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EXPONENTIAL ORDERING FOR NONAUTONOMOUS NEUTRAL FUNCTIONAL DIFFERENTIAL EQUATIONS*

SYLVIA NOVO†, RAFAEL OBAYA†, AND VÍCTOR M. VILLARRAGUT†

Abstract. We study monotone skew-product semiflows generated by families of nonautonomous neutral functional differential equations with infinite delay and stable D-operator, when the exponential ordering is considered. Under adequate hypotheses of stability for the order on bounded sets, we show that the omega-limit sets are copies of the base to explain the long-term behavior of the trajectories. The application to the study of the amount of material within the compartments of a neutral compartmental system with infinite delay shows the improvement with respect to the standard ordering.

Key words. nonautonomous dynamical systems, monotone skew-product semiflows, exponential ordering, neutral functional differential equations, compartmental systems

AMS subject classifications. 37B55, 34K40, 34K14

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1. Introduction.

Dynamical methods for monotone autonomous differential equations have been extensively studied for several decades (see Hirsch [13], Matano [20], Poláčik [24], and Smith [27], among many others). A critical problem in this theory concerns the long-term behavior of relatively compact trajectories. In an adequate dynamical scenario these papers prove that the relatively compact trajectories of a strongly monotone semiflow converge generically to the set of equilibria. This property, referred to as *generic quasi convergence*, has a significative theoretical and practical interest.

Following the above theoretical framework, Smith and Thieme [28, 29] studied the dynamical properties of semiflows induced by functional differential equations with finite delay, which are monotone for the exponential ordering. This ordering is technically more complicated than the standard one and can be employed in several important situations where the standard quasi-monotone condition fails, but a new nonstandard monotone condition is satisfied. Assuming that the vector field satisfies a strong version of the nonstandard monotonicity condition and a supplementary irreducibility assumption, Smith and Thieme proved that the induced semiflow is eventually strongly monotone in the phase space of Lipschitz functions, and strongly order preserving in the phase space of continuous functions, from which they deduced the quasi-convergence property. These results have been extended in several directions: Krisztin and Wu [19] studied the dynamical properties of scalar neutral functional differential equations with finite delay which induce a monotone semiflow for the exponential ordering, and Wu and Zhao [34] analyzed the same questions in a class of evolutionary equations with applications to reaction-diffusion models.

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†Departmento de Matemática Aplicada, E.T.S. de Ingenieros Industriales, Universidad de Valladolid, 47011 Valladolid, Spain (sylnov@wmatem.eis.uva.es, rafoba@wmatem.eis.uva.es, vicmun@wmatem.eis.uva.es). The first and second author's research was partly supported by Junta de Castilla y León and MEC under projects VA024A06 and MTM2008-00700. The third author's research was partly supported by MEC under project MTM2008-00700 and MICINN under FPU grant AP2006-00874.

More recently a new dynamical theory for deterministic and random monotone nonautonomous differential equations has been developed (see Shen and Yi [26], Jiang and Zhao [17], Novo, Obaya, and Sanz [23], and Chueshov [5], among others). Now the generic quasi-convergence property fails in general and new dynamical situations have to be considered. Assuming adequate hypotheses including the boundedness, relative compactness, and uniform stability of the trajectories, this theory ensures the convergence of the orbits to solutions which reproduce exactly the dynamics exhibited by the time variation of the equation. The aim of this paper is to extend the applicability of this theory to functional differential equations (FDEs) with infinite delay and neutral functional differential equations (NFDEs) with infinite delay and D-stable operator, which are monotone for the exponential ordering. The obtained results are applied to the study of the long-term behavior of the amount of material within the compartments of some classes of nonautonomous and closed compartmental systems extensively studied in the literature.

Compartmental systems have been used as mathematical models for the study of the dynamical behavior of many processes in biological and physical sciences, which depend on local mass balance conditions (see Jacquez [14] and Jacquez and Simon [15, 16] and the references therein). Some initial results for models described by FDEs with finite and infinite delay can be found in Györi [7] and Györi and Eller [8]. NFDEs represent systems where the compartments produce or swallow material. The papers by Arino and Haourigui [2], Györi and Wu [9], Wu and Freedman [33], and Muñoz-Villarragut, Novo, and Obaya [21] are significant predecessors in the use of monotone methods to analyze this kind of problem. On the other hand, supplementary results for the scalar NFDE

$$\frac{d}{dt}[x(t) - c(t)\,x(t-\tau)] = -h(t, x(t)) + h(t-\sigma, x(t-\sigma)) \tag{1.1}$$

have been obtained. Arino and Bourad [1] and Wu [32] studied this equation for $\tau = \sigma$ and time-independent, time-periodic, or almost-periodic $c(t)$, $h(t, x)$. Krisztin [18] observed that if $c = \sqrt{2} - 1$, $\tau = \pi/4$, $\sigma = 7\pi/4$, and $h(t, x) = x$, then (1.1) has the periodic solution $x(t) = \sin(t)$, and hence the convergence to equilibria is not always true when $\tau \neq \sigma$. Krisztin and Wu [19] pointed out that, for some values $\tau \neq \sigma$ with $c(t)$, $h(t, x)$ time-periodic, (1.1) generates a monotone semiflow for the exponential ordering, which becomes eventually strongly monotone in the phase space of Lipschitz continuous functions. In this space the asymptotic periodicity of the solutions is deduced.

In this paper we extend the conclusions of [19] to nonautonomous closed systems of NFDEs with finite or infinite delay. The total mass in closed compartmental systems defines an invariant function, which is uniformly continuous for the metric on bounded sets. This implies a property of stability for pairs of trajectories starting at ordered initial data. We will refer to this property as *stability for the order on bounded sets*; this will be the hypothesis we assume throughout the paper, determining the complete presentation of the theory. It is important to mention that in the case of the exponential ordering, this is a weak version of stability and it is not clear how to deduce the standard stability of any particular trajectory.

We assume some recurrence properties on the temporal variation of the NFDE; thus, its solutions induce a skew-product semiflow with a minimal flow on the base. In particular, the uniform almost-periodic and almost-automorphic cases are included in this formulation. The skew-product formalism permits the analysis of the dynamical properties of the trajectories using methods of ergodic theory and topological dynamics. In this paper we study nonautonomous NFDEs with infinite delay and autonomous D-stable operator, satisfying the same hypotheses considered in [21], which in particular imply the invertibility of D. We introduce the phase space $BU \subset C((-\infty,0],\mathbb{R}^m)$ of bounded and uniformly continuous functions with the supremum norm where the standard theory provides existence, uniqueness, and continuous dependence of the solutions. In our present setting every bounded trajectory is relatively compact for the compact open topology, the restriction of the semiflow to its omega-limit set is continuous for the metric, and it admits a flow extension. We also prove that every solution of the NFDE lying on an omega-limit set is differentiable everywhere. In addition, if two functions belong to the same omega-limit set and they are close in metric, then the same happens to their derivatives, due to the continuous variation of the vector field on metric compact sets. From this, for each pair of elements x, y of each omega-limit set, we deduce the existence of a supremum and an infimum, which depend continuously on x, y for the metric. These facts are essential in the proofs of the main results of the paper.

We now briefly describe the structure of the paper. Basic notions in topological dynamics, used throughout the rest of the sections, are stated in section 2. In section 3 we define the exponential ordering in the space BU to be used for the study of FDEs and NFDEs with infinite delay. For FDEs two cases arise: in Case I we use the exponential ordering on a compact subinterval of $(-\infty,0]$ and the standard ordering on its complementary subset, and in Case II the exponential ordering is used on the complete semi-interval $(-\infty,0]$. Case III refers to NFDEs, where we consider the exponential ordering on $(-\infty,0]$. The theory provides different dynamical consequences for each choice. We formulate a nonstandard monotone condition for the vector field which implies the monotonicity of the corresponding semiflow.

Section 4 deals with the study of the structure of the omega-limit set of a relatively compact trajectory which is uniformly stable for the order in bounded sets. We make use of the arguments included in section 3 of Novo, Obaya, and Sanz [23]; here some supplementary work is required because the conditions of stability are weaker. We prove that this compact set is uniformly stable for the order in bounded sets and the restriction of the semiflow to it is uniformly stable. We finally prove that this compact set is a minimal set assuming that the trajectory starts at a Lipschitz initial value, as a supplementary condition in Cases II and III.

Section 5 contains the main statement of the paper, which ensures the convergence of the trajectories and follows the arguments of Jiang and Zhao [17] and Novo, Obaya, and Sanz [23]. We describe an adequate dynamical scenario where the trajectories with Lipschitz initial data are relatively compact and uniformly stable for the order on bounded sets. We assume that the vector field satisfies a strong version of the nonstandard monotonicity condition for each component. We now fix a relatively compact trajectory uniformly stable for the order in bounded sets and prove that its omega-limit set is a minimal set given by a 1-cover of the base flow. Again, in Cases II and III the Lipschitz character of the initial data is assumed. Notice that no irreducibility condition is required, and hence the conclusions can be applied to more general problems under natural physical conditions.

In section 6 we apply the previous results to some nonautonomous and closed compartmental systems with finite or infinite delay. We extend a part of the dynamical description detailed in [21] to some models which are nonmonotone for the standard ordering. In particular, we provide a nonautonomous vectorial version of the results of Krisztin and Wu [19].

2. Some preliminaries. Let (Ω, d) be a compact metric space. A real *continuous flow* $(\Omega, \sigma, \mathbb{R})$ is defined by a continuous mapping $\sigma : \mathbb{R}\times\Omega \to \Omega,\ (t,\omega) \mapsto \sigma(t,\omega)$ satisfying

(i) $\sigma_0 = \mathrm{Id}$,

(ii) $\sigma_{t+s} = \sigma_t \circ \sigma_s$ for each $s,\, t \in \mathbb{R}$,

where $\sigma_t(\omega) = \sigma(t,\omega)$ for all $\omega \in \Omega$ and $t \in \mathbb{R}$. The set $\{\sigma_t(\omega) \mid t \in \mathbb{R}\}$ is called the *orbit* or the *trajectory* of the point ω. We say that a subset $\Omega_1 \subset \Omega$ is σ*-invariant* if $\sigma_t(\Omega_1) = \Omega_1$ for every $t \in \mathbb{R}$. A subset $\Omega_1 \subset \Omega$ is called *minimal* if it is compact and σ-invariant and its only nonempty compact σ-invariant subset is itself. Every compact and σ-invariant set contains a minimal subset; in particular it is easy to prove that a compact σ-invariant subset is minimal if and only if every trajectory is dense. We say that the continuous flow $(\Omega, \sigma, \mathbb{R})$ is *recurrent* or *minimal* if Ω is minimal.

The flow $(\Omega, \sigma, \mathbb{R})$ is *distal* if for any two distinct points $\omega_1,\, \omega_2 \in \Omega$ the orbits keep at a positive distance, that is, $\inf_{t\in\mathbb{R}} d(\sigma(t,\omega_1), \sigma(t,\omega_2)) > 0$. The flow $(\Omega, \sigma, \mathbb{R})$ is *almost periodic* when for every $\varepsilon > 0$ there is a $\delta > 0$ such that if $\omega_1,\, \omega_2 \in \Omega$ with $d(\omega_1,\omega_2) < \delta$, then $d(\sigma(t,\omega_1), \sigma(t,\omega_2)) < \varepsilon$ for every $t \in \mathbb{R}$. If $(\Omega, \sigma, \mathbb{R})$ is almost periodic, it is distal. The converse is not true; even if $(\Omega, \sigma, \mathbb{R})$ is minimal and distal, it does not need to be almost periodic. For the basic properties of almost-periodic and distal flows we refer the reader to Ellis [6] and Sacker and Sell [25].

A *flow homomorphism* from another continuous flow $(Y, \Psi, \mathbb{R})$ to $(\Omega, \sigma, \mathbb{R})$ is a continuous map $\pi\colon Y \to \Omega$ such that $\pi(\Psi(t,y)) = \sigma(t, \pi(y))$ for every $y \in Y$ and $t \in \mathbb{R}$. If π is also bijective, it is called a *flow isomorphism.* Let $\pi : Y \to \Omega$ be a surjective flow homomorphism and suppose $(Y, \Psi, \mathbb{R})$ is minimal (then, so is $(\Omega, \sigma, \mathbb{R})$). $(Y, \Psi, \mathbb{R})$ is said to be an *almost-automorphic extension* of $(\Omega, \sigma, \mathbb{R})$ if there is $\omega \in \Omega$ such that $\mathrm{card}(\pi^{-1}(\omega)) = 1$. Then, actually $\mathrm{card}(\pi^{-1}(\omega)) = 1$ for ω in a residual subset $\Omega_0 \subseteq \Omega$; in the nontrivial case $\Omega_0 \subsetneq \Omega$ the dynamics can be very complicated. A minimal flow $(Y, \Psi, \mathbb{R})$ is *almost automorphic* if it is an almost-automorphic extension of an almost-periodic minimal flow $(\Omega, \sigma, \mathbb{R})$. We refer the reader to the work of Shen and Yi [26] for a survey of almost-periodic and almost-automorphic dynamics.

Let E be a complete metric space and $\mathbb{R}^+ = \{t \in \mathbb{R} \,|\, t \geq 0\}$. A *semiflow* $(E, \Phi, \mathbb{R}^+)$ is determined by a continuous map $\Phi : \mathbb{R}^+ \times E \to E,\ (t,x) \mapsto \Phi(t,x)$ which satisfies

(i) $\Phi_0 = \mathrm{Id}$,

(ii) $\Phi_{t+s} = \Phi_t \circ \Phi_s$ for all $t,\, s \in \mathbb{R}^+$,

where $\Phi_t(x) = \Phi(t,x)$ for each $x \in E$ and $t \in \mathbb{R}^+$. The set $\{\Phi_t(x) \mid t \geq 0\}$ is the *semiorbit* of the point x. A subset E_1 of E is *positively invariant* (or just Φ*-invariant*) if $\Phi_t(E_1) \subset E_1$ for all $t \geq 0$. A semiflow $(E, \Phi, \mathbb{R}^+)$ admits a *flow extension* if there exists a continuous flow $(E, \widetilde{\Phi}, \mathbb{R})$ such that $\widetilde{\Phi}(t,x) = \Phi(t,x)$ for all $x \in E$ and $t \in \mathbb{R}^+$. A compact and positively invariant subset admits a flow extension if the semiflow restricted to it admits one.

Write $\mathbb{R}^- = \{t \in \mathbb{R} \,|\, t \leq 0\}$. A *backward orbit* of a point $x \in E$ in the semiflow $(E, \Phi, \mathbb{R}^+)$ is a continuous map $\psi : \mathbb{R}^- \to E$ such that $\psi(0) = x$ and for each $s \leq 0$ it holds that $\Phi(t, \psi(s)) = \psi(s+t)$ whenever $0 \leq t \leq -s$. If for $x \in E$ the semiorbit $\{\Phi(t,x) \mid t \geq 0\}$ is relatively compact, we can consider the *omega-limit set* of x,

$$\mathcal{O}(x) = \bigcap_{s\geq 0} \mathrm{closure}\{\Phi(t+s,x) \mid t \geq 0\}\,,$$

which is a nonempty compact connected and Φ-invariant set. Namely, it consists of the points $y \in E$ such that $y = \lim_{n\to\infty} \Phi(t_n, x)$ for some sequence $t_n \uparrow \infty$. It is well known that every $y \in \mathcal{O}(x)$ admits a backward orbit inside this set. Actually, a compact positively invariant set M admits a flow extension if every point in M admits a unique backward orbit which remains inside the set M (see Shen and Yi [26, part II]).

A compact positively invariant set M for the semiflow $(E, \Phi, \mathbb{R}^+)$ is *minimal* if it does not contain any other nonempty compact positively invariant set than itself. If E is minimal, we say that the semiflow is minimal.

