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Sequential Plastic Method for 2D frames limit analysis

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Abstract

This work focuses on the limit analysis or plastic calculation of slender beam planar frames using a new method. Sufficient static equilibrium equations are proposed by the Principle of Virtual Displacements (PVD) and kinematic compatibility equations are proposed by the Principle of Virtual Forces (PVF). The load level is increased and the structure is solved step by step until the plastic collapse. Equilibrium equations are posed by virtual problems in displacements and compatibility equations are posed by simple virtual problems in equilibrium. The method provides the following results: the collapse load factor, the final collapse mechanism, the bending moments and the accumulated rotations in the plastic hinges in each load step of the structure. This method has advantages over the classical methods; first, with respect to the step-by-step method based on matrix formulation, especially in the case of beams and/or columns with uniform distributed load, and secondly, with respect to the kinematic direct method, since the sequential method provides more information on the quantities involved in the plasticization process of the structure and also goes directly to the calculation of the collapse mechanism without the need to test or combine possible mechanisms.

Key words: collapse, plastic hinges, sequential method, 2D frames, uniform distributed load

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1. Introduction

Researchers and engineers devote their time to determining appropriate structural solutions concerning safety, serviceability and cost saving. A huge volume of steel is used in construction every year. Over the last 40 years, the theories of plasticity and computing technology have enabled great achievements and the framed structures are, as always, the test bench.

The plastic-zone or the plastic-hinge approach is adopted to capture the material inelasticity of a framed structure. In the plastic-zone method, a structure member is discretized into a mesh of finite elements. However, the plastic-zone method is still considered an expensive method requiring a considerable computing burden. On the other hand, in the plastic-hinge approach, only one beam-column element per physical member is considered, which leads to a significant reduction in the computation time.

According to many authors, the notion of the plastic hinge and the collapse mechanism were first pointed out by Kazinczy[1]. The terminology plastic hinge is used to indicate a section (zero-length) on which all points are within the plastic range. The collapse mechanism refers to the final state of a frame, where a deformable geometric system is considered. The aim of the analysis is to determine the maximum safe load for a frame that is fully specified.

The two fundamental theorems of limit analysis, the static, kinematic and uniqueness theorems, were first established by Gvozdev[2]. At the same time, the static shakedown theorem was first proved by Melan. Twenty years later, the kinematic theorem for the shakedown analysis of frames was derived by Neal [3].

Fundamental theorems lead to the static and kinematic approaches called direct methods, and which are based on combining mechanisms[4–6]. The terminology "direct" means that the load multiplier is directly found without any intermediate structural state [7–10].

The kinematic or direct method has important drawbacks from the point of view of its practical application: first, it is not systematic or general; and second, it requires possible collapse mechanisms to be tested. Even with only a few plastic hinges being involved, this implies many possible collapse mechanisms that will have to be tested and verified. On the other hand, the step-by-step methods, based on the matrix formulation, are systematic and efficient for concentrated load cases at the nodes of the structure. They are, however, very inefficient and imprecise for analyzing structures with uniform distributed loads because position of the active interelement hinges change continously during the structural load process [11–13].

These drawbacks have been solved in this work by using a step-by-step method and as many equilibrium and compatibility equations as critical sections. A set of cases has been solved, both with point loads and with a uniformly distributed load, in reduced calculation times.

The entire formulation has been simplified by means of a vector method that systematizes the equations required in the direct kinematic method (equilibrium equations and compatibility equations) to know the progress of the formation of plastic hinges. These equations are applied for each step of the load application.

Although the computational capacity of a desktop computer has increased considerably in recent years, one could think of performing a suitable plastic zone analysis using FE software, the practice is that such an approach tends to give numerical instability problems sometimes even after the formation of the first plastic hinge and is difficult to solve. However, even for medium and large structures, the application of the proposed method is a practical and useful solution.

This paper has been organized as follows: after this brief introduction, the methodology is presented, which is then verified by solving two cases, one with concentrated loads and the other with a uniformly distributed load. It is then applied to two practical cases: the first is a gabled frame with point loads and distributed loads, while the second consists of a double gabled frame with distributed loads. Finally, the main conclusions and contributions of the work are summarized.

2. Methodology

The most important work on the matrix method based on the plastic hinge analysis is due to V. Hoang [10, 13]. The advantages and disadvantages are known: the main advantage is that the calculation is direct from the initial structure to the final collapse mechanism, and the disadvantage is the calculation effort required.

The formulation developed in this work combines the two classical methods indicated in the Introduction: it is a step-by-step method, but all the equations necessary to solve the plastic problem are proposed by means of the the Principle of Virtual Displacements (PVD) and the Principle of Virtual Forces (PVF)[14, 15]. It should be clarified that the work of this paper proposes a vector method, which works with force vectors and not displacements, with which the calculation effort required is less than in matrix methods.

2.1. Hypothesis

- The beams and columns are assumed to be slender rectilinear lines of constant section.
- They are assumed to be free of residual or initial stresses.

- Plastic collapse implies unlimited displacement at constant load, and the level of load that causes it is called the collapse load.
- The plastic moment depends on the material and the section, and its possible reduction is neglected due to the effect of the rest of the forces transmitted by, for instance, the section, axial and shear forces.
- The formation of each plastic hinge is supposed to take place in a sudden and concentrated way in the section where the bending moment reaches the value of the plastic moment.
- The hypothesis of small displacements and rotations of the sections of the structure at the moment of collapse is assumed; therefore, the accumulated rotations between bars in the plastic hinges must also be small.

2.2. Principle of Virtual Displacements (PVD)

The Principle of Virtual Displacements (PVD) makes use of a virtual or auxiliary problem in displacements, the integral expression provides equilibrium equations for the analyzed structure[7]:

$$\sum_{j=1}^{nP} P_j \delta v_j + \sum_{l=1}^{nq} \int_0^{L_l} q_l(x) \delta v_l dx = \sum_{i=1}^{npPH} M_i \delta\theta_i \qquad (1)$$

where nP is the number of sections with point loads, P_j is the point load, δv_j is the transverse displacement that depends on the accumulated rotations in the plastic hinges, nq is the number of beams and/or columns with uniform distributed load, q_l is the value of the distributed load, $npPH$ is the number of possible plastic hinges in the structure, Mi is the bending moment and $\delta\theta_i$ is the accumulated rotation in those sections in the virtual problem[8].

As virtual problems, it is common to use problems with rigid body movements that imply zero deformations[16, 17].

