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NEUTRAL FUNCTIONAL DIFFERENTIAL EQUATIONS WITH APPLICATIONS TO COMPARTMENTAL SYSTEMS*

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Abstract. We study the monotone skew-product semiflow generated by a family of neutral functional differential equations with infinite delay and stable D-operator. The stability properties of D allow us to introduce a new order and to take the neutral family to a family of functional differential equations with infinite delay. Next, we establish the 1-covering property of omega-limit sets under the componentwise separating property and uniform stability. Finally, the obtained results are applied to the study of the long-term behavior of the amount of material within the compartments of a neutral compartmental system with infinite delay.

Key words. nonautonomous dynamical systems, monotone skew-product semiflows, neutral functional differential equations, infinite delay, compartmental systems

AMS subject classifications. 37B55, 34K40, 34K14

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1. Introduction. After the pioneering work of Hale and Meyer [11], the theory of neutral functional differential equations (NFDE) aroused considerable interest and a fast development ensued. At present a wide collection of theoretical and practical results make up the main body of the theory of NFDEs (see Hale [10], Hale and Verduyn Lunel [12], Kolmanovskii and Myshkis [18], and Salamon [22], among many others). In particular, a substantial number of results for delayed functional differential equations (FDEs) have been generalized for NFDEs solving new and challenging problems in these extensions.

In this paper we provide a dynamical theory for nonautonomous monotone NFDEs with infinite delay and autonomous stable D-operator along the lines of the results by Jiang and Zhao [17] and Novo, Obaya, and Sanz [20]. We assume some recurrence properties on the temporal variation of the NFDE. Thus, its solutions induce a skew-product semiflow with minimal flow on the base. In particular, the uniform almost periodic and almost automorphic cases are included in this formulation. The skew-product formalism permits the analysis of the dynamical properties of the trajectories using methods of ergodic theory and topological dynamics.

Novo et al. [20] study the existence of recurrent solutions of nonautonomous FDEs with infinite delay using the phase space $BU \subset C((-\infty,0],\mathbb{R}^m)$ of bounded and uniformly continuous functions with the supremum norm. Assuming some technical conditions on the vector field, it is shown that every bounded solution is relatively compact for the compact-open topology, and its omega-limit set admits a flow extension. An alternative method for the study of recurrent solutions of almost periodic FDEs with infinite delay makes use of a *fading memory* Banach phase space (see Hino, Murakami, and Naiko [13] for an axiomatic definition and main properties). Since this kind of space contains BU and, under natural assumptions, the restriction of the norm

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topology to the closure of a bounded solution agrees with the compact-open topology, it seems that the approach considered in Novo et al. [20] becomes natural in many cases of interest.

In this paper we consider NFDEs with linear autonomous operator D defined on BU which is continuous for the norm, continuous for the compact-open topology on bounded sets, and atomic at zero. We obtain an integral representation of D by means of Riesz theorem $Dx = \int_{-\infty}^{0} [\, d\mu]\, x$, where μ is a real Borel measure with finite total variation. The convolution operator $\widehat{D}$ defined by $\widehat{D}x(s) = \int_{-\infty}^{0} [d\mu(\theta)]\, x(\theta + s)$ maps BU into BU. We prove that if D is stable in the sense of Hale [10], then $\widehat{D}$ is an isomorphism of BU which is continuous for the norm and continuous for the compact-open topology on bounded sets. Moreover, $\widehat{D}^{-1}$ inherits these same properties and is associated to a linear stable operator D^*. In fact, the mentioned behavior of $\widehat{D}^{-1}$ characterizes the stability of the operator D. The proofs are self-contained and require only quantitative estimates associated to the stability of the operator D.

Staffans [24] shows that every NFDE with finite delay and autonomous stable D-operator can be written as an FDE with infinite delay in an appropriate fading memory space. A more systematic study on the inversion of the convolution operator $\widehat{D}$ can be found in the work of Gripenberg, Londen, and Staffans [4]. The papers by Haddock et al. [8, 9], Arino and Bourad [1], among others, make a systematic use of these ideas which have a theoretical and practical interest. We give a version of the above results for infinite delay NFDEs. It is obvious that the inversion of the convolution operator $\widehat{D}$ on BU allows us to transform the original equation into a retarded nonautonomous FDE with infinite delay. In addition, we transfer the dynamical theory of Jiang and Zhao [17] and Novo et al. [20] to nonautonomous monotone NFDEs with infinite delay and autonomous stable D-operator. In an appropriate dynamical framework we assume that the trajectories are bounded, uniformly stable, and satisfy a componentwise separating property, and we show that the omega-limit sets are all copies of the base. It is important to mention that no conditions of strong monotonicity are required, which permits the application of the results under natural physical conditions.

In this paper we provide a detailed description of the long-term behavior of the dynamics in some classes of compartmental systems extensively studied in the literature. Compartmental systems have been used as mathematical models for the study of the dynamical behavior of many processes in biological and physical sciences, which depend on local mass balance conditions (see Jacquez [14], Jacquez and Simon [15, 16], and the references therein). Some initial results for models described by FDEs with finite and infinite delay can be found in Györi [5] and Györi and Eller [6]. The paper by Arino and Haourigui [2] proves the existence of almost periodic solutions for compartmental systems described by almost periodic finite delay FDEs. NFDEs represent systems where the compartments produce or swallow material. Györi and Wu [7] and Wu and Freedman [28] study autonomous NFDEs with finite and infinite delay similar to those considered in this paper. We provide a nonautonomous version, under more general assumptions, of the monotone theory for NFDEs included in Wu and Freedman [28] and Wu [27]. More precise results for the case of scalar NFDEs can be found in Arino and Bourad [1] and Krisztin and Wu [19].

We study the dynamics of monotone compartmental systems in terms of the geometrical structure of the pipes connecting the compartments. Irreducible subsets of the set of indices detect the occurrence of subsystems on the complete system which reduce the dimension of the problem being studied. When the system is closed, the total mass is an invariant continuous function which implies the stability and boundedness of the solutions. In particular, the omega-limit set of every solution is a minimal set and a copy of the base. In a general compartmental system the existence of a bounded solution assures that every solution is bounded and uniformly stable. We first check that when there is no inflow of material then all the compartments of each irreducible subset with some outflow of material are empty on minimal subsets. On the contrary, when the solutions remain bounded and there is an inflow of material in some compartment of an irreducible subset, then for the indices of this irreducible subset all the minimal sets agree, and all the compartments contain some material. Finally, we describe natural physical conditions which ensure the existence of a unique minimal set asymptotically stable.

This paper is organized as follows. Basic notions in topological dynamics, used throughout the rest of the sections, are stated in section 2. Section 3 is devoted to the study of general and stability properties of linear autonomous operators from BU to $\mathbb{R}^m$, as well as the behavior of solutions of the corresponding homogeneous and nonhomogeneous equations given by them. In section 4, we study the monotone skew-product semiflow generated by a family of NFDEs with infinite delay and stable D-operator. In particular, we establish the 1-covering property of omega-limit sets under the componentwise separating property and uniform stability. These results are applied in section 5 to show that the solutions of a compartmental system given by a monotone NFDE with infinite delay are asymptotically of the same type as the transport functions. Finally, section 6 deals with the long-term behavior of its solutions in terms of the geometrical structure of the pipes, as explained above.

2. Some preliminaries. Let (Ω, d) be a compact metric space. A real *continuous flow* $(\Omega,\sigma,\mathbb{R})$ is defined by a continuous mapping $\sigma:\mathbb{R}\times\Omega\to\Omega,\ (t,\omega)\mapsto\sigma(t,\omega)$ satisfying

(i) $\sigma_0 = \mathrm{Id}$,

(ii) $\sigma_{t+s}=\sigma_t\circ\sigma_s$ for each $s,\, t\in\mathbb{R}$,

where $\sigma_t(\omega)=\sigma(t,\omega)$ for all $\omega\in\Omega$ and $t\in\mathbb{R}$. The set $\{\sigma_t(\omega)\mid t\in\mathbb{R}\}$ is called the *orbit* or the *trajectory* of the point ω. We say that a subset $\Omega_1\subset\Omega$ is σ-*invariant* if $\sigma_t(\Omega_1)=\Omega_1$ for every $t\in\mathbb{R}$. A subset $\Omega_1\subset\Omega$ is called *minimal* if it is compact, σ-invariant, and its only nonempty compact σ-invariant subset is itself. Every compact and σ-invariant set contains a minimal subset; in particular it is easy to prove that a compact σ-invariant subset is minimal if and only if every trajectory is dense. We say that the continuous flow $(\Omega,\sigma,\mathbb{R})$ is *recurrent* or *minimal* if Ω is minimal.

The flow $(\Omega,\sigma,\mathbb{R})$ is *distal* if for any two distinct points $\omega_1,\,\omega_2\in\Omega$ the orbits keep at a positive distance, that is, $\inf_{t\in\mathbb{R}} d(\sigma(t,\omega_1),\sigma(t,\omega_2))>0$. The flow $(\Omega,\sigma,\mathbb{R})$ is *almost periodic* when for every $\varepsilon>0$ there is a $\delta>0$ such that if $\omega_1,\,\omega_2\in\Omega$ with $d(\omega_1,\omega_2)<\delta$, then $d(\sigma(t,\omega_1),\sigma(t,\omega_2))<\varepsilon$ for every $t\in\mathbb{R}$. If $(\Omega,\sigma,\mathbb{R})$ is almost periodic, it is distal. The converse is not true; even if $(\Omega,\sigma,\mathbb{R})$ is minimal and distal, it does not need to be almost periodic. For the basic properties of almost periodic and distal flows we refer the reader to Ellis [3] and Sacker and Sell [21].

A *flow homomorphism* from another continuous flow $(Y,\Psi,\mathbb{R})$ to $(\Omega,\sigma,\mathbb{R})$ is a continuous map $\pi\colon Y\to\Omega$ such that $\pi(\Psi(t,y))=\sigma(t,\pi(y))$ for every $y\in Y$ and $t\in\mathbb{R}$. If π is also bijective, it is called a *flow isomorphism*. Let $\pi:Y\to\Omega$ be a surjective flow homomorphism and suppose $(Y,\Psi,\mathbb{R})$ is minimal (then so is $(\Omega,\sigma,\mathbb{R})$). $(Y,\Psi,\mathbb{R})$ is said to be an *almost automorphic extension* of $(\Omega,\sigma,\mathbb{R})$ if there is $\omega\in\Omega$ such that $\mathrm{card}(\pi^{-1}(\omega))=1$. Then actually $\mathrm{card}(\pi^{-1}(\omega))=1$ for ω in a residual subset $\Omega_0\subseteq\Omega$; in the nontrivial case $\Omega_0\subsetneq\Omega$ the dynamics can be very complicated. A minimal flow

$(Y, \Psi, \mathbb{R})$ is *almost automorphic* if it is an almost automorphic extension of an almost periodic minimal flow $(\Omega, \sigma, \mathbb{R})$. We refer the reader to the work of Shen and Yi [23] for a survey of almost periodic and almost automorphic dynamics.

Let E be a complete metric space and $\mathbb{R}^+ = \{t \in \mathbb{R} \,|\, t \geq 0\}$. A *semiflow* $(E, \Phi, \mathbb{R}^+)$ is determined by a continuous map $\Phi : \mathbb{R}^+ \times E \to E,\ (t, x) \mapsto \Phi(t, x)$ which satisfies

(i) $\Phi_0 = \mathrm{Id}$,

(ii) $\Phi_{t+s} = \Phi_t \circ \Phi_s$ for all $t,\, s \in \mathbb{R}^+$,

where $\Phi_t(x) = \Phi(t, x)$ for each $x \in E$ and $t \in \mathbb{R}^+$. The set $\{\Phi_t(x) \mid t \geq 0\}$ is the *semiorbit* of the point x. A subset E_1 of E is *positively invariant* (or just Φ-*invariant*) if $\Phi_t(E_1) \subset E_1$ for all $t \geq 0$. A semiflow $(E, \Phi, \mathbb{R}^+)$ admits a *flow extension* if there exists a continuous flow $(E, \widetilde{\Phi}, \mathbb{R})$ such that $\widetilde{\Phi}(t, x) = \Phi(t, x)$ for all $x \in E$ and $t \in \mathbb{R}^+$. A compact and positively invariant subset admits a flow extension if the semiflow restricted to it admits one.

Write $\mathbb{R}^- = \{t \in \mathbb{R} \,|\, t \leq 0\}$. A *backward orbit* of a point $x \in E$ in the semiflow $(E, \Phi, \mathbb{R}^+)$ is a continuous map $\psi : \mathbb{R}^- \to E$ such that $\psi(0) = x$ and for each $s \leq 0$ it holds that $\Phi(t, \psi(s)) = \psi(s + t)$ whenever $0 \leq t \leq -s$. If for $x \in E$ the semiorbit $\{\Phi(t, x) \mid t \geq 0\}$ is relatively compact, we can consider the *omega-limit set* of x,

$$\mathcal{O}(x) = \bigcap_{s \geq 0} \mathrm{closure}\{\Phi(t + s, x) \mid t \geq 0\}\,,$$

which is a nonempty compact connected and Φ-invariant set. Namely, it consists of the points $y \in E$ such that $y = \lim_{n\to\infty} \Phi(t_n, x)$ for some sequence $t_n \uparrow \infty$. It is well known that every $y \in \mathcal{O}(x)$ admits a backward orbit inside this set. Actually, a compact positively invariant set M admits a flow extension if every point in M admits a unique backward orbit which remains inside the set M (see Shen and Yi [23, part II]).

A compact positively invariant set M for the semiflow $(E, \Phi, \mathbb{R}^+)$ is *minimal* if it does not contain any nonempty compact positively invariant set other than itself. If E is minimal, we say that the semiflow is minimal.