A semiflow is *of skew-product type* when it is defined on a vector bundle and has a triangular structure; more precisely, a semiflow $(\Omega \times X, \tau, \mathbb{R}^+)$ is a *skew-product* semiflow over the product space $\Omega \times X$, for a compact metric space (Ω, d) and a complete metric space (X, d), if the continuous map τ is as follows:

$$\begin{array}{rccl} \tau\colon & \mathbb{R}^+ \times \Omega \times X & \longrightarrow & \Omega \times X, \\ & (t, \omega, x) & \mapsto & (\omega{\cdot}t, u(t, \omega, x))\,, \end{array} \tag{2.1}$$

where $(\Omega, \sigma, \mathbb{R})$ is a real continuous flow $\sigma : \mathbb{R} \times \Omega \to \Omega$, $(t, \omega) \mapsto \omega{\cdot}t$, called the *base flow.* The skew-product semiflow (2.1) is *linear* if $u(t, \omega, x)$ is linear in x for each $(t, \omega) \in \mathbb{R}^+ \times \Omega$.

3. Exponential ordering. We will study nonautonomous NFDEs with infinite delay and autonomous and D-stable operator. First, we describe the conditions on the vector field and the neutral operator D assumed throughout the paper.

We consider the Fréchet space $X = C((-\infty, 0], \mathbb{R}^m)$ endowed with the compact-open topology, i.e., the topology of uniform convergence over compact subsets, which is a metric space for the distance

$$\mathsf{d}(x, y) = \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{\|x - y\|_n}{1 + \|x - y\|_n}\,, \quad x, y \in X\,,$$

where $\|x\|_n = \sup_{s \in [-n,0]} \|x(s)\|$, and $\|\cdot\|$ denotes the maximum norm in $\mathbb{R}^m$.

Let $BU \subset X$ be the Banach space

$$BU = \{x \in X \mid x \text{ is bounded and uniformly continuous}\}$$

with the supremum norm $\|x\|_\infty = \sup_{s \in (-\infty,0]} \|x(s)\|$. Given $k > 0$, we will denote

$$B_k = \{x \in BU \mid \|x\|_\infty \le k\}\,.$$

As usual, given $I = (-\infty, a] \subset \mathbb{R}$, $t \in I$, and a continuous function $x : I \to \mathbb{R}^m$, x_t will denote the element of X defined by $x_t(s) = x(t + s)$ for $s \in (-\infty, 0]$.

Let $D\colon BU \to \mathbb{R}^m$ be a linear operator satisfying the following hypotheses:

(D1) D is linear and continuous for the norm.

(D2) For each $k > 0$, $D\colon B_k \to \mathbb{R}^m$ is continuous when we take the restriction of the compact-open topology to B_k; i.e., if $x_n \xrightarrow{\mathsf{d}} x$ as $n \to \infty$ with x_n, $x \in B_k$, then $\lim_{n\to\infty} Dx_n = Dx$.

(D3) D is atomic at 0 (see the definition in Hale [10] or Hale and Verduyn Lunel [11]).

(D4) D is *stable*; i.e., there is a continuous function $c \in C([0, \infty), \mathbb{R}^+)$ with $\lim_{t\to\infty} c(t) = 0$ such that, for each $\varphi \in BU$ with $D\varphi = 0$, the solution of

$$\begin{cases} Dx_t = 0\,, & t \ge 0, \\ x_0 = \varphi & \end{cases}$$

satisfies $\|x(t)\| \le c(t)\, \|\varphi\|_\infty$ for each $t \ge 0$.

From (D1)–(D2), as shown in Muñoz-Villarragut, Novo, and Obaya [21], if $x \in BU$,

$$Dx = \int_{-\infty}^{0} [d\mu(s)]\, x(s),$$

where $\mu = [\mu_{ij}]_{i,j\in\{1,\dots,m\}}$ and μ_{ij} is a real regular Borel measure with finite total variation $|\mu_{ij}|(-\infty, 0] < \infty$ for all $i,\ j \in \{1,\dots,m\}$. From (D3), without loss of generality, we may assume that

$$Dx = x(0) - \int_{-\infty}^{0} [d\nu(s)]\, x(s), \tag{3.1}$$

where $\nu = [\nu_{ij}]_{i,j\in\{1,\dots,m\}}$, ν_{ij} is a real regular Borel measure with finite total variation, and $|\nu_{ij}|(\{0\}) = 0$ for all $i,\ j \in \{1,\dots,m\}$. For any measurable set $E \subset (-\infty, 0]$ we will denote by $|\nu|(E)$ the $m \times m$ matrix $[\,|\nu_{ij}|(E)\,]$ and by $\|\nu\|_\infty(E)$ the corresponding matricial norm. We define the linear operator

$$\begin{array}{rccccl} \widehat{D}\colon & BU & \longrightarrow & BU, & & \\ & x & \mapsto & \widehat{D}x\colon (-\infty,0] & \to & \mathbb{R}^m, \\ & & & s & \mapsto & Dx_s; \end{array} \tag{3.2}$$

that is, $\widehat{D}x(s) = x(s) - \int_{-\infty}^{0} [d\nu(\theta)]\, x(\theta + s)$ for each $s \in (-\infty, 0]$. From (D4) (see Theorem 3.8 of [21]), $\widehat{D}$ is invertible, and $\widehat{D}^{-1}$ is bounded for the norm and uniformly continuous when we take the restriction of the compact-open topology to B_k; i.e., given $\varepsilon > 0$, there is a $\delta(k) > 0$ such that $\mathsf{d}(\widehat{D}^{-1}h_1, \widehat{D}^{-1}h_2) < \varepsilon$ for all $h_1,\ h_2 \in B_k$ with $\mathsf{d}(h_1, h_2) < \delta(k)$. Hence the linear operator $T\colon BU \to \mathbb{R}^m$, $x \mapsto (\widehat{D}^{-1}x)(0)$ also satisfies (D1)–(D4), and has a representation

$$Tx = \int_{-\infty}^{0} [d\widehat{\mu}(s)]\, x(s)\,, \tag{3.3}$$

where $\widehat{\mu} = [\widehat{\mu}_{ij}]$ and $\widehat{\mu}_{ij}$ is a real regular Borel measure with finite total variation.

Let $\mathcal{L}$ be the space of linear operators $\mathcal{L} = \{D\colon BU \to \mathbb{R}^m \mid \text{ (D1)–(D2) hold}\}$, which is complete for the operator norm. As a consequence of the next result it can be checked that $\mathcal{U} = \{D \in \mathcal{L} \mid \text{ (D3)–(D4) hold}\}$ is open in $\mathcal{L}$.

PROPOSITION 3.1. *Let us assume that D satisfies* (D1)–(D4) *and it is given by* (3.1)*. Then, there is an $\varepsilon > 0$ such that for any $\nu^* = [\nu^*_{ij}]$, where ν^*_{ij} is a real regular Borel measure with finite total variation, $|\nu^*_{ij}|(\{0\}) = 0$ for each $i,\ j \in 1,\dots,m$, and $\|\nu - \nu^*\|_\infty(-\infty, 0] < \varepsilon$, the linear operator $D^*\colon BU \to \mathbb{R}^m$ given by*

$$D^*x = x(0) - \int_{-\infty}^{0} [d\nu^*(s)]\, x(s)$$

is stable.

Proof. As shown in Theorem 3.9 of [21], it is enough to check that the linear operator $\widehat{D^*}\colon BU \to BU$, defined from D^* as in (3.2), is invertible and $(\widehat{D^*})^{-1}$ is continuous for the restriction of the compact-open topology to B_k. Since $\widehat{D}$ is invertible, we define $X\colon BU \to BU$ by

$$X = \widehat{D}^{-1} \circ (\widehat{D} - \widehat{D^*})\,.$$

Hence, taking $\varepsilon < 1/\|\widehat{D}^{-1}\|$, from $\|\widehat{D} - \widehat{D^*}\| < \varepsilon$ we deduce that $\|X\| < 1$ and $\mathrm{Id} - X$ is invertible, and consequently $\widehat{D^*} = \widehat{D} \circ (\mathrm{Id} - X)$ is also invertible. From $(\widehat{D^*})^{-1} = (\mathrm{Id} - X)^{-1} \circ \widehat{D}^{-1} = (\sum_{n=0}^{\infty} X^n) \circ \widehat{D}^{-1}$ it is easy to check the continuity for the restriction of the compact-open topology to B_k, and the proof is finished. ∎

Let $(\Omega, \sigma, \mathbb{R})$ be a minimal flow over a compact metric space (Ω, d) and denote $\sigma(t,\omega) = \omega{\cdot}t$ for all $\omega \in \Omega$ and $t \in \mathbb{R}$. Let $F\colon \Omega \times BU \to \mathbb{R}^m$, $(\omega, x) \mapsto F(\omega, x)$ be a function satisfying the following conditions:

(F1) F is continuous on $\Omega \times BU$ and locally Lipschitz in x for the norm $\|\cdot\|_\infty$.

(F2) For each $k > 0$, $F(\Omega \times B_k)$ is a bounded subset of $\mathbb{R}^m$.

(F3) For each $k > 0$, $F\colon \Omega \times B_k \to \mathbb{R}^m$ is continuous when we take the restriction of the compact-open topology to B_k; i.e., if $\omega_n \to \omega$ and $x_n \xrightarrow{\mathsf{d}} x$ as $n \to \infty$ with $x_n, x \in B_k$, then $\lim_{n\to\infty} F(\omega_n, x_n) = F(\omega, x)$.

We consider the families of nonautonomous FDEs with infinite delay

$$z'(t) = F(\omega{\cdot}t, z_t)\,, \quad t \geq 0\,,\ \omega \in \Omega\,, \tag{3.4$_\omega$}$$

and nonautonomous NFDEs with infinite delay and stable D-operator

$$\frac{d}{dt} D z_t = F(\omega{\cdot}t, z_t)\,, \quad t \geq 0\,,\ \omega \in \Omega\,, \tag{3.5$_\omega$}$$

which obviously include $(3.4)_\omega$ when $\nu \equiv 0$.

From hypothesis (F1), the standard theory of FDEs (resp., NFDEs) with infinite delay (see Hino, Murakami, and Naito [12] (resp., Wang and Wu [30] and Wu [31])), assures that for each $x \in BU$ and each $\omega \in \Omega$ the system $(3.4)_\omega$ (resp., $(3.5)_\omega$) locally admits a unique solution $z(t, \omega, x)$ with initial value x; i.e., $z(s, \omega, x) = x(s)$ for each $s \in (-\infty, 0]$. Therefore, the family $(3.4)_\omega$ (resp., $(3.5)_\omega$) induces a local skew-product semiflow

$$\begin{aligned} \tau\ \ : \mathbb{R}^+ \times \Omega \times BU &\longrightarrow \Omega \times BU, \\ (t, \omega, x) &\mapsto (\omega{\cdot}t, u(t, \omega, x))\,, \end{aligned} \tag{3.6}$$

where $u(t, \omega, x) \in BU$ and $u(t, \omega, x)(s) = z(t + s, \omega, x)$ for $s \in (-\infty, 0]$.

As shown in Muñoz-Villarragut, Novo, and Obaya [21], the change of variable $y = \widehat{D} z$ takes $(3.5)_\omega$ to

$$y'(t) = G(\omega{\cdot}t, y_t)\,, \quad t \geq 0\,,\ \omega \in \Omega\,, \tag{3.7$_\omega$}$$

with $G\colon \Omega \times BU \to \mathbb{R}^m$, $(\omega, x) \mapsto G(\omega, x) = F(\omega, \widehat{D}^{-1} x)$. This family induces a local skew-product semiflow

$$\begin{aligned} \widehat{\tau}\ \ : \mathbb{R}^+ \times \Omega \times BU &\longrightarrow \Omega \times BU, \\ (t, \omega, x) &\mapsto (\omega{\cdot}t, \widehat{u}(t, \omega, x))\,, \end{aligned}$$

where $\widehat{u}(t, \omega, x) \in BU$ and $\widehat{u}(t, \omega, x)(s) = y(t + s, \omega, x)$ for $s \in (-\infty, 0]$, and it is related to the previous one (3.6) by

$$\widehat{u}(t, \omega, x) = \widehat{D}\, u(t, \omega, \widehat{D}^{-1} x)\,. \tag{3.8}$$

The above change of variable is often used to deduce the regularity of the solutions of $(3.5)_\omega$. In particular, if the initial value x is Lipschitz, then it is easy to check that $\widehat{D}x$ is Lipschitz, and as a consequence, $\widehat{u}(t,\omega,\widehat{D}x)$ and $u(t,\omega,x)$ are also Lipschitz. Besides, as it is well known [10, 11], the solution of a NFDE is not necessarily differentiable at every point. Next, we show that this indeed holds, provided that the trajectory is bounded and admits a backward orbit.

PROPOSITION 3.2. *We consider the skew-product semiflow* (3.6) *induced by* $(3.5)_\omega$. *Assume that* $(\omega,x)\in\Omega\times BU$ *admits a backward orbit extension and that there is a* $k_1>0$ *such that* $u(t,\omega,x)\in B_{k_1}$ *for each* $t\in\mathbb{R}$. *Then* $z(t)=z(t,\omega,x)$, *the solution of* $(3.5)_\omega$ *with initial value* x, *belongs to* $C^1(\mathbb{R},\mathbb{R}^m)$.

Proof. From (3.3), we have for each $h\neq 0$

$$\frac{z(t+h)-z(t)}{h}=\int_{-\infty}^{0}[d\widehat{\mu}(\theta)]\,\frac{y(t+\theta+h)-y(t+\theta)}{h}\,,$$

where $y(t)=y(t,\omega,\widehat{D}x)$ is the solution of $(3.7)_\omega$ with initial value $\widehat{D}x$, which is defined for each $t\in\mathbb{R}$, it belongs to $C^1(\mathbb{R},\mathbb{R}^m)$, and it is bounded by $k_2=\|\widehat{D}\|\,k_1$. From (F2) there is a $c>0$ such that $\|G(\Omega\times B_{k_2})\|\le c$; then an application of the dominated convergence theorem shows that z is differentiable,

$$z'(t)=\int_{-\infty}^{0}[d\widehat{\mu}(\theta)]\,y'(t+\theta)\,, \tag{3.9}$$

and $\|z'(t)\|\le c\,\|\widehat{\mu}\|_\infty(-\infty,0]$ for each $t\in\mathbb{R}$.

Finally, we check the continuity of z'. Given t_0 and $\varepsilon>0$, we find $k>0$ such that $\|\widehat{\mu}\|_\infty(-\infty,-k]<\varepsilon/(4\,c)$. Since y' is uniformly continuous on $I_0=[t_0-k-1,t_0+1]$, there is $0<\delta<1$ such that $\|y'(s)-y'(\theta)\|<\varepsilon/(2\,\|\widehat{\mu}\|_\infty[-k,0]+1)$ for each s, $\theta\in I_0$ with $|s-u|<\delta$. Hence, from (3.9), we deduce that if $|t-t_0|<\delta$,

$$\begin{aligned}\|z'(t)-z'(t_0)\|&\le\frac{\varepsilon}{2}+\left\|\int_{-k}^{0}[d\widehat{\mu}(\theta)]\,(y'(t+\theta)-y'(t_0+\theta))\right\|\\&\le\frac{\varepsilon}{2}+\|\widehat{\mu}\|_\infty[-k,0]\sup_{\theta\in[-k,0]}\|y'(t+\theta)-y'(t_0+\theta)\|\le\frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon\end{aligned}$$

because $t+\theta$ and $t_0+\theta$ belong to I_0 for $\theta\in[-k,0]$, and the proof is finished. ◻

Next, we introduce two exponential orderings in the space BU and will formulate a simultaneous quasi-monotone condition for FDEs and NFDEs.

As usual, in $\mathbb{R}^m$ we take the partial order relation

$$\begin{aligned}&v\le w\quad\Longleftrightarrow\quad v_j\le w_j\quad\text{for } j=1,\dots,m\,,\\&v<w\quad\Longleftrightarrow\quad v\le w\quad\text{and}\quad v_j<w_j\quad\text{for some } j\in\{1,\dots,m\}\,,\end{aligned}$$

and if $x\in BU$, we denote $x\ge 0$ when $x(s)\ge 0$ for each $s\in(-\infty,0]$.