2.3. Principle of Virtual Forces (PVF)

The Principle of Virtual Forces (PVF) is posed using a virtual or auxiliary problem in equilibrium, then the integral expression provides compatibility equations[7].

$$\sum_{j=1}^{nP} \delta P_j v_j = \sum_{k=1}^{nb} \int_0^{L_k} M(x) \frac{\delta m(x)}{EI_y} dx + \sum_{i=1}^{npPH} \delta m_i \theta_i \quad (2)$$

where nP is the number of sections with point loads, δP_j are the virtual point loads, v_j is the transverse displacement, nb is the number of beams and columns in the structure, L_k is the length of the element, beam or column, $M(x)$ is the bending moment in the beams and columns of the structure, $\delta m(x)$ is the bending moment of the auxiliary or virtual problem, EI_y is the bending stiffness of the beam/column and θ_i is the accumulated rotation in the plastic hinges[8].

As virtual problems, it is common to use problems with only point loads and even with zero external loads[16, 17].

2.4. Case a: Concentrated loads

In the case of point loads (see figure 1), it is known that the sections of the structure that are candidates for forming a possible plastic hinge are: the nodes (joints between bars), the fixed supports, the section of application of the loads and section changes, and the total number of possible plastic hinges is called (npPH).

The steps to follow in the Sequential Plastic Method (SPM) are listed below:

1. Equilibrium Equations (EEs)
   The Principle of Virtual Displacements (PVD) requires auxiliary problems to be posed in compatible displacements, which can be mechanisms that involve null deformations and stresses (hypothesis of small displacements and rotations has been assumed).

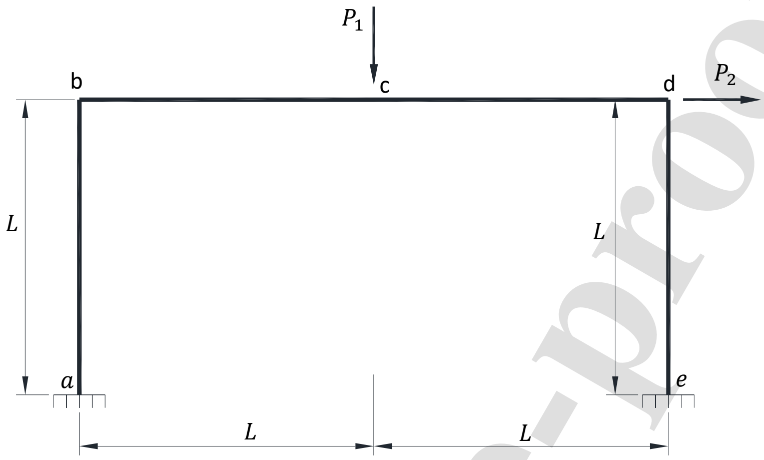

Figure 1: Methodology. Concentrated loads case

A total of (nEE) equilibrium equations is required, this number is obtained from $(npPH - nCE)$, where (nCE) is the degree of hyperstaticity of the structure.

According to the methodology of this work, the simplest thing is to formulate the (PVD) with mechanisms as auxiliary problems. For the case of figure 1, it is necessary to propose (nEE) different mechanisms (M_i), while the mechanism (M_1) is independent of the mechanism (M_2), see figures 2 and 3.

The equations of equilibrium are:

$$\begin{aligned} P_1 L &= -M_b + 2M_c - M_d \\ P_2 L &= -M_a + M_b - M_d + M_e \end{aligned} \tag{3}$$

2. Compatibility Equations (CEs)

The Principle of Virtual Forces (PVF) requires auxiliary problems in balance to be posed. This then is simple, at least in principle, as it only includes concentrated forces and/or moments, a total of (nCE)

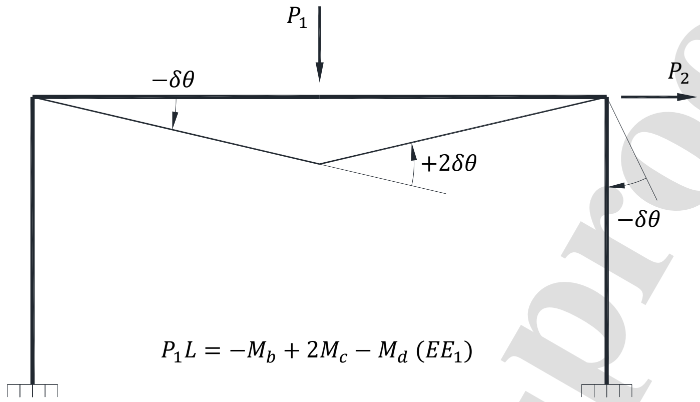

Figure 2: Virtual problem in displacements. Mechanism 1 (M_1)

compatibility-behavior equations is also required.

Taking advantage of the formulation, a system of homogeneous equations results if the terms of loads are made null in the equilibrium equations (3). The virtual problems in forces must satisfy the previous equilibrium equations with zero loads:

$$\begin{aligned} 0 &= \delta m_b - 2\delta m_c + \delta m_d \\ 0 &= \delta m_a - \delta m_b + \delta m_d - \delta m_e \end{aligned} \tag{4}$$

It can be expressed in a matrix form and the matrix of coefficients (A) can then be defined:

$$A \cdot \delta m_i^T = \begin{pmatrix} 0 & 1 & -2 & 1 & 0 \\ 1 & -1 & 0 & 1 & -1 \end{pmatrix} \cdot \begin{pmatrix} \delta m_{i,a} \\ \delta m_{i,b} \\ \delta m_{i,c} \\ \delta m_{i,d} \\ \delta m_{i,e} \end{pmatrix}; \ i = 1 \ldots nCE; \tag{5}$$

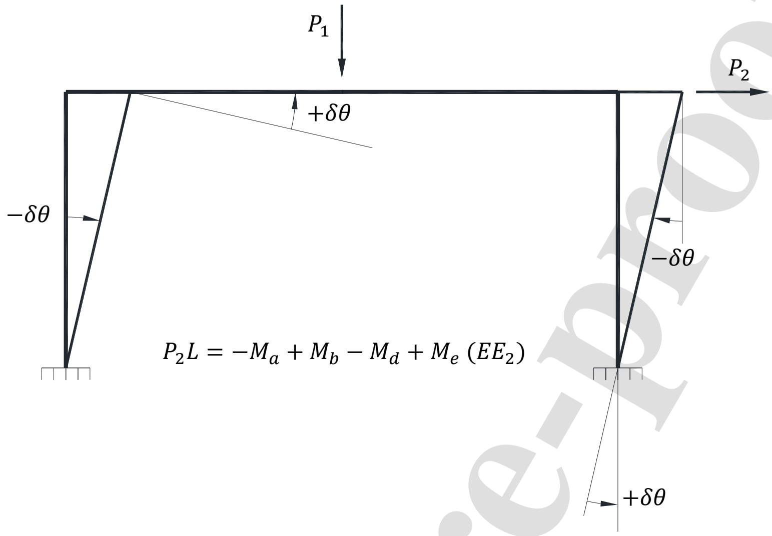

Figure 3: Virtual problem in displacements. Mechanism 2 (M_2)

$$\begin{pmatrix} \delta m_{1,j} \\ \delta m_{2,j} \\ \delta m_{3,j} \end{pmatrix} = N(A) = \begin{pmatrix} 1 & 0 & 0 & 0 & 1 \\ -2 & -1 & 0 & 1 & 0 \\ 2 & 2 & 1 & 0 & 0 \end{pmatrix}; \ nCE = 3; \ j = a, b, c, d, e; \tag{6}$$