A semiflow is of *skew-product type* when it is defined on a vector bundle and has a triangular structure; more precisely, a semiflow $(\Omega \times X, \tau, \mathbb{R}^+)$ is a *skew-product* semiflow over the product space $\Omega \times X$, for a compact metric space (Ω, d) and a complete metric space (X, d), if the continuous map τ is as follows:

$$\begin{aligned} \tau : \quad \mathbb{R}^+ \times \Omega \times X \quad &\longrightarrow \quad \Omega \times X \\ (t, \omega, x) \quad &\mapsto \quad (\omega{\cdot}t, u(t, \omega, x))\,, \end{aligned} \tag{2.1}$$

where $(\Omega, \sigma, \mathbb{R})$ is a real continuous flow $\sigma : \mathbb{R} \times \Omega \to \Omega,\ (t, \omega) \mapsto \omega{\cdot}t$, called the *base flow*. The skew-product semiflow (2.1) is *linear* if $u(t, \omega, x)$ is linear in x for each $(t, \omega) \in \mathbb{R}^+ \times \Omega$.

Now, we introduce some definitions concerning the stability of the trajectories. A forward orbit $\{\tau(t, \omega_0, x_0) \,|\, t \geq 0\}$ of the skew-product semiflow (2.1) is said to be *uniformly stable* if for every $\varepsilon > 0$ there is a $\delta(\varepsilon) > 0$, called the *modulus of uniform stability*, such that if $s \geq 0$ and $\mathsf{d}(u(s, \omega_0, x_0), x) \leq \delta(\varepsilon)$ for certain $x \in X$, then for each $t \geq 0$,

$$\mathsf{d}(u(t + s, \omega_0, x_0), u(t, \omega_0{\cdot}s, x)) = \mathsf{d}(u(t, \omega_0{\cdot}s, u(s, \omega_0, x_0)), u(t, \omega_0{\cdot}s, x)) \leq \varepsilon\,.$$

A forward orbit $\{\tau(t, \omega_0, x_0) \,|\, t \geq 0\}$ of the skew-product semiflow (2.1) is said to be *uniformly asymptotically stable* if it is uniformly stable and there is a $\delta_0 > 0$ with the following property: for each $\varepsilon > 0$ there is a $t_0(\varepsilon) > 0$ such that if $s \geq 0$ and $\mathsf{d}(u(s, \omega_0, x_0), x) \leq \delta_0$, then

$$\mathsf{d}(u(t+s, \omega_0, x_0), u(t, \omega_0{\cdot}s, x)) \leq \varepsilon \quad \text{for each } t \geq t_0(\varepsilon)\,.$$

3. Stable *D*-operators. We consider the Fréchet space $X = C((-\infty, 0], \mathbb{R}^m)$ endowed with the compact-open topology, i.e., the topology of uniform convergence over compact subsets, which is a metric space for the distance

$$\mathsf{d}(x, y) = \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{\|x-y\|_n}{1 + \|x-y\|_n}\,, \quad x, y \in X\,,$$

where $\|x\|_n = \sup_{s\in[-n,0]} \|x(s)\|$ and $\|\cdot\|$ denotes the maximum norm on $\mathbb{R}^m$.

Let $BU \subset X$ be the Banach space

$$BU = \{x \in X \mid x \text{ is bounded and uniformly continuous}\}$$

with the supremum norm $\|x\|_\infty = \sup_{s\in(-\infty,0]} \|x(s)\|$. Given $r > 0$, we will denote

$$B_r = \{x \in BU \mid \|x\|_\infty \leq r\}\,.$$

As usual, given $I = (-\infty, a] \subset \mathbb{R}$, $t \in I$, and a continuous function $x : I \to \mathbb{R}^m$, x_t will denote the element of X defined by $x_t(s) = x(t+s)$ for $s \in (-\infty, 0]$.

This section is devoted to the study of general and stability properties of linear autonomous operators $D\colon BU \to \mathbb{R}^m$, as well as the behavior of solutions of the corresponding homogeneous equation $Dx_t = 0$, $t \geq 0$, and nonhomogeneous equations $Dx_t = h(t)$, $t \geq 0$, for $h \in C([0, \infty), \mathbb{R}^m)$. We will assume the following:

(D1) D is linear and continuous for the norm.

(D2) For each $r > 0$, $D\colon B_r \to \mathbb{R}^m$ is continuous when we take the restriction of the compact-open topology to B_r; i.e., if $x_n \xrightarrow{\mathsf{d}} x$ as $n \to \infty$ with $x_n, x \in B_r$, then $\lim_{n\to\infty} Dx_n = Dx$.

(D3) D is atomic at 0 (see the definition in Hale [10] or Hale and Verduyn Lunel [12]).

From (D1) and (D2) we obtain the following representation.

Proposition 3.1. *If $D\colon BU \to \mathbb{R}^m$ satisfies* (D1) *and* (D2)*, then for each $x \in BU$*

$$Dx = \int_{-\infty}^{0} [d\mu(s)]\, x(s),$$

where $\mu = [\mu_{ij}]$ and μ_{ij} is a real regular Borel measure with finite total variation $|\mu_{ij}|(-\infty, 0] < \infty$ for all $i, j \in \{1, \ldots, m\}$.

Proof. From Riesz representation theorem we obtain the above relation for each x whose components are of compact support. Moreover, if $x \in BU$, there are an $r > 0$ and a sequence of functions of compact support $\{x_n\}_{n\in\mathbb{N}} \subset B_r$ with $\|x_n\|_\infty \leq \|x\|_\infty$ such that $x_n \xrightarrow{\mathsf{d}} x$ as $n \to \infty$ and, from hypothesis (D2), $\lim_{n\to\infty} Dx_n = Dx$. However,

$$Dx_n = \int_{-\infty}^{0} [d\mu(s)]\, x_n(s),$$

and the Lebesgue-dominated convergence theorem yields

$$\lim_{n\to\infty} Dx_n = \int_{-\infty}^{0} [d\mu(s)]\, x(s)\,,$$

which finishes the proof. ∎

Since in addition D is atomic at 0, $\det[\mu_{ij}(\{0\})] \neq 0$, and without loss of generality, we may assume that

$$Dx = x(0) - \int_{-\infty}^{0} [d\nu(s)]\, x(s), \tag{3.1}$$

where $\nu = [\nu_{ij}]_{i,j \in \{1,\dots,m\}}$, ν_{ij} is a real regular Borel measure with finite total variation, and $|\nu_{ij}|(\{0\}) = 0$ for all $i,\, j \in \{1,\dots,m\}$. We will denote by $|\nu|[-r,0]$ the $m \times m$ matrix $[\,|\nu_{ij}|[-r,0]\,]$ and by $\|\nu\|_\infty[-r,0]$ the corresponding matricial norm.

From now on, we will assume that the operator D satisfying (D1)–(D3) has the form (3.1). First, it is easy to check the following result, whose proof is omitted.

PROPOSITION 3.2. *For all $h \in C([0,\infty),\mathbb{R}^m)$ and $\varphi \in BU$ with $D\varphi = h(0)$, the nonhomogeneous equation*

$$\begin{cases} Dx_t = h(t)\,, \quad t \geq 0\,, \\ x_0 = \varphi \end{cases} \tag{3.2}$$

has a solution defined for all $t \geq 0$.

Next we obtain a bound for the solution in a finite interval $[0,T]$, in terms of the initial data and the independent term h, which in particular implies the uniqueness of the solution of (3.2).

LEMMA 3.3. *Given $T > 0$, there are positive constants k_T^1, k_T^2 such that if x is a solution of* (3.2)*, then for each $t \in [0,T]$*

$$\|x_t\|_\infty \leq k_T^1 \sup_{0\leq u\leq t} \|h(u)\| + k_T^2\, \|\varphi\|_\infty\,. \tag{3.3}$$

Proof. Since $|\nu_{ij}|[-r,0] \to 0$ as $r \to 0$, for each $i,\, j \in \{1,\dots,m\}$, there is an $r > 0$ such that $\|\nu\|_\infty[-r,0] < 1/2$. Let x be a solution of (3.2). From (3.1),

$$x(t) = h(t) + \int_{-t}^{0} [d\nu(s)]\, x(t+s) + \int_{-\infty}^{-t} [d\nu(s)]\, \varphi(t+s)$$

for each $t \geq 0$. Consequently, if $t \in [0,r]$,

$$\|x(t)\| \leq \|h(t)\| + \frac{1}{2} \sup_{0\leq u\leq t} \|x(u)\| + \|\varphi\|_\infty \|\nu\|_\infty(-\infty,0]\,,$$

from which we deduce that if $t \in [0,r]$,

$$\sup_{0\leq u\leq t} \|x(u)\| \leq 2 \sup_{0\leq u\leq t} \|h(u)\| + 2\,a\,\|\varphi\|_\infty\,, \tag{3.4}$$

where $a = \|\nu\|_\infty(-\infty,0]$. Next, let $y(t) = x(t+r)$, which is a solution of

$$\begin{cases} Dy_t = h(t+r)\,, \quad t \geq 0\,, \\ y_0 = x_r\,. \end{cases}$$

As above, we conclude that if $t \in [0,r]$,

$$\sup_{0\leq u\leq t} \|y(u)\| \leq 2 \sup_{0\leq u\leq t} \|h(u+r)\| + 2\,a\,\|x_r\|_\infty\,,$$

which together with $\|x_r\|_\infty \le \|\varphi\|_\infty + \sup_{0\le u\le r}\|x(u)\|$ and (3.4) yields

$$\sup_{0\le u\le t}\|x(u)\| \le b \sup_{0\le u\le t}\|h(u)\| + c\,\|\varphi\|_\infty$$

for $t \in [r, 2\,r]$ and some positive constants b and c independent of h and φ. This way, the result is obtained in a finite number of steps. $\square$

Following Hale [10], we introduce the concept of stability for the operator D. Although the initial definition is given for the homogeneous equation, it is easy to deduce quantitative estimates in terms of the initial data for the solution of a nonhomogeneous equation.

Definition 3.4. *The linear operator D is said to be* stable *if there is a continuous function $c \in C([0,\infty), \mathbb{R}^+)$ with $\lim_{t\to\infty} c(t) = 0$ such that, for each $\varphi \in BU$ with $D\varphi = 0$, the solution of the homogeneous problem*

$$\begin{cases} Dx_t = 0\,, & t \ge 0,\\ x_0 = \varphi \end{cases}$$

satisfies $\|x(t)\| \le c(t)\,\|\varphi\|_\infty$ for each $t \ge 0$.

Proposition 3.5. *Let us assume that D is stable. Then there is a positive constant $d > 0$ such that, for each $h \in C([0,\infty), \mathbb{R}^m)$ with $h(0) = 0$, the solution of*

$$\begin{cases} Dx_t = h(t)\,, & t \ge 0\,,\\ x_0 = 0 \end{cases}$$

satisfies $\|x(t)\| \le d \sup_{0\le u\le t}\|h(u)\|$ for each $t \ge 0$.

Proof. Let $\{e_1, \ldots, e_m\}$ be the canonical basis of $\mathbb{R}^m$. A proof similar to that of Lemma 3.2 of Hale [10, sec. 12] shows that there are m functions $\phi_1, \ldots, \phi_m \in BU$ such that $D\phi_j = e_j$ for each $j \in \{1, \ldots, m\}$. We will denote by Φ the $m \times m$ matrix function $\Phi = [\phi_1, \ldots, \phi_m]$ and by $\|\Phi\|_\infty$ the matricial norm corresponding to the norm $\|\cdot\|_\infty$ on BU.

Let $c \in C([0,\infty), \mathbb{R}^m)$ be the function given in Definition 3.4. Assume that c is decreasing and take $T > 0$ such that $c(T) < 1$. From Lemma 3.3, $\|x(t)\| \le k_T^1 \sup_{0\le u\le t}\|h(u)\|$, provided that $t \in [0, T]$.

If $t \ge T$, there is a $j \in \mathbb{N}$ such that $t \in [j\,T, (j+1)\,T]$ and it is easy to check that $x(t) = x^1(t-(j-1)\,T) + x^2(t-(j-1)\,T)$, where x^1 and x^2 are the solutions of

$$\begin{cases} Dx_t^1 = 0\,, & t \ge 0\,,\\ x_0^1 = x_{(j-1)\,T} - \Phi\, h((j-1)\,T)\,, \end{cases} \qquad \begin{cases} Dx_t^2 = h(t+(j-1)\,T)\,, & t \ge 0\,,\\ x_0^2 = \Phi\, h((j-1)\,T)\,, \end{cases}$$

respectively. From the stability of D and Lemma 3.3, we deduce that

$$\|x(t)\| \le c(t-(j-1)\,T)\,\|x_{(j-1)\,T} - \Phi\, h((j-1)\,T)\|_\infty + k_{2T} \sup_{(j-1)\,T\le u\le t}\|h(u)\|\,.$$

In addition, since $t-(j-1)\,T \ge T$ and c is decreasing we conclude that

$$\|x(t)\| \le c(T)\,c_j + (c(T)\,\|\Phi\|_\infty + k_{2T}) \sup_{0\le u\le t}\|h(u)\|\,, \quad t \in [j\,T, (j+1)\,T], \tag{3.5}$$

where $c_j = \|x_{jT}\|_\infty = \sup_{0\le u\le jT}\|x(u)\|$.