We write $A\le B$ for $m\times m$ matrices $A=[a_{ij}]$ and $B=[b_{ij}]$ if and only if $a_{ij}\le b_{ij}$ for all i,j. Let A be an $m\times m$ quasi-positive matrix; that is, there is a $\lambda>0$ such that $A+\lambda I\ge 0$.

Remark 3.3. If A is a quasi-positive matrix, then e^{At} is a nonnegative matrix with the entries in the main diagonal being strictly positive for each $t\ge 0$.

We introduce the new families of positive cones with empty interior on BU

$$\begin{aligned}BU^+_{A,r}&=\{x\in BU\mid x\ge 0\ \text{ and }\ x(t)\ge e^{A(t-s)}x(s)\quad\text{for }-r\le s\le t\le 0\}\,,\\BU^+_{A,\infty}&=\{x\in BU\mid x\ge 0\ \text{ and }\ x(t)\ge e^{A(t-s)}x(s)\quad\text{for }-\infty< s\le t\le 0\}\,,\end{aligned}$$

which, respectively, induce the following partial order relations on BU:

$$x \le_{A,r} y \iff x \le y \text{ and } y(t)-x(t) \ge e^{A(t-s)}(y(s)-x(s))\,, -r \le s \le t \le 0\,,$$

$$x <_{A,r} y \iff x \le_{A,r} y \quad\text{and}\quad x \ne y\,, \text{ and}$$

$$x \le_{A,\infty} y \iff x \le y \text{ and } y(t)-x(t) \ge e^{A(t-s)}(y(s)-x(s))\,, -\infty < s \le t \le 0\,,$$

$$x <_{A,\infty} y \iff x \le_{A,\infty} y \text{ and } \quad x \ne y\,.$$

As in [29] for the case of finite delay, the following result can be checked.

Remark 3.4. A smooth function (resp., a Lipschitz continuous function) x belongs to $BU^+_{A,r}$ if and only if

$$x \ge 0 \quad\text{and}\quad x'(s) \ge A\,x(s) \quad \text{for each (resp., almost every) } s \in [-r,0]$$

and belongs to $BU^+_{A,\infty}$ if and only if

$$x \ge 0 \quad\text{and}\quad x'(s) \ge A\,x(s) \quad \text{for each (resp., almost every) } s \in (-\infty,0]\,.$$

In the rest of the paper $\le_A$ will denote any of the above order relations and we assume the following quasi-monotone condition:

(F4) If $x, y \in BU$ with $x \le_A y$, then $F(\omega, y) - F(\omega, x) \ge A\,(Dy - Dx)$ for each $\omega \in \Omega$.

This generates a monotone skew-product semiflow in the following cases:

Case I: FDEs $(3.4)_\omega$ under (F1)–(F4) and order $\le_{A,r}$.
Case II: FDEs $(3.4)_\omega$ under (F1)–(F4) and order $\le_{A,\infty}$.
Case III: NFDEs $(3.5)_\omega$, under (F1)–(F4), (D1)–(D5), and order $\le_{A,\infty}$.

D is of the form (3.1), $\nu \equiv 0$ in Cases I–II, and the following is satisfied:

(D5) The measures ν_{ij} in (3.1) are positive for all i, $j \in \{1,\dots,m\}$.

THEOREM 3.5. *The skew-product semiflow* (3.6) *induced by* $(3.5)_\omega$ *is monotone; that is, for each* $\omega \in \Omega$ *and* $x, y \in BU$ *such that* $x \le_A y$ *it holds that*

$$u(t,\omega,x) \le_A u(t,\omega,y)$$

whenever they are defined.

Proof. We omit the proof of Cases I–II, which is completely analogous to the one given in Proposition 3.1 of [29].

Case III. First, we show the result when D has the form

$$Dx = x(0) - \int_{-\infty}^{-\rho} [d\nu(s)]\,x(s) \tag{3.10}$$

for some $\rho > 0$. Let $\mathbf{1} = (1,\dots,1)^T \in \mathbb{R}^m$. For $\varepsilon > 0$ let $z_\varepsilon(t,\omega,y)$ be the solution of

$$\frac{d}{dt} Dz_t = F(\omega{\cdot}t, z_t) + \varepsilon\,\mathbf{1}\,, \quad t \ge 0\,,\ \omega \in \Omega\,,$$

with initial value y; i.e., $z_\varepsilon(s,\omega,y) = y(s)$ for each $s \in (-\infty,0]$. There exist $\varepsilon_0 > 0$ and $T > 0$ such that for each $\varepsilon \in [0,\varepsilon_0)$, $z(t) = z(t,\omega,x)$ and $y^\varepsilon(t) = z_\varepsilon(t,\omega,y)$ are defined on $[0,T]$ and denote $z^\varepsilon(t) = y^\varepsilon(t) - z(t)$.

Let $t_1 \in [0,T]$ be the greatest time such that $z^\varepsilon_t \geq_A 0$ (i.e., $z_t \leq_A y^\varepsilon_t$); that is, $z^\varepsilon(t) \geq 0$ and $z^\varepsilon(t) \geq e^{A(t-s)} z^\varepsilon(s)$ for each $s \leq t \leq t_1$. We claim that $t_1 = T$. Assume on the contrary that $t_1 < T$. From (F4)

$$\frac{d}{dt} D z^\varepsilon_t|_{t=t_1} - AD z^\varepsilon_{t_1} = [F(\omega, y^\varepsilon_{t_1}) - F(\omega, z_{t_1}) - AD(y^\varepsilon_{t_1} - z_{t_1})] + \varepsilon\, \mathbf{1} \geq \varepsilon\, \mathbf{1}\,,$$

and there is $h > 0$ with $h < \rho$ such that

$$\frac{d}{dt} D z^\varepsilon_t - AD z^\varepsilon_t \geq 0\,, \quad t \in [t_1, t_1 + h]\,.$$

Hence, $D z^\varepsilon_t \geq e^{A(t-s)} D z^\varepsilon_s$ for each $t_1 \leq s \leq t \leq t_1 + h$, and from (3.10)

$$z^\varepsilon(t) - e^{A(t-s)} z^\varepsilon(s) \geq \int_{-\infty}^{-\rho} [d\nu(\theta)](z^\varepsilon(t+\theta) - e^{A(t-s)} z^\varepsilon(s+\theta))\,.$$

Since $h < \rho$ and $t \in [t_1, t_1 + h]$ we have $s + \theta \leq t + \theta \leq t_1 + h - \rho \leq t_1$ and thus $z^\varepsilon(t+\theta) - e^{A(t-s)} z^\varepsilon(s+\theta) \geq 0$ if $\theta \in (-\infty, -\rho)$. Consequently, from (D5)

$$z^\varepsilon(t) - e^{A(t-s)} z^\varepsilon(s) \geq 0$$

if $t_1 \leq s \leq t \leq t_1 + h$, which contradicts the definition of t_1. Then $t_1 = T$, and letting ε go to 0 the result is proved.

Next, since D is stable, from Proposition 3.1 there is an $\varepsilon_0 > 0$ such that

$$D_\varepsilon : BU \to \mathbb{R}^m\,, \quad x \mapsto D_\varepsilon x = x(0) - \int_{-\infty}^{-\varepsilon} [d\nu(s)]\, x(s)$$

is stable for each $\varepsilon \leq \varepsilon_0$. Then the skew-product semiflow

$$\begin{array}{rccl} \tau_\varepsilon & : \mathbb{R}^+ \times \Omega \times BU & \longrightarrow & \Omega \times BU, \\ & (t, \omega, x) & \mapsto & (\omega{\cdot}t, u_\varepsilon(t, \omega, x))\,, \end{array}$$

induced by

$$(3.11)_\omega \qquad \frac{d}{dt} D_\varepsilon z_t = F(\omega{\cdot}t, z_t) - A(D z_t - D_\varepsilon z_t)\,, \quad t \geq 0\,,\ \omega \in \Omega\,,$$

satisfies the corresponding hypotheses (F1)–(F4) for D_ε, and we deduce that

$$u_\varepsilon(t, \omega, x) \leq_A u_\varepsilon(t, \omega, y),$$

provided that $x \leq_A y$. The continuous dependence of solutions of NFDEs, or the continuous dependence of solutions of FDEs after taking $(3.11)_\omega$ to a FDE system as above, shows that $\lim_{\varepsilon\to 0} u_\varepsilon(t, \omega, x) = u(t, \omega, x)$, and we conclude that $u(t, \omega, x) \leq_A u(t, \omega, y)$, as stated. ∎

Notice that the restriction provided by Krisztin and Wu in [19] on A ($-\mu$ in their paper) to show the theorem in the scalar neutral case with finite delay has been removed in the previous theorem.

4. Stability of omega-limit sets. Throughout this section we will study the omega-limit sets for the monotone skew-product semiflows (3.6) induced by $(3.5)_\omega$ in the following cases:

Case I: FDEs $(3.4)_\omega$ under (F1)–(F4), and order $\leq_{A,r}$.

Case II: FDEs $(3.4)_\omega$ under (F1)–(F4), (A1), and order $\leq_{A,\infty}$.

Case III: NFDEs $(3.5)_\omega$, under (F1)–(F4), (A1), (D1)–(D5), and order $\leq_{A,\infty}$.

D is of the form (3.1), $\nu \equiv 0$ in Cases I–II, and the following is satisfied:

(A1) All the eigenvalues of A have strictly negative real part.

We introduce different concepts of stability for the metric topology of trajectories and compact invariant sets. In section 6, we will illustrate the practical interest of these concepts for the study of compartmental systems which are monotone for the exponential ordering.

Definition 4.1. *Let K be a compact positively τ-invariant set of $\Omega \times BU$. It is said that $(K, \tau, \mathbb{R}^+)$ is* uniformly stable *if for any $\varepsilon > 0$ there exists a $\delta > 0$, called the* modulus of uniform stability*, such that if $(\omega, x), (\omega, y) \in K$ are such that $\mathsf{d}(x, y) < \delta$, then $\mathsf{d}(u(t, \omega, x), u(t, \omega, y)) \leq \varepsilon$ for all $t \geq 0$.*

Definition 4.2. *Given $k > 0$, a forward orbit $\{\tau(t, \omega_0, x_0) \mid t \geq 0\}$ of the skew-product semiflow (3.6) is said to be* uniformly stable for the order $\leq_A$ in B_k *if for every $\varepsilon > 0$ there is a $\delta > 0$, called the* modulus of uniform stability*, such that if $s \geq 0$ and $\mathsf{d}(u(s, \omega_0, x_0), x) \leq \delta$ for certain $x \in B_k$ with $x \leq_A u(s, \omega_0, x_0)$ or $u(s, \omega_0, x_0) \leq_A x$, then for each $t \geq 0$,*

$$\mathsf{d}(u(t+s, \omega_0, x_0), u(t, \omega_0{\cdot}s, x)) = \mathsf{d}(u(t, \omega_0{\cdot}s, u(s, \omega_0, x_0)), u(t, \omega_0{\cdot}s, x)) \leq \varepsilon\,.$$

If this happens for each $k > 0$, the forward orbit is said to be uniformly stable for the order $\leq_A$ in bounded sets*.*

Definition 4.3. *Given $k > 0$, a positively τ-invariant set $K \subset \Omega \times BU$ is* uniformly stable for the order $\leq_A$ in B_k *if for any $\varepsilon > 0$ there exists a $\delta > 0$, called the* modulus of uniform stability*, such that if $(\omega, x) \in K$, $(\omega, y) \in \Omega \times B_k$ are such that $\mathsf{d}(x, y) < \delta$ with $x \leq_A y$ or $y \leq_A x$, then $\mathsf{d}(u(t, \omega, x), u(t, \omega, y)) \leq \varepsilon$ for all $t \geq 0$. If this happens for each $k > 0$, K is said to be* uniformly stable for the order $\leq_A$ in bounded sets*.*

From Proposition 4.1 of [23] and Proposition 4.2 of [21], if $z(t, \omega_0, x_0)$ is bounded, we deduce that the family $\{u(t, \omega_0, x_0) \mid t \geq 0\}$ is equicontinuous on $[-k, 0]$, and we can define the omega-limit set of the trajectory of the point (ω_0, x_0) as

$$\mathcal{O}(\omega_0, x_0) = \{(\omega, x) \in \Omega \times BU \mid \exists\, t_n \uparrow \infty \;\text{ with } \omega_0{\cdot}t_n \to \omega\,,\; u(t_n, \omega_0, x_0) \xrightarrow{\mathsf{d}} x\}\,.$$

Notice that the omega-limit set of a pair $(\omega_0, x_0) \in \Omega \times BU$ makes sense whenever $\mathrm{cls}_X\{u(t, \omega_0, x_0) \mid t \geq 0\}$ is a compact set. As shown in Proposition 4.4 of [23] and [21], $\mathcal{O}(\omega_0, x_0)$ is a positively invariant compact subset admitting a flow extension.

The next statement shows that the omega-limit set inherits and improves the stability properties of certain relatively compact trajectories.

Proposition 4.4. *Let $(\omega_0, x_0) \in \Omega \times BU$ with forward orbit $\{\tau(t, \omega_0, x_0) \mid t \geq 0\}$ relatively compact for the product metric topology and uniformly stable for $\leq_A$ in bounded sets. Let K denote the omega-limit set of (ω_0, x_0). Then the following hold:*

(i) *K is uniformly stable for $\leq_A$ in bounded sets.*

(ii) *$(K, \tau, \mathbb{R}^+)$ is uniformly stable.*

Proof. (i) Let $k_0 > 0$ be such that $\mathrm{cls}_{\Omega\times X}\{\tau(t,\omega_0,x_0) \mid t \geq 0\} \subset \Omega \times B_{k_0}$. Given $k > 0$, we check that K is uniformly stable for $\leq_A$ in B_k. Thus, we fix an $\varepsilon > 0$ and take $\delta > 0$ as the modulus of uniform stability of the trajectory for $\leq_A$ in B_{2k_0+k}. Let $(\omega,x) \in K$ and $(\omega,y) \in \Omega \times B_k$ with $\mathsf{d}(x,y) < \delta$ and $x \leq_A y$ or $y \leq_A x$, and take a sequence $\{t_n\}_{n\in\mathbb{N}}$ such that $\lim_{n\to\infty} \tau(t_n,\omega_0,x_0) = (\omega,x)$. Since $u(t_n,\omega_0,x_0) \leq_A u(t_n,\omega_0,x_0) + y - x$ or $u(t_n,\omega_0,x_0) + y - x \leq_A u(t_n,\omega_0,x_0)$, $u(t_n,\omega_0,x_0)+y-x \in B_{2k_0+k}$, and $\mathsf{d}(u(t_n,\omega_0,x_0), u(t_n,\omega_0,x_0)+y-x) < \delta$, we deduce that for each $n \in \mathbb{N}$ and $t \geq 0$

$$\mathsf{d}(u(t,\omega_0{\cdot}t_n,u(t_n,\omega_0,x_0)), u(t,\omega_0{\cdot}t_n,u(t_n,\omega_0,x_0)+y-x)) \leq \varepsilon\,.$$

Finally, from Proposition 4.2 of [23], and the analogous one for the neutral case, which ensures certain continuity of the semiflow when the compact-open topology is considered on BU, as $n \uparrow \infty$ we get $\mathsf{d}(u(t,\omega,x),u(t,\omega,y)) \leq \varepsilon$ for each $t \geq 0$, and (i) is proved.