From a mathematical point of view, it is known as the null space of a matrix A (see Annex B). This allows (nCE) sets of values of linearly independent virtual moments to be obtained. When taken to the integral expression of the PVF, this then provides the compatibility equations necessary to calculate the accumulated rotations in the plastic hinges:

$$\frac{L(2M_a + M_b + M_d + 2M_e)}{EI_y} + 6(\theta_a + \theta_e) = 0$$
$$\frac{L(4M_d + M_e - 5M_a - 6M_b)}{6EI_y} + \theta_d - 2\theta_a - \theta_b = 0 \quad (7)$$
$$\frac{L(6M_a + 11M_b + 6M_c + M_d)}{EI_y} + 6(2\theta_a + 2\theta_b + \theta_c) = 0$$

3. Step 0: elastic-linear analysis ($\theta_i = 0$)

   This step consists of solving the $(npPH)$ equations with zero accumulated rotations (it is assumed that no plastic hinge has yet been formed). Therefore, the unknowns are the values of the bending moments in the sections $(M_{i,0})$ considered for the nominal loads.

4. Step 1: first Plastic Hinge (PH_1)

   In the previous step, the maximum moment occurs in section **a**, for example, so that is where the first plastic hinge is formed. In that section **a**, the value of the bending moment is $M_{a,1} = sign(M_{a,0}) \cdot M_{y,Rk}$, and the system of equations can be solved again. Notice that now we have one more equation and one more unknown; after solving it we obtain values for $M_{i,1}$ and θ_a (for i=1 ... npPH).

5. Step 2: second Plastic Hinge (PH_2)

   Again, the maximum value of the bending moment is sought. This results in $M_{max} = |M_{b,1}|$, and the following plastic hinge is formed in **b**. We can then add the equation $M_{b,2} = sign(M_{b,0}) \cdot M_{y,Rk}$, and solve to get $M_{i,2}$, θ_a and θ_b.

6. Step 3: third Plastic Hinge (PH_3)

   Ditto for section **d**, $M_{d,3} = sign(M_{d,0}) \cdot M_{y,Rk}$, and it is solved to obtain $M_{i,3}$, θ_a, θ_b and θ_d.

7. Step n: nth Plastic Hinge (PH_n)

   The last plastic hinge is formed in section **e**, the value of the bending moment in this section reaches the maximum value $M_{e,4} = sign(M_{e,0}) \cdot$

$M_{y,Rk}$, but it does not have time to accumulate rotation, therefore, $\theta_e = 0$. Solving the system of equations gives a solution to the following unknowns: $M_{i,4}$, θ_a, θ_b, θ_d and θ_e, which is null.

In this case $n = 4$, the collapse mechanism is complete because it involves the formation of $nCE + 1$ plastic hinges.

2.5. Case b: Uniform distributed loads

In the case of beams with distributed load (see figure 4), additionally, plastic hinges can be formed in the intermediate sections of the beams with applied uniform distributed load. Logically, it is then necessary to carry out the corresponding checks from the bending moments calculated at the nodes of the structure.

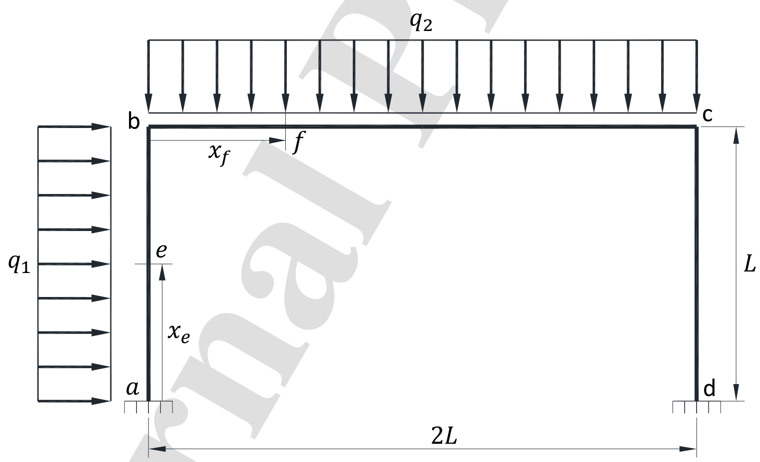

Figure 4: Methodology. Uniform distributed loads case

The Sequential Plastic Method raises a minimum number of equations, in this case $npPH = 4$, which corresponds to the values of the bending moments at the nodes of the structure, sections a, b, c and d.

The steps to follow to carry out the plastic analysis are the same as in the case of concentrated loads. However, it is necessary to keep in mind:

1. The correct calculation of the external work of the applied loads when raising the PVD.
2. To check the values of the bending moments in intermediate sections of the beams with uniform distributed load, because a plastic hinge can also form in them during the advance of plasticization:

$$\begin{aligned} M_e &= f(q_1, M_a, M_b); \quad x_e = g(q_1, M_a, M_b) \\ M_f &= f(q_2, M_b, M_c); \quad x_f = g(q_2, M_b, M_c) \end{aligned} \tag{8}$$

It is important to bear in mind that if a plastic hinge is produced in an intermediate section (for example in section e) in step 2, then its location at the beam (given by parameter x_e) can be modified during the plasticizing process, up until the formation of the collapse mechanism.

For the last state of the structure, it can be verified (in case of doubt) that it is a mechanism called Collapse Mechanism (CM) using the matrix method of calculation of structures, by means of the Working Model software or others.

In addition, the solution must be compatible, that is, it requires that the bending moments and accumulated rotations in the plastic hinges to have the same sign ($M \cdot \theta \geq 0$), a condition that ensures that the energy is dissipated in the plastic hinges (a monotonous loading process is assumed).