Let $a_T = \max\{k_T^1,\, c(T)\,\|\Phi\|_\infty + k_{2T}\}$. We have $c_1 \le a_T \sup_{0\le u\le T}\|h(u)\|$ and from (3.5), if $j\ge 2$,

$$c_j \le \max\left\{c_{j-1}, c(T)\,c_{j-1} + a_T \sup_{0\le u\le j\,T}\|h(u)\|\right\}.$$

Hence, we check that for each $j\ge 2$

$$c_j \le a_T\left(1+c(T)+\cdots+c(T)^{j-1}\right)\sup_{0\le u\le j\,T}\|h(u)\|\,,$$

and again from (3.5) we finally deduce that for $t\ge 0$ (and hence $t\in[j\,T,(j+1)\,T]$ for some $j\ge 0$)

$$\|x(t)\| \le a_T\sum_{k=0}^{j} c(T)^k \sup_{0\le u\le t}\|h(u)\| \le \frac{a_T}{1-c(T)}\sup_{0\le u\le t}\|h(u)\|\,,$$

which finishes the proof. ∎

THEOREM 3.6. *Let us assume that D is stable. Then there is a continuous function $c\in C([0,\infty),\mathbb{R}^+)$ with $\lim_{t\to\infty} c(t)=0$ and a positive constant $k>0$ such that the solution of* (3.2) *satisfies*

$$\|x(t)\| \le c(t)\,\|\varphi\|_\infty + k\sup_{0\le u\le t}\|h(u)\|$$

for each $t\ge 0$.

Proof. It is not hard to check that $x(t)=x^1(t)+x^2(t)$, where x^1 and x^2 are the solutions of

$$\begin{cases} Dx_t^1 = \psi(t)\,h(t)\,, & t\ge 0\,,\\ x_0^1=\varphi\,, \end{cases} \qquad \begin{cases} Dx_t^2 = (1-\psi(t))\,h(t)\,, & t\ge 0\,,\\ x_0^2=0\,, \end{cases}$$

respectively, and

$$\begin{array}{rccl} \psi\colon & [0,\infty) & \longrightarrow & \mathbb{R}\\ & t & \mapsto & \psi(t)=\begin{cases}1-t, & 0\le t\le 1\,,\\ 0\,, & 1\le t\,.\end{cases} \end{array}$$

Moreover, since $y(t)=x^1(t+1)$ satisfies $Dy_t=0$, $t\ge 0$, with $y_0=x_1^1$, the result follows from the application of Definition 3.4, Lemma 3.3, and Proposition 3.5 to y, x^1 on $[0,1]$, and x^2, respectively. ∎

The conclusions of Theorem 3.6 are essential in what follows. In particular, it allows us to estimate the norm of a function x in terms of the norm of the function $(-\infty,0]\to\mathbb{R}^m$, $s\mapsto Dx_s$.

PROPOSITION 3.7. *Let us assume that D is stable. Then there is a positive constant $k>0$ such that $\|x^h\|_\infty \le k\,\|h\|_\infty$ for all $h\in BU$ and $x^h\in BU$ satisfying $Dx_s^h=h(s)$ for $s\le 0$.*

Proof. Let $x(t)$ be the solution of

$$\begin{cases} Dx_t = h(0)\,, & t\ge 0\,,\\ x_0 = x^h\,, \end{cases}$$

$$\widetilde{h}(t) = \begin{cases} h(t)\,, & t \le 0\,, \\ h(0)\,, & t \ge 0\,, \end{cases}$$

and for $s \le 0$ we define

$$y^s(t) = \begin{cases} x(t+s)\,, & t+s \ge 0\,, \\ x^h(t+s)\,, & t+s \le 0\,. \end{cases}$$

Then

$$\begin{cases} Dy^s_t = \widetilde{h}(t+s)\,, & t \ge 0\,, \\ y^s_0 = x^h_s\,, \end{cases}$$

and Theorem 3.6 yields

$$\|y^s(t)\| \le c(t)\,\|x^h_s\|_\infty + k \sup_{0\le u\le t} \|\widetilde{h}(u+s)\|_\infty \le c(t)\,\|x^h\|_\infty + k\,\|h\|_\infty$$

for all $t \ge 0$ and $s \le 0$. Hence, $\|x^h(s)\| = \|y^{s-t}(t)\| \le c(t)\,\|x^h\|_\infty + k\,\|h\|_\infty$, and as $t \to \infty$ we prove the result. ∎

Let D be stable and given by (3.1). We define the linear operator

$$(3.6)\qquad \begin{array}{rcccccl} \widehat{D}\colon & BU & \longrightarrow & BU & & & \\ & x & \mapsto & \widehat{D}x\colon (-\infty,0] & \to & \mathbb{R}^m & \\ & & & s & \mapsto & Dx_s\,, & \end{array}$$

that is, $\widehat{D}x(s) = x(s) - \int_{-\infty}^0 [d\nu(\theta)]\,x(\theta+s)$ for each $s \in (-\infty,0]$, which is well defined, i.e., $h = \widehat{D}x \in BU$, provided that $x \in BU$ because D is bounded and $h(s+\tau) - h(s) = D\,(x_{s+\tau} - x_s)$ for all τ, $s \le 0$. Moreover, it is easy to check that $\widehat{D}$ is bounded for the norm and uniformly continuous when we take the restriction of the compact-open topology to B_r; i.e., given $\varepsilon > 0$ there is a $\delta(r) > 0$ such that $\mathsf{d}(\widehat{D}x_1, \widehat{D}x_2) < \varepsilon$ for all x_1, $x_2 \in B_r$ with $\mathsf{d}(x_1,x_2) < \delta(r)$. The next result shows, after proving that $\widehat{D}$ is invertible, that the same happens for $\widehat{D}^{-1}$.

Theorem 3.8. *Let us assume that D is stable. Then $\widehat{D}$ is invertible, and $\widehat{D}^{-1}$ is bounded for the norm and uniformly continuous when we take the restriction of the compact-open topology to B_r; i.e., given $\varepsilon > 0$ there is a $\delta(r) > 0$ such that $\mathsf{d}(\widehat{D}^{-1}h_1, \widehat{D}^{-1}h_2) < \varepsilon$ for all h_1, $h_2 \in B_r$ with $\mathsf{d}(h_1,h_2) < \delta(r)$.*

Proof. $\widehat{D}$ is injective because from Proposition 3.7 the only solution of $Dx_s = 0$ for $s \le 0$ is $x = 0$. To show that $\widehat{D}$ is onto, let $h \in BU$ and $\{h_n\}_{n\in\mathbb{N}} \subset B_r$, for some $r > 0$, be a sequence of continuous functions whose components are of compact support such that $h_n \xrightarrow{\mathsf{d}} h$ as $n \uparrow \infty$. Moreover, it is easy to choose them with the same modulus of uniform continuity as h. It is not hard to check that for each $n \in \mathbb{N}$ there is an $x^n \in BU$ such that $\widehat{D}x^n = h_n$, that is, $Dx^n_s = h_n(s)$ for $s \le 0$ and $n \in \mathbb{N}$. From Proposition 3.7, $x^n \in B_{kr}$ because $\|x^n\|_\infty \le k\,\|h_n\|_\infty \le k\,r$ and $\|x^n - x^n_\tau\|_\infty \le k\,\|h_n - (h_n)_\tau\|_\infty$ for each $\tau \le 0$ and $n \in \mathbb{N}$, which implies that $\{x_n\}_{n\in\mathbb{N}}$ is equicontinuous, and hence relatively compact for the compact-open topology. Hence, there is a convergent subsequence, let us assume the whole sequence; i.e., there is a continuous function x such that $x^n \xrightarrow{\mathsf{d}} x$ as $n \uparrow \infty$. From this, $x^n_s \xrightarrow{\mathsf{d}} x_s$ for each $s \le 0$ and (3.1) yields $Dx^n_s = h_n(s) \to Dx_s$, i.e., $Dx_s = h(s)$ for $s \le 0$ and $\widehat{D}x = h$. It is immediate to check that $x \in BU$ and then $\widehat{D}$ is onto, as claimed.

Since $\widehat{D}$ is linear, bounded for the norm, and bijective, the continuity of $\widehat{D}^{-1}$ for the norm is immediate. However, it also follows from Proposition 3.7 which reads as $\|\widehat{D}^{-1}h\|_\infty \le k\,\|h\|_\infty$. Finally, since $\widehat{D}^{-1}$ is linear, to check the uniform continuity for the metric on each B_r, it is enough to prove the continuity at 0, i.e., $\widehat{D}^{-1}x_n \xrightarrow{\mathsf{d}} 0$ as $n\uparrow\infty$, whenever $x_n \xrightarrow{\mathsf{d}} 0$ as $n\uparrow\infty$ and $\{x_n\}_{n\in\mathbb{N}}\subset B_r$. Let $y^n=\widehat{D}^{-1}x_n\in B_{kr}$. We extend the definition of y^n to $t\ge 0$ as the solution of

$$\begin{cases} Dy^n_t = x_n(0)\,, & t\ge 0\,,\\ y^n_0 = y^n\,. \end{cases}$$

The stability of D provides

$$\|y^n(t+s)\| \le c(t)\,\|y^n\|_\infty + k \sup_{s\le u\le 0}\|x_n(u)\| \tag{3.7}$$

for all $t\ge 0$ and $s\le 0$. Now we check that $\{y^n\}_{n\in\mathbb{N}}$ converges uniformly to 0 on each compact set $K=[-a,0]$. Given $\varepsilon>0$, there is a $t_0>0$ such that $c(t_0)\,\|y^n\|_\infty<\varepsilon/2$ for each $n\in\mathbb{N}$. Moreover, $x_n\to 0$ in $\widetilde{K}=[-a-t_0,0]$, and hence there is an n_0 such that for each $n\ge n_0$ we have $k\,\|x_n\|_{\widetilde{K}}<\varepsilon/2$. Therefore, from (3.7) we deduce that for all $u\in K=[-a,0]$ and $n\ge n_0$,

$$\|y^n(u)\| = \|y^n(t_0+u-t_0)\| < \frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon\,,$$

that is, $\|y^n\|_K<\varepsilon$, and $\widehat{D}^{-1}x_n=y^n\xrightarrow{\mathsf{d}}0$ as $n\uparrow\infty$, which finishes the proof. □

A more systematic study of the properties of the linear operator $\widehat{D}$ defined in (3.6) can be found in Staffans [25]. The next result provides a necessary and sufficient condition for a continuous operator D to be stable. In particular, if $\widehat{D}$ is invertible and $\widehat{D}^{-1}$ is continuous for the restriction of the compact-open topology to B_r, then D is stable.

THEOREM 3.9. *Let $D\colon BU\to\mathbb{R}^m$ be given by* (3.1) *and let $\widehat{D}$ be the linear operator in BU defined in* (3.6). *The following statements are equivalent:*

(i) *D is stable.*

(ii) *For each $r>0$ and each sequence $\{x_n\}_{n\in\mathbb{N}}$ in BU such that $\|\widehat{D}x_n\|_\infty\le r$ and $\widehat{D}x_n\xrightarrow{\mathsf{d}}0$ as $n\uparrow\infty$, $x_n(0)\to 0$ as $n\uparrow\infty$.*

Proof. (i) ⇒ (ii) is a consequence of Theorem 3.8.

(ii) ⇒ (i) For each $T>0$ we define $\mathcal{L}_T\colon\{\varphi\in BU\mid D\varphi=0\}\to\mathbb{R}^m$, $\varphi\mapsto x(T)$, where x is the solution of

$$\begin{cases} Dx_t = 0\,, & t\ge 0\,,\\ x_0=\varphi\,. \end{cases}$$

It is easy to check that $\mathcal{L}_T$ is well defined and linear. In addition, from (3.3) we deduce that $\|\mathcal{L}_T(\varphi)\|\le\|x_T\|_\infty\le k_T^2\,\|\varphi\|_\infty$, and hence it is bounded.

Next we check that $\|\mathcal{L}_T\|_\infty\to 0$ as $T\to\infty$, which shows the stability of D because $\|x(T)\|\le c(T)\,\|\varphi\|_\infty$ for $c(T)=\|\mathcal{L}_T\|_\infty$. Let us assume, on the contrary, that there exist $\delta>0$, a sequence $T_n\uparrow\infty$, and a sequence $\{\varphi_n\}_{n\in\mathbb{N}}$ with $\|\varphi_n\|_\infty\le 1$ and $D\varphi_n=0$ such that $\|\mathcal{L}_{T_n}(\varphi_n)\|\ge\delta$ for each $n\in\mathbb{N}$. That is, $\|x^n(T_n)\|\ge\delta$, where x^n is the solution of

$$\begin{cases} Dx^n_t = 0\,, & t\ge 0\,,\\ x^n_0=\varphi_n\,. \end{cases}$$

Therefore,

$$\begin{cases} D((x^n_{T_n})_s) = D(x^n_{T_n+s}) = 0 & \text{if } s \in [-T_n, 0]\,, \\ D((x^n_{T_n})_s) = D((\varphi_n)_{T_n+s}) & \text{if } s \le -T_n\,, \end{cases}$$

and taking $r = \|D\|_\infty$, the sequence $\{x^n_{T_n}\}_{n\in\mathbb{N}} \subset BU$ satisfies $\|\widehat{D}x_n\|_\infty \le r$ and $\widehat{D}x^n_{T_n} \xrightarrow{\mathsf{d}} 0$ as $n \uparrow \infty$. Consequently, $x^n_{T_n}(0) = x^n(T_n) \to 0$ as $n \uparrow \infty$, which contradicts the fact that $\|x^n(T_n)\| \ge \delta$ and finishes the proof. ∎

PROPOSITION 3.10. *Let $D\colon BU \to \mathbb{R}^m$ be a stable operator given by* (3.1) *and let $\widehat{D}$ be the linear operator in BU defined in* (3.6). *Then*

$$\begin{array}{rccc} D^*\colon & BU & \longrightarrow & \mathbb{R}^m \\ & x & \mapsto & \widehat{D}^{-1}x(0) \end{array}$$

is also stable and satisfies (D1)–(D3).