(ii) Case I, that is, FDEs when $\leq_A$ denotes the ordering $\leq_{A,r}$. Let (ω,x) and $(\omega,y) \in K$. Since K is a positively invariant compact subset admitting a flow extension, from Proposition 3.2 we deduce that $x,\, y \in C^1((-\infty,0],\mathbb{R}^m)$. We define

$$\begin{aligned} h\colon [-r,0] &\longrightarrow \mathbb{R}^m,\\ s &\mapsto \inf\{x'(s) - A\,x(s), y'(s) - A\,y(s)\}\,, \text{ and}\\ a_{x,y}\colon (-\infty,0] &\longrightarrow \mathbb{R}^m,\\ s &\mapsto \begin{cases} i(s) = \inf\{x(s),y(s)\}\,, & s \leq -r,\\ e^{A\,(s+r)} i(-r) + \int_{-r}^{s} e^{A(s-\tau)} h(\tau)\,d\tau\,, & -r \leq s. \end{cases}\end{aligned} \tag{4.1}$$

It is easy to check that $a_{x,y} \in BU$ and $a_{x,y} \leq x$. Moreover, if $-r \leq s \leq 0$,

$$a'_{x,y}(s) = A\,a_{x,y}(s) + h(s)\,,$$

from which we deduce that $x'(s) - a'_{x,y}(s) \geq A(x(s) - a_{x,y}(s))$ and hence, as stated in Remark 3.4, $a_{x,y} \leq_A x$. Analogously, $a_{x,y} \leq_A y$.

Moreover, since there is a $k_0 > 0$ such that $\mathrm{cls}_X\{u(t,\omega_0,x_0) \mid t \geq 0\} \subset B_{k_0}$, $x'(s) = F(\omega{\cdot}s, x_s)$ for each $s \in (-\infty,0]$ and $(\omega,x) \in K$, and there is $k > 0$ such that $\|F(\omega,x)\| \leq k$ for each $(\omega,x) \in \Omega \times B_{k_0}$, we deduce that there is a $k_1 > 0$ such that $a_{x,y} \in B_{k_1}$ for each (ω,x), $(\omega,y) \in K$.

Let $\delta_1 > 0$ be the modulus of uniform stability of K for $\leq_A$ in B_{k_1} for $\varepsilon/2$. From the definition of $a_{x,y}$ and (F3), given $\delta_1 > 0$, we can find $\delta > 0$ such that if $\mathsf{d}(x,y) \leq \delta$, then $\mathsf{d}(x,a_{x,y}) \leq \delta_1$ and $\mathsf{d}(y,a_{x,y}) \leq \delta_1$. We omit the details, which are similar to those of Cases II–III sketched below. Consequently, whenever $\mathsf{d}(x,y) \leq \delta$, since $a_{x,y} \leq_A x$, $a_{x,y} \leq_A y$, and $a_{x,y} \in B_{k_1}$, the uniform stability of K for the order $\leq_A$ yields

$$\mathsf{d}(u(t,\omega,x),u(t,\omega,y)) \leq \mathsf{d}(u(t,\omega,x),u(t,\omega,a_{x,y})) + \mathsf{d}(u(t,\omega,a_{x,y}),u(t,\omega,y)) \leq \varepsilon$$

and $(K,\tau,\mathbb{R}^+)$ is uniformly stable.

Cases II–III, that is, FDEs and NFDEs when $\leq_A$ denotes the ordering $\leq_{A,\infty}$. Both cases can be studied together by taking $D\colon BU \to \mathbb{R}^m$, $x \mapsto x(0)$ in Case II. As in the previous case, given (ω,x) and $(\omega,y) \in K$, we know that $x,\, y \in C^1((-\infty,0],\mathbb{R}^m)$, and from (A1) we can define

$$
\begin{aligned}
a_{x,y}\colon (-\infty,0] \quad &\longrightarrow \quad \mathbb{R}^m, \\
s \quad &\mapsto \quad \int_{-\infty}^{s} e^{A(s-\tau)} \inf\{x'(\tau)-A\,x(\tau), y'(\tau)-A\,y(\tau)\}\,d\tau\,,
\end{aligned}
\tag{4.2}
$$

which satisfies $a_{x,y}\in BU$, $a_{x,y}\le_A x$, and $a_{x,y}\le_A y$. Let $\widehat{x}=\widehat{D}x$ and $\widehat{y}=\widehat{D}y$. As in Proposition 3.2,

$$
x'(s)=\int_{-\infty}^{0}[d\widehat{\mu}(\theta)]\,\widehat{x}\,'(s+\theta)\,,\quad y'(s)=\int_{-\infty}^{0}[d\widehat{\mu}(\theta)]\,\widehat{y}\,'(s+\theta)\,,
\tag{4.3}
$$

and $\widehat{x}\,'(s)=G(\omega{\cdot}s,\widehat{x}_s)$, $\widehat{y}\,'(s)=G(\omega{\cdot}s,\widehat{y}_s)$ for each $s\in(-\infty,0]$.

Hence, since there is a $k_0>0$ such that $\mathrm{cls}_X\{u(t,\omega_0,x_0)\mid t\ge 0\}\subset B_{k_0}$ and $k>0$ such that $\|G(\omega,\widehat{x})\|\le k$ for each $(\omega,\widehat{x})\in\Omega\times B_{k_0\,\|\widehat{D}\|}$, we deduce that there is a $k_1>0$ such that $a_{x,y}\in B_{k_1}$ for each (ω,x), $(\omega,y)\in K$.

As in Case I, in order to finish the proof it is enough to check that given $\delta_1>0$, there is a $\delta>0$ such that $\mathsf{d}(a_{x,y},x)\le\delta_1$ and $\mathsf{d}(a_{x,y},y)\le\delta_1$, provided that $\mathsf{d}(x,y)\le\delta$. Assume on the contrary that there are $\delta_1>0$ and sequences $\{(\omega,x_n)\}_{n\in\mathbb{N}}$, $\{(\omega,y_n)\}_{n\in\mathbb{N}}\subset K$ such that $\lim_{n\to\infty}\mathsf{d}(x_n,y_n)=0$ and $\mathsf{d}(a_{x_n,y_n},x_n)>\delta_1$. However, $\|a_{x_n,y_n}(s)-x_n(s)\|\le\|a_{x_n,y_n}(s)-b_{x_n,y_n}(s)\|$ with

$$
b_{x_n,y_n}(s)=\int_{-\infty}^{s} e^{A(s-\tau)}\sup\{x_n'(\tau)-A\,x_n(\tau),y_n'(\tau)-A\,y_n(\tau)\}\,d\tau\,,
$$

from which we deduce that

$$
\|a_{x_n,y_n}(s)-x_n(s)\|\le\int_{-\infty}^{s}\big\|e^{A(s-\tau)}\big\|(\|x_n'(\tau)-y_n'(\tau)\|+\|A\|\,\|x_n(\tau)-y_n(\tau)\|)\,d\tau\,.
\tag{4.4}
$$

Moreover, from relation (4.3) and $G(\omega,x)=F(\omega,\widehat{D}^{-1}x)$ we have

$$
\|x_n'(\tau)-y_n'(\tau)\|\le\left\|\int_{-\infty}^{0}[d\widehat{\mu}(\theta)]\,(F(\omega{\cdot}(\tau+\theta),(x_n)_{\tau+\theta})-F(\omega{\cdot}(\tau+\theta),(y_n)_{\tau+\theta}))\right\|\,.
$$

Hence, from the total finite variation of $\widehat{\mu}$, hypothesis (F2), the uniform continuity of F on K, implied by hypothesis (F3), hypothesis (A1), and $\lim_{n\to\infty}\mathsf{d}(x_n,y_n)=0$, we deduce that $a_{x_n,y_n}-x_n$ tends to 0 uniformly on compact sets of $(-\infty,0]$, which contradicts that $\mathsf{d}(a_{x_n,y_n},x_n)>\delta_1$ and finishes the proof for Cases II–III. ∎

The next result shows that, under the previous assumptions, the omega-limit set K is always a minimal subset in Case I, and more regularity in the initial data is needed for Cases II and III.

First, we recall the definition of the section map of a compact set $M\subset\Omega\times X$. We introduce the projection set of M into the fiber space

$$
M_X=\{x\in X\mid \text{ there exists }\omega\in\Omega\text{ such that }(\omega,x)\in M\}\subseteq X\,.
$$

From the compactness of M it is immediate to show that M_X is also a compact subset of X. Let $\mathcal{P}_c(M_X)$ denote the set of closed subsets of M_X, endowed with the Hausdorff metric ρ; that is, for any two sets C, $B\in\mathcal{P}_c(M_X)$,

$$
\rho(C,B)=\sup\{\alpha(C,B),\ \alpha(B,C)\}\,,
$$

where $\alpha(C,B)=\sup\{r(c,B)\mid c\in C\}$ and $r(c,B)=\inf\{\mathsf{d}(c,b)\mid b\in B\}$. Then, define the so-called *section map*

$$\begin{aligned}\Omega &\longrightarrow \mathcal{P}_c(M_X),\\ \omega &\mapsto M_\omega=\{x\in X\mid(\omega,x)\in M\}\,.\end{aligned}\tag{4.5}$$

Due to the minimality of Ω and the compactness of M, the set M_ω is nonempty for every $\omega\in\Omega$; besides, the map is trivially well defined.

THEOREM 4.5. *Let $(\omega_0,x_0)\in\Omega\times BU$ with forward orbit $\{\tau(t,\omega_0,x_0)\mid t\geq 0\}$ relatively compact for the product metric topology and uniformly stable for $\leq_A$ in bounded sets. Let K denote the omega-limit set of (ω_0,x_0). Then, K is a minimal subset in* Case I*, and in Cases* II *and* III*, provided that x_0 is Lipschitz.*

Proof. Let M be a minimal subset such that $M\subseteq K$. We just need to show that $K\subseteq M$. So, take an element $(\omega,x)\in K$ and let us prove that $(\omega,x)\in M$. As M is in particular closed, it suffices to see that for any fixed $\varepsilon>0$ there exists $(\omega,x^*)\in M$ such that $\mathsf{d}(x,x^*)\leq\varepsilon$.

First, there exists $s_n\uparrow\infty$ such that $\lim_{n\to\infty}(\omega_0{\cdot}s_n,u(s_n,\omega_0,x_0))=(\omega,x)$. Now, take a pair $(\omega,\widetilde{x})\in M\subseteq K$. Then, there exists a sequence $t_n\uparrow\infty$ such that

$$(\omega,\widetilde{x})=\lim_{n\to\infty}(\omega_0{\cdot}t_n,u(t_n,\omega_0,x_0))\,.$$

Since from Proposition 4.4 $(M,\tau,\mathbb{R}^+)$ is uniformly stable, we can apply Theorem 3.4 of [23] so that the section map (4.5) turns out to be continuous at any point. As $\omega_0{\cdot}t_n\to\omega$, we deduce that $M_{\omega_0\cdot t_n}\to M_\omega$ in the Hausdorff metric. Therefore, for $\widetilde{x}\in M_\omega$ there exists a sequence $x_n\in M_{\omega_0\cdot t_n}$, $n\geq 1$, such that $x_n\to\widetilde{x}$ as $n\uparrow\infty$. From Proposition 3.2 we deduce that $x_n\in C^1((-\infty,0],\mathbb{R}^m)$ for each $n\in\mathbb{N}$, and denoting $y_n=u(t_n,\omega_0,x_0)$, we have $\mathsf{d}(x_n,y_n)\to 0$ as $n\uparrow\infty$. The rest of the proof depends on the case we are dealing with.

Case I, that is, FDEs when $\leq_A$ denotes the ordering $\leq_{A,r}$. Let n_0 be such that $t_n>r$ for each $n\geq n_0$. Then $y_n\in C^1([-r,0],\mathbb{R}^m)$ and we can define a_{x_n,y_n} as in (4.1) for each $n\geq n_0$. Moreover, as in Proposition 4.4, we check that $a_{x_n,y_n}\in BU$, $a_{x_n,y_n}\leq_A x_n$, and $a_{x_n,y_n}\leq_A y_n$ and there is a $k_1>0$ such that $a_{x_n,y_n}\in B_{k_1}$ for each $n\geq n_0$.

Next, let $\delta>0$ be the modulus of uniform stability of K for $\leq_A$ in B_{k_1} for $\varepsilon/2$. Since $\lim_{n\to\infty}\mathsf{d}(x_n,y_n)=0$, from the definition of a_{x_n,y_n} and hypothesis (F3) there is an $n_1\geq n_0$ such that $\mathsf{d}(x_n,a_{x_n,y_n})<\delta$ and $\mathsf{d}(y_n,a_{x_n,y_n})<\delta$ for each $n\geq n_1$. Hence, the uniform stability of K for the order $\leq_A$ in B_{k_1} yields

$$\mathsf{d}(u(t+t_{n_1},\omega_0,x_0),u(t,\omega_0{\cdot}t_{n_1},x_{n_1}))=\mathsf{d}(u(t,\omega_0{\cdot}t_{n_1},y_{n_1}),u(t,\omega_0{\cdot}t_{n_1},x_{n_1}))\leq\varepsilon$$

for each $t\geq 0$. In particular, if n_2 is such that $s_n-t_{n_1}\geq 0$ for $n\geq n_2$, we obtain

$$\mathsf{d}(u(s_n,\omega_0,x_0),u(s_n-t_{n_1},\omega_0{\cdot}t_{n_1},x_{n_1}))\leq\varepsilon\quad\text{for each }\ n\geq n_2\,.\tag{4.6}$$

Now, it remains to notice that, as $(\omega_0{\cdot}t_{n_1},x_{n_1})\in M$, also $\tau(s_n-t_{n_1},\omega_0{\cdot}t_{n_1},x_{n_1})=(\omega_0{\cdot}s_n,u(s_n-t_{n_1},\omega_0{\cdot}t_{n_1},x_{n_1}))\in M$ for all $n\geq n_2$. Therefore, there is a convergent subsequence towards a pair $(\omega,x^*)\in M$, and taking limits in (4.6), we deduce that $\mathsf{d}(x,x^*)\leq\varepsilon$, as we wanted.

Cases II–III, that is, FDEs and NFDEs when $\leq_A$ denotes the ordering $\leq_{A,\infty}$. Both cases can be studied together by taking $D\colon BU\to\mathbb{R}^m$, $x\mapsto x(0)$ in Case II.

Remember that $x_n \in C^1((-\infty,0],\mathbb{R}^m)$, and since x_0 is Lipschitz, it is not hard to check that $y_n = u(t_n,\omega_0,x_0)$ is also Lipschitz. Therefore,

$$\inf\{x_n'(\tau) - A\,x_n(\tau), y_n'(\tau) - A\,y_n(\tau)\}$$

is defined for almost every $\tau \in (-\infty,0]$ and we can define a_{x_n,y_n} as in (4.2). As in Proposition 4.4, we check that $a_{x_n,y_n} \in BU$, $a_{x_n,y_n} \le_A x_n$, and $a_{x_n,y_n} \le_A y_n$ and there is a $k_1 > 0$ such that $a_{x_n,y_n} \in B_{k_1}$ for each $n \in \mathbb{N}$.