3. Numerical results and discussion

In this section, the methodology is verified through two simple examples and then the sequential plastic method is applied to the study of the gabled frames of industrial buildings, one simple and the other double.

3.1. Numerical data

All the columns and beams have the same mechanical and geometric properties, the numerical data in common for all the problems are: $P = 1.0\,kN$; $q = 1.0\,kN/m$; $E = 2.1 \cdot 10^8\,kN/m^2$; $I_y = 8360.0 \cdot 10^{-8}\,m^4$; $W_{pl,y} = 628.0 \cdot 10^{-6}\,m^3$; $f_y = 275.0 \cdot 10^3\,kN/m^2$; $M_{y,Rk} = W_{pl,y} \cdot f_y = 172.7\,kNm$, where P is the value of the concentrated loads, q the intensity of the distributed load, E is Young's module, I_y the moment of inertia, $W_{pl,y}$ the section plastic module, f_y the yield strength of the steel, and $M_{y,Rk}$ the characteristic value of resistance to bending moments about the y-y axis.

3.2. Validation Problems (VP)

In this section, a basic frame fixed ended in the base of both columns and two load cases are solved: a first case includes only concentrated loads (see figure 5), while in the second case, uniform load is applied in the left column of the frame (see figure 6). Solution by the direct kinematic method for both cases can be consulted in Annex C.

3.2.1. VPa: concentrated loads

This case consists of a basic frame fixed ended in the base of both columns (both columns of length L), with the beam length ($2L$), whose loads are as indicated in figure 5. Both concentrated loads are of the P value type. The additional data are: $L = 4\,m$.

The methodology outlined in section 2 is applied using a Mathematica notebook that systematically solves the plastic problem. To do so, it first calculates the equilibrium equations ($nEE = 2$, see equations (9)):

$$\begin{aligned} PL &= -M_b + 2M_c - M_d \\ PL &= -M_a + M_b - M_d + M_e \end{aligned} \tag{9}$$

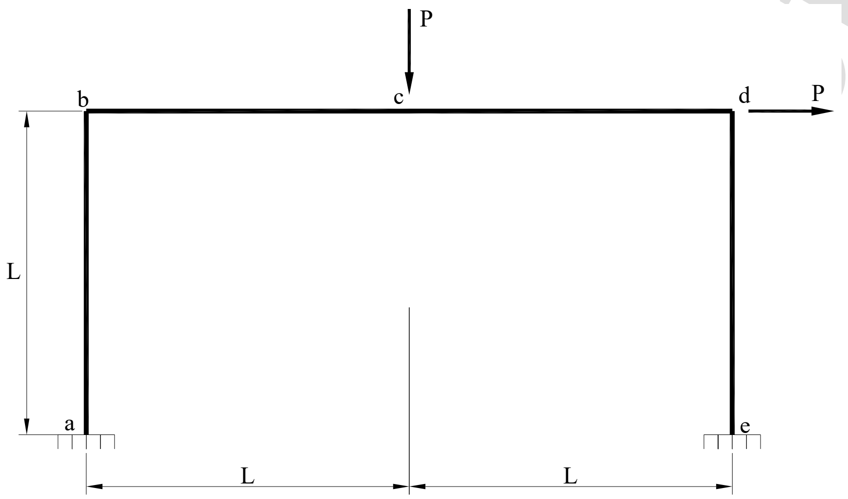

Figure 5: Fixed-fixed frame. Concentrated loads case

And the compatibility equations ($nCE = 3$, see equations (10)) for this problem are:

$$\begin{aligned}
&\frac{L(2M_a + M_b + M_d + 2M_e)}{EI_y} + 6(\theta_a + \theta_e) = 0 \\
&\frac{L(4M_d + M_e - 5M_a - 6M_b)}{6EI_y} + \theta_d - 2\theta_a - \theta_b = 0 \\
&\frac{L(6M_a + 11M_b + 6M_c + M_d)}{EI_y} + 6(2\theta_a + 2\theta_b + \theta_c) = 0
\end{aligned} \tag{10}$$

The derivation of the equilibrium and compatibility equations can be consulted in the book by Doblaré-Gracia [4].

The first step of the method provides the elastic solution, and the development of the first plastic hinge is simple, it consists in equalling the maximum absolute value of the bending moments in the candidate sections to the plastic moment and obtaining the corresponding load factor.

For the rest of the plastic hinges, it is necessary to take into account the final value of the bending moments of the previous step and the size of the increase of said values in the next step, in order to know which will be the next plastic hinge and to evaluate the value of the next load factor.

The plastic hinge process can be followed step by step until the Collapse Mechanism (CM) is reached. In this case, the collapse mechanism of the structure involves the formation of plastic hinges in sections **e, d, c** and **a**. Table 1 shows the value of the bending moment and Table 2 shows the accumulated rotation in the plastic hinges for each loading step.

step	PH	M_a	M_b	M_c xM$_{y,Rk}$ (kN·m)	M_d	M_e	P (kN)
0	-	-0.2125	-0.01250	0.3000	-0.3875	0.4125	1
1	**e**	-0.5152	-0.03030	0.7273	-0.9394	1	104.6
2	**d**	-0.5821	-0.01493	0.7761	-1	1	110.8
3	**c**	-0.9130	0.04347	1	-1	1	127.6
4	**a**	-1	0	1	-1	1	**129.5**

Table 1: VPa. Concentrated loads. Plastic Hinges process. Bending moments

step	PH	θ_a	θ_b	θ_c	θ_d (rad)	θ_e	P (kN)
0	-	0	0	0	0	0	1
1	**e**	0	0	0	0	0	104.6
2	**d**	0	0	0	0	0.001175	110.8
3	**c**	0	0	0	-0.008554	0.005132	127.6
4	**a**	0	0	0.006558	-0.0132	0.006558	**129.5**

Table 2: VPa. Concentrated loads. Plastic Hinges process. Rotations in plastic hinges

3.2.2. VPb: uniform distributed load

This section shows the methodology in the case of applying a distributed load on the structure (see figure 6). This case is interesting because the loads include a uniform distributed load along the total length of the lefthand column, which can, for example, simulate the action of the wind. The data in this case are: $L_p = 3\,m$; $L_d = 5\,m$, where L_p is the height of the column and L_d is the beam length.