Proof. From Theorem 3.8, we deduce that D^* satisfies (D1)–(D2). Hence as in Proposition 3.1, there is a real regular Borel measure μ^* with finite total variation such that $D^*x = \int_{-\infty}^0 [d\mu^*(s)]\,x(s)$ for each $x \in BU$. We can write $\mu^* = A\,\delta - \nu^*$ with $A = [\mu^*_{ij}(\{0\})]$. We claim that D^* is atomic at 0, i.e., $\det A \ne 0$. Assume, on the contrary, that $\det A = 0$ and let $v \in \mathbb{R}^m$ be a unitary vector with $A\,v = 0$. For each $\varepsilon > 0$ we take $\varphi_\varepsilon\colon (-\infty, 0] \to \mathbb{R}$ with $\|\varphi_\varepsilon\|_\infty = \varphi_\varepsilon(0) = 1$ and $\varphi_\varepsilon(s) = 0$ for each $s \in (-\infty, -\varepsilon]$. Let $x^\varepsilon \in BU$ be defined by $x^\varepsilon(s) = \varphi_\varepsilon(s)\,v$. The continuity of $\widehat{D}$ yields

$$1 = \|x^\varepsilon\|_\infty \le c\,\|\widehat{D}^{-1}x^\varepsilon\|_\infty\,. \tag{3.8}$$

However, for each $s \in (-\infty, 0]$

$$\widehat{D}^{-1}x^\varepsilon(s) = D^*x^\varepsilon_s = \varphi_\varepsilon(s)A\,v - \int_{-\infty}^0 [d\nu^*(\theta)]\,\varphi_\varepsilon(\theta+s)\,v$$

and, consequently, $\|\widehat{D}^{-1}x^\varepsilon\|_\infty \le \|\nu^*\|_\infty(-\varepsilon, 0]$, which tends to 0 as $\varepsilon \to 0$, contradicts (3.8) and shows that D^* is atomic at 0. Finally, D^* is stable as a consequence of Theorem 3.9. Notice that μ^* is the inverse of the measure μ for the convolution defining the operator $\widehat{D}$. ∎

4. Monotone NFDEs. Throughout this section, we will study the monotone skew-product semiflow generated by a family of NFDEs with infinite delay and stable D-operator. In particular, we establish the 1-covering property of omega-limit sets under the componentwise separating property and uniform stability, as in Jiang and Zhao [17] for FDEs with finite delay, and Novo, Obaya, and Sanz [20] for infinite delay. The main tool in the proof of the result is the transformation of the initial family of NFDEs into a family of FDEs with infinite delay in whose study the results of Novo et al. [20] turn out to be useful.

Let $(\Omega, \sigma, \mathbb{R})$ be a minimal flow over a compact metric space (Ω, d) and denote $\sigma(t, \omega) = \omega{\cdot}t$ for all $\omega \in \Omega$ and $t \in \mathbb{R}$. In $\mathbb{R}^m$, we take the maximum norm $\|v\| = \max_{j=1,\dots,m} |v_j|$ and the usual partial order relation

$$\begin{aligned} v \le w \quad &\Longleftrightarrow \quad v_j \le w_j \quad \text{for } j = 1, \dots, m\,, \\ v < w \quad &\Longleftrightarrow \quad v \le w \quad \text{and} \quad v_j < w_j \quad \text{for some } j \in \{1, \dots, m\}\,. \end{aligned}$$

As in section 3, we consider the Fréchet space $X = C((-\infty, 0], \mathbb{R}^m)$ endowed with the compact-open topology, i.e., the topology of uniform convergence over compact subsets, and $BU \subset X$ the Banach space of bounded and uniformly continuous functions with the supremum norm $\|x\|_\infty = \sup_{s\in(-\infty,0]} \|x(s)\|$.

Let $D\colon BU \to \mathbb{R}^m$ be an autonomous and stable linear operator satisfying hypotheses (D1)–(D3) and given by relation (3.1). The subset

$$BU_D^+ = \{x \in BU \mid Dx_s \geq 0 \quad \text{for each} \quad s \in (-\infty, 0]\}$$

is a positive cone in BU, because it is a nonempty closed subset $BU_D^+ \subset BU$ satisfying $BU_D^+ + BU_D^+ \subset BU_D^+$, $\mathbb{R}^+ BU_D^+ \subset BU_D^+$, and $BU_D^+ \cap (-BU_D^+) = \{0\}$. As usual, a partial order relation on BU is induced, given by

$$x \leq_D y \iff Dx_s \leq Dy_s \quad \text{for each } s \in (-\infty, 0]\,,$$

$$x <_D y \iff x \leq_D y \quad \text{and} \quad x \neq y\,.$$

Remark 4.1. Notice that if we denote the usual partial order of BU

$$x \leq y \iff x(s) \leq y(s) \quad \text{for each } s \in (-\infty, 0]\,,$$

we have that $x \leq_D y$ if and only if $\widehat{D}\,x \leq \widehat{D}\,y$, where $\widehat{D}$ is defined by relation (3.6). Although in some cases they may coincide, this new order is different from the one given by Wu and Freedman in [28].

We consider the family of nonautonomous NFDEs with infinite delay and stable D-operator

$$\frac{d}{dt} Dz_t = F(\omega{\cdot}t, z_t)\,, \quad t \geq 0\,,\ \omega \in \Omega\,, \tag{4.1$_\omega$}$$

defined by a function $F\colon \Omega \times BU \to \mathbb{R}^m$, $(\omega, x) \mapsto F(\omega, x)$ satisfying the following conditions:

(F1) F is continuous on $\Omega \times BU$ and locally Lipschitz in x for the norm $\|\cdot\|_\infty$.

(F2) For each $r > 0$, $F(\Omega \times B_r)$ is a bounded subset of $\mathbb{R}^m$.

(F3) For each $r > 0$, $F\colon \Omega \times B_r \to \mathbb{R}^m$ is continuous when we take the restriction of the compact-open topology to B_r; i.e., if $\omega_n \to \omega$ and $x_n \xrightarrow{\mathsf{d}} x$ as $n \to \infty$ with $x \in B_r$, then $\lim_{n\to\infty} F(\omega_n, x_n) = F(\omega, x)$.

(F4) If $x, y \in BU$ with $x \leq_D y$ and $D_j\, x = D_j\, y$ holds for some $j \in \{1, \dots, m\}$, then $F_j(\omega, x) \leq F_j(\omega, y)$ for each $\omega \in \Omega$.

From hypothesis (F1), the standard theory of NFDEs with infinite delay (see Wang and Wu [26] and Wu [27]) assures that for each $x \in BU$ and each $\omega \in \Omega$ the system $(4.1)_\omega$ locally admits a unique solution $z(t, \omega, x)$ with initial value x, i.e., $z(s, \omega, x) = x(s)$ for each $s \in (-\infty, 0]$. Therefore, the family $(4.1)_\omega$ induces a local skew-product semiflow

$$\begin{aligned} \tau \quad : \mathbb{R}^+ \times \Omega \times BU &\longrightarrow \Omega \times BU \\ (t, \omega, x) &\mapsto (\omega{\cdot}t, u(t, \omega, x))\,, \end{aligned} \tag{4.2}$$

where $u(t, \omega, x) \in BU$ and $u(t, \omega, x)(s) = z(t + s, \omega, x)$ for $s \in (-\infty, 0]$.

As proved in Theorem 3.8, the operator $\widehat{D}$ defined by relation (3.6) is an isomorphism of BU. Hence, the change of variable $y = \widehat{D}z$ takes $(4.1)_\omega$ to

$$y'(t) = G(\omega{\cdot}t, y_t)\,, \quad t \geq 0\,,\ \omega \in \Omega\,, \tag{4.3$_\omega$}$$

with $G\colon \Omega \times BU \to \mathbb{R}^m$, $(\omega, x) \mapsto G(\omega, x) = F(\omega, \widehat{D}^{-1}x)$ satisfying the following conditions:

(H1) G is continuous on $\Omega \times BU$ and locally Lipschitz in x for the norm $\|\cdot\|_\infty$.

(H2) For each $r > 0$, $G(\Omega \times B_r)$ is a bounded subset of $\mathbb{R}^m$.

(H3) For each $r > 0$, $G\colon \Omega \times B_r \to \mathbb{R}^m$ is continuous when we take the restriction of the compact-open topology to B_r, i.e., if $\omega_n \to \omega$ and $x_n \xrightarrow{\mathsf{d}} x$ as $n \to \infty$ with $x \in B_r$, then $\lim_{n\to\infty} G(\omega_n, x_n) = G(\omega, x)$.

(H4) If $x, y \in BU$ with $x \le y$ and $x_j(0) = y_j(0)$ holds for some $j \in \{1, \ldots, m\}$, then $G_j(\omega, x) \le G_j(\omega, y)$ for each $\omega \in \Omega$.

From hypothesis (H1), the standard theory of infinite delay FDEs (see Hino, Murakami, and Naiko [13]) assures that for each $x \in BU$ and each $\omega \in \Omega$ the system $(4.3)_\omega$ locally admits a unique solution $y(t, \omega, x)$ with initial value x, i.e., $y(s, \omega, x) = x(s)$ for each $s \in (-\infty, 0]$. Therefore, the new family $(4.3)_\omega$ induces a local skew-product semiflow

$$
\begin{array}{rlcl}
\widehat{\tau} & : \mathbb{R}^+ \times \Omega \times BU & \longrightarrow & \Omega \times BU \\
 & (t, \omega, x) & \mapsto & (\omega{\cdot}t, \widehat{u}(t, \omega, x))\,,
\end{array}
\tag{4.4}
$$

where $\widehat{u}(t, \omega, x) \in BU$ and $\widehat{u}(t, \omega, x)(s) = y(t + s, \omega, x)$ for $s \in (-\infty, 0]$, and it is related to the previous one, (4.2), by

$$
\widehat{u}(t, \omega, x) = \widehat{D}\, u(t, \omega, \widehat{D}^{-1}\, x)\,. \tag{4.5}
$$

As a consequence, most of the results obtained in Novo et al. [20] for the skew-product semiflow (4.4) can now be translated to (4.2).

From hypotheses (F1) and (F2), each bounded solution $z(t, \omega_0, x_0)$ provides a relatively compact trajectory, as deduced from Proposition 4.1 of Novo et al. [20].

Proposition 4.2. *Let $z(t, \omega_0, x_0)$ be a bounded solution of $(4.1)_{\omega_0}$, that is, $r = \sup_{t\in\mathbb{R}} \|z(t, \omega_0, x_0)\| < \infty$. Then* $\operatorname{closure}_X\{u(t, \omega_0, x_0) \mid t \ge 0\}$ *is a compact subset of BU for the compact-open topology.*

From hypotheses (F1), (F2), and (F3) and Proposition 4.2 and Corollary 4.3 of Novo et al. [20] for the skew-product semiflow (4.4), we can deduce the continuity of the semiflow (4.2) restricted to some compact subsets $K \subset \Omega \times BU$ when the compact-open topology is considered on BU.

Proposition 4.3. *Let $K \subset \Omega \times BU$ be a compact set for the product metric topology and assume that there is an $r > 0$ such that $\tau_t(K) \subset \Omega \times B_r$ for all $t \ge 0$. Then the map*

$$
\begin{array}{rlcl}
\tau & : \mathbb{R}^+ \times K & \longrightarrow & \Omega \times BU \\
 & (t, \omega, x) & \mapsto & (\omega{\cdot}t, u(t, \omega, x))
\end{array}
$$

is continuous when the product metric topology is considered.

From Proposition 4.2, when $z(t, \omega_0, x_0)$ is bounded we can define the omega-limit set of the trajectory of the point (ω_0, x_0) as

$$
\mathcal{O}(\omega_0, x_0) = \{(\omega, x) \in \Omega \times BU \mid \exists\, t_n \uparrow \infty \ \text{ with } \omega_0{\cdot}t_n \to \omega\,,\ u(t_n, \omega_0, x_0) \xrightarrow{\mathsf{d}} x\}\,.
$$

Notice that the omega-limit set of a pair $(\omega_0, x_0) \in \Omega \times BU$ makes sense whenever $\operatorname{closure}_X\{u(t, \omega_0, x_0) \mid t \ge 0\}$ is a compact set, because then $\{u(t, \omega_0, x_0)(0) = z(t, \omega_0, x_0) \mid t \ge 0\}$ is a bounded set. Proposition 4.3 implies that the restriction of the semiflow (4.2) to $\mathcal{O}(\omega_0, x_0)$ is continuous for the compact-open topology. The following result is a consequence of Proposition 4.4 of Novo et al. [20].

PROPOSITION 4.4. *Let* $(\omega_0, x_0) \in \Omega \times BU$ *be such that* $\sup_{t\geq 0} \|z(t, \omega_0, x_0)\| < \infty$. *Then* $K = \mathcal{O}(\omega_0, x_0)$ *is a positively invariant compact subset admitting a flow extension.*

From hypothesis (F4), the monotone character of the semiflow (4.2) is deduced.

PROPOSITION 4.5. *For all* $\omega \in \Omega$ *and* $x, y \in BU$ *such that* $x \leq_D y$ *it holds that*

$$u(t, \omega, x) \leq_D u(t, \omega, y)$$

whenever they are defined.