Moreover, from relation (3.8), $y_n = \widehat{D}^{-1}\widehat{u}(t_n,\omega_0,\widehat{D}x_0) = \widehat{D}^{-1}\widehat{u}(0,\omega_0{\cdot}t_n,\widehat{D}y_n)$ and

$$(\widehat{D}y_n)'(s) = G(\omega_0{\cdot}(t_n+s),(\widehat{D}y_n)_s)\,, \quad -t_n \le s \le 0\,.$$

Therefore, we deduce from (3.3) that

$$y_n(s) = \int_{-\infty}^{0} [d\widehat{\mu}(\theta)]\,\widehat{D}y_n(s+\theta)\,, \quad s \in (-\infty,0]\,,$$

which yields

$$\frac{y_n(s+h)-y_n(s)}{h} = \int_{-\infty}^{-t_n-s} [d\widehat{\mu}(\theta)]\left[\frac{\widehat{D}y_n(s+\theta+h)-\widehat{D}y_n(s+\theta)}{h}\right] + \int_{-t_n-s}^{0} [d\widehat{\mu}(\theta)]\left[\frac{1}{h}\int_{s+\theta}^{s+\theta+h} G(\omega_0{\cdot}(t_n+t),(\widehat{D}y_n)_t)\,dt\right] \tag{4.7}$$

for $-t_n < s \le s+h \le 0$. Analogously, from

$$\frac{d}{ds}D(x_n)_s = F(\omega_0{\cdot}(t_n+s),(x_n)_s)\,, \quad s \in (-\infty,0],$$

and $G(\omega,x) = F(\omega,\widehat{D}^{-1}x)$ we deduce that

$$\frac{x_n(s+h)-x_n(s)}{h} = \int_{-\infty}^{-t_n-s} [d\widehat{\mu}(\theta)]\left[\frac{\widehat{D}x_n(s+\theta+h)-\widehat{D}x_n(s+\theta)}{h}\right] + \int_{-t_n-s}^{0} [d\widehat{\mu}(\theta)]\left[\frac{1}{h}\int_{s+\theta}^{s+\theta+h} G(\omega_0{\cdot}(t_n+t),(\widehat{D}x_n)_t)\,dt\right] \tag{4.8}$$

for $-t_n < s \le s+h \le 0$. Moreover, since x_n' and y_n' exist almost everywhere and $\widehat{D}y_n$ and $\widehat{D}x_n$ are Lipschitz on $(-\infty,0]$ with the same constant for all $n \in \mathbb{N}$, from relations (4.7), (4.8), and $G(\omega,x) = F(\omega,\widehat{D}^{-1}x)$ we deduce that there is a constant $c_0 \ge 0$ such that

$$\|x_n'(\tau) - y_n'(\tau)\| \le c_0\,\|\widehat{\mu}\|_\infty(-\infty,-t_n-\tau] + \left\|\int_{-t_n-\tau}^{0} [d\widehat{\mu}(\theta)]\,(F(\omega_0{\cdot}(t_n+\tau+\theta),(x_n)_{\tau+\theta}) - F(\omega_0{\cdot}(t_n+\tau+\theta),(y_n)_{\tau+\theta}))\right\|$$

for almost every $\tau \geq -t_n$. Hence, from (4.4), the total finite variation of $\widehat{\mu}$, hypotheses (F2), the uniform continuity of F in $\mathrm{cls}_{\Omega\times X}\{\tau(t,\omega_0,x_0)\,|\,t\geq 0\}$, implied by hypothesis (F3), $t_n \uparrow \infty$, hypothesis (A1), and $\lim_{n\to\infty} \mathsf{d}(x_n,y_n)=0$, we deduce that

$$\lim_{n\to\infty} \mathsf{d}(x_n, a_{x_n,y_n}) = 0 \quad \text{ and } \quad \lim_{n\to\infty} \mathsf{d}(y_n, a_{x_n,y_n}) = 0\,.$$

Consequently, if $\delta > 0$ is the modulus of uniform stability of K for $\leq_A$ in B_{k_1} for $\varepsilon/2$, there is an n_1 such that $\mathsf{d}(x_n, a_{x_n,y_n}) < \delta$ and $\mathsf{d}(y_n, a_{x_n,y_n}) < \delta$ for each $n \geq n_1$, and the rest of the proof is identical to that of Case I. $\square$

5. Topological structure of omega-limit sets. As in the previous section we consider the monotone skew-product semiflows (3.6) induced by $(3.5)_\omega$ in Cases I–III. First, we extend to this setting results of Novo, Núñez, and Obaya [22] and Novo, Obaya, and Sanz [23] ensuring the presence of almost-automorphic dynamics from the existence of a semicontinuous semiequilibrium.

Definition 5.1. *A map $a:\Omega\to BU$ such that $u(t,\omega,a(\omega))$ is defined for any $\omega\in\Omega$, $t\geq 0$ is*

(a) *an* equilibrium *if $a(\omega{\cdot}t) = u(t,\omega,a(\omega))$ for any $\omega\in\Omega$ and $t\geq 0$,*
(b) *a* superequilibrium *if $a(\omega{\cdot}t) \geq_A u(t,\omega,a(\omega))$ for any $\omega\in\Omega$ and $t\geq 0$, and*
(c) *a* subequilibrium *if $a(\omega{\cdot}t) \leq_A u(t,\omega,a(\omega))$ for any $\omega\in\Omega$ and $t\geq 0$.*

We will use semiequilibrium *to refer to either a super- or a subequilibrium.*

Definition 5.2. *A superequilibrium (resp., subequilibrium) $a:\Omega\to BU$ is* semicontinuous *if the following properties hold:*

(1) *$\Gamma_a = \mathrm{cls}_X\{a(\omega)\mid \omega\in\Omega\}$ is a compact subset of X for the compact-open topology.*
(2) *$C_a = \{(\omega,x)\mid x\leq_A a(\omega)\}$ (resp., $C_a = \{(\omega,x)\mid x\geq_A a(\omega)\}$) is a closed subset of $\Omega\times X$ for the product metric topology.*

An equilibrium is semicontinuous *in any of these cases.*

As shown in Proposition 4.8 of [23] a semicontinuous equilibrium does always have a residual subset of continuity points. This theory requires topological properties of semicontinuous maps stated in Aubin and Frankowska [3] and Choquet [4]. The next result shows that a semicontinuous semiequilibrium provides an almost-automorphic extension of the base if a relatively compact trajectory exists. We omit its proof, analogous to the one of Proposition 4.9 of [23], once Proposition 4.4 is proved.

Proposition 5.3. *Let $a:\Omega\to BU$ be a semicontinuous semiequilibrium and assume that there is an $\omega_0\in\Omega$ such that $\mathrm{cls}_X\{u(t,\omega_0,a(\omega_0))\mid t\geq 0\}$ is a compact subset of X for the compact-open topology. Then the following hold:*

(i) *The omega-limit set $\mathcal{O}(\omega_0,a(\omega_0))$ contains a unique minimal set, which is an almost-automorphic extension of the base flow.*
(ii) *If the orbit $\{\tau(t,\omega_0,a(\omega_0))\mid t\geq 0\}$ is uniformly stable for $\leq_A$ in bounded sets, then $\mathcal{O}(\omega_0,a(\omega_0))$ is a copy of the base.*

If the semicontinuous semiequilibrium satisfies some supplementary and somehow natural compactness conditions, a semicontinuous equilibrium is obtained. As shown in Proposition 4.10 of [23] for infinite delay, provided that $\Gamma_a\subset BU$ and $\sup_{\omega\in\Omega}\|a(\omega)\|_\infty<\infty$, it can be proved that the following conditions are equivalent:

(C1) $\Gamma = \mathrm{cls}_X\{u(t,\omega,a(\omega))\mid t\geq 0,\,\omega\in\Omega\}$ is a compact subset of BU for the compact-open topology.
(C2) For each $\omega\in\Omega$, the $\mathrm{cls}_X\{u(t,\omega,a(\omega))\mid t\geq 0\}$ is a compact subset of BU for the compact-open topology.
(C3) There is an $\omega_0\in\Omega$ such that the $\mathrm{cls}_X\{u(t,\omega_0,a(\omega_0))\mid t\geq 0\}$ is a compact subset of BU for the compact-open topology.

Consequently, an easy adaptation of the proof of Theorem 4.11 of [23] proves the following result.

THEOREM 5.4. *Let us assume the existence of a semicontinuous semiequilibrium* $a : \Omega \to BU$ *satisfying* $\sup_{\omega\in\Omega} \|a(\omega)\|_\infty < \infty$, $\Gamma_a \subset BU$, *and one of the equivalent conditions* (C1)–(C3). *Then the following hold:*

(i) *There exists a semicontinuous equilibrium* $c : \Omega \to BU$ *with* $c(\omega) \in \Gamma$ *for any* $\omega \in \Omega$.

(ii) *Let* ω_1 *be a continuity point for* c. *Then, the restriction of the semiflow* τ *to the minimal set*

$$K^* = \mathrm{cls}_{\Omega\times X}\{(\omega_1{\cdot}t, c(\omega_1{\cdot}t)) \mid t \geq 0\} \subset C_a$$

is an almost-automorphic extension of the base flow $(\Omega, \sigma, \mathbb{R})$.

(iii) K^* *is the only minimal set contained in the omega-limit set* $\mathcal{O}(\widehat{\omega}, a(\widehat{\omega}))$ *for each point* $\widehat{\omega} \in \Omega$.

(iv) *If there is a point* $\widetilde{\omega} \in \Omega$ *such that the trajectory* $\{\tau(t, \widetilde{\omega}, a(\widetilde{\omega})) \mid t \geq 0\}$ *is uniformly stable for* $\leq_A$ *in bounded sets, then for each* $\widehat{\omega} \in \Omega$,

$$\mathcal{O}(\widehat{\omega}, a(\widehat{\omega})) = K^* = \{(\omega, c(\omega)) \mid \omega \in \Omega\};$$

i.e., it is a copy of the base determined by the equilibrium c *of* (i), *which is a continuous map.*

For the rest of the section we will assume the following stability assumption for the trajectories with initial data in a ball:

(F5) There is a $k_0 > 0$ such that all the trajectories with Lipschitz initial data in $B_{\widehat{k}_0}$ are relatively compact for the product metric topology and uniformly stable for $\leq_A$ in bounded sets, where

$$\widehat{k}_0 = \frac{\|\widehat{D}^{-1}\| \, \sup\{\|F(\omega, x)\| \mid (\omega, x) \in \Omega \times \widehat{D}B_{k_0}\}}{\|A\|} + k_0\,. \tag{5.1}$$

The following result provides a continuous superequilibrium for every compact, positively invariant set contained in $\Omega\times B_{k_0}$. It requires basic properties of the exponential ordering obtained in section 4.

THEOREM 5.5. *Let* $(\omega_0, x_0) \in \Omega \times BU$ *with forward orbit* $\{\tau(t, \omega_0, x_0) \mid t \geq 0\}$ *relatively compact for the product metric topology, uniformly stable for* $\leq_A$ *in bounded sets and* $\mathrm{cls}_X\{u(t, \omega_0, x_0) \mid t \geq 0\} \subset B_{k_0}$. *Let* $K = \mathcal{O}(\omega_0, x_0) \subset \Omega \times B_{k_0}$ *be its omega-limit set. For each* $\omega \in \Omega$ *we define the map* $a(\omega)$ *on* $(-\infty, 0]$ *by*

$$\textit{Case I:}\quad a(\omega)(s) = \begin{cases} i(\omega)(s) = \inf\{x(s) \mid (\omega, x) \in K\}\,, & s \leq -r\,,\\ e^{A\,(s+r)} i(\omega)(-r) + \int_{-r}^{s} e^{A(s-\tau)}\, h(\omega)(\tau)\, d\tau\,, & -r \leq s\,,\end{cases}$$

$$\textit{Cases II–III:}\quad a(\omega)(s) = \int_{-\infty}^{s} e^{A(s-\tau)}\, h(\omega)(\tau)\, d\tau\,, \quad s \leq 0\,,$$

$$\begin{array}{llll} \textit{where} & h(\omega)\colon (-\infty, 0] & \longrightarrow & \mathbb{R}^m,\\ & \tau & \mapsto & \inf\{x'(\tau) - A\,x(\tau) \mid (\omega, x) \in K\}\,. \end{array}$$

Then, $a(\omega)$ *is Lipschitz for every* $\omega \in \Omega$, $\sup_{\omega\in\Omega} \|a(\omega)\|_\infty \leq \widehat{k}_0$, *and the map* $a\colon \Omega \to BU$, $\omega \mapsto a(\omega)$ *is well defined, it is a continuous superequilibrium, and it satisfies the equivalent conditions* (C1)–(C3).

Proof. Since K is a positively invariant compact subset admitting a flow extension, from Proposition 3.2 we deduce that $x \in C^1((-\infty,0],\mathbb{R}^m)$ for each $(\omega,x) \in K$. We claim that there is a positive constant $L>0$ such that x is a Lipschitz function with constant L for each $(\omega,x)\in K$. In Cases I and II this assertion follows from

$$z'(t,\omega_0,x_0) = F(\omega_0{\cdot}t, u(t,\omega_0,x_0))\,, \quad t\geq 0\,,$$

and (F2) with $L=\sup\{\|F(\omega,x)\| \mid (\omega,x)\in\Omega\times B_{k_0}\}$. In Case III, from relation (3.8)

$$u(t,\omega_0,x_0) = \widehat{D}^{-1}\widehat{u}(t,\omega_0,\widehat{D}x_0)\,,$$

and we deduce the result with $L = \|\widehat{D}^{-1}\|\,\sup\{\|F(\omega,x)\| \mid (\omega,x)\in\Omega\times\widehat{D}B_{k_0}\}$. From this, as in [23] we deduce that $i(\omega)$ is also Lipschitz on $(-\infty,-r]$ with the same constant.

Since $\|x\|_\infty \leq k_0$ and $\|x'\|_\infty\leq L$ for each $x\in K_\omega$, the function $h(\omega)$ is well defined for each $\omega\in\Omega$. Moreover, since the family $\{x'(\tau)-A\,x(\tau) \mid (\omega,x)\in K\}$ is equicontinuous on $[-k,0]$, we can check that $h(\omega)\in X=C((-\infty,0],\mathbb{R}^m)$ and $\sup_{\omega\in\Omega}\|h(\omega)\|_\infty \leq L+\|A\|\,k_0$.

Then, $a(\omega)$ is well defined for each $\omega\in\Omega$, $a(\omega)\in X$, and $\sup_{\omega\in\Omega}\|a(\omega)\|_\infty<\widehat{k}_0$, for $\widehat{k}_0=k_0+L/\|A\|$, as defined in (5.1). In addition, $a(\omega)$ belongs to $C^1([-r,0],\mathbb{R}^m)$ in Case I, and to $C^1((-\infty,0],\mathbb{R}^m)$ in Cases II–III, with

$$a(\omega)'(s) = A\,a(\omega)(s)+h(\omega)(s)\,, \qquad \begin{array}{rl} -r\leq s\leq 0\,, & \text{Case I}\,.\\ -\infty< s\leq 0\,, & \text{Cases II–III}\,.\end{array}$$

From this fact, the uniform boundedness of $a(\omega)$ and $h(\omega)$, and the uniform Lipschitz character of $i(\omega)$ on $(-\infty,-r]$, we deduce that there is a positive constant $\widehat{L}>0$ such that $a(\omega)$ is a Lipschitz function with constant $\widehat{L}$ for each $\omega\in\Omega$. Hence, $a(\omega)$ belongs to BU; i.e., a is well defined. Moreover, $\Gamma_a=\mathrm{cls}_X\{a(\omega)\mid\omega\in\Omega\}$ is a compact subset of X, and actually $\Gamma_a\subset BU$.

Let us check that a defines a superequilibrium. Since $a(\omega)\in B_{\widehat{k}_0}$, it follows from hypothesis (F5) that $u(t,\omega,a(\omega))$ exists for any $\omega\in\Omega$ and $t\geq 0$. It is easy to prove, as in Proposition 4.4, that $a(\omega)\leq_A x$ for each $x\in K_\omega$. Next, we claim that if $z\in BU$ with $z\leq_A x$ for each $x\in K_\omega$, then $z\leq_A a(\omega)$, provided that z is Lipschitz on $[-r,0]$ in Case I and on $(-\infty,0]$ in Cases II–III.

From the definition of $a(\omega)$ it is easy to check that $z(s)\leq a(\omega)(s)$ for each $s\leq 0$. Moreover, since $x\in C^1((-\infty,0],\mathbb{R}^m)$, $z\leq_A x$, and z is Lipschitz, we deduce from Remark 3.4 that

$$z'(s)-A\,z(s)\leq x'(s)-A\,x(s)$$

for almost every $s\in[-r,0]$ in Case I, almost every $s\in(-\infty,0]$ in Cases II–III, and every $x\in K_\omega$. Hence, the definition of $h(\omega)$ provides at these points

$$z'(s)-A\,z(s)\leq h(\omega)(s) = a(\omega)'(s)-A\,a(\omega)(s)\,,$$

and we conclude that $z\leq_A a(\omega)$, as claimed.

