big(B(\mathbf{w}_{1,h}^n, \mathbf{w}_{1,h}^n) - B(\mathbf{w}_{2,h}^n, \mathbf{w}_{2,h}^n)\big),$$

for $n = 0, 1, \ldots, N$, we have that the left hand side of (107) can be bounded by

$$\|\mathbf{w}_h^0\|_0^2 + \Delta t \sum_{j=1}^{n} \frac{1}{\nu} \|\boldsymbol{\tau}_h^j\|_{-1}^2 + \Delta t \sum_{j=0}^{n-1} \frac{1}{\nu} \left\| B(\mathbf{w}_{1,h}^j, \mathbf{w}_{1,h}^j) - B(\mathbf{w}_{2,h}^j, \mathbf{w}_{2,h}^j) \right\|_{-1}^2. \quad (108)$$

We will now show that for some positive $\gamma_0, \ldots, \gamma_{N-1}$ and $L > 0$ satisfying

$$\Delta t \sum_{j=0}^{n-1} \gamma_j \leq L,$$

the last sum in (108) can be bounded as

$$\Delta t \sum_{j=0}^{n-1} \frac{1}{\nu} \left\| B(\mathbf{w}_{1,h}^j, \mathbf{w}_{1,h}^j) - B(\mathbf{w}_{2,h}^j, \mathbf{w}_{2,h}^j) \right\|_{-1}^2 \leq \Delta t \frac{1}{\nu} \sum_{j=0}^{n-1} \gamma_j \|\mathbf{w}_h^j\|_0^2, \quad (109)$$

so that applying Lemma 4 the proof will be finished. We do this for the more difficult case $d = 3$. For $\boldsymbol{\phi} \in H_0^1(\Omega)^3$, applying (102) we have

$$\begin{aligned} \big(B(\mathbf{w}_{1,h}^j, \mathbf{w}_{1,h}^j) - B(\mathbf{w}_{2,h}^j, \mathbf{w}_{2,h}^j), \boldsymbol{\phi}\big) \leq \|\mathbf{w}_h^j\|_0 \Big(&\big(\|\mathbf{w}_{1,h}^j\|_\infty + \|\mathbf{w}_{2,h}^j\|_\infty\big) \|\nabla \boldsymbol{\phi}\|_0 \\ &+ \big(\|\nabla \mathbf{w}_{1,h}^j\|_{L^{2d/(d-1)}} + \|\nabla \mathbf{w}_{2,h}^j\|_{L^{2d/(d-1)}}\big) \|\boldsymbol{\phi}\|_{L^{2d}} \Big). \end{aligned}$$

Applying Sobolev's inequality we have that $\|\boldsymbol{\phi}\|_{L^{2d}} \leq c_1 \|\boldsymbol{\phi}\|_1$, so that, we can take

$$\gamma_j = 2\big(\|\mathbf{w}_{1,h}^j\|_\infty^2 + \|\mathbf{w}_{2,h}^j\|_\infty^2\big) + c_1^2\big(\|\nabla \mathbf{w}_{1,h}^j\|_{L^{2d/(d-1)}} + \|\nabla \mathbf{w}_{2,h}^j\|_{L^{2d/(d-1)}}\big),$$

which, in view of Lemma 6 and the threshold condition (106) we see that (109) follows with

$$L = 4\big(c_{\mathrm{r}} \Gamma_1^2 \nu^{-1} \Lambda + C_{\mathrm{th}}\big)(1 + c_1^2).$$

Thus, we have that the statement of the Lemma holds with

$$S = \exp(L/\nu). \quad (110)$$

□

To proof the convergence of the numerical approximation $(\tilde{\mathbf{u}}_h^n)_{n=0}^N$ we will apply the following result due to Stetter [26, Lemma 1.2.2].

Lemma 8 *Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be two normed linear spaces with the same finite dimension. Let $F : X \to Y$ be a mapping continuous in $B_X(x_u, r_1) = \{x \in X \mid \|x - x_u\|_X < r_1\}$, for which there exist $S > 0$ and $r_2 > 0$ such that*

$$\|x_1 - x_2\|_X \leq S \|F(x_1) - F(x_2)\|_Y, \tag{111}$$

for every $x_1, x_2 \in B_X(x_u, r_1)$ satisfying $\|F(x_j) - F(x_u)\|_Y \leq r_2$, for $j = 1, 2$.

Then, for $r_0 = \min(r_2, r_1/S)$ the mapping F^{-1} exists and is Lipschitz-continuous in $B_Y(F(x_u), r_0)$ with Lipschitz constant equal to S.

Before applying Lemma 8 we need to prove a consistency result.