Proof. From $x \leq_D y$ we know that $\widehat{D}\, x \leq \widehat{D}\, y$, and since (F4) $\Rightarrow$ (H4), from Proposition 4.5 of Novo et al. [20] we deduce that $\widehat{u}(t, \omega, \widehat{D}\, x) \leq \widehat{u}(t, \omega, \widehat{D}\, y)$ whenever they are defined, that is,

$$u(t, \omega, x) = \widehat{D}^{-1}\, \widehat{u}(t, \omega, \widehat{D}\, x) \leq_D \widehat{D}^{-1} \widehat{u}(t, \omega, \widehat{D}\, y) = u(t, \omega, y)\,,$$

as stated. $\square$

We establish the 1-covering property of omega-limit sets when in addition to hypotheses (F1)–(F4) the componentwise separating property and uniform stability are assumed:

(F5) If $x, y \in BU$ with $x \leq_D y$ and $D_i\, x < D_i\, y$ holds for some $i \in \{1, \dots, m\}$, then $D_i\, z_t(\omega, x) < D_i\, z_t(\omega, y)$ for all $t \geq 0$ and $\omega \in \Omega$.

(F6) There is an $r > 0$ such that all the trajectories with initial data in $\widehat{D}^{-1} B_r$ are uniformly stable in $\widehat{D}^{-1} B_{r'}$ for each $r' > r$, and relatively compact for the product metric topology.

From relation (4.5) we deduce that the transformed skew-product semiflow (4.4) satisfies the following:

(H5) If $x, z \in BU$ with $x \leq z$ and $x_i(0) < z_i(0)$ holds for some $i \in \{1, \dots, m\}$, then $y_i(t, \omega, x) < y_i(t, \omega, z)$ for all $t \geq 0$ and $\omega \in \Omega$.

(H6) There is an $r > 0$ such that all the trajectories with initial data in B_r are uniformly stable in $B_{r'}$ for each $r' > r$, and relatively compact for the product metric topology.

Finally, from Theorem 5.3 of Novo et al. [20] applied to the skew-product semiflow (4.4) satisfying hypotheses (H1)–(H6), we obtain the next result for NFDEs with infinite delay.

THEOREM 4.6. *Assume that hypotheses* (F1)–(F6) *hold and let* $(\omega_0, x_0) \in \Omega \times \widehat{D}^{-1} B_r$ *be such that* $K = \mathcal{O}(\omega_0, x_0) \subset \Omega \times \widehat{D}^{-1} B_r$. *Then* $K = \{(\omega, c(\omega)) \mid \omega \in \Omega\}$ *is a copy of the base and*

$$\lim_{t\to\infty} \mathsf{d}(u(t, \omega_0, x_0), c(\omega_0 {\cdot} t)) = 0\,,$$

where $c : \Omega \to BU$ *is a continuous equilibrium, i.e.,* $c(\omega{\cdot}t) = u(t, \omega, c(\omega))$ *for any* $\omega \in \Omega$, $t \geq 0$, *and it is continuous for the compact-open topology on* BU.

Remark 4.7. It is easy to check that it is enough to ask for property (F5) (and for (H5) in the case of FDEs with infinite delay) for initial data in BU whose trajectories are globally defined on $\mathbb{R}$.

5. Compartmental systems. We consider compartmental models for the mathematical description of processes in which the transport of material between compartments takes a nonnegligible length of time, and each compartment produces or swallows material. We provide a nonautonomous version, without strong monotonicity assumptions, of previous autonomous results by Wu and Freedman [28] and Wu [26].

First, we introduce the model with which we are going to deal as well as some notation. Let us suppose that we have a system formed by m compartments $C_1, \dots, C_m$. Denote by C_0 the environment surrounding the system, and by $z_i(t)$ the amount of material within compartment C_i at time t for each $i \in \{1, \dots, m\}$. Material flows from compartment C_j into compartment C_i through a pipe P_{ij} having a transit time distribution given by a positive regular Borel measure μ_{ij} with finite total variation $\mu_{ij}(-\infty, 0] = 1$ for each $i, j \in \{1, \dots, m\}$. Let $\widetilde{g}_{ij} : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be the so-called *transport function* determining the volume of material flowing from C_j to C_i given in terms of the time t and the value of $z_j(t)$ for $i \in \{0, \dots, m\}, j \in \{1, \dots, m\}$. For each $i \in \{1, \dots, m\}$, we will assume that there exists an incoming flow of material $\widetilde{I}_i$ from the environment into compartment C_i which depends only on time. For each $i \in \{1, \dots, m\}$, at time time $t \geq 0$, the compartment C_i produces material itself at a rate $\sum_{j=1}^m \int_{-\infty}^0 z_j'(t+s)\, d\nu_{ij}(s)$, where ν_{ij} is a positive regular Borel measure with finite total variation $\nu_{ij}(-\infty, 0] < \infty$ and $\nu_{ij}(\{0\}) = 0$ for all $i, j \in \{1, \dots, m\}$.

Once the destruction and creation of material is taken into account, the change of the amount of material of any compartment C_i, $1 \leq i \leq m$, equals the difference between the amount of total influx into and total outflux out of C_i, and we obtain a model governed by the following system of infinite delay NFDEs:

$$\begin{aligned} \text{(5.1)} \quad \frac{d}{dt}\left[z_i(t) - \sum_{j=1}^m \int_{-\infty}^0 z_j(t+s)\, d\nu_{ij}(s) \right] &= -\widetilde{g}_{0i}(t, z_i(t)) - \sum_{j=1}^m \widetilde{g}_{ji}(t, z_i(t)) \\ &\quad + \sum_{j=1}^m \int_{-\infty}^0 \widetilde{g}_{ij}(t+s, z_j(t+s))\, d\mu_{ij}(s) + \widetilde{I}_i(t), \end{aligned}$$

$i = 1, \dots, m$. For simplicity, we denote $\widetilde{g}_{i0} : \mathbb{R} \times \mathbb{R} \to \mathbb{R}^+$, $(t, v) \mapsto \widetilde{I}_i(t)$ for $i \in \{1, \dots, m\}$ and let $\widetilde{g} = (\widetilde{g}_{ij})_{i,j} : \mathbb{R} \times \mathbb{R} \to \mathbb{R}^{m(m+2)}$. We will assume that

- (C1) $\widetilde{g}$ is C^1-*admissible*, i.e., $\widetilde{g}$ is C^1 in its second variable and $\widetilde{g}$, $\frac{\partial}{\partial v}\widetilde{g}$ are uniformly continuous and bounded on $\mathbb{R} \times \{v_0\}$ for all $v_0 \in \mathbb{R}$; all its components are monotone in the second variable, and $\widetilde{g}_{ij}(t, 0) = 0$ for each $t \in \mathbb{R}$;
- (C2) $\widetilde{g}$ is a recurrent function, i.e., its *hull* is minimal;
- (C3) $\mu_{ij}(-\infty, 0] = 1$ and $\int_{-\infty}^0 |s|\, d\mu_{ij}(s) < \infty$;
- (C4) $\nu_{ij}(\{0\}) = 0$ and $\sum_{j=1}^m \nu_{ij}(-\infty, 0] < 1$, which implies that the operator $D \colon BU \to \mathbb{R}^m$, with $D_i\, x = x_i(0) - \sum_{j=1}^m \int_{-\infty}^0 x_j(s)\, d\nu_{ij}(s)$, $i = 1, \dots, m$, is stable and satisfies (D1)–(D3);
- (C5) the measures $d\eta_{ij} = c_{ij}\, d\mu_{ij} - \sum_{k=0}^m d_{ki}\, d\nu_{ij}$ are positive, where

$$c_{ij} = \inf_{(t,v) \in \mathbb{R}^2} \frac{\partial \widetilde{g}_{ij}}{\partial v}(t, v) \quad \text{and} \quad d_{ij} = \sup_{(t,v) \in \mathbb{R}^2} \frac{\partial \widetilde{g}_{ij}}{\partial v}(t, v)\,.$$

In practical cases, in which the solutions with physical interest belong to the positive cone and the functions g_{ij} are only defined on $\mathbb{R} \times \mathbb{R}^+$, we can extend them to $\mathbb{R} \times \mathbb{R}$ by $g_{ij}(t, -v) = -g_{ij}(t, v)$ for all $v \in \mathbb{R}^+$. Note that (C5) is a condition for controlling the material produced in the compartments in terms of the material transported through the pipes.

The above formulation includes some particularly interesting cases. When the measures ν_{ij} and μ_{ij} are concentrated on a compact set, then (5.1) is an NFDE with finite delay. When the measures $\nu_{ij} \equiv 0$, then (5.1) is a family of FDEs with finite or infinite delay.

As usual, we include the nonautonomous system (5.1) in a family of nonautonomous NFDEs with infinite delay and stable D-operator of the form $(4.1)_\omega$ as follows.

Let Ω be the *hull* of $\widetilde{g}$, namely, the closure of the set of mappings $\{\widetilde{g}_t \mid t \in \mathbb{R}\}$, with $\widetilde{g}_t(s,v) = \widetilde{g}(t+s,v)$, $(s,v) \in \mathbb{R}^2$, with the topology of uniform convergence on compact sets, which from (C1) is a compact metric space (more precisely from the admissibility of $\widetilde{g}$; see Hino et al. [13]). Let $(\Omega,\sigma,\mathbb{R})$ be the continuous flow defined on Ω by translation, $\sigma : \mathbb{R}\times\Omega \to \Omega$, $(t,\omega) \mapsto \omega{\cdot}t$, with $\omega{\cdot}t(s,v) = \omega(t+s,v)$. By hypothesis (C2), the flow $(\Omega,\sigma,\mathbb{R})$ is minimal. In addition, if $\widetilde{g}$ is almost periodic (resp., almost automorphic) the flow will be almost periodic (resp., almost automorphic). Notice that these two cases are included in our formulation.

Let $g : \Omega\times\mathbb{R} \to \mathbb{R}^{m(m+2)}$, $(\omega,v) \mapsto \omega(0,v)$, continuous on $\Omega\times\mathbb{R}$ and denote $g = (g_{ij})_{i,j}$. It is easy to check that, for all $\omega = (\omega_{ij})_{i,j} \in \Omega$ and all $i \in \{1,\dots,m\}$, ω_{i0} is a function dependent only on t; thus, we can define $I_i = \omega_{i0}$, $i \in \{1,\dots,m\}$. Let $F : \Omega\times BU \to \mathbb{R}^m$ be the map defined by

$$F_i(\omega,x) = -g_{0i}(\omega,x_i(0)) - \sum_{j=1}^m g_{ji}(\omega,x_i(0)) + \sum_{j=1}^m \int_{-\infty}^0 g_{ij}(\omega{\cdot}s,x_j(s))\,d\mu_{ij}(s) + I_i(\omega)$$

for $(\omega,x) \in \Omega\times BU$ and $i \in \{1,\dots,m\}$. Hence, the family

$$(5.2)_\omega \qquad \frac{d}{dt}Dz_t = F(\omega{\cdot}t,z_t)\,, \quad t \ge 0\,,\ \omega\in\Omega\,,$$

where the stable operator D is defined in (C4) and satisfies (D1)–(D3), includes system (5.1) when $\omega = \widetilde{g}$.

It is easy to check that this family satisfies hypotheses (F1)–(F3). The following lemma will be useful when proving (F4) and (F5). We omit its proof, which is analogous to the one given in Wu and Freedman [28] for the autonomous case with finite delay.

Lemma 5.1. *For all $\omega\in\Omega$, x, $y \in BU$ with $x \le_D y$, and $i = 1,\dots,m$*

$$(5.3) \quad F_i(\omega,y) - F_i(\omega,x) \ge -\sum_{j=0}^m d_{ji}\,[D_i y - D_i x] + \sum_{j=1}^m \int_{-\infty}^0 (y_j(s)-x_j(s))\,d\eta_{ij}(s),$$

where the measures η_{ij} are defined in (C5).

Condition (C5) is essential to proving the monotone character of the semiflow. It can be improved in some cases (see Arino and Bourad [1] for the scalar case).

Proposition 5.2. *Under assumptions* (C1)–(C5)*, the family* $(5.2)_\omega$ *satisfies hypotheses* (F4)*,* (F5) *and $\Omega\times BU_D^+$ is positively invariant.*

Proof. Let x, $y \in BU$ with $x \le_D y$ and $D_i\,x = D_i\,y$ for some $i \in \{1,\dots,m\}$. From (C4), apart from the stability of the operator D, it is easy to prove that the inverse operator of $\widehat{D}$ defined by (3.6) is positive. Hence, from $x \le_D y$, that is, $\widehat{D}\,x \le \widehat{D}\,y$, we also deduce that $x \le y$, which, together with $D_i\,x = D_i\,y$, relation (5.3), and hypothesis (C5), yields $F_i(\omega,y) \ge F_i(\omega,x)$, that is, hypothesis (F4) holds.

Next, we check hypothesis (F5). Let x, $y \in BU$ with $x \le_D y$ and $D_i\,x < D_i\,y$ for some $i \in \{1,\dots,m\}$. Since (F4) holds, from Proposition 4.5 $u(t,\omega,x) \le_D u(t,\omega,y)$ and, as before, we deduce in this case that $u(t,\omega,x) \le u(t,\omega,y)$, i.e., $z_t(\omega,x) \le z_t(\omega,y)$ for all $t \ge 0$ and $\omega\in\Omega$. Let $h(t) = D_i z_t(\omega,y) - D_i z_t(\omega,x)$. From $(5.2)_\omega$ and

Lemma 5.1

$$h'(t) = F_i(\omega{\cdot}t, z_t(\omega,y)) - F_i(\omega{\cdot}t, z_t(\omega,x))$$
$$\geq -\sum_{j=0}^{m} d_{ji}h(t) + \sum_{j=1}^{m}\int_{-\infty}^{0} (z_j(t+s,\omega,y) - z_j(t+s,\omega,x))\, d\eta_{ij}(s)\,,$$

and again from hypothesis (C5) we deduce that $h'(t) \geq -dh(t)$ for some $d \geq 0$, which together with $h(0) > 0$ yields $h(t) = D_i z_t(\omega,y) - D_i z_t(\omega,x) > 0$ for each $t \geq 0$ and (F5) holds. Finally, since $I_i(\omega) \geq 0$ for each $\omega \in \Omega$ and $i \in \{1,\dots,m\}$, and the semiflow is monotone, a comparison argument shows that $\Omega \times BU_D^+$ is positively invariant, as stated. □

Next we will study some cases in which hypothesis (F6) is satisfied. In order to do this, we define $M\colon \Omega \times BU \to \mathbb{R}$, the *total mass* of the system $(5.2)_\omega$, as

$$M(\omega,x) = \sum_{i=1}^{m} D_i x + \sum_{i=1}^{m}\sum_{j=1}^{m}\int_{-\infty}^{0}\left(\int_{s}^{0} g_{ji}(\omega{\cdot}\tau, x_i(\tau))\,d\tau\right) d\mu_{ji}(s) \tag{5.4}$$

for all $\omega \in \Omega$ and $x \in BU$, which is well defined from condition (C3). The next result shows the continuity properties of M and its variation along the flow.