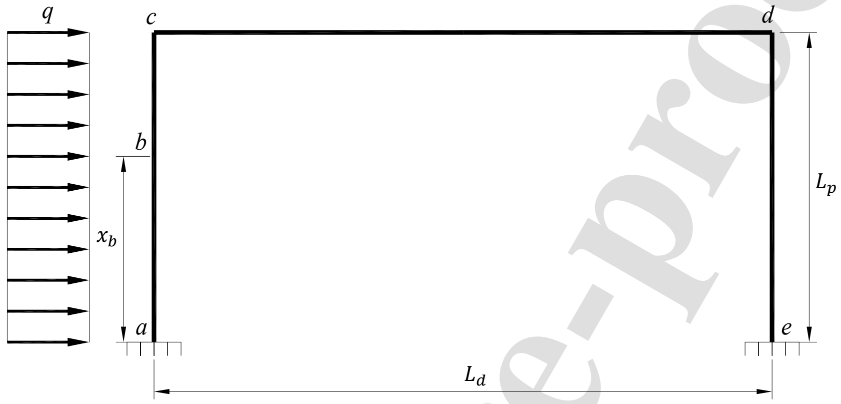

Figure 6: Fixed-fixed frame. Uniform distributed load

As pointed out above, this example is very interesting because, when introducing a distributed type load, an intermediate plastic hinge may originate in the column. A priori, it is not known how the plastification will progress, in section b for example. One more unknown (x_b, position of the plastic hinge) appears for each additional distributed load[3].

According to the methodology of this work, the following sections are considered as candidates for the formation of a plastic hinge: a, c, d and e ($npPH = 4$, $nCE = 3$, $nEE = 1$). Once the values of the bending moments in these sections are known, the relative maximum or minimum value of the moment in column ac will be checked.

The methodology systematically solves the plastic problem. The equilibrium (11) and compatibility equations (12) for this case are:

$$\frac{qL_p^2}{2} = -M_a + M_c - M_d + M_e \tag{11}$$

where to obtain this equation the virtual/auxiliary problem indicated in figure

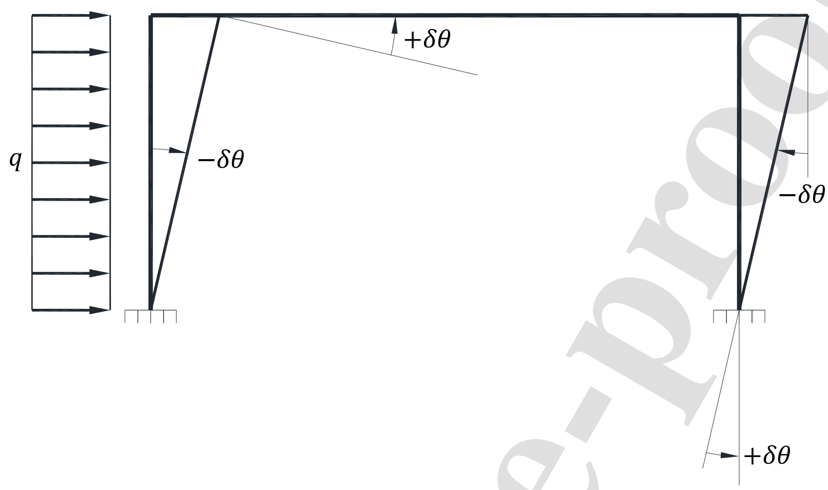

Figure 7: Fixed-fixed frame. Virtual problem in displacements. Mechanism 1

7 is used.

$$
\begin{aligned}
&\frac{L_p\left(\frac{L_p^2 q}{2}+3M_a+3M_c\right)}{6EI_y}+\frac{L_d(2M_c+M_d)}{6EI_y}+\theta_a+\theta_b+\theta_c=0\\
&\frac{L_d(M_c+2M_d)}{6EI_y}+\frac{L_p(3M_d+3M_e)}{6EI_y}+\theta_d+\theta_e=0\\
&\frac{L_p\left(\frac{L_p^2 q}{4}+2M_a+M_c\right)}{6EI_y}+\frac{L_p(M_d+2M_e)}{6EI_y}+\theta_a+\theta_b\left(1-\frac{x}{L_p}\right)+\theta_e=0
\end{aligned}
\tag{12}
$$

The value of the bending moment in section b is expressed as a function of the bending moments at the ends of the column (M_a, M_c) and the applied load (q), and the coordinate x of its location is updated at each step:

$$
\begin{aligned}
&M_b=M_a+\frac{x(M_c-M_a)}{L_p}+\frac{L_p q x}{2}-\frac{q x^2}{2}\\
&x=\frac{qL_p^2-2M_a+2M_c}{2L_p q};\ 0\le x\le L_p
\end{aligned}
\tag{13}
$$

The problem is well posed, so the algorithm itself is in charge of looking for the minimum collapse load that causes the plastic collapse of the structure of interest. In this case, the mechanism involves plastic hinges in the following sections: **a, e, b** and **d**. Again, the final mechanism formed is a complete mechanism. Note that the plastic hinges in sections **b** and **d** are formed simultaneously.

step	PH	M_a	M_b	M_c	M_d	M_e	q (kN/m)
		x$M_{y,Rk}$ (kN·m)					
0	-	-0.01264	0.003960	0.002898	-0.003900	0.006623	1.0
1	**a**	-1	0.3134	0.2293	-0.3086	0.5241	79.14
2	**e**	-1	0.5691	0.3591	-0.5682	1	112.3
3	**b,d**	-1	1	0.7321	-1	1	**143.2**

Table 3: VPb. Uniform distributed load. Plastic hinges process. Bending moments

step	PH	θ_a	θ_b	θ_c	θ_d	θ_e	q (kN/m)
		(rad)					
0	-	0	0	0	0	0	1.0
1	**a**	0	0	0	0	0	79.14
2	**e**	-0.006171	0	0	0	0	112.3
3	**b,d**	-0.01822	0	0	0	0.01036	**143.2**

Table 4: VPb. Uniform distributed load. Plastic hinges process. Rotations in plastic hinges

The intermediate section (section b) in the element requested by the distributed load is $x_b = 2.196\,\mathrm{m}$.

It is important to highlight that the methodology allows problems to be solved both with loads concentrated in certain sections and with uniform distributed load on some beams and/or columns.

3.3. Application Problems (AP)

3.3.1. APa: gable frame

The resolution of the practical problem of a gable type frame with distributed loads and point loads is considered in this section, see figure 8. The data of the problem are: $L_p = 4\,m$; $L_d = 6\,m$ and $\beta = 10\,°$, where L_p is the height of the columns, L_d is half the distance between columns and β is the roof's angle of inclination.

After applying the same methodology as in the examples in the previous section, a collapse load factor is obtained. A complete plastic collapse mechanism is formed that involves the formation of hinges in sections **a, d, f** and **g**, see Tables 5 and 7.