Fix $\omega\in\Omega$, $t\geq 0$ and consider any $y\in K_{\omega\cdot t}$, i.e., $(\omega{\cdot}t,y)\in K$. As we have a flow on K, $\tau(-t,\omega{\cdot}t,y)=(\omega,u(-t,\omega{\cdot}t,y))\in K$ and, therefore, $a(\omega)\leq_A u(-t,\omega{\cdot}t,y)$. Applying monotonicity, $u(t,\omega,a(\omega))\leq_A y$. As this happens for any $y\in K_{\omega\cdot t}$, and

$u(t,\omega,a(\omega))$ is Lipschitz on $(-\infty,0]$, we get that $u(t,\omega,a(\omega))\leq_A a(\omega{\cdot}t)$ and a is a superequilibrium, as stated.

Now let us prove that a is continuous on Ω. From the definition of a it is enough to check the continuity of h on Ω, as well as the continuity of i in Case I. The continuity of i is shown in Proposition 5.2 of [23], so let us prove that h is continuous on Ω. Fix $\omega\in\Omega$ and assume that $\omega_n\to\omega$ and $h(\omega_n)\xrightarrow{\mathsf{d}} y$ as $n\uparrow\infty$. First, we check that $h(\omega)\leq y$. Fix $s\in(-\infty,0]$ and $i\in\{1,\dots,m\}$. From the definition of h there are $(\omega_n,x_n)\in K$, depending on s and i, although dropped from the notation, such that

$$\big|h_i(\omega_n)(s)-(x'_{n,i}(s)-(A\,x_n)_i(s))\big|<\frac{1}{n}\,,$$

where $x_{n,i}$ and $(A\,x_n)_i$ denote the corresponding components of x_n and $A\,x_n$. This implies that $\lim_{n\to\infty}(x'_{n,i}(s)-(A\,x_n)_i(s))=y_i(s)$ for $i\in\{1,\dots,m\}$, that is,

$$\lim_{n\to\infty}x'_n(s)-A\,x_n(s)=y(s)\,.$$

Moreover, from the compactness of K, an adequate subsequence (ω_{n_j},x_{n_j}) tends to some $(\omega,x)\in K$ in the product metric topology. From this, it can be shown that $\lim_{n\to\infty}x'_{n_j}(s)=x'(s)$; hence $y(s)=x'(s)-A\,x(s)$, and again, from the definition of $h(\omega)$ we conclude that $h(\omega)(s)\leq y(s)$. As this happens for each $s\in(-\infty,0]$, we deduce that $h(\omega)\leq y$. On the other hand, from Proposition 4.4 we know that $(K,\tau,\mathbb{R}^+)$ is uniformly stable, and then Theorem 3.4 of [23] asserts that the section map for K, $\omega\in\Omega\mapsto K_\omega$ is continuous at every $\omega\in\Omega$, which implies that $K_{\omega_n}\to K_\omega$ in the Hausdorff metric. Therefore, for any $z\in K_\omega$ there exist $z_n\in K_{\omega_n}$, $n\geq 1$, such that $z_n\xrightarrow{\mathsf{d}} z$. Then, $(\omega_n,z_n)\in K$ implies that $h(\omega_n)(s)\leq z'_n(s)-A\,z_n(s)$ and, taking limits, $y(s)\leq z'(s)-A\,z(s)$ for each $s\in(-\infty,0]$). As this happens for any $z\in K_\omega$, we conclude that $y\leq h(\omega)$. In all, $h(\omega)=y$, as wanted. Consequently, $\Gamma_a=\{a(\omega)\,|\,\omega\in\Omega\}$.

Finally, since $a(\omega)\in B_{\widehat{k}_0}$ for each $\omega\in\Omega$, from hypothesis (F5), Proposition 4.1 of [23], and Proposition 4.2 of [21], we deduce that the equivalent conditions (C1)–(C3) hold, and the proof is complete. Notice that $a(\omega)$ is the infimum among the Lipschitz functions of the set K_ω. ∎

For the main theorem of the paper, in which the 1-covering property of omega-limit sets is established, we add the following hypothesis:

(F6) If (ω,x), $(\omega,y)\in\Omega\times BU$ admit a backward orbit extension, $x\leq_A y$, and there is a subset $J\subset\{1,\dots,m\}$ such that

$$x_i=y_i\quad\text{for each } i\notin J\,,$$
$$x_i(s)<y_i(s)\quad\text{for each } i\in J\ \text{ and } s\leq 0,$$

then $F_i(\omega,y)-F_i(\omega,x)-(A\,(Dy-Dx))_i>0$ for each $i\in J$ and $\omega\in\Omega$,

for all the cases, and the following ones for Case III:

(D6) The measure $\widehat{\mu}$ of (3.3) is positive; i.e., $\widehat{D}^{-1}$ is a positive operator.

(D7) $AD\,x=DA\,x$ for each $x\in BU$.

THEOREM 5.6. *We consider the monotone skew-product semiflows* (3.6) *induced by* $(3.5)_\omega$ *in the following cases:*

Case I: *FDEs* $(3.4)_\omega$ *under* (F1)–(F6) *and order* $\leq_{A,r}$.

Case II: *FDEs* $(3.4)_\omega$ *under* (F1)–(F6), (A1), *and order* $\leq_{A,\infty}$.

Case III: *NFDEs* $(3.5)_\omega$ *under* (F1)–(F6), (A1), (D1)–(D7), *and order* $\leq_{A,\infty}$.

D is of the form (3.1), *and $\nu \equiv 0$ in* Cases I–II. *Let $(\omega_0, x_0) \in \Omega \times B_{k_0}$ with forward orbit $\{\tau(t, \omega_0, x_0) \mid t \geq 0\}$ relatively compact for the product metric topology, and uniformly stable for $\leq_A$ in bounded sets, be such that its omega-limit set $K = \mathcal{O}(\omega_0, x_0) \subset \Omega \times B_{k_0}$. In addition, in* Cases II *and* III *assume that x_0 is Lipschitz.*

Then $K = \mathcal{O}(\omega_0, x_0) = \{(\omega, c(\omega)) \mid \omega \in \Omega\}$ is a copy of the base and

$$\lim_{t\to\infty} \mathsf{d}(u(t, \omega_0, x_0), c(\omega_0{\cdot}t)) = 0\,,$$

where $c : \Omega \to BU$ is a continuous equilibrium.

Proof. We apply Theorem 5.5 to obtain a continuous superequilibrium a satisfying (C1)–(C3) with $a(\omega)$ Lipschitz for each $\omega \in \Omega$. Then, from (F5) and Theorem 5.4 we deduce that there is a continuous equilibrium $c : \Omega \to BU$ such that for each $\widehat{\omega} \in \Omega$,

$$\mathcal{O}(\widehat{\omega}, a(\widehat{\omega})) = K^* = \{(\omega, c(\omega)) \mid \omega \in \Omega\}\,. \tag{5.2}$$

The definition of a yields $a(\omega) \leq_A x$ for each $(\omega, x) \in K$ and hence $c(\omega) \leq_A x$ by the construction of c. As in Jiang and Zhao [17] and Novo, Obaya, and Sanz [23] we prove that there is a subset $J \subset \{1, \dots, m\}$ such that

$$c_i(\omega) = x_i \quad \text{for each } (\omega, x) \in K \text{ and } i \notin J\,,$$

$$c_i(\omega) < x_i \quad \text{for each } (\omega, x) \in K \text{ and } i \in J\,.$$

It is enough to check that if $c_i(\widetilde{\omega})(0) = \widetilde{x}_i(0)$ for some $i \in \{1, \dots, m\}$ and $(\widetilde{\omega}, \widetilde{x}) \in K$, then $c_i(\omega) = x_i$ for any $(\omega, x) \in K$. We first notice that $c_i(\widetilde{\omega}) = \widetilde{x}_i$. Otherwise, there would be an $s_0 \in (-\infty, 0]$ with $c_i(\widetilde{\omega})(s_0) < \widetilde{x}_i(s_0)$.

Cases II–III: from $c(\widetilde{\omega}) \leq_A \widetilde{x}$ we know that

$$\widetilde{x}(0) - c(\widetilde{\omega})(0) \geq e^{-As_0}(x(s_0) - c(\widetilde{\omega})(s_0)),$$

which implies from Remark 3.3 that $c_i(\widetilde{\omega})(0) < \widetilde{x}_i(0)$, a contradiction.

Case I: in this case $\leq_A$ denotes $\leq_{A,r}$. Since K admits a flow extension, for each $s \leq 0$ we know that $(\widetilde{\omega}{\cdot}s, u(s, \widetilde{\omega}, \widetilde{x})) = (\widetilde{\omega}{\cdot}s, \widetilde{x}_s) \in K$. Hence, $c(\widetilde{\omega}{\cdot}s) \leq_A \widetilde{x}_s$ and

$$\widetilde{x}_s(t) - c(\widetilde{\omega}{\cdot}s)(t) \geq e^{A(t-\theta)}(\widetilde{x}_s(\theta) - c(\widetilde{\omega}{\cdot}s)(\theta))\,, \quad -r \leq \theta \leq t \leq 0\,,\ s \leq 0\,.$$

Therefore, as above, from $c_i(\widetilde{\omega})(s_0) < \widetilde{x}_i(s_0)$ and Remark 3.3, in a finite number of steps, we deduce that $c_i(\widetilde{\omega})(0) < \widetilde{x}_i(0)$, a contradiction.

Therefore, $c_i(\widetilde{\omega}) = \widetilde{x}_i$. Next, from Theorem 4.5 we know that K is minimal. Thus we take $(\omega, x) \in K$ and a sequence $s_n \downarrow -\infty$ such that $\widetilde{\omega}{\cdot}s_n \to \omega$ and $u(s_n, \widetilde{\omega}, \widetilde{x}) \xrightarrow{\mathsf{d}} x$. Then,

$$\begin{aligned} x_i(0) &= \lim_{n\to\infty} u_i(s_n, \widetilde{\omega}, \widetilde{x})(0) = \lim_{n\to\infty} \widetilde{x}_i(s_n) \\ &= \lim_{n\to\infty} c_i(\widetilde{\omega})(s_n) = \lim_{n\to\infty} c_i(\widetilde{\omega}{\cdot}s_n)(0) = c_i(\omega)(0)\,, \end{aligned}$$

and as before this implies that $c_i(\omega) = x_i$, as wanted.

Let $(\omega, x) \in K$ and define $x_\alpha = (1-\alpha)\, a(\omega) + \alpha\, x \in B_{\widehat{k}_0} \subset BU$ for $\alpha \in [0, 1]$, and

$$L = \{\alpha \in [0, 1] \mid \mathcal{O}(\omega, x_\alpha) = K^*\}\,.$$

If we prove that $L = [0,1]$, then $K = K^*$, and the proof is finished. From the monotone character of the semiflow and since $\mathcal{O}(\omega, a(\omega)) = K^*$, it is immediate to check that if $0 < \alpha \in L$, then $[0,\alpha] \subset L$.

Next, we show that L is closed; that is, if $[0,\alpha) \subset L$, then $\alpha \in L$. Since $\{\tau(t,\omega,x_\alpha) \mid t \geq 0\}$ is uniformly stable for $\leq_A$ in bounded sets, let $\delta(\varepsilon) > 0$ be the modulus of uniform stability for $\varepsilon > 0$ in $B_{\widehat{k}_0}$. Thus, we take $\beta \in [0,\alpha)$ with $\mathsf{d}(x_\alpha, x_\beta) < \delta(\varepsilon)$ and obtain $\mathsf{d}(u(t,\omega,x_\alpha), u(t,\omega,x_\beta)) < \varepsilon$ for each $t \geq 0$. Moreover, $\mathcal{O}(\omega, x_\beta) = K^*$, and hence there is a t_0 such that $\mathsf{d}(u(t,\omega,x_\beta), c(\omega{\cdot}t)) < \varepsilon$ for each $t \geq t_0$. Then, we deduce that $\mathsf{d}(u(t,\omega,x_\alpha), c(\omega{\cdot}t)) < 2\,\varepsilon$ for each $t \geq t_0$ and $\mathcal{O}(\omega, x_\alpha) = K^*$, i.e., $\alpha \in L$, as claimed.

Finally, we prove that the case $L = [0,\alpha]$ with $0 \leq \alpha < 1$ is impossible. For each $i \in J$ we consider the continuous maps

$$K \longrightarrow (0,\infty)\,, \quad (\widetilde{\omega}, \widetilde{x}) \mapsto \widetilde{x}_i(0) - c_i(\widetilde{\omega})(0)\,,$$

$$K \longrightarrow (0,\infty)\,, \quad (\widetilde{\omega}, \widetilde{x}) \mapsto F_i(\widetilde{\omega}, \widetilde{x}) - F_i(\widetilde{\omega}, c(\widetilde{\omega})) - (A(D\widetilde{x} - Dc(\widetilde{\omega})))_i\,.$$

As explained above, $\widetilde{x}_i(0) - c_i(\widetilde{\omega})(0) > 0$. Moreover, from $c(\widetilde{\omega}) \leq_A \widetilde{x}$ and (F6) we deduce that $F_i(\widetilde{\omega}, \widetilde{x}) - F_i(\widetilde{\omega}, c(\widetilde{\omega})) - (A(D\widetilde{x} - Dc(\widetilde{\omega})))_i > 0$. Hence, there is an $\varepsilon > 0$ such that $\widetilde{x}_i(0) - c_i(\widetilde{\omega})(0) \geq \varepsilon$ and $F_i(\widetilde{\omega}, \widetilde{x}) - F_i(\widetilde{\omega}, c(\widetilde{\omega})) - (A(D\widetilde{x} - Dc(\widetilde{\omega})))_i > \varepsilon$ for each $(\widetilde{\omega}, \widetilde{x}) \in K$. Besides, since $(\widetilde{\omega}{\cdot}s, u(s,\widetilde{\omega},\widetilde{x})) \in K$, $u(s,\widetilde{\omega},\widetilde{x})(0) = \widetilde{x}(s)$ for each $s \leq 0$ because K admits a flow extension, and $c(\widetilde{\omega})(s) = c(\widetilde{\omega}{\cdot}s)(0)$, we deduce that $\widetilde{x}_i(s) - c_i(\widetilde{\omega})(s) \geq \varepsilon$ for each $s \in (-\infty, 0]$ and $(\widetilde{\omega}, \widetilde{x}) \in K$.

It is not hard to check that $\cup_{\beta\in[0,1]} \operatorname{cls}_{\Omega\times X}\{\tau(t,\omega,x_\beta) \mid t \geq 0\}$ is a compact set. Hence, since $\{\tau(t,\omega,x_\alpha) \mid t \geq 0\}$ is uniformly stable for $\leq_A$ in $B_{\widehat{k}_0}$, $(\omega, x_\beta) \in B_{\widehat{k}_0}$ for each $\beta \in [0,1]$, and hypotheses (D2) and (F3) hold, we deduce that there is a $\delta > 0$ such that $\|u(t,\omega,x_\gamma)(0) - u(t,\omega,x_\alpha)(0)\| < \varepsilon/4$ and

$$\|F(\omega{\cdot}t, u(t,\omega,x_\gamma)) - F(\omega{\cdot}t, u(t,\omega,x_\alpha)) - A(Du(t,\omega,x_\gamma) - Du(t,\omega,x_\alpha))\| < \frac{\varepsilon}{4}$$

for each $t \geq 0$ and $\gamma \in (\alpha, 1]$ with $\mathsf{d}(x_\alpha, x_\gamma) < \delta$. Besides, $\alpha \in L$; i.e., $\mathcal{O}(\omega, x_\alpha) = K^*$ and there is a $t_0 \geq 0$ such that $\|u(t,\omega,x_\alpha)(0) - c(\omega{\cdot}t)(0)\| < \varepsilon/4$ and

$$\|F(\omega{\cdot}t, u(t,\omega,x_\alpha)) - F(\omega{\cdot}t, c(\omega{\cdot}t)) - A(Du(t,\omega,x_\alpha) - Dc(\omega{\cdot}t))\| < \frac{\varepsilon}{4}$$

for each $t \geq t_0$. Consequently, for each $t \geq t_0$

$$\begin{aligned} &\|u(t,\omega,x_\gamma)(0) - c(\omega{\cdot}t)(0)\| < \frac{\varepsilon}{2}\,, \\ &\|F(\omega{\cdot}t, u(t,\omega,x_\gamma)) - F(\omega{\cdot}t, c(\omega{\cdot}t)) - A(Du(t,\omega,x_\gamma) - Dc(\omega{\cdot}t))\| < \frac{\varepsilon}{2}\,. \end{aligned} \tag{5.3}$$

Let $(\widetilde{\omega}, \widetilde{x}) \in \mathcal{O}(\omega, x_\gamma)$; i.e., $(\widetilde{\omega}, \widetilde{x}) = \lim_{n\to\infty}(\omega{\cdot}t_n, u(t_n,\omega,x_\gamma))$ for some $t_n \uparrow \infty$. The monotonicity and $c(\omega) \leq_A x_\gamma$ imply that $c(\omega{\cdot}t_n) \leq_A u(t_n,\omega,x_\gamma)$, which yields $c(\widetilde{\omega}) \leq_A \widetilde{x}$. From (5.3) there is an n_0 such that for each $i \in \{1,\ldots,m\}$

$$0 \leq u_i(t_n,\omega,x_\gamma)(0) - c_i(\omega{\cdot}t_n)(0) < \varepsilon/2\,,$$

$$F_i(\omega{\cdot}t_n, u(t_n,\omega,x_\gamma)) - F_i(\omega{\cdot}t_n, c(\omega{\cdot}t_n)) - (A(Du(t_n,\omega,x_\alpha) - Dc(\omega{\cdot}t_n)))_i < \varepsilon/2$$

for each $n \geq n_0$. Hence, $0 \leq \widetilde{x}_i(0) - c_i(\widetilde{\omega})(0) \leq \varepsilon/2$ and

$$0 \leq F_i(\widetilde{\omega}, \widetilde{x}) - F_i(\widetilde{\omega}, c(\widetilde{\omega})) - (A(D\widetilde{x} - Dc(\widetilde{\omega})))_i \leq \frac{\varepsilon}{2}\,.$$

As before, since this is true for each $(\widetilde{\omega},\widetilde{x})\in\mathcal{O}(\omega,x_\gamma)$, admitting a flow extension, and $(\widetilde{\omega}{\cdot}s,\widetilde{x}_s)\in\mathcal{O}(\omega,x_\gamma)$, we deduce that

$$\begin{aligned} &0\le \widetilde{x}_i(s)-c_i(\widetilde{\omega})(s)\le\frac{\varepsilon}{2}\,,\\ &0\le F_i(\widetilde{\omega}{\cdot}s,\widetilde{x}_s)-F_i(\widetilde{\omega}{\cdot}s,c(\widetilde{\omega}{\cdot}s))-(A(D\widetilde{x}_s-Dc(\widetilde{\omega}{\cdot}s)))_i\le\frac{\varepsilon}{2} \end{aligned}\tag{5.4}$$

for each $s\in(-\infty,0]$ and $i\in\{1,\dots,m\}$. Given any $(\widetilde{\omega},z)\in K$, as shown above,

$$\begin{aligned} &z_i(s)-c_i(\widetilde{\omega})(s)\ge\varepsilon\,,\\ &F_i(\widetilde{\omega}{\cdot}s,z_s)-F_i(\widetilde{\omega}{\cdot}s,c(\widetilde{\omega}{\cdot}s))-(A(Dz_s-Dc(\widetilde{\omega}{\cdot}s)))_i\ge\varepsilon \end{aligned}$$

for each $s\in(-\infty,0]$ and $i\in J$, which combined with (5.4) yields

$$\widetilde{x}_i(s)\le z_i(s)\quad\text{and}\quad F_i(\widetilde{\omega}{\cdot}s,z_s)-F_i(\widetilde{\omega}{\cdot}s,\widetilde{x}_s)-(A(Dz_s-D\widetilde{x}_s))_i\ge 0\tag{5.5}$$

for each $s\in(-\infty,0]$ and $i\in J$. In addition, we deduce from Proposition 3.2 that $c(\omega)$, z, and $\widetilde{x}\in C^1(\mathbb{R},\mathbb{R}^m)$.

Next, we claim that if $i\notin J$, the equality holds in (5.5). We know that $c_i(\widetilde{\omega})=z_i$,

$$F_i(\widetilde{\omega},z)-F_i(\widetilde{\omega},c(\widetilde{\omega}))\ge(A(Dz-Dc(\widetilde{\omega})))_i\,,$$

and $(A(z-c(\widetilde{\omega})))_i=0$ because $0=z_i'-c_i(\widetilde{\omega})'\ge(A(z-c(\widetilde{\omega})))_i\ge 0$. Hence, from (D7) and (3.1) we deduce that

$$(A(Dz-Dc(\widetilde{\omega})))_i=-\int_{-\infty}^0([d\nu(s)]\,A(z(s)-c(\widetilde{\omega})(s)))_i\,.\tag{5.6}$$

Moreover, $z'-c(\widetilde{\omega})'\ge A(z-c(\widetilde{\omega}))$ and (D5) yield

$$\int_{-\infty}^0(\,[d\nu(s)](z'(s)-c(\widetilde{\omega})'(s)))_i\ge\int_{-\infty}^0(\,[d\nu(s)]\,A(z(s)-c(\widetilde{\omega})(s)))_i\,,$$

which together with (5.6) provides

$$\begin{aligned}(A(Dz-Dc(\widetilde{\omega})))_i&\ge-\int_{-\infty}^0(\,[d\nu(s)](z'(s)-c(\widetilde{\omega})'(s)))_i=\frac{d}{ds}\,(D(z-c(\widetilde{\omega})))_i\\ &=F_i(\widetilde{\omega},z)-F_i(\widetilde{\omega},c(\widetilde{\omega}))\ge(A(Dz-Dc(\widetilde{\omega})))_i\,,\end{aligned}$$

that is,

$$F_i(\widetilde{\omega},z)-F_i(\widetilde{\omega},c(\widetilde{\omega}))=(A(Dz-Dc(\widetilde{\omega})))_i\,.$$

From $c(\widetilde{\omega})\le_A\widetilde{x}$ and $c_i(\widetilde{\omega})=\widetilde{x}_i$ we also deduce that

$$F_i(\widetilde{\omega},\widetilde{x})-F_i(\widetilde{\omega},c(\widetilde{\omega}))=(A(D\widetilde{x}-Dc(\widetilde{\omega})))_i\,,$$

to conclude that $F_i(\widetilde{\omega},z)-F_i(\widetilde{\omega},\widetilde{x})=(A(Dz-D\widetilde{x}))_i$. Hence, if $i\notin J$ and $s\le 0$,

$$\widetilde{x}_i(s)=z_i(s)\quad\text{and}\quad F_i(\widetilde{\omega}{\cdot}s,z_s)-F_i(\widetilde{\omega}{\cdot}s,\widetilde{x}_s)=(A(Dz_s-D\widetilde{x}_s))_i\,,$$

as stated, which together with (5.5) provide $\widetilde{x} \le z$ and

$$\frac{d}{ds} Dz_s - \frac{d}{ds} D\widetilde{x}_s - A(Dz_s - D\widetilde{x}_s) \ge 0\,, \quad s \le 0;$$

that is, $\widehat{D}(z' - Az - (\widetilde{x}' - A\widetilde{x})) \ge 0$ from (D7). Finally, since $\widehat{D}^{-1}$ is obviously positive in Cases I–II, and in Case III because of (D6), we conclude that $\widetilde{x}' - Ax \le z' - Az$, that is, $\widetilde{x} \le_A z$.

Since this holds for each $(\widetilde{\omega}, z) \in K$, the definition of a yields $c(\widetilde{\omega}) \le_A \widetilde{x} \le_A a(\widetilde{\omega})$. From (5.2) we know that $\mathcal{O}(\widetilde{\omega}, a(\widetilde{\omega})) = K^*$ and therefore $\mathcal{O}(\widetilde{\omega}, \widetilde{x}) = K^* \subseteq \mathcal{O}(\omega, x_\gamma)$. Finally, from (F5) and Theorem 4.5 we know that $\mathcal{O}(\omega, x_\gamma)$ is a minimal set, and we conclude that $\mathcal{O}(\widetilde{\omega}, \widetilde{x}) = \mathcal{O}(\omega, x_\gamma) = K^*$, that is, $\gamma \in L$, a contradiction. Therefore, $L = [0, 1]$ and $\mathcal{O}(\omega_0, x_0) = K^*$, as stated. ∎

6. Applications to compartmental systems. In this section we apply the previous results to the study of compartmental models for the mathematical description of processes in which the transport of material among compartments takes a nonnegligible length of time, and each compartment produces or swallows material.

Let us suppose that we have a system formed by m compartments $C_1, \dots, C_m$, and denote by $z_i(t)$ the amount of material within compartment C_i at time t for each $i \in \{1, \dots, m\}$. Material flows from compartment C_j into compartment C_i through a pipe having a transit time distribution given by a positive regular Borel measure μ_{ij} with finite total variation $\mu_{ij}(-\infty, 0] = 1$, for each i, $j \in \{1, \dots, m\}$, whereas the outcome of material from C_i to C_j is assumed to be instantaneous. Let $\widetilde{g}_{ij} : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be the so-called *transport function* determining the volume of material flowing from C_j to C_i given in terms of the time t and the value of $z_j(t)$ for $i, j \in \{1, \dots, m\}$. For each $i \in \{1, \dots, m\}$, at time $t \ge 0$, the compartment C_i itself produces material at a rate $\int_{-\infty}^0 z_i'(t+s)\, d\nu_i(s)$, where ν_i is a positive regular Borel measure with finite total variation $\nu_i(-\infty, 0] < \infty$ and $\nu_i(\{0\}) = 0$. We will assume that the system is closed; that is, there is not inflow or outflow of material from or to the environment surrounding the system.

Once the destruction and creation of material is taken into account, the change of the amount of material of any compartment C_i, $1 \le i \le m$, equals the difference between the amount of total influx into and total outflux out of C_i, and we obtain a model governed by the following system of NFDEs:

$$\begin{aligned}(6.1)\quad \frac{d}{dt}\left[z_i(t) - \int_{-\infty}^0 z_i(t+s)\, d\nu_i(s) \right] &= -\sum_{j=1}^m \widetilde{g}_{ji}(t, z_i(t)) \\ &\quad + \sum_{j=1}^m \int_{-\infty}^0 \widetilde{g}_{ij}(t+s, z_j(t+s))\, d\mu_{ij}(s)\,,\end{aligned}$$

$i = 1, \dots, m$, where $\widetilde{g} = (\widetilde{g}_{ij})_{i,j} : \mathbb{R} \times \mathbb{R} \to \mathbb{R}^{m \times m}$, and we will assume that

- (H1) $\widetilde{g}$ is C^1-*admissible*; that is, $\widetilde{g}$ is C^1 in its second variable and $\widetilde{g}$, $\frac{\partial}{\partial v}\widetilde{g}$ are uniformly continuous and bounded on $\mathbb{R} \times \{v_0\}$ for all $v_0 \in \mathbb{R}$;
- (H2) $\widetilde{g}_{ij}(t, 0) = 0$ and $\widetilde{g}_{ij}(t, v)$ is increasing in v for each $t \in \mathbb{R}$ and $i, j = 1, \dots, m$;
- (H3) $\widetilde{g}$ is a recurrent function; i.e., its *hull* is minimal;
- (H4) $\nu_i(\{0\}) = 0$, $\sum_{i=1}^m \nu_i(-\infty, 0] < 1$, $\mu_{ij}(-\infty, 0] = 1$, and $\int_{-\infty}^0 |s|\, d\mu_{ij}(s) < \infty$ for $i, j = 1, \dots, m$;

(H5) for each $i = 1, \dots, m$, $L_i^+ = \sum_{j=1}^m l_{ji}^+ < +\infty$ and there is a $\beta_i > 0$ such that $\beta_i(1 - \int_{-\infty}^0 e^{-\beta_i s}\, d\nu_i(s)) > L_i^+$, where $l_{ji}^+ = \sup_{(t,\,v)\in\mathbb{R}^2} \frac{\partial \widetilde{g}_{ji}}{\partial v}(t,v)$.

As usual, we include the nonautonomous equation (6.1) in a family of nonautonomous NFDEs as follows. Let Ω be the *hull* of $\widetilde{g}$, namely, the closure of the set of mappings $\{\widetilde{g}_t \mid t \in \mathbb{R}\}$, with $\widetilde{g}_t(s,v) = \widetilde{g}(t+s,v)$, $(s,v) \in \mathbb{R}^2$, with the topology of uniform convergence on compact sets, which from (H1) is a compact metric space (see Hino, Murakami, and Naito [12]). Let $(\Omega, \sigma, \mathbb{R})$ be the continuous flow defined on Ω by translation, $\sigma : \mathbb{R} \times \Omega \to \Omega$, $(t,\omega) \mapsto \omega{\cdot}t$, with $\omega{\cdot}t(s,v) = \omega(t+s,v)$. From hypothesis (H3), the flow $(\Omega, \sigma, \mathbb{R})$ is minimal. In addition, if $\widetilde{g}$ is almost periodic (resp., almost automorphic), the flow will be almost periodic (resp., almost automorphic). Notice that these two cases are included in our formulation.

Let $g : \Omega \times \mathbb{R} \to \mathbb{R}^{m\times m}$, $(\omega, v) \mapsto \omega(0,v)$, continuous on $\Omega \times \mathbb{R}$, and denote $g = (g_{ij})_{i,j}$. Let $F : \Omega \times BU \to \mathbb{R}^m$ be the map defined by

$$F_i(\omega,x) = -\sum_{j=1}^m g_{ji}(\omega, x_i(0)) + \sum_{j=1}^m \int_{-\infty}^0 g_{ij}(\omega{\cdot}s, x_j(s))\, d\mu_{ij}(s) \tag{6.2}$$

for $(\omega, x) \in \Omega \times BU$ and $i \in \{1, \dots, m\}$. Hence, the family

$$\frac{d}{dt} Dz_t = F(\omega{\cdot}t, z_t)\,, \quad t \ge 0\,,\ \omega \in \Omega\,, \tag*{$(6.3)_\omega$}$$

is of the form $(3.5)_\omega$ with $D_i x = x_i(0) - \int_{-\infty}^0 x_i(s)\, d\nu_i(s)$, $i = 1, \dots, m$, and includes system (6.1) when $\omega = \widetilde{g}$.

In [21], Muñoz-Villarragut, Novo, and Obaya study NFDEs $(6.3)_\omega$, which are monotone for the standard ordering. This requires the positivity of the measures $d\eta_i = l_{ii}^-\, d\mu_{ii} - \sum_{k=0}^m l_{ki}^+\, d\nu_i$, where

$$l_{ii}^- = \inf_{(t,\,v)\in\mathbb{R}^2} \frac{\partial \widetilde{g}_{ii}}{\partial v}(t,v) \ \text{ and } \ l_{ij}^+ = \sup_{(t,\,v)\in\mathbb{R}^2} \frac{\partial \widetilde{g}_{ij}}{\partial v}(t,v)\,, \quad i,j = 1, \dots, m.$$

We now remove this condition, which controls the material produced in the compartments in terms of the material transported through the pipes. In order to verify the monotonicity for the exponential ordering, we assume hypothesis (H5), which imposes a strong restriction on the size of the neutral term. The next statement shows that the main conclusions obtained in [21] remain valid for $(6.3)_\omega$.

THEOREM 6.1. *Assume that* (H1)–(H5) *hold and let* $(\omega_0, x_0) \in \Omega \times BU$ *with* x_0 *Lipschitz. Then the solution* $z(t, \omega_0, x_0)$ *of* $(6.3)_{\omega_0}$ *with initial value* x_0 *is bounded and its omega-limit set is a copy of the base; that is,* $\mathcal{O}(\omega_0, x_0) = \{(\omega, c(\omega)) \mid \omega \in \Omega\}$ *for a continuous equilibrium* $c\colon \Omega \to BU$ *and*

$$\lim_{t\to\infty} \mathsf{d}(u(t, \omega_0, x_0), c(\omega_0{\cdot}t)) = 0\,.$$

Proof. It is easy to check that assumptions (D1)–(D6) and (F1)–(F3) hold, and we consider the local skew-product semiflow (3.6) induced by $(6.3)_\omega$. Next, we check the monotonicity assumptions (F4) and (F6) for the exponential ordering $\le_{A,\infty}$, where A is the quasi-positive diagonal matrix with diagonal entries $(-\beta_1, \dots, -\beta_m)$ given in (H5). Notice that (A1) and (D7) also hold. As usual, $\le_{A,\infty}$ will be denoted by $\le_A$.

Let x, $y \in BU$ with $x \le_A y$; that is, $x \le y$ and $y(t) - x(t) \ge e^{A(t-s)}(y(s) - x(s))$ if $-\infty < s \le t \le 0$. Then, from (H2) g_{ij} are increasing in their second variable, and from (6.2) we deduce that

$$F_i(\omega,y)-F_i(\omega,x)+\beta_i(D_iy-D_ix)\geq \sum_{j=1}^{m}[g_{ji}(\omega,x_i(0))-g_{ji}(\omega,y_i(0))]$$

$$+\,\beta_i\left(y_i(0)-\int_{-\infty}^{0} y_i(s)\,d\nu_i(s)-x_i(0)+\int_{-\infty}^{0} x_i(s)\,d\nu_i(s)\right)$$

for each $i=1,\dots,m$. In addition, $g_{ji}(\omega,x_i(0))-g_{ji}(\omega,y_i(0))\geq -l^+_{ji}\,(y_i(0)-x_i(0))$, and from $x\leq_A y$ we deduce that for each $s\leq 0$

$$(y_i(s)-x_i(s))\,e^{\beta_i s}\leq y_i(0)-x_i(0)\,,$$

which yields

$$F_i(\omega,y)-F_i(\omega,x)+\beta_i(D_iy-D_ix) \geq \left[-L^+_i+\beta_i\left(1-\int_{-\infty}^{0} e^{-\beta_i s}\,d\nu_i(s)\right)\right](y_i(0)-x_i(0)) \tag{6.4}$$

for each $i=1,\dots,m$. Hence from hypothesis (H5) we deduce that (F4) and (F6) hold, as claimed.

In order to check hypothesis (F5) we define $M\colon\Omega\times BU\to\mathbb{R}$, the *total mass* of the system $(6.3)_\omega$, as

$$M(\omega,x):=\sum_{i=1}^{m} D_ix+\sum_{i=1}^{m}\sum_{j=1}^{m}\int_{-\infty}^{0}\left(\int_{s}^{0} g_{ji}(\omega{\cdot}\tau,x_i(\tau))\,d\tau\right)d\mu_{ji}(s)\,,$$

which is well defined from condition (H4). It is easy to check, as shown in [21], that M is a uniformly continuous function on all the sets of the form $\Omega\times B_k$ with $k>0$ for the product metric topology. Moreover, it is constant along the trajectories because from $(6.3)_\omega$

$$\frac{d}{dt}M(\tau(t,\omega,x))=\frac{d}{dt}M(\omega{\cdot}t,z_t(\omega,x))=0;$$

that is, $M(\omega{\cdot}t,z_t(\omega,x))=M(\omega,x)$ for each $t\geq 0$, where $z(t,\omega,x)$ is defined.

Let $x,\ y\in BU$ with $x\leq_A y$. We can apply Theorem 3.5 to deduce that the induced semiflow is monotone and, hence, $u(t,\omega,x)\leq_A u(t,\omega,y)$ whenever they are defined. Thus, since the transport functions g_{ji} are monotone and the measures μ_{ji} are positive,

$$\int_{-\infty}^{0}\left(\int_{s}^{0}[g_{ji}(\omega{\cdot}(t+\tau),z_i(t+\tau,\omega,y))-g_{ji}(\omega{\cdot}(t+\tau),z_i(t+\tau,\omega,x))]\,d\tau\right)d\mu_{ji}(s)\geq 0,$$

and we deduce that

$$\sum_{i=1}^{m}(D_iz_t(\omega,y)-D_iz_t(\omega,x))\leq M(\omega{\cdot}t,z_t(\omega,y))-M(\omega{\cdot}t,z_t(\omega,x))$$

$$=M(\omega,y)-M(\omega,x)\,.$$

Next, we check that given $\varepsilon > 0$, there is a $\delta > 0$ such that if $x,\, y \in B_k$ with $x \leq_A y$ and $\mathsf{d}(x,y) < \delta$, then $\|z(t,\omega,y) - z(t,\omega,x)\| \leq \varepsilon$ whenever they are defined. From hypothesis (H4) we define $\gamma = \sum_{i=1}^m \nu_i(-\infty,0] < 1$. From the uniform continuity of M, given $\varepsilon_0 = \varepsilon\,(1-\gamma) > 0$, there exists $0 < \delta < \varepsilon_0$ such that if $x,\, y \in B_k$ with $\mathsf{d}(x,y) < \delta$, then $|M(\omega,y) - M(\omega,x)| < \varepsilon_0$. Therefore, if $x,\, y \in B_k$ and $x \leq_A y$, we have $\sum_{i=1}^m (D_i z_t(\omega,y) - D_i z_t(\omega,x)) < \varepsilon_0$. Consequently, from the definition of D_i,

$$
\begin{aligned}
0 \leq z_i(t,\omega,y) - z_i(t,\omega,x) &\leq \sum_{j=1}^m (z_j(t,\omega,y) - z_j(t,\omega,x)) \\
&< \varepsilon_0 + \sum_{j=1}^m \int_{-\infty}^0 (z_j(t+s,\omega,y) - z_j(t+s,\omega,x))\, d\nu_j(s) \\
&\leq \varepsilon_0 + \gamma\, \|z_t(\omega,y) - z_t(\omega,x)\|_\infty\,,
\end{aligned}
$$

from which we deduce that $\|z_t(\omega,y) - z_t(\omega,x)\|_\infty (1-\gamma) < \varepsilon_0 = \varepsilon(1-\gamma)$, that is, $\|z(t,\omega,y) - z(t,\omega,x)\| \leq \varepsilon$ whenever they are defined, as claimed.

From this and since 0 is a solution of $(6.3)_\omega$ for each $\omega \in \Omega$ because $g_{ij}(\omega,0) = 0$ for each $i,j = 1,\dots,m$, we deduce that each solution $z(t,\omega,x)$ with $x \geq_A 0$ or $x \leq_A 0$ is globally defined and bounded. As a consequence, and together with the monotone character of the semiflow, we conclude that all the solutions with Lipschitz initial data are globally defined and bounded because given $x \in BU$ and Lipschitz, it is easy to find $c_1 < 0$ and $c_2 > 0$ such that $c_1\,\mathbf{1} \leq_A x \leq_A c_2\,\mathbf{1}$.

Let $(\omega,x) \in \Omega \times BU$ with x Lipschitz, and $k > 0$ such that $z_t(\omega,x) \in B_k$ for all $t \geq 0$. Then, as above, we deduce that given $\varepsilon > 0$, there exists a $\delta > 0$ such that

$$
\|z(t+s,\omega,x) - z(t,\omega{\cdot}s,y)\| = \|z(t,\omega{\cdot}s,z_s(\omega,x)) - z(t,\omega{\cdot}s,y)\| < \varepsilon
$$

for all $t \geq 0$ whenever $y \in B_k$, $y \leq_A z_s(\omega,x)$, or $z_s(\omega,x) \leq_A y$ and $\mathsf{d}(z_s(\omega,x),y) < \delta$, which shows the uniform stability of the trajectories for the order $\leq_A$ in B_k for each $k > 0$, and hypothesis (F5) holds. Hence, the result follows from Theorem 5.6. ∎

Concerning the solutions of the original compartmental system, we obtain the following result. Although the theorem is stated in the almost-periodic case, similar conclusions are obtained changing almost periodicity for periodicity, almost automorphy, or recurrence; that is, all solutions are asymptotically of the same type as the transport functions.

THEOREM 6.2. *Under Assumptions* (H1)–(H5) *and in the almost-periodic case, there are infinitely many almost-periodic solutions of system* (6.1) *and all the solutions with Lipschitz initial data are asymptotically almost periodic.*

Proof. Let $\omega_0 = \widetilde{g}$ and a Lipschitz initial value x_0. The omega-limit set of each solution $z(t,\omega_0,x_0)$ is a copy of the base $\mathcal{O}(\omega_0,x_0) = \{(\omega,x(\omega)) \mid \omega \in \Omega\}$, and hence $z(t,\omega_0,x(\omega_0)) = x(\omega_0{\cdot}t)(0)$ is an almost-periodic solution of (6.1) with

$$
\lim_{t\to\infty} \|z(t,\omega_0,x_0) - z(t,\omega_0,x(\omega_0))\| = 0\,.
$$

Next, we check that there are infinitely many minimal subsets. From the definition of the mass and (H4),

$$
M(\omega_0,k\,\mathbf{1}) = \sum_{i=1}^m k\,(1-\nu_i(-\infty,0]) + \sum_{i=1}^m \sum_{j=1}^m \int_{-\infty}^0 \left(\int_s^0 g_{ji}(\omega_0{\cdot}\tau,k)\, d\tau \right) d\mu_{ji}(s)
$$

tends to $+\infty$ as k goes to $+\infty$, and thus, given $c > 0$, there is a $k_c > 0$ such that $M(\omega_0, k_c\,\mathbf{1}) = c$. Finally, since the mass is constant along the trajectories, $\mathcal{O}(\omega_0, k_c\mathbf{1})$ provides a different minimal set and hence a different almost-periodic solution for each $c > 0$. ◻

Next, we consider the particular case where the compartmental systems are described by NFDEs with finite delay

$$\frac{d}{dt}(z_i(t) - \gamma_i\, z_i(t-\alpha_i)) = -\sum_{j=1}^{m} \widetilde{g}_{ji}(t, z_i(t)) + \sum_{j=1}^{m} \widetilde{g}_{ij}(t-\rho_{ij}, z_j(t-\rho_{ij}))\,, \tag{6.5}$$

$i = 1, \ldots, m$, where $\widetilde{g} = (\widetilde{g}_{ij})_{i,j} : \mathbb{R} \times \mathbb{R} \to \mathbb{R}^{m\times m}$, and we will assume that

(G1) $\widetilde{g}$ is C^1-*admissible*; that is, $\widetilde{g}$ is C^1 in its second variable and $\widetilde{g}$, $\frac{\partial}{\partial v}\widetilde{g}$ are uniformly continuous and bounded on $\mathbb{R} \times \{v_0\}$ for all $v_0 \in \mathbb{R}$;

(G2) $\widetilde{g}_{ij}(t, 0) = 0$ and $\widetilde{g}_{ij}(t, v)$ is increasing in v for each $t \in \mathbb{R}$ and $i, j = 1, \ldots, m$;

(G3) $\widetilde{g}$ is a recurrent function; i.e., its *hull* is minimal;

(G4) $\sum_{i=1}^{m} \gamma_i < 1$, $\alpha_i > 0$, and $\rho_{ij} \ge 0$, $i, j = 1, \ldots, m$;

(G5) for each $i = 1, \ldots, m$ one of the following hypotheses holds:

(G5.1) $L_i^+ = \sum_{j=1}^{m} l_{ji}^+ < +\infty$ and there is a $\beta_i > 0$ such that

$$\beta_i(1 - \gamma_i\, e^{\beta_i\alpha_i}) > L_i^+\,;$$

(G5.2) $L_i^+ = \sum_{j=1}^{m} l_{ji}^+ < +\infty$, $\alpha_i \ge \rho_{ii}$, and there is a $\beta_i \ge L_i^+$ such that

$$\beta_i\,(1 - \gamma_i\, e^{\beta_i\alpha_i}) + e^{\beta_i\rho_{ii}}\, l_{ii}^- > L_i^+\,,$$

where $l_{ij}^+ = \sup_{(t,\,v)\in\mathbb{R}^2} \frac{\partial\widetilde{g}_{ij}}{\partial v}(t, v)$ and $l_{ii}^- = \inf_{(t,v)\in\mathbb{R}^2} \frac{\partial\widetilde{g}_{ii}}{\partial v}(t, v)$.

As before, the family

$$\frac{d}{dt}Dz_t = F(\omega{\cdot}t, z_t)\,, \quad t \ge 0\,,\ \omega \in \Omega\,, \tag{6.6$_\omega$}$$

with

$$F_i(\omega, x) = -\sum_{j=1}^{m} g_{ji}(\omega, x_i(0)) + \sum_{j=1}^{m} g_{ij}(\omega{\cdot}(-\rho_{ij}), x_j(t-\rho_{ij}))\,, \tag{6.7}$$

is of the form $(3.5)_\omega$ with $D_i x = x_i(0) - \gamma_i\, x(-\alpha_i)$, $i = 1, \ldots, m$, and includes system (6.1) when $\omega = \widetilde{g}$.

Under these assumptions we deduce the convergence of the solutions with Lipschitz initial data. Notice that hypothesis (G5.2), in this case of finite delay, improves the conditions of applicability of the theory stated in Theorem 6.1.

THEOREM 6.3. *Assume that* (G1)–(G5) *hold and let* $(\omega_0, x_0) \in \Omega \times BU$ *with* x_0 *Lipschitz. Then the solution* $z(t, \omega_0, x_0)$ *of* $(6.6)_{\omega_0}$ *with initial value* x_0 *is bounded and its omega-limit set is a copy of the base; that is,* $\mathcal{O}(\omega_0, x_0) = \{(\omega, c(\omega)) \mid \omega \in \Omega\}$ *for a continuous equilibrium* $c\colon \Omega \to BU$ *and*

$$\lim_{t\to\infty} \mathsf{d}(u(t, \omega_0, x_0), c(\omega_0{\cdot}t)) = 0\,.$$

Proof. The only difference with the proof of Theorem 6.1 is to check (F4) and (F6) for the components $i \in \{1, \ldots, m\}$ for which (G5.2) holds.

Thus, let $i \in \{1, \dots, m\}$ satisfy (G5.2), and let x, $y \in BU$ with $x \leq_A y$. Then, from (6.7), $g_{ii}(\omega, y(-\rho_{ii})) - g_{ii}(\omega, x(-\rho_{ii})) \geq l_{ii}^- \,(y_i(-\rho_{ii}) - x_i(-\rho_{ii}))$ and

$$(y_i(-\alpha_i) - x_i(-\alpha_i))\, e^{-\beta_i \alpha_i} \leq (y(-\rho_{ii}) - x(-\rho_{ii}))e^{-\beta_i \rho_{ii}};$$

because $\alpha_i \geq \rho_{ii}$ and $x \leq_A y$, we deduce that

$$\begin{aligned} F_i(\omega, y) - F_i(\omega, x) + \beta_i(D_i y - D_i x) \geq\ & (\beta_i - L_i^+)\,(y_i(0) - x_i(0)) \\ & + (l_{ii}^- - \beta_i\, \gamma_i\, e^{\beta_i(\alpha_i - \rho_{ii})})\,(y_i(-\rho_{ii}) - x_i(-\rho_{ii}))\,. \end{aligned}$$

In addition, from $\beta_i \geq L_i^+$, $(y(-\rho_{ii}) - x(-\rho_{ii}))\, e^{-\beta_i \rho_{ii}} \leq y_i(0) - x_i(0)$, and (G5.2) we conclude that

$$\begin{aligned} F_i(\omega, y) - F_i(\omega, x) + \beta_i(D_i y - D_i x) \geq (y_i(-\rho_{ii}) - x_i(-\rho_{ii}))\big[&(\beta_i - L_i^+)\, e^{-\beta_i \rho_{ii}} \\ & + l_{ii}^- - \beta_i\, \gamma_i\, e^{\beta_i(\alpha_i - \rho_{ii})}\big]\,, \end{aligned}$$

and (F4) and (F6) hold. □

As before, concerning the solutions of the original compartmental system, we obtain the following result.

THEOREM 6.4. *Under Assumptions* (G1)–(G5) *and in the almost-periodic case, there are infinitely many almost-periodic solutions of system* (6.5) *and all the solutions with Lipschitz initial data are asymptotically almost periodic.*

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