PROPOSITION 5.3. *The total mass M is a continuous function on all the sets of the form $\Omega \times B_r$ with $r > 0$ for the product metric topology. Moreover, for each $t \geq 0$*

$$\frac{d}{dt}M(\tau_t(\omega,x)) = \sum_{i=1}^{m}\left[I_i(\omega{\cdot}t) - g_{0i}(\omega{\cdot}t, z_i(t,\omega,x))\right]. \tag{5.5}$$

Proof. The continuity follows from (D2), (C1), and (C3). A straightforward computation similar to that given in Wu and Freedman [28] shows that

$$M(\omega{\cdot}t, z_t(\omega,x)) = M(\omega,x) + \sum_{i=1}^{m}\int_{0}^{t}\left[I_i(\omega{\cdot}s) - g_{0i}(\omega{\cdot}s, z_i(s,\omega,x))\right] ds\,, \tag{5.6}$$

from which (5.5) is deduced. □

The following lemma is essential in the proof of the stability of solutions.

LEMMA 5.4. *Let x, $y \in BU$ with $x \leq_D y$. Then*

$$0 \leq D_i z_t(\omega,y) - D_i z_t(\omega,x) \leq M(\omega,y) - M(\omega,x)$$

for each $i = 1,\dots,m$ and whenever $z(t,\omega,x)$ and $z(t,\omega,y)$ are defined.

Proof. From Propositions 5.2 and 4.5 the skew-product semiflow induced by $(5.2)_\omega$ is monotone. Hence, if $x \leq_D y$, then $u(t,\omega,x) \leq_D u(t,\omega,y)$ whenever they are defined. From this, as before, since $\widehat{D}^{-1}$ is positive we also deduce that $x \leq y$ and $u(t,\omega,x) \leq u(t,\omega,y)$. Therefore, $D_i z_t(\omega,x) \leq D_i z_t(\omega,y)$ and $z_i(t,\omega,x) \leq z_j(t,\omega,y)$ for each $i = 1,\dots,m$. In addition, the monotonicity of transport functions yields $g_{ij}(\omega, z_j(t,\omega,x)) \leq g_{ij}(\omega, z_j(t,\omega,y))$ for each $\omega \in \Omega$. From all these inequalities and (5.4) and (5.6) we deduce that

$$0 \leq D_i z_t(\omega,y) - D_i z_t(\omega,x) \leq \sum_{i=1}^{m}\left[D_i z_t(\omega,y) - D_i z_t(\omega,x)\right]$$
$$\leq M(\omega{\cdot}t, z_t(\omega,y)) - M(\omega{\cdot}t, z_t(\omega,x)) \leq M(\omega,y) - M(\omega,x)\,,$$

as stated. □

PROPOSITION 5.5. *Fix $r > 0$. Then given $\varepsilon > 0$ there exists $\delta > 0$ such that if x, $y \in B_r$ with $\mathsf{d}(x, y) < \delta$, then $\|z(t, \omega, x) - z(t, \omega, y)\| \le \varepsilon$ whenever they are defined.*

Proof. Let $c = \max_i \sum_{j=1}^m \nu_{ij}(-\infty, 0] < 1$. From the continuity of M, given $\varepsilon_0 = \varepsilon\,(1-c) > 0$ there exists $0 < \delta < \varepsilon_0$, such that if x, $y \in B_r$ with $\mathsf{d}(x, y) < \delta$, then $|M(\omega, y) - M(\omega, x)| < \varepsilon_0$. Therefore, if x, $y \in B_r$ and $x \le_D y$, from Lemma 5.4 we deduce that $0 \le D_i z_t(\omega, y) - D_i z_t(\omega, x) < \varepsilon_0$ whenever $\mathsf{d}(x, y) < \delta$. The definition of D_i yields

$$\begin{aligned} 0 \le z_i(t, \omega, y) - z_i(t, \omega, x) &< \varepsilon_0 + \sum_{j=1}^m \int_{-\infty}^0 [z_j(t+s, \omega, y) - z_j(t+s, \omega, x)]\, d\nu_{ij}(s) \\ &\le \varepsilon_0 + \|z_t(\omega, y) - z_t(\omega, x)\|_\infty \sum_{j=1}^m \nu_{ij}(-\infty, 0]\,, \end{aligned}$$

from which we deduce that $\|z_t(\omega, y) - z_t(\omega, x)\|_\infty (1 - c) < \varepsilon_0 = \varepsilon(1 - c)$, that is, $\|z(t, \omega, x) - z(t, \omega, y)\| \le \varepsilon$ whenever they are defined. The case in which x and y are not ordered follows easily from this one. ∎

As a consequence, from the existence of a bounded solution for one of the systems of the family, the boundedness of all solutions is inferred, and this is the case in which hypothesis (F6) holds.

THEOREM 5.6. *Under assumptions* (C1)–(C5)*, if there exists $\omega_0 \in \Omega$ such that* $(5.2)_{\omega_0}$ *has a bounded solution, then all solutions of* $(5.2)_\omega$ *are bounded as well, hypothesis* (F6) *holds, and all omega-limit sets are copies of the base.*

Proof. The boundedness of all solutions is an easy consequence of the previous proposition and the continuity of the semiflow. Let $(\omega, x) \in \Omega \times BU$ and $r' > 0$ such that $z_t(\omega, x) \in B_{r'}$ for all $t \ge 0$. Then also from Proposition 5.5, we deduce that given $\varepsilon > 0$ there exists a $\delta > 0$ such that

$$\|z(t+s, \omega, x) - z(t, \omega{\cdot}s, y)\| = \|z(t, \omega{\cdot}s, z_s(\omega, x)) - z(t, \omega{\cdot}s, y)\| < \varepsilon$$

for all $t \ge 0$ whenever $y \in B_{r'}$ and $\mathsf{d}(z_s(\omega, x), y) < \delta$, which shows the uniform stability of the trajectories in $B_{r'}$ for each $r' > 0$. Moreover, for each $r > 0$ there is an $r' > 0$ such that $\widehat{D}^{-1} B_r \subset B_{r'}$. Hence, hypothesis (F6) holds for all $r > 0$ and Theorem 4.6 applies for all initial data, which finishes the proof. ∎

Concerning the solutions of the original compartmental system, we obtain the following result providing a nontrivial generalization of the autonomous case, in which the asymptotical constancy of the solutions was shown (see Wu and Freedman [28]). Although the theorem is stated in the almost periodic case, similar conclusions are obtained changing almost periodic to periodic, almost automorphic, or recurrent, that is, all solutions are asymptotically of the same type as the transport functions.

THEOREM 5.7. *Under assumptions* (C1)–(C5) *and in the almost periodic case, if there is a bounded solution of* (5.1)*, then there is at least an almost periodic solution and all the solutions are asymptotically almost periodic. For closed systems, i.e., $\widetilde{I}_i \equiv 0$ and $\widetilde{g}_{0i} \equiv 0$ for each $i = 1, \ldots, m$, there are infinitely many almost periodic solutions and the rest of them are asymptotically almost periodic.*

Proof. The first statement is an easy consequence of the previous theorem. Let $\omega_0 = \widetilde{g}$. The omega-limit of each solution $z(t, \omega_0, x_0)$ is a copy of the base $\mathcal{O}(\omega_0, x_0) = \{(\omega, x(\omega)) \mid \omega \in \Omega\}$, and hence $z(t, \omega_0, x(\omega_0)) = x(\omega_0{\cdot}t)(0)$ is an almost periodic solution of (5.1) and

$$\lim_{t\to\infty} \|z(t, \omega_0, x_0) - z(t, \omega_0, x(\omega_0))\| = 0\,.$$

The statement for closed systems follows in addition from (5.6), which implies that the mass is constant along the trajectories. Hence, there are infinitely many minimal subsets because from the definition of the mass and (C4), given $c > 0$ there is an $(\omega_0, x_0) \in \Omega \times BU_D^+$ such that $M(\omega_0, x_0) = c$ and hence $M(\omega, x) = c$ for each $(\omega, x) \in \mathcal{O}(\omega_0, x_0)$. ∎

6. Long-term behavior of compartmental systems.

This section deals with the long-term behavior of the amount of material within the compartments of the compartmental system (5.1) satisfying hypotheses (C1)–(C5). As in the previous section, the study of the minimal sets for the corresponding skew-product semiflow (4.2) induced by the family $(5.2)_\omega$ will be essential. In addition to hypotheses (C1)–(C5) we will assume the following hypothesis:

(C6) Given $i \in \{0, \dots, m\}$ and $j \in \{1, \dots, m\}$ either $\widetilde{g}_{ij} \equiv 0$ on $\mathbb{R} \times \mathbb{R}^+$ (and hence $g_{ij} \equiv 0$ on $\Omega \times \mathbb{R}^+$), i.e., *there is not a pipe from compartment C_j to compartment C_i*, or for each $v > 0$ there is a $\delta_v > 0$ such that $\widetilde{g}_{ij}(t, v) \geq \delta_v$ for all $t \in \mathbb{R}$ (and hence $g_{ij}(\omega, v) > 0$ for all $\omega \in \Omega$ and $v > 0$). In this case we will say that the pipe P_{ij} *carries material (or that there is a pipe from compartment C_j to compartment C_i).*

Let $I = \{1, \dots, m\}$. $\mathcal{P}(I)$ denotes, as usual, the set of all subsets of I.

DEFINITION 6.1. *Let $\zeta : \mathcal{P}(I) \to \mathcal{P}(I)$, $J \mapsto \cup_{j\in J}\{i \in I \mid P_{ij}$ carries material$\}$. A subset J of I is said to be* irreducible *if $\zeta(J) \subset J$ and no proper subset of J has that property. System* (5.1) *is* irreducible *if the whole set I is irreducible.*

Note that $\zeta(I) \subset I$, so there is always some irreducible subset of I. Irreducible sets detect the occurrence of dynamically independent subsystems. Our next result gives a useful property of irreducible sets with more than one element.

PROPOSITION 6.2. *If a subset J of I is irreducible, then, for all $i, j \in J$ with $i \neq j$, there exist $p \in \mathbb{N}$ and $i_1, \dots, i_p \in J$ such that $P_{i_1 i}$, $P_{i_2 i_1}, \dots, P_{i_p i_{p-1}}$, and $P_{j i_p}$ carry material.*

Proof. Let us assume, on the contrary, that $j \notin \cup_{n=1}^{\infty} \zeta^n(\{i\}) = \widetilde{J}_i$. Then $\widetilde{J}_i \subsetneq J$ and, obviously, $\zeta(\widetilde{J}_i) \subset \widetilde{J}_i$, which contradicts the fact that J is irreducible. ∎

Let $J_1, \dots, J_k$ be all the irreducible subsets of I and let $J_0 = I \setminus \cup_{l=1}^{k} J_l$. These sets reflect the geometry of the compartmental system in a good enough way as to describe the long-term behavior of the solutions, as we will see below.

Let K be any minimal subset of $\Omega \times BU$ for the skew-product semiflow induced by $(5.2)_\omega$. From Theorem 5.6, K is of the form $K = \{(\omega, x(\omega)) \mid \omega \in \Omega\}$, where x is a continuous map from Ω into BU. All of the subsequent results give qualitative information about the long-term behavior of the solutions. Let us see that, provided that we are working on a minimal set K, if there is no inflow from the environment, then the total mass is constant on K, all compartments out of an irreducible subset are empty, and, in an irreducible subset, either all compartments are empty or all are never empty. In particular, in any irreducible subset *with some outflow of material*, all compartments are empty.

THEOREM 6.3. *Assume that $\widetilde{I}_i \equiv 0$ for each $i \in I$ and let $K = \{(\omega, x(\omega)) \mid \omega \in \Omega\}$ be a minimal subset of $\Omega \times BU$ with $K \subset \Omega \times BU_D^+$. Then the following hold:*

(i) *There exists $c \geq 0$ such that $M|_K \equiv c$.*

(ii) *$x_i \equiv 0$ for each $i \in J_0$.*

(iii) *If, for some $l \in \{1, \dots, k\}$, there exists $j_l \in J_l$ such that $x_{j_l} \equiv 0$, then $x_i \equiv 0$ for each $i \in J_l$. In particular, this happens if there is a $j_l \in J_l$ such that there is an outflow of material from C_{j_l}.*

Proof. We first suppose that the system is closed, i.e., $\widetilde{g}_{0i} \equiv 0$, $\widetilde{I}_i \equiv 0$ for all $i \in I$, from which we deduce $g_{0i} \equiv 0$ and $I_i \equiv 0$ for all $i \in I$.