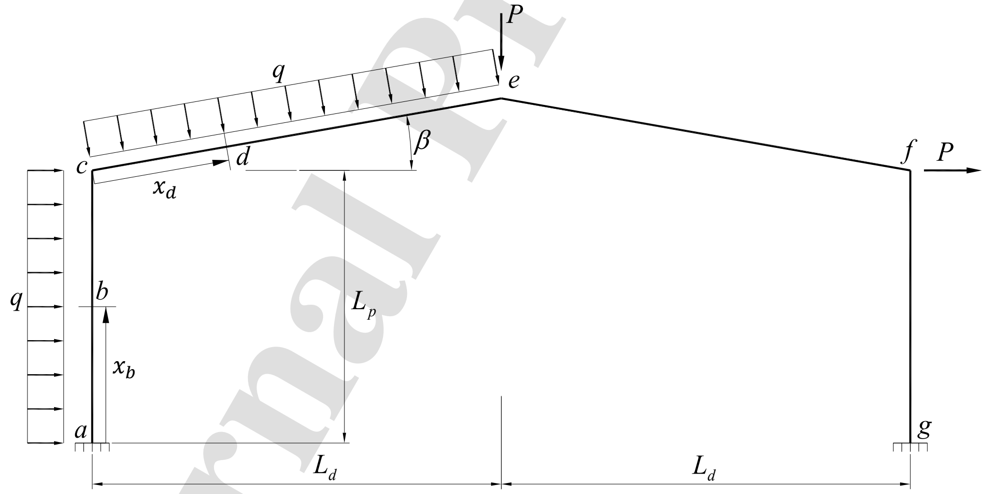

Figure 8: Gable frame

It is necessary to clarify that in this case of geometry, loads and supports do not originate a plastic hinge in section b, although they do in section **d** due to the distributed load (see tables 5 and 6), and the intermediate plastic hinge is:

step	PH	M_a	M_b	M_c	M_d xM$_{y,Rk}$ (kN·m)	M_e	M_f	M_g	λ
0	-	-0.02133	-0.010974	-0.02385	0.02879	0.01959	-0.04092	0.05558	1
1	**g**	-0.3838	0.1858	-0.4290	0.4742	0.3525	-0.7362	1	17.99
2	**f**	-0.5623	-0.2030	-0.3921	0.7907	0.5645	-1	1	23.09
3	**d**	-0.8699	-0.3018	-0.4207	1	0.7803	-1	1	26.06
4	**a**	-1	-0.3863	-0.4939	1	0.7906	-1	1	**26.66**

Table 5: APa. Gable frame. Plastic hinges process. Bending moments

step	PH	x_b (m)	x_d	λ
0	-	1.891	4.297	1
1	**g**	1.994	3.116	17.99
2	**f**	2.318	4.239	23.09
3	**d**	2.744	4.373	26.06
4	**a**	2.820	4.433	**26.66**

Table 6: APa. Gable frame. Plastic hinges process. Location x_i

step	PH	θ_a	θ_b	θ_c	θ_d	θ_e (rad)	θ_f	θ_g	λ
0	-	0	0	0	0	0	0	0	1
1	**g**	0	0	0	0	0	0	0	17.99
2	**f**	0	0	0	0	0	0	0.005558	23.09
3	**d**	0	0	0	0	0	-0.01356	0.01238	26.06
4	**a**	0	0	0	0.008760	0	-0.02059	0.01624	**26.66**

Table 7: APa. Gable frame. Plastic hinges process. Rotations in plastic hinges

$$x_d = 4.433\,\mathrm{m} \tag{14}$$

Finally, comment that once the method has been explained and verified for one type of section, it is easy to extend it to other types of sections or consider different sections in columns and beams.

3.3.2. APb: double gable frame

Finally, the methodology is applied to a double gable frame. The data of the problem are: $L_p = 5\,m$; $L_d = 10\,m$ and $H = 7\,m$, where L_p is the height of the columns, L_d is half the distance between columns and H is the maximum height of the frame (see Figure 9).

It is verified that the method allows the behavior of the structure to be known as the load increases. It facilitates the study of real cases, since more bars and loads can be used (in this case up to four bars with distributed load).

From the results of Tables 8 and 9, it is very immediate to obtain the safety factor of the elastic-linear design of the structure:

$$\eta_e = \frac{q_5}{q_1} = \frac{10.51}{6.411} = 1.64 \tag{15}$$

It is by definition the quotient between a plastic design and an elastic design, the latter understood according to the philosophy of Eurocode 3, value of the load (q_1) for which the first plastic hinge is formed.

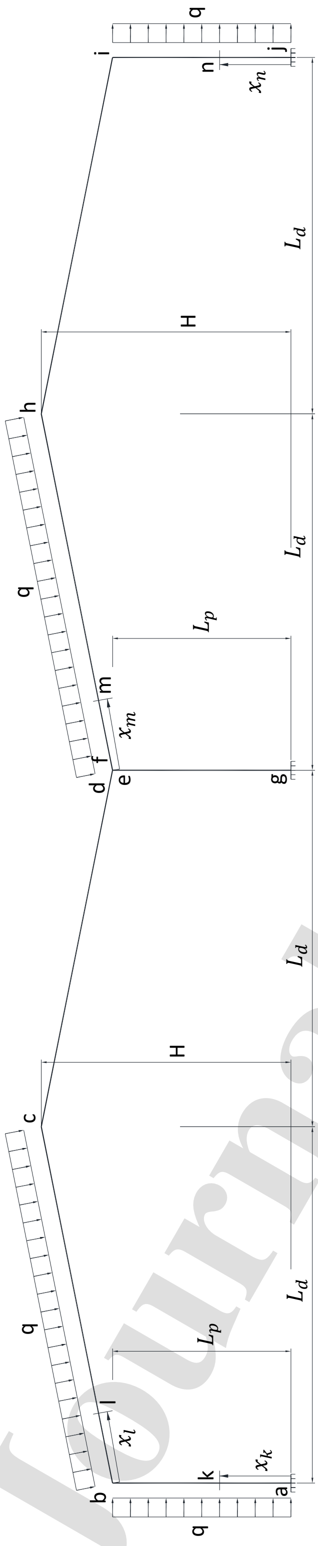

Figure 9: Double gable frame

step	PH	M_a	M_b	M_c	M_d	M_e	M_f	M_g	q
		x$M_{y,Rk}$ (kN·m)							(kN/m)
0	-	-0.1560	-0.01725	0.01725	-0.03027	-0.0324	0.004461	0.03244	1
1	**a**	-1	-0.1106	0.1106	-0.19406	-0.2080	0.02860	0.2080	6.411
2	**h,j**	-1	-0.05363	0.1158	-0.2057	-0.3056	0.1578	0.3056	9.191
3	**l**	-1	0.2040	0.2810	-0.3141	-0.3051	0.3355	0.3051	10.06
4	**m**	-1	0.1969	0.2298	-0.3923	-0.2670	0.8707	0.2670	10.45
5	**d,e,g**	-1	0.3458	0.05915	-1	-1	0.8671	1	**10.51**