Lemma 9 *Let $(\mathbf{u}, p)$ be the solution of (93) and let $(\tilde{\mathbf{v}}_h^n, q_h^n)$ be the solution of (35)-(36) with $\boldsymbol{g}$ defined in (96) and $(\tilde{\mathbf{v}}_h^0, q_h^0) = (\mathbf{s}_h^0, z_h^0)$. Then, there exists a positive constant C_B, depending on the Sobolev's constant c_1, c_{A} in (7), C_{th} in Lemma 6, M_1, M_2 in (28) and the ratio Λ in (1), such that the truncation error*

$$\boldsymbol{\tau}_h^n = P_{V_h}\big(B(\tilde{\mathbf{v}}_h^{n-1}, \tilde{\mathbf{v}}_h^{n-1}) - B(\mathbf{u}(t_n), \mathbf{u}(t_n))\big), \qquad n = 0, \ldots, N-1$$

satisfies the following bounds

$$\Delta t \sum_{j=1}^{n} \|\boldsymbol{\tau}_h^j\|_0^2 \leq C_B^2 \left(\Delta t \sum_{j=0}^{n-1} \|\nabla(\tilde{\mathbf{v}}_h^j - \mathbf{u}(t_j))\|_0^2 + \nu^2 \Delta t^2 K_{3,2}^2 \right),$$

$$\Delta t \sum_{j=1}^{n} \|\boldsymbol{\tau}_h^j\|_{-1}^2 \leq C_B^2 \left(t_n \max_{0 \leq j \leq n-1} \|\tilde{\mathbf{v}}_h^j - \mathbf{u}(t_j)\|_0^2 + \nu^2 \Delta t^2 K_{2,2}^2 \right).$$

Proof We concentrate on the more difficult case $d = 3$. We write $\boldsymbol{\tau}_h^n = \boldsymbol{\tau}_{1,h}^n + \boldsymbol{\tau}_{2,h}^n$, were

$$\begin{aligned} \boldsymbol{\tau}_{1,h}^n &= P_{V_h}\big(B(\tilde{\mathbf{v}}_h^{n-1}, \tilde{\mathbf{v}}_h^{n-1}) - B(\mathbf{u}(t_{n-1}), \mathbf{u}(t_{n-1}))\big), \\ \boldsymbol{\tau}_{2,h}^n &= P_{V_h}\big(B(\mathbf{u}(t_{n-1}), \mathbf{u}(t_{n-1})) - B(\mathbf{u}(t_n), \mathbf{u}(t_n))\big). \end{aligned}$$

For $\boldsymbol{\tau}_{1,h}^n$, applying Lemma 5 and denoting $\mathbf{e} = \tilde{\mathbf{v}}_h^{n-1} - \mathbf{u}(t_{n-1})$, we have that it can be bounded by

$$\begin{aligned} \|\boldsymbol{\tau}_{1,h}\|_0 \leq \|\mathbf{e}\|_{L^{2d}} \left(\|\nabla \mathbf{u}(t_{n-1})\|_{L^{2d/(d-1)}} + \|\nabla \tilde{\mathbf{v}}_h^{n-1}\|_{L^{2d/(d-1)}} \right) \\ + \|\nabla \mathbf{e}\|_0 \left(\|\mathbf{u}(t_{n-1})\|_\infty + \left\|\tilde{\mathbf{v}}_h^{n-1}\right\|_\infty \right). \end{aligned}$$

Arguing similarly with $\boldsymbol{\tau}_{2,h}^n$, and denoting $\hat{\mathbf{e}} = \tilde{\mathbf{u}}(t_{n-1}) - \mathbf{u}(t_n)$, it can be bounded by

$$\begin{aligned}\|\boldsymbol{\tau}_{2,h}\|_0 \leq & \|\hat{\mathbf{e}}\|_{L^{2d}}\big(\|\nabla \mathbf{u}(t_n)\|_{L^{2d/(d-1)}} + \|\nabla \mathbf{u}(t_{n-1})\|_{L^{2d/(d-1)}}\big) \\ & + \|\nabla \hat{\mathbf{e}}\|_0 \left(\|\mathbf{u}(t_n)\|_\infty + \|\mathbf{u}(t_{n-1})\|_\infty\right).\end{aligned}$$

Applying Agmon's inequality (7), (9) and Lemma 6, and noticing that due to Hölder's inequality we can write

$$\|\nabla \hat{\mathbf{e}}\|_0 = \left\|\int_{t_{n-1}}^{t_n} \nabla \mathbf{u}_t(t)\, dt\right\|_0 \leq \Delta t^{1/2}\left(\int_{t_{n-1}}^{t_n} \|\nabla \mathbf{u}_t\|_0^2\, dt\right)^{1/2},$$

then it follows that

$$\|\boldsymbol{\tau}_h^n\|_0 \leq C_B^0 \left(\|\nabla(\tilde{\mathbf{v}}_h^{n-1} - \mathbf{u}(t_{n-1}))\|_0 + \Delta t^{1/2}\left(\int_{t_{n-1}}^{t_n} \|\nabla \mathbf{u}_t\|_0^2\, dt\right)^{1/2}\right),$$

where

$$C_B^0 = 2(c_{\mathrm{A}} + c_1^{3/2})(M_1 M_2)^{1/2} + C_{th}(1 + c_1). \tag{112}$$

Recalling now the definition of $K_{3,2}$ in (31), the bound for the L^2 norm follows with constant $C_B = C_B^0\sqrt{2}$.

To prove the estimate in the negative norm, we recall that due to Sobolev's inequality we have that for $\boldsymbol{\phi} \in H^1(\Omega)^3$, we have that $\|P_{V_h}\boldsymbol{\phi}\|_{L^6} \leq c_1 \|P_{V_h}\boldsymbol{\phi}\|_1$. Taking into account that $\|P_{V_h}\boldsymbol{\phi}\|_1 \leq C\|\boldsymbol{\phi}\|_1$ from (102) it follows that

$$\begin{aligned}\|\boldsymbol{\tau}_{1,h}^n\|_{-1} \leq & C\,\|\mathbf{e}\|_0 \left(c_1 \|\nabla \mathbf{u}(t_{n-1})\|_{L^{2d/(d-1)}} + \|\mathbf{u}(t_{n-1})\|_\infty\right) \\ & + \|\mathbf{e}\|_0 \left(\|\tilde{\mathbf{v}}_h^{n-1}\|_\infty + c_1\|\nabla \tilde{\mathbf{v}}_h^{n-1}\|_{L^{2d/(d-1)}}\right),\end{aligned}$$

and a similar result for $\boldsymbol{\tau}_{2,h}^n$ with $\tilde{\mathbf{v}}_h^{n-1}$ replaced by $\mathbf{u}(t_n)$, from where the result for the negative norm follows easily with a constant C_B^1 proportional to C_B^0 in (112). The proof of the lemma finishes taking $C_B = \max(C_B^0, C_B^1)$. □

Theorem 4 *Under the assumptions of Lemma 6, assuming also (104), then the solution* $(\tilde{\mathbf{u}}_h^n, p_h^n)$ *of (94) with initial condition* $(\tilde{\mathbf{u}}_h^0, p_h^0) = (\mathbf{s}_h^0, z_h^0)$ *satisfies the following bounds for* $n = 1, \ldots, N$ *and for* h *small enough:*

$$t_n\|\tilde{\mathbf{u}}_h^n - \mathbf{u}(t_n)\|_0^2 \ \leq \ \hat{C}_1 t_n \Delta t^2 + \hat{C}_2 t_n (h^4 + (\nu\delta)^2)) \tag{113}$$

where

$$\hat{C}_1 = \bigl(1 + (SC_B)^2 T\nu^{-1}\bigr)C_1 + \nu(SC_B)^2 K_{2,2}^2 \tag{114}$$
$$\hat{C}_2 = \bigl(1 + (SC_B)^2 T\nu^{-1}\bigr)C_2, \tag{115}$$

C_1 and C_2 being the constants in (48) and (49). Also, assuming for simplicity that (46) holds,

$$\Delta t \sum_{j=1}^{n} \bigl(\nu\|\nabla(\tilde{\mathbf{u}}_h^j - \mathbf{u}(t_j))\|_0^2 + \delta\|\nabla(q_h^j - q(t_j))\|_0^2\bigr) \le \tilde{C}_1\Delta t + \tilde{C}_2(h^2 + \nu\delta) + \hat{C}_1\Delta t^2 + \hat{C}_2(h^4 + (\nu\delta)^2), \tag{116}$$

where $\tilde{C}_1$ and $\tilde{C}_2$ are the constants in (51) and (52).

Proof We apply Lemma 8 with $X = Y = V_h^{N+1}$, $\|\cdot\|_X = |\!|\!| \cdot |\!|\!|_{\delta,\Delta t}$,

$$\|(\boldsymbol{\tau}_h^n)_{n=0}^N\|_Y = \Bigl(\|\boldsymbol{\tau}_h^0\|_0^2 + \Delta t \sum_{j=1}^{N} \frac{1}{\nu}\|\boldsymbol{\tau}_h^j\|_{-1}^2\Bigr)^{1/2}.$$

and $x_u = (\tilde{\mathbf{v}}_h^n)_{n=0}^N$ where $(\tilde{\mathbf{v}}_h^n, q_h^n)$ is the solution of (35)-(36) with $\boldsymbol{g}$ defined in (96) and initial condition $(\tilde{\mathbf{v}}_h^0, q_h^0) = (\mathbf{s}_h^0, z_h^0)$. We take $r_2 = 1$ and $r_1 = \Gamma_1 h^{1/2}$, and F defined by $F((\mathbf{w}_h^n)_{n=0}^N) = (\mathbf{F}_h^n)_{n=0}^N$ where

$$\mathbf{F}_h^0 = \mathbf{w}_h^0 - \tilde{\mathbf{v}}_h^0, \tag{117}$$

and for $n = 1, \ldots, N$, $\mathbf{F}_h^n$ is the element in V_h satisfying

$$(\mathbf{F}_h^n, \boldsymbol{\chi}_h) = \Bigl(\frac{\mathbf{w}_h^n - \mathbf{w}_h^{n-1}}{\Delta t}, \boldsymbol{\chi}_h\Bigr) + \nu(\nabla\mathbf{w}_h^n, \nabla\boldsymbol{\chi}_h) + (\nabla Z_h\mathbf{w}_h^{n-1}, \boldsymbol{\chi}_h) + \bigl(B(\mathbf{w}_h^{n-1}, \mathbf{w}_h^{n-1}) - \boldsymbol{f}(t_n), \boldsymbol{\chi}_h\bigr), \qquad \boldsymbol{\chi}_h \in V_h. \tag{118}$$

We notice that the truncation error $(\boldsymbol{\tau}_h^n)_{n=0}^N = F((\tilde{\mathbf{v}}_h^n)_{n=0}^N)$ is

$$\boldsymbol{\tau}_h^0 = 0,$$
$$\boldsymbol{\tau}_h^n = P_{V_h}\bigl(B(\tilde{\mathbf{v}}_h^{n-1}, \tilde{\mathbf{v}}_h^{n-1}) - B(\mathbf{u}(t_n), \mathbf{u}(t_n))\bigr), \qquad n = 1, \ldots, N.$$

The assumption (111) holds due to Lemma 7.

For the mapping F defined in (117)-(118) from Lemma 9 and (98) it follows that

$$\|F((\tilde{\mathbf{v}}_h^n)_{n=0}^N)\|_Y \le C_B\Big((T\nu^{-1})\big(C_1\Delta t^2 + C_2(h^4 + (\nu\delta)^2)\big) + \nu\Delta t^2 K_{2,2}^2\Big)^{1/2}.$$

Then, in view of condition (104), we have

$$\|F((\tilde{\mathbf{v}}_h^n)_{n=0}^N)\|_Y \le C_B\Big((T\nu^{-1})\big(C_1+C_2\nu^2\big)+\nu K_{2,2}^2\Big)^{1/2}\mathrm{diam}(\Omega)c_M\nu^{-1}h+O(h^2).$$

and, thus, decays faster with h than $r_1 = \Gamma_1 h^{1/2}$. Consequently, for h sufficiently small, the null element in V_h^{N+1} belongs to the ball centered at $F((\tilde{\mathbf{v}}_h^n)_{n=0}^N)$ and with radius $\min(r_2, r_1/S)$. Observe that the null element is the image by F of the numerical approximation with initial condition $(\tilde{\mathbf{u}}_h^0, p_h^0) = (\mathbf{s}_h^0, z_h^0)$, that is $0 = F((\tilde{\mathbf{u}}_h^n)_{n=0}^N)$. Since, according to Lemma 8, the mapping F has inverse in this ball, then the differences $\tilde{\boldsymbol{\epsilon}}_h^n = \tilde{\mathbf{u}}_h^n - \tilde{\mathbf{v}}_h^n$, $n = 0, \ldots, N$ satisfy the bound

$$\begin{aligned}|||(\tilde{\boldsymbol{\epsilon}}_h^n)_{n=0}^N|||_{\delta,\Delta t} &\le S\|F((\tilde{\mathbf{v}}_h^n)_{n=0}^N) - 0\|_Y \\ &\le SC_B\Big((T\nu^{-1})\big(C_1\Delta t^2 + C_2(h^4 + (\nu\delta)^2)\big) + \nu\Delta t^2\ K_{2,2}^2\Big)^{1/2}\end{aligned} \tag{119}$$

Let us denote by $\tilde{\boldsymbol{\epsilon}}^n = \tilde{\mathbf{u}}_h^n - \mathbf{u}(t_n)$ and $\varrho^n = p_h^n - p(t_n)$, $n = 0, \ldots, N$. The proof is finished by writing $\tilde{\boldsymbol{\epsilon}}^n = \tilde{\boldsymbol{\epsilon}}_h^n + (\tilde{\mathbf{v}}_h^n - \mathbf{u}(t_n))$, and $\varrho_h^n = \varrho^n + (q_n^n - p(t_n))$, $n = 0, 1, \ldots, N$ and applying the bounds (98) and (99). $\square$

Remark 7 Although we have analyzed a semi implicit method, the analysis, with some minor changes that we now comment, applies also to the fully implicit backward Euler method. First, using Lemma 5.1 in [16] instead of Lemma 4, Lemma 7 can be easily extended to the fully implicit method. Also, in the case of the fully implicit method, the truncation error $\boldsymbol{\tau}_h^n$ in Lemma 9 would reduce to $\boldsymbol{\tau}_{1,h}^{n+1}$, so that the constant C_B can be taken smaller. However, in the case of the fully implicit method, existence of the numerical solution has to be proved, but this, as the arguments leading to (119) above show, would be a consequence of the null element belonging to the ball in V_h^{N+1} centered at $F((\tilde{\mathbf{v}}_h^n)_{n=0}^N)$ and with radius $\min(r_2, r_1/S)$, where the inverse of F exists. Taking into account these three details, the reader will find no difficulty in extending the results of this paper to the fully implicit method.

Remark 8 Let us observe that for $(\tilde{\mathbf{u}}_h^n, p_h^n)$ we take as initial condition $(\tilde{\mathbf{u}}_h^0, p_h^0) = (\tilde{\mathbf{v}}_h^0, q_h^0)$. Although for simplicity we have assumed in Theorem 3 that $\tilde{\mathbf{v}}_h^0 = \mathbf{s}_h^0$, $q_h^0 = z_h^0$, with $(\mathbf{s}_h^n, q_h^n)$ the stabilized Stokes approximation defined in (11)-(12) with $\boldsymbol{g}$ defined in (97) other initial approximations can be chosen. For example, we can choose $\tilde{\mathbf{v}}_h^0 = I_h \mathbf{u}_0$, the interpolant of the initial velocity, and define q_h^0 as the pressure obtained solving (36) for $n = -1$. With this choice the initial pressure error $r_h^0 = q_h^0 - z_h^0$ satisfies

$$(\nabla \cdot (I_h \mathbf{u}_0 - \mathbf{s}_h^0), \psi_h) = \delta(\nabla r_h^0, \nabla \psi_h), \quad \forall \psi_h \in Q_h,$$

from which $\|\nabla r_h^0\|_0 \leq \delta^{-1} \|I_h \mathbf{u}_0 - \mathbf{s}_h^0\|_0$. Now, since both δ and $\|I_h \mathbf{u}_0 - \mathbf{s}_h^0\|_0$ are $O(h^2)$ we get $\|\nabla r_h^0\|_0$ is bounded by a constant. In view of (47)-(50) this choice keeps the optimal rate of convergence for the approximation $(\tilde{\mathbf{v}}_h^n, q_h^n)$ since the term $\Delta t^2 \|\nabla r_h^0\|_0^2$ is $O(\Delta t^2)$.

To conclude we obtain a bound for the pressure. We need a previous result that we now state

Lemma 10 *Let $(\tilde{\mathbf{u}}_h^n, p_h^n)$ be the solution of (94) with $(\tilde{\mathbf{u}}_h^0, p_h^0) = (\mathbf{s}_h^0, z_h^0)$ and assume δ satisfies condition (104). Then, there exists a positive constant C_B^*, depending on the Sobolev's constant c_1, $c_{\rm A}$ in (7), C_{th} and $c_{\rm r}$ in Lemma 6, M_1, M_2 in (28), the ratio Λ in (1) and the constants $\hat{C}_1^n$ and $\hat{C}_2$ in (114–115), such that that the error*

$$(\boldsymbol{\tau}^*)_h^n = P_{V_h}\big(B(\tilde{\mathbf{u}}_h^{n-1}, \tilde{\mathbf{u}}_h^{n-1}) - B(\mathbf{u}(t_n), \mathbf{u}(t_n))\big), \qquad n = 1, \ldots, N \quad (120)$$

satisfies the following bounds

$$\Delta t \sum_{j=1}^{n} \|(\boldsymbol{\tau}^*)_h^j\|_0^2 \leq (C_B^*)^2 \left(\Delta t \sum_{j=0}^{n-1} \|\nabla(\tilde{\mathbf{u}}_h^j - \mathbf{u}(t_j))\|_0^2 + \nu^2 \Delta t^2 K_{3,2}^2 \right),$$

$$\Delta t \sum_{j=1}^{n} \|(\boldsymbol{\tau}^*)_h^j\|_{-1}^2 \leq (C_B^*)^2 \left(t_n \max_{0 \leq j \leq n-1} \|\tilde{\mathbf{u}}_h^j - \mathbf{u}(t_j)\|_0^2 + \nu^2 \Delta t^2 K_{2,2}^2 \right).$$

Proof We concentrate on the more difficult case $d = 3$. Arguing exactly as in the proof of Lemma 9 and using (119) we have that the result for the L^2 norm holds with the constant C_B^* replaced by

$$(c_{\rm A} + c_1^{3/2})(M_1 M_2)^{1/2} + (1 + c_1) \left(C_{th} + c_{\rm r} h^{-1} \Lambda \big(\hat{C}_1 \Delta t^2 + \hat{C}_2 (h^4 + (\nu\delta)^2) \big) \right),$$

which taking into account that $\Delta t \le \delta < Ch$ can be bounded by

$$(C_B^{*,0})^2 = (c_{\rm A} + c_1^{3/2})(M_1M_2)^{1/2} + (1+c_1)\left(C_{th} + c_{\rm r}\Lambda\big(\nu^{-2}\hat C_1 + \hat C_2\big)c_M^2 {\rm diam}(\Omega)^3\right),$$

The result for the negative norm follows also arguing as in Lemma 9, for an appropriate constant $C_B^{*,1}$. The proof concludes taking $C_B^* = \max(C_B^{*,0}, C_B^{*,1})$. □

Theorem 5 *Under the assumptions of Theorem 4 the following bound holds*

$$\Delta t \sum_{j=1}^n t_j \|p_h^j - p(t_j)\|_0^2 \le \hat C_3 \Delta t + \hat C_4 (h^2 + \nu\delta), \qquad (121)$$

where $\hat C_3$ and $\hat C_4$ are defined by

$$\begin{aligned}
\hat C_3 &= C_3 + CT\big(c_0\lambda^{-1}(C_B^*)^2\nu^{-1}\tilde C_1 + (C_B^*)^2\nu^2 K_{2,2}^2 T\big) \\
&\quad + C(c_0\lambda^{-1} + \nu T)(1 + (C_B^*)^2\nu^{-1}T)\hat C_1 T, \\
\hat C_3 &= C_4 + CT\big(c_0\lambda^{-1}(C_B^*)^2\nu^{-1}\tilde C_2 \\
&\quad + C(c_0\lambda^{-1} + \nu t_{n+1})(1 + (C_B^*)^2\nu^{-1}t_{n+1})\hat C_2({\rm diam}(\Omega^2)(1 + c_M),
\end{aligned}$$

where $\tilde C_1$, $\tilde C_2$, C_3, C_4, $\hat C_1$ and $\hat C_2$ and C_6 in (51), (52), (81), (82), (114) and (115) respectively, C_B^* is the constant in Lemma 10, and c_M is the constant in (104)

Proof For the proof we argue as in the proof of Theorem 2. We first observe that $\tilde{\boldsymbol\epsilon}_h^n = \tilde{\mathbf u}_h^n - \tilde v_h^n$ and $\varrho_h^n = p_n^n - q_h^n$ satisfy the following relations

$$\begin{aligned}
&\Big(\frac{\tilde{\boldsymbol\epsilon}_h^{n+1} - \tilde{\boldsymbol\epsilon}_h^n}{\Delta t}, \boldsymbol\chi_h\Big) + \nu(\nabla\tilde{\boldsymbol\epsilon}_h^{n+1}, \nabla\boldsymbol\chi_h) + (\nabla\varrho_h^n, \boldsymbol\chi_h) = ((\boldsymbol\tau^*)_h^{n+1}, \boldsymbol\chi_h), \quad \forall \boldsymbol\chi_h \in V_h \\
&(\nabla\cdot\tilde{\boldsymbol\epsilon}_h^{n+1}, \psi_h) + \delta(\nabla\varrho_h^{n+1}, \nabla\psi_h) = 0, \qquad \forall\psi_h \in Q_h, \qquad (122)
\end{aligned}$$

where $(\boldsymbol\tau^*)_h^{n+1}$ is defined in (120). Applying Lemma 1 and (122) it is easy to obtain

$$\begin{aligned}
\Delta t\sum_{j=1}^n t_j\|\varrho_h^j\|_0^2 \le & C\Delta t\sum_{j=1}^n t_j\nu\delta\|\nabla\varrho_h^j\|_0^2 + C\Delta t\sum_{j=1}^n t_j\left\|\frac{\tilde{\boldsymbol\epsilon}_h^{j+1} - \tilde{\boldsymbol\epsilon}_h^j}{\Delta t}\right\|_{-1}^2 \\
&+ C\Delta t\sum_{j=1}^n t_j\|(\boldsymbol\tau^*)_h^j\|_{-1}^2 + C\Delta t\sum_{j=1}^n t_j\nu^2\|\nabla\tilde{\boldsymbol\epsilon}_h^{j+1}\|_0^2. \qquad (123)
\end{aligned}$$

We will bound all the terms on the right-hand side of (123). We first observe that the first and forth terms can be bounded by

$$C\nu t_n \Delta t \sum_{j=0}^{n} \left(\nu \|\nabla \tilde{\boldsymbol{\epsilon}}_h^{j+1}\|_0^2 + \delta \|\nabla \varrho_h^j\|_0^2\right)$$

and then applying (119) and taking into account the value of the constants $\hat{C}_1$ and $\hat{C}_2$ in (114) and (115) we have

$$C\Delta t \sum_{j=1}^{n} t_j \nu\delta \|\nabla \varrho_h^j\|_0^2 + C\Delta t \sum_{j=1}^{n} t_j \nu^2 \|\nabla \tilde{\boldsymbol{\epsilon}}_h^{j+1}\|_0^2 \leq C\nu t_n \left(\hat{C}_1 \Delta t^2 + \hat{C}_2 (h^4 + (\nu\delta)^2)\right). \tag{124}$$

To bound the third term we apply Lemma 10 and (113) to get

$$\begin{aligned} \Delta t \sum_{j=1}^{n} t_j \|(\boldsymbol{\tau}^*)_h^j\|_{-1}^2 &\leq (C_B^*)^2 t_n \left(t_n \max_{0\leq j\leq n-1} \|\tilde{\mathbf{u}}_h^j - \mathbf{u}(t_j)\|_0^2 + \nu^2 \Delta t^2 K_{2,2}^2\right) \quad (125) \\ &\leq (C_B^*)^2 t_n \left(\hat{C}_1 t_n \Delta t^2 + \hat{C}_2 t_n (h^4 + (\nu\delta)^2) + \nu^2 \Delta t^2 K_{2,2}^2\right). \end{aligned}$$

To conclude we will bound the second term on the right-hand side of (123). Since $t_j \leq t_{j+1}$ and taking into account (4) we can write

$$\begin{aligned} \Delta t \sum_{j=1}^{n} t_j \left\| \frac{\tilde{\boldsymbol{\epsilon}}_h^{j+1} - \tilde{\boldsymbol{\epsilon}}_h^j}{\Delta t} \right\|_{-1}^2 &\leq \Delta t \sum_{j=1}^{n} t_{j+1} \left\| \frac{\tilde{\boldsymbol{\epsilon}}_h^{j+1} - \tilde{\boldsymbol{\epsilon}}_h^j}{\Delta t} \right\|_{-1}^2 \\ &\leq C\lambda^{-1} \Delta t \sum_{j=1}^{n} t_{j+1} \left\| \frac{\tilde{\boldsymbol{\epsilon}}_h^{j+1} - \tilde{\boldsymbol{\epsilon}}_h^j}{\Delta t} \right\|_0^2. \end{aligned}$$

Applying (45) we get

$$\begin{aligned} \lambda^{-1} \Delta t \sum_{j=1}^{n} t_{j+1} \left\| \frac{\tilde{\boldsymbol{\epsilon}}_h^{j+1} - \tilde{\boldsymbol{\epsilon}}_h^j}{\Delta t} \right\|_0^2 \leq c_0 \lambda^{-1} \Bigg(\Delta t \sum_{j=1}^{n} t_{j+1} \|(\boldsymbol{\tau}^*)_h^j\|_0^2 \\ + \Delta t \nu \sum_{j=1}^{n} \|\nabla \tilde{\boldsymbol{\epsilon}}_h^j\|_0^2 + \Delta t \delta \sum_{j=1}^{n} \|\nabla \varrho_h^j\|_0^2 \Bigg). \end{aligned} \tag{126}$$

To conclude we will bound the three terms on the right-hand side of (126).

For the first one we apply Lemma 10 and (116) to get

$$
\begin{aligned}
\Delta t \sum_{j=1}^{n} t_{j+1} \|(\boldsymbol{\tau}^*)_h^j\|_0^2 \leq & (C_B^*)^2 t_{n+1} \left(\Delta t \sum_{j=0}^{n-1} \|\nabla(\tilde{\mathbf{u}}_h^j - \mathbf{u}(t_j))\|_0^2 + \nu^2 \Delta t^2 K_{3,2}^2 \right) \\
\leq & (C_B^*)^2 \frac{t_{n+1}}{\nu} \big(\tilde{C}_1 \Delta t + \tilde{C}_2 (h^2 + \nu\delta) \\
& + \hat{C}_1 \Delta t^2 + \hat{C}_2 (h^4 + (\nu\delta)^2) \big),
\end{aligned} \tag{127}
$$

where $\tilde{C}_1$, $\tilde{C}_2$, $\hat{C}_1$ and $\hat{C}_2$ are the constants in (51), (52), (114) and (115) respectively. Finally to bound the last two terms on the right-hand side of (126) we apply (119) to obtain

$$
\Delta t \nu \sum_{j=1}^{n} \|\nabla \tilde{\boldsymbol{\epsilon}}_h^j\|_0^2 + \Delta t \delta \sum_{j=1}^{n} \|\nabla \varrho_h^j\|_0^2 \leq \hat{C}_1 \Delta t^2 + \hat{C}_2 (h^4 + (\nu\delta)^2). \tag{128}
$$

Inserting (127) and (128) in (126) we get

$$
\begin{aligned}
\Delta t \sum_{j=1}^{n} t_j \left\| \frac{\tilde{\boldsymbol{\epsilon}}_h^{j+1} - \tilde{\boldsymbol{\epsilon}}_h^j}{\Delta t} \right\|_{-1}^2 \leq c_0 \lambda^{-1} \Big(& (C_B^*)^2 \nu^{-1} t_{n+1} \big(\tilde{C}_1 \Delta t + \tilde{C}_2 (h^2 + \nu\delta) \big) \\
& + \big(1 + (C_B^*)^2 \nu^{-1} t_{n+1} \big) \big(\hat{C}_1 \Delta t^2 + \hat{C}_2 (h^4 + (\nu\delta)^2) \big) \Big).
\end{aligned} \tag{129}
$$

Inserting (124), (125) and (129) into (123) we get

$$
\begin{aligned}
\Delta t \sum_{j=1}^{n} t_j \|\varrho_h^j\|_0^2 \leq & C c_0 \lambda^{-1} (C_B^*)^2 \nu^{-1} t_{n+1} \big(\tilde{C}_1 \Delta t + \tilde{C}_2 (h^2 + \nu\delta) \big) \\
& + C (c_0 \lambda^{-1} + \nu t_{n+1}) (1 + (C_B^*)^2 \nu^{-1} t_{n+1}) \big(\hat{C}_1 \Delta t^2 + \hat{C}_2 (h^4 + (\nu\delta)^2) \big) \\
& + C (C_B^*)^2 t_n \nu^2 K_{2,2}^2 \Delta t^2.
\end{aligned}
$$

Applying triangle inequality together with (80) we finally reach (121). □

References

[1] R. A. Adams. *Sobolev spaces*. Academic Press [A subsidiary of Harcourt Brace Jovanovich, Publishers], New York-London, 1975. Pure and Applied Mathematics, Vol. 65.

[2] S. Agmon, *Lectures on Elliptic Boundary Value Problems.* Prepared for publication by B. Frank Jones, Jr. with the assistance of George W. Batten, Jr. Revised edition of the 1965 original. AMS Chelsea Publishing, Providence, RI, 2010.

[3] S. Badia & R. Codina, Convergence analysis of the FEM approximation of the first order projection method for incompressible flows with and without the inf-sup condition, Numer. Math. 107, (2007) 533–557.

[4] E. Burman & M. A. Fernández, Analysis of the PSPG method for the transient Stokes' problem, Comput. Methods Appl. Mech. Engrg. 200, (2011) 2882–2890.

[5] A. J. Chorin, Numerical solution of the Navier-Stokes equations, Math. Comput. 22, (1968) 745–762.

[6] Philippe G. Ciarlet. *The finite element method for elliptic problems*, North-Holland Publishing Co., Amsterdam, 1978.

[7] R. Codina, Pressure Stability in Fractional Step Finite Element Methods for Incompressible Flows, J. Comput. Physics 170, (2001), 112–140.

[8] P. Constantin & C. Foias, *Navier-Stokes Equations*, Chicago Lectures in Mathematics. University of Chicago Press, Chicago, IL, 1988.

[9] J. de Frutos, B. García-Archilla & J. Novo, Error analysis of projection methods for non inf-sup stable mixed finite elements. The evolutionary Stokes problem, submitted.

[10] G. P. Galdi, *An Introduction to the Mathematical Theory of the Navier-Stokes Equations. Volumne I. Linearized Steady Problems*. Sprinber-Vergla, New-York, 1994

[11] J. L. Guermond & L. Quartapelle, On stability and convergence of projection methods based on pressure Poisson equation, Inter. J. Numer. Methods Fluids, 26 (1998) 1039–1053.

[12] J. L. Guermond, P. Minev & J. Shen, An overview of projection methods for incompressible flows, Comput. Methods Appl. Mech. Engrg. 195 (2006) 6011–6045.

[13] J. L. Guermond & L. Quartapelle, On the approximation of the unsteady Navier-Stokes equations by finite element projection methods, Numer. Math. 80, (1998) 207–238.

[14] A. Prohl, On Pressure Approximation Via Projection Methods for Nonstationary Incompressible Navier-Stokes Equations, SIAM J. Numer. Anal., 47 (2008) 158–180.

[15] J. G. Heywood and R. Rannacher, Finite element approximation of the nonstationary Navier–Stokes problem. I. Regularity of solutions and second-order error estimates for spatial discretization, SIAM J. Numer. Anal., 19 (1982), pp. 275–311.

[16] J. G. Heywood and R. Rannacher, Finite element approximation of the nonstationary Navier–Stokes problem. IV: Error analysis for second-order time dis discretizations, SIAM J. Numer. Anal., 25 (1988), pp. 489–512.

[17] V. John & J. Novo, Analysis of the PSPG Stabilization for the Evolutionary Stokes Equations Avoiding Time-Step Restrictions, SIAM J. Numer. Anal. 53 (2015) 1005–1031.

[18] J. C. López-Marcos & J. M. Sanz-Serna, A definition of stability for nonlinear problems. In Numerical treatment of differential equations (Halle, 1987), 216226, Teubner-Texte Math., 104, Teubner, Leipzig, 1988.

[19] J. C. López-Marcos & J. M. Sanz-Serna, Stability and convergence in numerical analysis. III. Linear investigation of nonlinear stability, *IMA J. Numer. Anal.*, 8 (1988), 71–84.

[20] P. D. Minev, A stabilized incremental projection scheme for the incompressible Navier-Stokes equations, Inter. J. Numer. Methods Fluids, 36 (2001) 441–464.

[21] A. Prohl, Projection and Quasi-Compressibility Methids for Solving the Incompressible Navier-Stokes equations, *Advances in Numerical Mathematics*, B. G. Teubner, Stuttgart, 1997.

[22] R. Rannacher, On Chorin's projection method for the incompressible Navier-Stokes equations, *Lecture Notes in Mathematics*, 1530, Springer, Berlin, 1992, 167–183.

[23] J. Shen, On error estimates of projection methods for Navier-Stokes equations: first-order schemes, SIAM J. Numer. Anal., 29 (1992) 57–77.

[24] J. Shen, Remarks on the pressure error estimates for the projection methods, Numer. Math., 67 (1994) 513–520.

[25] R. Temam, Sur lapproximation de la solution des equations de Navier-Stokes par la methode des pas fractionnaires ii, Arch. Ration. Mech. Anal. 33 (1969) 377–385.

[26] H. J. Stetter, *Analysis of Discretization Methods for Ordinary Differential Equations*, Springer-Verlag, Berlin, 1973.