(i) From (5.6) the total mass M is constant along the trajectories, and hence $M(\omega{\cdot}t, x(\omega{\cdot}t)) = M(\omega, x(\omega))$ for all $t \geq 0$ and $\omega \in \Omega$, which together with the fact that Ω is minimal and M continuous shows the statement.

(ii) Let $i \in J_0$. The set $\widetilde{J}_i = \cup_{n=1}^{\infty} \zeta^n(\{i\})$ satisfies $\zeta(\widetilde{J}_i) \subset \widetilde{J}_i$ and hence contains an irreducible set J_l for some $l \in \{1, \ldots, k\}$. Consequently, there are $i_1, \ldots, i_p \in J_0$ and $j_l \in J_l$ such that $P_{j_l i_p}$ carry material.

It is easy to prove that there is an $r > 0$ such that $\|x(\omega)\|_\infty \leq r$ for each $\omega \in \Omega$. We define $M_l \colon \Omega \times BU \to \mathbb{R}$, the *mass restricted to* J_l, as

$$M_l(\omega, y) = \sum_{i \in J_l} D_i y + \sum_{i,j \in J_l} \int_{-\infty}^{0} \left(\int_{s}^{0} g_{ji}(\omega{\cdot}\tau, y_i(\tau))\, d\tau \right) d\mu_{ji}(s)\,, \tag{6.1}$$

which is continuous on $\Omega \times B_r$. From $x(\omega) \geq_D 0$, which also implies $x(\omega) \geq 0$, and (C1), we have $0 \leq M_l(\omega, x(\omega)) \leq M(\omega, x(\omega)) = c$ for each $\omega \in \Omega$.

Since J_l is irreducible, for all $i \in J_l$ and $\omega \in \Omega$

$$\frac{d}{dt} D_i x(\omega{\cdot}t) = -\sum_{j \in J_l} g_{ji}(\omega{\cdot}t, x_i(\omega{\cdot}t)(0)) + \sum_{j \in J_l \cup J_0} \int_{-\infty}^{0} g_{ij}(\omega{\cdot}(s+t), x_j(\omega{\cdot}t)(s))\, d\mu_{ij}(s)$$

because the rest of the terms vanish. Consequently,

$$\frac{d}{dt} M_l(\omega{\cdot}t, x(\omega{\cdot}t)) = \sum_{i \in J_l} \sum_{j \in J_0} \int_{-\infty}^{0} g_{ij}(\omega{\cdot}(s+t), x_j(\omega{\cdot}t)(s))\, d\mu_{ij}(s) \geq 0 \tag{6.2}$$

for each $\omega \in \Omega$. We claim that $M_l(\omega, x(\omega))$ is constant for each $\omega \in \Omega$. Assume, on the contrary, that there are ω_1, $\omega_2 \in \Omega$ such that $M_l(\omega_1, x(\omega_1)) < M_l(\omega_2, x(\omega_2))$, and let $t_n \uparrow \infty$ such that $\lim_{n \to \infty} \omega_2{\cdot}t_n = \omega_1$. From (6.2) we deduce that $M_l(\omega_2, x(\omega_2)) \leq M_l(\omega_2{\cdot}t_n, x(\omega_2{\cdot}t_n))$ for each $n \in \mathbb{N}$, and taking limits as $t \to \infty$ we conclude that $M_l(\omega_2, x(\omega_2)) \leq M_l(\omega_1, x(\omega_1))$, a contradiction. Hence $M_l(\omega, x(\omega))$ is constant and from (6.2)

$$\sum_{i \in J_l} \sum_{j \in J_0} \int_{-\infty}^{0} g_{ij}(\omega{\cdot}(s+t), x_j(\omega{\cdot}t)(s))\, d\mu_{ij}(s) = 0\,. \tag{6.3}$$

Next we check that $x_{i_p} \equiv 0$. From (6.3) we deduce that for each $\omega \in \Omega$

$$\int_{-\infty}^{0} g_{j_l i_p}(\omega{\cdot}s, x_{i_p}(\omega)(s))\, d\mu_{j_l i_p}(s) = 0\,. \tag{6.4}$$

Assume that there is an $\omega_0 \in \Omega$ such that $x_{i_p}(\omega_0)(0) > 0$. Hence there is an $\varepsilon > 0$ with $x_{i_p}(\omega_0)(s) > 0$ for each $s \in (-\varepsilon, 0]$, and since $P_{j_l i_p}$ carries material $g_{j_l i_p}(\omega_0{\cdot}s, x_{i_p}(\omega_0)(s)) > 0$ for $s \in (-\varepsilon, 0]$. In addition, from $\mu_{j_l i_p}(-\infty, 0] = 1$ there is a $b \leq 0$ such that $\mu_{j_l i_p}(b - \varepsilon, b] > 0$. Hence, denoting $\omega_0{\cdot}(-b) = \omega_1$ we deduce that

$$\int_{b-\varepsilon}^{b} g_{j_l i_p}(\omega_1{\cdot}s, x_{i_p}(\omega_1)(s))\, d\mu_{j_l i_p}(s) > 0\,,$$

which contradicts (6.4) and shows that $x_{i_p} \equiv 0$, as claimed. Since $x(\omega) \geq_D 0$, we have $D_{i_p} x(\omega) \geq 0$ and from the definition of D_{i_p} we deduce that $D_{i_p} x(\omega) = 0$ for each

$\omega \in \Omega$. Therefore,

$$0 = \frac{d}{dt} D_{i_p} x(\omega{\cdot}t) = \sum_{j=1}^{m} \int_{-\infty}^{0} g_{i_p j}(\omega{\cdot}(t+s), x_j(\omega{\cdot}t)(s))\, d\mu_{i_p j}(s)\,,$$

from which $\int_{-\infty}^{0} g_{i_p i_{p-1}}(\omega{\cdot}s, x_{i_{p-1}}(\omega)(s))\, d\mu_{i_p i_{p-1}}(s) = 0$, and as before $x_{i_{p-1}} \equiv 0$. In a finite number of steps we check that $x_i \equiv 0$, as stated.

(iii) From Proposition 6.2, given i, $j_l \in J_l$ there exist $p \in \mathbb{N}$ and $i_1, \dots, i_p \in J_l$ such that $P_{i_1 i}$, $P_{i_2 i_1}, \dots, P_{i_p i_{p-1}}$, and $P_{j_l i_p}$ carry material. If $x_{j_l} \equiv 0$, the same argument given in the last part of (ii) shows that $x_i \equiv 0$, which finishes the proof for closed systems.

Next we deal with the case when $\tilde{I}_i \equiv 0$ for each $i \in I$ but the system is not necessarily closed. We also have $I_i \equiv 0$ and from (5.5) we deduce that the total mass M is decreasing along the trajectories. In particular,

$$\frac{d}{dt} M(\omega{\cdot}t, x(\omega{\cdot}t)) = -\sum_{i=1}^{n} g_{0i}(\omega{\cdot}t, x_i(\omega{\cdot}t)(0)) \le 0\,. \tag{6.5}$$

Assume that there are ω_1, $\omega_2 \in \Omega$ such that $M(\omega_1, x(\omega_1)) < M(\omega_2, x(\omega_2))$, and let $t_n \uparrow \infty$ such that $\lim_{n\to\infty} \omega_1{\cdot}t_n = \omega_2$. From relation (6.5) we deduce that $M(\omega_1{\cdot}t_n, x(\omega_1{\cdot}t_n)) \le M(\omega_1, x(\omega_1))$ for each $n \in \mathbb{N}$, and taking limits as $n \uparrow \infty$ we conclude that $M(\omega_2, x(\omega_2)) \le M(\omega_1, x(\omega_1))$, a contradiction, which shows that M is constant on K, as stated in (i). Consequently, the derivative in (6.5) vanishes and $g_{0i}(\omega{\cdot}t, x_i(\omega{\cdot}t)(0)) = 0$ for all $i \in I$, $\omega \in \Omega$, and $t \ge 0$. This means that $z(t, \omega, x(\omega)) = x(\omega{\cdot}t)(0)$ is a solution of a closed system, and (ii) and the first part of (iii) follow from the previous case.

Finally, let $j_l \in J_l$ be such that there is an outflow of material from C_{j_l}, that is, $g_{0j_l}(\omega, v) > 0$ for all $\omega \in \Omega$ and $v > 0$. Moreover, as before, $g_{0j_l}(\omega, x_{j_l}(\omega)(0)) = 0$ for each $\omega \in \Omega$, which implies that $x_{j_l} \equiv 0$ and completes the proof. □

Remark 6.4. Notice that, concerning the solutions of the family of systems $(5.2)_\omega$ and hence the solutions of the original system (5.1) when $\omega = \widetilde{g}$, we deduce that in the case of no inflow from the environment, $\lim_{t\to\infty} z_i(t, \omega, x_0) = 0$ for all $i \in J_0$, $i \in J_l$ for compartments J_l with some outflow, and each $x_0 \ge_D 0$.

Remark 6.5. If there is no inflow from the environment and for all $l \in \{1, \dots, k\}$ there is a $j_l \in J_l$ such that there is outflow of material from C_{j_l}, then the only minimal set in $\Omega \times BU_D^+$ is $K = \{(\omega, 0) \mid \omega \in \Omega\}$ and all the solutions $z(t, \omega, x_0)$ with initial data $x_0 \ge_D 0$ tend to 0 as $t \to \infty$.

In a nonclosed system, that is, a system which may have any inflow and any outflow of material, if there exists a bounded solution, i.e., all solutions are bounded as shown above, and an irreducible set which has *some inflow*, then, working on a minimal set, all compartments of that irreducible set are nonempty and there must be some outflow from the irreducible set.

THEOREM 6.6. *Assume that there exists a bounded solution of* (5.1) *and let* $K = \{(\omega, x(\omega)) \mid \omega \in \Omega\}$ *be a minimal subset of* $\Omega \times BU_D^+$. *If, for some* $l \in \{1, \dots, k\}$, *there is a* $j_l \in J_l$ *such that* $\widetilde{I}_{j_l} \ne 0$, *i.e., there is some inflow into* C_{j_l}, *then*

(i) $x_i \not\equiv 0$ *for each* $i \in J_l$, *and*

(ii) *there is a* $j \in J_l$ *such that there is outflow of material from* C_j.

Proof. (i) Let us assume, on the contrary, that there is an $i \in J_l$ such that $x_i \equiv 0$. Then since $x(\omega) \ge_D 0$ we have that $0 \le D_i x(\omega)$, and from the definition of D_i given in (C4) we deduce that $D_i x(\omega) = 0$ for each $\omega \in \Omega$. Therefore,

$$0 = \frac{d}{dt} D_i x(\omega{\cdot}t) = \sum_{j=1}^{m} \int_{-\infty}^{0} g_{ij}(\omega{\cdot}(t+s), x_j(\omega{\cdot}t)(s))\, d\mu_{ij}(s) + I_i(\omega{\cdot}t) \tag{6.6}$$

for all $\omega \in \Omega$, $t \geq 0$, and, as in (ii) of Theorem 6.3, we check that $x_{j_l} \equiv 0$. However, since $\widetilde{I}_{j_l} \not\equiv 0$, there is an $\omega_0 \in \Omega$ such that $I_{j_l}(\omega_0) > 0$, which contradicts (6.6) for $\omega = \omega_0$, $i = j_l$ at $t = 0$.

(ii) Assume, on the contrary, that $g_{0j} \equiv 0$ for each $j \in J_l$. Then if we consider (6.1) the restriction of the mass to J_l, we check that

$$\frac{d}{dt} M_l(\omega{\cdot}t, x(\omega{\cdot}t)) = \sum_{i \in J_l} \left[I_i(\omega{\cdot}t) + \sum_{j \in J_0} \int_{-\infty}^{0} g_{ij}(\omega{\cdot}(s+t), x_j(\omega{\cdot}t)(s))\, d\mu_{ij}(s) \right] \geq 0$$

for all $\omega \in \Omega$ and $t \geq 0$. A similar argument to the one given in (ii) of Theorem 6.3 shows that $M_l(\omega, x(\omega))$ is constant for each $\omega \in \Omega$, which contradicts the fact that the above derivative is strictly positive for $\omega = \omega_0$ at $t = 0$ and proves the statement. ∎

Finally, we will change hypothesis (C6) to the following, slightly stronger one.

(C6)* Given $i \in \{0, \dots, m\}$ and $j \in \{1, \dots, m\}$ either $\widetilde{g}_{ij} \equiv 0$ on $\mathbb{R} \times \mathbb{R}^+$ (and hence $g_{ij} \equiv 0$ on $\Omega \times \mathbb{R}^+$), i.e., *there is not a pipe from compartment C_j to compartment C_i*, or for each $v \geq 0$ there is a $\delta_v > 0$ such that $\frac{\partial}{\partial v}\widetilde{g}_{ij}(t, v) \geq \delta_v$ for each $t \in \mathbb{R}$ (and hence $\frac{\partial}{\partial v} g_{ij}(\omega, v) > 0$ for all $\omega \in \Omega$ and $v \geq 0$). In this case we will say that the pipe P_{ij} *carries material (or that there is a pipe from compartment C_j to compartment C_i)*.

In this case, we are able to prove that if there exists a bounded solution, then all the minimal sets coincide both on irreducible sets having *some outflow* and out of irreducible sets. Concerning the solutions of the initial compartmental system (5.1),

$$\lim_{t \to \infty} |z_i(t, x_0) - z_i(t, y_0)| = 0$$

for all $i \in J_0$, $i \in J_l$ for compartments J_l with some outflow, and all x_0, $y_0 \geq_D 0$.

THEOREM 6.7. *Let us assume that hypotheses* (C1)–(C5) *and* (C6)* *hold and that there exists a bounded solution of system* (5.1). *Let* $K_1 = \{(\omega, x(\omega)) \mid \omega \in \Omega\}$ *and* $K_2 = \{(\omega, y(\omega)) \mid \omega \in \Omega\}$ *be two minimal subsets of* $\Omega \times BU_D^+$. *Then*

(i) *$x_i \equiv y_i$ for each $i \in J_0$;*

(ii) *if, for some $l \in \{1, \dots, k\}$, there is a $j_l \in J_l$ such that there is outflow of material from C_{j_l}, then $x_i \equiv y_i$ for each $i \in J_l$.*

Proof. For each $i \in \{0, \dots, m\}$ and each $j \in \{1, \dots, m\}$ we define $h_{ij} \colon \Omega \to \mathbb{R}^+$ as

$$h_{ij}(\omega) = \int_0^1 \frac{\partial g_{ij}}{\partial v}(\omega, s\, x_j(\omega)(0) + (1-s)\, y_j(\omega)(0))\, ds \geq 0\,,$$

and we consider the family of monotone linear compartmental systems

$$\begin{aligned} \frac{d}{dt} D_i \hat{z}_t = &-h_{0i}(\omega{\cdot}t)\, \hat{z}_i(t) - \sum_{j=1}^{m} h_{ji}(\omega{\cdot}t)\, \hat{z}_i(t) \\ &+ \sum_{j=1}^{m} \int_{-\infty}^{0} h_{ij}(\omega{\cdot}(s+t))\, \hat{z}_j(t+s)\, d\mu_{ij}(s)\,, \quad \omega \in \Omega, \end{aligned} \tag{6.7$_\omega$}$$

satisfying the corresponding hypotheses (C1)–(C4) and (C6). Moreover, (C5) for each of the systems $(6.7)_\omega$, follows from

$$\inf_{\omega\in\Omega} h_{ij}(\omega) \geq \inf_{v\geq 0\,,\,\omega\in\Omega} \frac{\partial g_{ij}}{\partial v}(\omega, v)\,, \quad \sup_{\omega\in\Omega} h_{ij}(\omega) \leq \sup_{v\geq 0\,,\,\omega\in\Omega} \frac{\partial g_{ij}}{\partial v}(\omega, v)\,,$$

and (C5) for (5.1). From the definition of h_{ij} and (C6)* we deduce that the irreducible sets for the families $(6.7)_\omega$ and $(5.2)_\omega$ coincide. Consequently, Theorem 6.3 (see Remark 6.4) applies to this case, and we deduce that if $z_0 \geq_D 0$ and J_l is a compartment with some outflow of material, then

$$\lim_{t\to\infty} \hat{z}_i(t, \omega, z_0) = 0 \qquad \text{for each } i \in J_0 \cup J_l\,.$$

The same happens for $z_0 \leq_D 0$ because the systems are linear.

Let $z(\omega) = x(\omega) - y(\omega)$ for each $\omega \in \Omega$. It is easy to check $\hat{z}(t, \omega, z(\omega)) = z(\omega{\cdot}t)(0)$ for all $\omega \in \Omega$ and $t \geq 0$. Moreover, we can find $z_1 \leq_D 0$ and $z_0 \geq_D 0$ such that $z_1 \leq_D z(\omega) \leq_D z_0$ for each $\omega \in \Omega$. Hence, the monotonicity of the induced skew-product semiflow and the positivity of $\widehat{D}^{-1}$ yields

$$\hat{z}(t, \omega, z_1) \leq z(\omega{\cdot}t)(0) \leq \hat{z}(t, \omega, z_0) \quad \text{for all } \omega \in \Omega \;\; t \geq 0\,,$$

from which we deduce that $z_i \equiv 0$ for all $i \in J_0$, $i \in J_l$ and (i) and (ii) follow. ∎

As a consequence, under the same assumptions of the previous theorem, when for all $l \in \{1, \ldots, k\}$ there is an outflow of material from one of the compartments in J_l, there is a unique minimal set $K = \{(\omega, x(\omega)) \mid \omega \in \Omega\}$ in $\Omega \times BU_D^+$ attracting all the solutions with initial data in BU_D^+; i.e.,

$$\lim_{t\to\infty} \|z(t, \omega, x_0) - x(\omega{\cdot}t)(0)\| = 0\,, \qquad \text{whenever } \; x_0 \geq_D 0\,.$$

Moreover, $x \not\equiv 0$ if and only if there is some $j \in \{1, \ldots, m\}$ such that $\widetilde{I}_j \neq 0$; i.e., there is some inflow into one of the compartments C_j.

For the next result, in addition to hypotheses (C1)–(C5) and (C6)* we will assume the following hypothesis:

(C7) If $K_1 = \{(\omega, x(\omega)) \mid \omega \in \Omega\}$ and $K_2 = \{(\omega, y(\omega)) \mid \omega \in \Omega\}$ are two minimal subsets of $\Omega \times BU_D^+$ such that $x(\omega) \leq_D y(\omega)$ and $D_i x(\omega_0) = D_i y(\omega_0)$ for some $\omega_0 \in \Omega$ and $i \in \{1, \ldots, m\}$, then $x(\omega) = y(\omega)$ for each $\omega \in \Omega$, i.e., $K_1 = K_2$.

Note that if $D_i x(\omega_0) = D_i y(\omega_0)$ holds for some $\omega_0 \in \Omega$ and $i \in \{1, \ldots, m\}$, then from hypothesis (F5) we deduce that it holds for each $\omega \in \Omega$.

Hypothesis (C7) is relevant when it applies to closed systems, and it holds in many cases studied in the literature. A closed system satisfying (C7) is irreducible. Systems with a unique compartment, studied by Arino and Bourad [1] and Krisztin and Wu [19], satisfy (C7). It follows from Theorem 6.3 that irreducible closed systems described by FDEs (see Arino and Haourigui [2]) satisfy (C7). Closed systems given by Wu [27], and Wu and Freedman [28] in the strongly ordered case, also satisfy (C7).

DEFINITION 6.8. *Let $K_1 = \{(\omega, x(\omega)) \mid \omega \in \Omega\}$ and $K_2 = \{(\omega, y(\omega)) \mid \omega \in \Omega\}$ be two minimal subsets. It is said that $K_1 <_D K_2$ if $x(\omega) <_D y(\omega)$ for each $\omega \in \Omega$.*

Hypothesis (C7) allows us to classify the minimal subsets in terms of the value of their total mass, as shown in the next result.

THEOREM 6.9. *Assume that system* (5.1) *is closed (i.e., $\widetilde{I}_i \equiv 0$ and $\widetilde{g}_{0i} \equiv 0$ for each $i \in \{1, \ldots, m\}$), and hypotheses* (C1)–(C5), (C6)*, *and* (C7) *hold. Then for* *each $c > 0$ there is a unique minimal subset K_c such that $M|_{K_c} = c$. Moreover, $K_c \subset \Omega \times BU_D^+$ and $K_{c_1} <_D K_{c_2}$ whenever $c_1 < c_2$.*

Proof. Since the minimal subsets are copies of the base, and the total mass (5.4) is constant along the trajectories and increasing for the D-order because $\widehat{D}^{-1}$ is positive, it is easy to check that given $c > 0$ there is a minimal subset $K_c \subset \Omega \times BU_D^+$ such that $M|_{K_c} = c$.

Let $\widehat{D}$ be the isomorphism of BU defined by the relation (3.6). For each $x \in BU$ we define $x^+ = \widehat{D}^{-1} \sup(0, \widehat{D}x)$. Hence $0 \leq_D x^+$, $x \leq_D x^+$, and if $y \in BU$ with $x \leq_D y$ and $0 \leq_D y$, then $x^+ \leq_D y$.

Since the semiflow is monotone, from $x \leq_D x^+$ we deduce that $u(t, \omega, x) \leq_D u(t, \omega, x^+)$. Since the system is closed, $u(t, \omega, 0) = 0$, and from $0 \leq_D x^+$ we check that $0 \leq_D u(t, \omega, x^+)$. Consequently $u(t, \omega, x)^+ \leq_D u(t, \omega, x^+)$ for each $t \geq 0$.

Next we check that if $K = \{(\omega, x(\omega)) \mid \omega \in \Omega\}$ is minimal, the same happens for $K^+ = \{(\omega, x(\omega)^+) \mid \omega \in \Omega\}$. Since $x(\omega{\cdot}t) = u(t, \omega, x)$ for each $t \geq 0$, we deduce that $x(\omega{\cdot}t)^+ = u(t, \omega, x(\omega))^+ \leq_D u(t, \omega, x(\omega)^+)$, and the fact that $\widehat{D}^{-1}$ is positive yields $x(\omega{\cdot}t)^+ \leq u(t, \omega, x(\omega)^+)$ for each $t \geq 0$. In addition, since the total mass (5.4) is constant along the trajectories and increasing for the D-order, we deduce that $M(\omega, x(\omega)^+) = M(\omega{\cdot}t, u(t, \omega, x(\omega)^+)) \geq M(\omega{\cdot}t, u(t, \omega, x(\omega))^+) = M(\omega{\cdot}t, x(\omega{\cdot}t)^+)$ for each $t \geq 0$. Moreover, since $x(\omega)^+$ is a continuous function in ω and Ω is minimal, a similar argument to the one given in (ii) of Theorem 6.3 shows that $M(\omega, x(\omega)^+)$ is constant on Ω and, consequently, $M(\omega{\cdot}t, u(t, \omega, x(\omega)^+) = M(\omega{\cdot}t, x(\omega{\cdot}t)^+)$ for each $\omega \in \Omega$ and $t \geq 0$. Hence, from (5.4) we conclude that

$$0 = \sum_{i=1}^{m} D_i(u(t, \omega, x(\omega)^+) - x(\omega{\cdot}t)^+)\,,$$

that is, $D(u(t, \omega, x(\omega)^+)) = D(x(\omega{\cdot}t)^+)$ for each $\omega \in \Omega$ and $t \geq 0$. In addition, it is easy to check that $(\varphi_s)^+ = (\varphi^+)_s$ whenever $\varphi \in BU$ and $s \leq 0$, from which we deduce that $D((u(t, \omega, x(\omega)^+))_s) = D((x(\omega{\cdot}t)^+)_s)$ for each $s \leq 0$, $t \geq 0$, and $\omega \in \Omega$. That is, $\widehat{D}(u(t, \omega, x(\omega)^+)) = \widehat{D}(x(\omega{\cdot}t)^+)$ for each $t \geq 0$ and $\omega \in \Omega$, and since $\widehat{D}$ is an isomorphism $u(t, \omega, x(\omega)^+) = x(\omega{\cdot}t)^+$ for each $t \geq 0$ and $\omega \in \Omega$, which shows that K^+ is a minimal subset, as stated. Let $K_1 = \{(\omega, x(\omega)) \mid \omega \in \Omega\}$ and $K_2 = \{(\omega, y(\omega)) \mid \omega \in \Omega\}$ be two minimal subsets such that $M|_{K_i} = c$ for $i = 1, 2$. We fix $\omega \in \Omega$. The change of variable $\widehat{z}(t) = z(t) - y(\omega{\cdot}t)$ takes $(5.2)_\omega$ to

$$\frac{d}{dt} D\widehat{z}_t = G(\omega{\cdot}t, \widehat{z}_t)\,, \quad t \geq 0\,,\ \omega \in \Omega\,,$$

where $G(\omega{\cdot}t, \widehat{z}_t) = F(\omega{\cdot}t, \widehat{z}_t + y(\omega{\cdot}t)) - F(\omega{\cdot}t, y(\omega{\cdot}t))$. It is not hard to check that this is a new family of compartmental systems satisfying the corresponding hypotheses (C1)–(C5) and (C6)*, and

$$\widehat{K} = \{(\omega, x(\omega) - y(\omega)) \mid \omega \in \Omega\}$$

is one of its minimal subsets. As before

$$\widehat{K}^+ = \{(\omega, (x(\omega) - y(\omega))^+) \mid \omega \in \Omega\}$$

is also a minimal subset, and hence

$$K^+ = \{(\omega, y(\omega) + (x(\omega) - y(\omega))^+) \mid \omega \in \Omega\} = \{(\omega, z(\omega)) \mid \omega \in \Omega\}$$

is a minimal set for the initial family. For each $\omega \in \Omega$ we have $z(\omega) \geq_D y(\omega)$.

Let us assume that $Dz(\omega) \gg Dy(\omega)$ for each $\omega \in \Omega$, which implies that $D((x(\omega) - y(\omega))^+) \gg 0$ for each $\omega \in \Omega$. Consequently, $D((x(\omega) - y(\omega))_s^+) = D(((x(\omega) - y(\omega))_s)^+) = D((x(\omega{\cdot}s) - y(\omega{\cdot}s))^+) \gg 0$ for each $s \leq 0$, and we deduce that $\widehat{D}x(\omega) > \widehat{D}y(\omega)$, i.e., $x(\omega) >_D y(\omega)$ for each $\omega \in \Omega$, and $M|_{K_1} > M|_{K_2}$, a contradiction. Hence, there are an $\omega_0 \in \Omega$ and an $i \in \{1, \ldots, m\}$ such that $D_i z(\omega_0) = D_i y(\omega_0)$, and hypothesis (C7) provides that $z(\omega) = y(\omega)$ for each $\omega \in \Omega$. That is, $(x(\omega) - y(\omega))^+ \equiv 0$ for each $\omega \in \Omega$, or equivalently $x(\omega) - y(\omega) \leq_D 0$ for each $\omega \in \Omega$. Finally, as before, from $M|_{K_1} = M|_{K_2}$ we conclude by contradiction that $x(\omega) = y(\omega)$ for each $\omega \in \Omega$, and the minimal set K_c is unique, as stated. The same argument shows that $K_{c_1} <_D K_{c_2}$ whenever $c_1 < c_2$ and finishes the proof. $\square$

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