Table 8: APb. Double gable frame. Plastic hinges process. Bending moments

step	PH	M_h	M_i	M_j	M_k	M_l	M_m	M_n	q
		x$M_{y,Rk}$ (kN·m)							(kN/m)
0	-	-0.09906	0.01984	0.07680	M_b	0.07626	0.03687	0.07762	1
1	**a**	-0.6350	0.1272	0.4924	M_b	0.4889	0.2364	0.4976	6.411
2	**h,j**	-1	0.1626	1	M_b	0.7256	0.3793	M_j	9.19
3	**l**	-1	0.1937	1	M_b	1	0.5720	M_j	10.06
4	**m**	-1	0.008029	1	M_b	1	1	M_j	10.45
5	**d,e,g**	-1	0.01375	1	M_b	1	1	M_j	**10.51**

Table 9: APb. Double gable frame. Plastic hinges process. Bending moments (cont.)

step	PH	x_k	x_l	x_m	x_n	q
		(m)				(kN/m)
0	-	5.0	5.683	3.346	4.467	1
1	**a**	5.0	5.683	3.346	4.467	6.411
2	**h,j**	5.0	5.411	2.885	0.0	9.191
3	**l**	5.0	5.229	2.850	0.0	10.06
4	**m**	5.0	5.152	2.067	0.0	10.45
5	**d,e,g**	5.0	4.637	2.090	0.0	**10.51**

Table 10: APb. Double gable frame. Plastic hinges process. Location x_i

step	PH	θ_a	θ_b	θ_c	θ_d (rad)	θ_e	θ_f	θ_g	q (kN/m)
0	-	0	0	0	0	0	0	0	1
1	**a**	0	0	0	0	0	0	0	6.411
2	**h,j**	-0.01609	0	0	0	0	0	0	9.191
3	**l**	-0.04436	0	0	0	0	0	0	10.06
4	**m**	-0.1182	0	0	0	0	0	0	10.45
5	**d,e,g**	-1.393	0	0	0	0	0	0	**10.51**

Table 11: APb. Double gable frame. Plastic hinges process. Rotations in plastic hinges

step	PH	θ_h	θ_i	θ_j	θ_k (rad)	θ_l	θ_m	θ_n	q (kN/m)
0	-	0	0	0	0	0	0	0	1
1	**a**	0	0	0	0	0	0	0	6.411
2	**h,j**	0	0	0	0	0	0	0	9.191
3	**l**	-0.04251	0	0.01501	0	0	0	0	10.06
4	**m**	-0.1323	0	0.06940	0	0.1015	0	0	10.45
5	**d,e,g**	-1.768	0	0.8853	0	1.753	0.4790	0	**10.51**

Table 12: APb. Double gable frame. Plastic hinges process. Rotations in plastic hinges (cont.)

There are intermediate plastic hinges in sections **l** and **m** (see tables 8, 9 and 10):

$$\begin{aligned} x_k &= 5.0\,m \text{ (section b)} \\ x_l &= 4.637\,m \\ x_m &= 2.090\,m \\ x_n &= 0.0\,m \text{ (section j)} \end{aligned} \tag{16}$$

4. Conclusions

The classic formulation for plastic methods of planar frames is very unsystematic. It is based on the Principle of Virtual Displacements (PVD) and the Principle of Virtual Forces (PVF) and use equilibrium equations to find the structure's collapse mechanism. To obtain these equilibrium equations, the PVD is formulated using virtual problems in displacements (virtual mechanisms).

The classical method is based on testing possible mechanisms until the collapse mechanism is found. This procedure can be successful if the structure's collapse mechanism is tested the first time, but in general, this is not known and may require testing many mechanisms, all of which result from combinations of possible plastic hinges. This involves many calculations and is complicated when distributed loads act, even if it is applied on a single bar.

However, this work applies static equilibrium and kinematic compatibility equations during the plastic progress (step by step). In order to avoid having to test possible mechanisms one by one, it leads directly to the collapse mechanism corresponding to the structure with given loads, geometry and boundary conditions.

This work systematizes the plastic analysis of the structure to obtain the equilibrium and compatibility equations necessary to completely solve the structure, in order to discover the bending moments and the accumulated rotations in the plastic hinges. The search for the final state of the structure (collapse mechanism) is not carried out by trial and error, but using a step by step method.

This work summarizes a two-dimensional method of great utility for the analysis of real industrial buildings, since they are large continuous section structures that can be studied by planar frames. The methodology of this work is useful to analyze the structure regardless of the type of load, whether it be point loads, uniform distributed loads or both types. Furthermore, the safety factor for a linear-elastic design of the structure under studycan be obtained quickly.

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Annex A. Compatibility Equations (CEs)

The Principle of Virtual Forces (PVF) requires auxiliary problems in balance to be posed. This then is simple, at least in principle, as it only includes concentrated forces and/or moments. By posing virtual problems with loads and/or specific moments, the calculation of the work of the external loads is simple and the calculation of the deformation energy can be systematized.

- Concentrated loads

  If the structure of the problem of interest only has concentrated loads, the following deformation energy expression results for each beam:

$$\int_0^L M(x)\frac{\delta m(x)}{EI_z}dx = \frac{L}{6EI_z}(\delta m_a(2M_a + M_b) + \delta m_b(M_a + 2M_b)) \tag{17}$$

- Uniform distributed load

  If the structure beam is requested by uniform distributed load, the following expression is as follows::

$$\begin{aligned} \int_0^L M(x)\frac{\delta m(x)}{EI_z}dx = \frac{L}{6EI_z}(\delta m_a(2M_a + M_b) + \\ + \delta m_b(M_a + 2M_b) + \frac{qL^2}{4}(\delta m_a + \delta m_b)) \end{aligned} \tag{18}$$

where $M(x)$ are the moments of the problem of interest, $\delta m(x)$ are the moments of the virtual problem and δm_a; δm_b, the bending moments at the end sections and q is the value of the uniform distributed load requested at the beam.

Annex B. Null space of a matrix

The solution sets of homogeneous linear systems provide an important source of vector spaces. Let A be an m by n matrix, and consider the homogeneous system:

$$A \cdot x = 0 \tag{19}$$

Since A is m by n, the set of all vectors x which satisfy this equation forms a subset of R^n (it clearly contains the zero vector). This subset is nonempty and forms a subspace of R^n, called the nullspace of the matrix A, and is denoted $N(A)$.

Thus, the solution set of a homogeneous linear system forms a vector space. Note carefully that if the system is not homogeneous, then the set of solutions is not a vector space, since the set will not contain the zero vector.

Annex C. Validation Problems (VP). Direct kinematic method

The direct kinematic method is applied to the validation problems of this work.

VPa: concentrated loads

When applying the direct kinematic method, the collapse mechanism must be tested and searched through trial and error, this process can be time consuming (see Table 13).

We assume that the collapse mechanism of the structure involves the formation of plastic hinges in sections e, d, c and a. If we substitute in the equilibrium equations (9):

PHs	$P \cdot \frac{L}{M_{y,Rk}}$
abc	14.0
abd	20.0
abe	6.4
acd	3.33
ace	3.5
ade	3.06
bcd	4.0
bce	8.5
bde	9.33
cde	-
abcd	4.0
abce	-
abde	4.0
acde	**3.0**
bcde	4.0

Table 13: Possible mechanisms

$$
\begin{aligned}
&M_a = -M_{y,Rk};\ M_c = +M_{y,Rk};\ M_d = -M_{y,Rk};\ M_e = +M_{y,Rk};\\
&\left.\begin{aligned} PL &= -M_b + 2M_c - M_d \\ PL &= -M_a + M_b - M_d + M_e \end{aligned}\right\} \rightarrow \quad \begin{aligned} &P_c = \tfrac{3M_{y,Rk}}{L} = 129.525 \text{ kN} \\ &M_b = 0.0 \text{ kNm} \end{aligned}
\end{aligned} \tag{20}
$$

After substituting the previous results in (10) the compatibility equations are:

$$
\begin{aligned}
-\frac{LM_{y,Rk}}{6EI_y} + \theta_a + \theta_e = 0\\
\frac{LM_{y,Rk}}{3EI_y} + \theta_d - 2\theta_a - \theta_b = 0\\
-\frac{LM_{y,Rk}}{6EI_y} + 2\theta_a + 2\theta_b + \theta_c) = 0
\end{aligned} \tag{21}
$$

Finally, if a new assumption is made that the last plastic hinge is formed in section a ($\theta_a = 0$). And it is already possible to know the accumulated rotation in the plastic hinges (θ_i):

$$\begin{aligned}\theta_c &= \frac{LM_{y,Rk}}{6EI_y} = 0.006558 \text{ rad} \\ \theta_d &= -\frac{LM_{y,Rk}}{3EI_y} = 0.013112 \,\text{rad} \\ \theta_e &= \frac{LM_{y,Rk}}{6EI_y} = 0.006558 \text{ rad}\end{aligned} \tag{22}$$

The assumptions are correct if it is found that energy is dissipated, that is $M_i \cdot \theta_i \geq 0$ must be satisfied in all the plastic hinges.

VPb: uniform distributed load

The direct kinematic method needs one more equilibrium equation to solve this example, that it is obtained by applying the PVD to the virtual problem in figure 10.

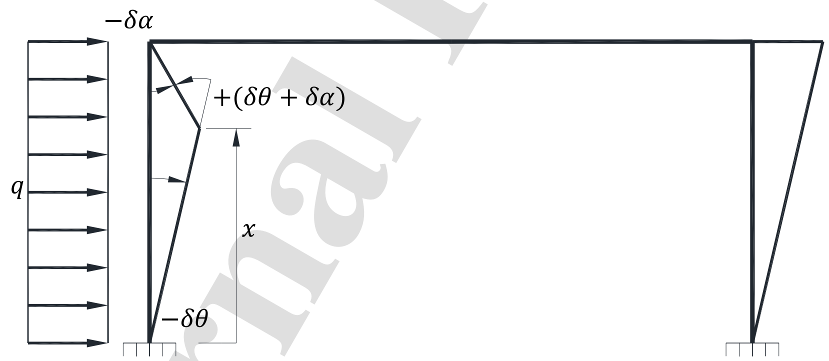

Figure 10: Fixed-fixed frame. Virtual problem in displacements. Mechanism 2

$$\frac{1}{2}q(L_p - x)^2\delta\alpha + \frac{1}{2}qx^2\delta\theta = M_a(-\delta\theta) + M_b(\delta\theta + \delta\alpha) + M_c(-\delta\alpha) \tag{23}$$

due to the distributed load on the left column, a plastic hinge can be produced in section b located at a distance x from section a and M_b is the bending moment in this section. The angles involved in figure 10 can be related according to:

$$\delta\alpha = \frac{L_p - x}{x}\delta\theta \tag{24}$$

The second equilibrium equation required by the method is obtained:

$$\frac{L_p(L_p^2 - 3xL_px + 3x^2)}{2x}q = M_c - M_a + \frac{L_p}{x}(M_b - M_c) \tag{25}$$

The equilibrium equations (11) and (25) allow us to apply the direct kinematic method, plastic hinges in sections a, b, d and e is assumed.

$$M_a = -M_{y,Rk};\ M_b = +M_{y,Rk};\ M_d = -M_{y,Rk};\ M_e = +M_{y,Rk};$$
$$\left.\begin{array}{l} \frac{qL_p^2}{2} = -M_a + M_c - M_d + M_e \\ \frac{L_p(L_p^2-3xL_px+3x^2)}{2x}q = M_c - M_a + \frac{L_p}{x}(M_b - M_c) \end{array}\right\} \rightarrow \begin{array}{l} q = \frac{4M_{y,Rk}(2L_p-x)}{L_p(L_p^2-x)} \\ M_c = \frac{1}{2}qL_p^2 - 3M_{y,Rk} \end{array} \tag{26}$$

The kinematic or minimum theorem states that the collapse load must be minimal:

$$\frac{\partial q}{\partial x} = 0 \ \rightarrow \ x = \left(\sqrt{3}-1\right)L_p \ \rightarrow \ \begin{cases} q_c = q(x) = \frac{2\left(2+\sqrt{3}\right)}{L_p^2}M_{y,Rk} \\ M_c = \left(\sqrt{3}-1\right)M_{y,Rk} \end{cases} \tag{27}$$

Substituting the data, the following numerical values result:

$$\begin{aligned} x &= 2.196 \text{ m} \\ q_c &= 143.2 \text{ kN/m} \\ M_c &= 0.7321 \text{ kNm} \end{aligned} \tag{28}$$

Declaration of interests

☒ The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

☐The authors declare the following financial interests/personal relationships which may be considered as potential competing interests: