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SOME QUESTIONS CONCERNING ATTRACTORS FOR NON-AUTONOMOUS DYNAMICAL SYSTEMS

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Abstract. We compare various concepts of attractor in the context of non-autonomous dynamical systems. Then, we prove an appropriate version of the Pliss reduction principle for non-autonomous differential systems with rapidly oscillating coefficients.

1. Introduction

An important theme in the theory of non-autonomous differential and discrete systems is that of integral manifolds and the invariant sets which they contain. Questions concerning the existence and properties of attracting invariant sets are of interest. In particular, it is important to know when an invariant set which is locally attracting with respect to the restriction of a differential system to the integral manifold, is actually locally attracting with respect to the full differential system. That is, one wants to know if the "Pliss reduction principle" (see [Pl]) is valid.

In this paper, we consider two topics concerning integral manifolds for non-autonomous differential systems and the attractors which they may contain. In our discussion, we will adopt the framework of skew-product flows ([Be],[Se]), which has been found to be convenient in the study of a wide spectrum of problems concerning non-autonomous differential systems.

The first topic is that of the relationship between diverse notions of "attractor" in the skew-product framework. We adopt a Lyapunov type definition of attractor as starting point, and compare it with a definition of *pullback attractor* which seems appropriate for skew-product flows. This last concept has been widely discussed and applied in the recent literature on non-autonomous dynamical systems (see e.g. [CRC], [S-H], [S]). Still another type of attractor is determined in certain circumstances as a fixed point of a contraction operator defined using a given skew-product flow; see e.g. [FJM], [JM] for examples of such "fixed-point" attractors. We will see that, under mild hypotheses, an attractor is of Lyapunov type if and only if it is of pullback type. We will also see that a fixed-point attractor is of pullback type, and so of Lyapunov type as well.

The second issue we address concerns the behavior of solutions of non-autonomous differential systems with *rapidly varying coefficients*, which lie in or near an integral manifold. The construction of integral manifolds for such systems was studied by Coppel and Palmer ([CP],[Pa1]). More recently this theme has been taken up by Cheban, Duan and Gherco [CDG], and by Fabbri, Johnson and Palmer [FJP]. Our point of departure is the theory presented in [FJP]. We consider differential systems with rapidly varying coefficients for which the origin is an equilibrium point of neutral type. We state and prove a result concerning the existence of the asymptotic phase for small solutions. We also state and prove a result which is analogous to the Pliss reduction principle for autonomous differential systems [Pl]. We follow the approach of Palmer [Pa2] to these

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questions, though we will see that some preparatory work is necessary in order to apply his arguments.

It will be clear that the methods which we apply can be used to state and prove results concerning the existence of the asymptotic phase and the validity of the Pliss reduction principle for other classes of non-autonomous dynamical systems. In this regard, we can refer to the contributions of Janglajew [Ja], Pötzsche [Pö], and Reinfelds-Janglajew [RJ].

This paper is organized as follows. In Section 2, we elaborate and compare the notions of attractor mentioned above. In Section 3, we consider the questions of the existence of the asymptotic phase and the validity of the Pliss reduction principle for differential systems with rapidly varying coefficients.

2. Attractors

As stated in the Introduction, we will work in the *skew-product* framework for non-autonomous differential systems. We now describe that framework.

Let $\mathfrak{P}$ be a compact metric space, and let $\{\tau_t : t \in \mathbb{R}\}$ be a flow on $\mathfrak{P}$. That is, for each $p \in \mathfrak{P}$, the map $\tau_t : \mathfrak{P} \to \mathfrak{P}$ is a homeomorphism and, moreover, the following conditions are satisfied:

(i) $\tau_0(p) = p$ for all $p \in \mathfrak{P}$;
(ii) $\tau_t \circ \tau_s = \tau_{t+s}$ for all $t, s \in \mathbb{R}$;
(iii) the map $\tau : \mathfrak{P} \times \mathbb{R} \to \mathfrak{P}$, $(p, t) \mapsto \tau_t(p)$, is continuous.

We introduce some terminology related to the flow concept. If $p \in \mathfrak{P}$, then the *orbit* through p is $\{\tau_t(p) : t \in \mathbb{R}\}$. The *positive* (resp. *negative*) *semiorbit* through p is $\{\tau_t(p) : t \geq 0\,(\text{resp. } t \leq 0)\}$. A subset $A \subset \mathfrak{P}$ is said to be *invariant* if $\tau_t(A) \subset A$ for all $t \in \mathbb{R}$; it is said to be *positively* (resp. *negatively*) *invariant* if $\tau_t(A) \subset A$ for all $t \geq 0$ (resp. $t \leq 0$). If $p \in \mathfrak{P}$, the *omega-limit set* $\omega(p) = \{p_1 \in \mathfrak{P}$: there exists a sequence $t_n \to \infty$ such that $\tau_{t_n}(p) \to p_1\}$.

Next, let $d \geq 1$ be an integer. We define the concept of *skew-product local flow* on $\mathfrak{P} \times \mathbb{R}^d$. By this, we mean a pair $(Y, \tilde{\tau})$ which satisfies the conditions given below. We write $Z = \mathfrak{P} \times \mathbb{R}^d$, and $z = (p, x)$ for a generic point of Z.

(i) Y is an open subset of $Z \times \mathbb{R}$ which contains the set $\{(z, 0) : z \in Z\}$, such that, if $(z, t) \in Y$, then $(z, s) \in Y$ for all $0 \leq s \leq t$ if $t \geq 0$, and for all $t \leq s \leq 0$ if $t \leq 0$. The map $\tilde{\tau} : Y \to Z$ is continuous.
(ii) $\tilde{\tau}(z, 0) = z$ for all $z \in Z$.
(iii) $\tilde{\tau}(\tilde{\tau}(z, s), t) = \tilde{\tau}(z, t + s)$ whenever $\tilde{\tau}(z, s)$ and $\tilde{\tau}(\tilde{\tau}(z, s), t)$ are defined.
(iv) If $z = (p, x) \in Z$ and if $\tilde{\tau}(z, t)$ is defined, then $\tilde{\tau}(z, t) = (\tau(p), x_t)$ where $x_t \in \mathbb{R}^d$.
(v) The pair $(Y, \tilde{\tau})$ is maximal with respect to the properties (i)-(iv).

Condition (iv) is called the *skew-product* property of $\tilde{\tau}$. We will sometimes write $\tilde{\tau}(z, t) = \tilde{\tau}_t(z)$ when $(z, t) \in Y$.

One can define a skew-product local flow beginning with an appropriate family of non-autonomous differential systems. Let $|\cdot|$ denote the Euclidean norm on $\mathbb{R}^d$. Let $f : \mathfrak{P} \times \mathbb{R}^d \to \mathbb{R}^d$ be a continuous map such that, for each compact set $K \subset \mathbb{R}^d$, the uniform Lipschitz constant

$$\mathrm{Lip}_K = \sup_{p \in \mathfrak{P}} \sup_{x_1 \neq x_2} \left\{ \frac{|f(p, x_1) - f(p, x_2)|}{|x_1 - x_2|} \right\}$$

is finite. Consider the family of differential systems

$$x' = f(\tau_t(p), x) \tag{2.1$_p$}$$

where p ranges over $\mathfrak{P}$. For each $p \in \mathfrak{P}$ and each $x_0 \in \mathbb{R}$, let $x(t)$ be the solution of $(2.1)_p$ such that $x(0) = x_0$. Set $z_0 = (p, x_0) \in Z$, and put $\tilde{\tau}(z_0, t) = (\tau_t(p), x(t))$ where $x(\cdot)$ is the maximal solution of $(2.1)_p$ with initial value x_0. Let $Y = \{(z, t) : \tilde{\tau} \text{ is well-defined}\} \subset Z \times \mathbb{R}$. Then $(Y, \tilde{\tau})$ is a skew-product local flow on $\mathfrak{P} \times \mathbb{R}^d$.

It is well-known that, if $\tilde{f} : \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}^d$ is uniformly continuous on $\mathbb{R} \times K$ for each compact set $K \subset \mathbb{R}^d$, and if $\tilde{f}$ is uniformly Lipschitz continuous in $x \in K$ for each such set $\mathbb{R} \times K$, then $\tilde{f}$ gives rise to a compact metric space $\mathfrak{P}$, a flow $(\mathfrak{P}, \{\tau_t\})$, and a function $f : \mathfrak{P} \times \mathbb{R}^d \to \mathbb{R}^d$ satisfying the above conditions. See e.g. [Se].

Let $\mathfrak{P}$ be a compact metric space, and let $(\mathfrak{P}, \{\tau_t\})$ be a flow. Let $\tilde{\tau}$ be a skew-product local flow on $\mathfrak{P} \times \mathbb{R}^d$, and let $\pi : \mathfrak{P} \times \mathbb{R}^d \to \mathfrak{P}$ be the projection.

Definition 2.1. A compact set $A \subset Z = \mathfrak{P} \times \mathbb{R}^d$ is said to be a *Lyapunov attractor* if the following conditions are satisfied.

(i) If $z = (p, x) \in A$, then $\tilde{\tau}(z, t)$ is defined for all $t \in \mathbb{R}$.
(ii) There is an open set $W \subset Z$ containing A such that, if $z \in W$,then $\tilde{\tau}_t(z)$ is defined for all $t \geq 0$, and the omega-limit set $\omega(z) \subset A$.
(iii) If V is an open set in Z containing A, then there is an open set $V_1 \subset V$, which contains A, such that, if $z \in V_1$, then $\tilde{\tau}_t(z)$ is defined and lies in V for all $t \geq 0$.
(iv) The projection $\pi(A) = \mathfrak{P}$.

The last condition is imposed for reasons of convenience. In practice, it will not usually entail any loss of generality, since one has the option of replacing $\mathfrak{P}$ by $\pi(A)$.

Our first goal is to show that a compact subset $A \subset \mathfrak{P} \times \mathbb{R}^d$ is a Lyapunov attractor if and only if it is a *pullback attractor* ([CRC], [CKS], [KR], [S-H], [S]). We give a definition of the latter concept which seems appropriate in the context of skew-product local flows. We make use of the concept of Hausdorff distance [KR]. Let (Z, d_Z) be a compact metric space, and let K_1, K_2 be two nonempty compact subsets of Z. Set $H_*(K_1, K_2) = \sup_{z_1 \in K_1} \inf\{d_Z(z_1, z_2) : z_2 \in K_2\}$, then set

$$H(K_1, K_2) = \max\{H_*(K_1, K_2), H_*(K_2, K_1)\}.$$

The quantity $H(K_1, K_2)$ is the *Hausdorff distance* between K_1 and K_2. If $z \in Z$ and if $K \subset Z$ is compact, we abuse notation slightly and write $H(z, K)$ in place of $H(\{z\}, K)$.

Let us introduce the following notation: if $Z = \mathfrak{P} \times \mathbb{R}^d$ and $X \subset Z$, then $X_p = X(p) = (\{p\} \times \mathbb{R}^d) \cap X$. Further, let $d_{\mathfrak{P}}$ be a metric on $\mathfrak{P}$ compatible with its topology, and define the metric d on $Z = \mathfrak{P} \times \mathbb{R}^d$ by $d(z_1, z_2) = d_{\mathfrak{P}}(p_1, p_2) + |x_1 - x_2|$ whenever $z_1 = (p_1, x_1)$ and $z_2 = (p_2, x_2)$.

Definition 2.2. Let $A \subset Z = \mathfrak{P} \times \mathbb{R}^d$ be a compact invariant set such that $\pi(A) = \mathfrak{P}$. Say that A is a *pullback attractor* if there is an open subset $W \subset Z$, which contains A, with the following property: let $D \subset W$ be a compact set such that $D_p \neq \emptyset$ for each $p \in \mathfrak{P}$; then

$$\lim_{t \to \infty} \sup_{p \in \mathfrak{P}} H(\tilde{\tau}_t(D_{\tau_{-t}(p)}), A) = 0.$$

Our definition is related to that given in [S-H], [S]. However, our "universe" of sets $\{D\}$ is chosen so as to take account of the skew-product framework in which we work.

The following result is quite natural; however, so far as we know it has not appeared in the literature.

Proposition 2.3. *Let $A \subset \mathfrak{P} \times \mathbb{R}^d$ be a compact invariant set. Then A is a Lyapunov attractor if and only if it is a pullback attractor.*

Proof. Suppose first that A is a Lyapunov attractor. Let $W \subset Z = \mathfrak{P} \times \mathbb{R}^d$ be an open neighborhood of A such that $\omega(z) \subset A$ for all $z = (p, x) \in W$. We claim that the pullback condition is satisfied by the family $\{D\}$ of compact subsets D of W such that $D_p \neq \emptyset$, $p \in \mathfrak{P}$.

To see this, let $D \subset W$ be a compact subset such that $D_p \neq \emptyset$, $p \in \mathfrak{P}$. Suppose for contradiction that there exist a number $\varepsilon > 0$ and sequences $t_n \to \infty$, $p_n \in \mathfrak{P}$ such that

$$H(\tilde{\tau}_{t_n}(D(\tau_{-t_n}(p_n))), A) \geq \varepsilon,\ n \in \mathbb{N}.$$

This means that, for each $n \geq 1$, there exists $x_n \in D(\tau_{-t_n}(p_n))$ such that, if $z_n = \tilde{\tau}_{t_n}(\tau_{-t_n}(p_n), x_n)$, then

$$H(z_n, A) \geq \varepsilon. \tag{*}$$

On the other hand, let $\delta > 0$ be a number such that, if $z \in Z$ and $H(z, A) \leq \delta$, then $H(\tilde{\tau}_t(z), A) \leq \varepsilon/2$ for all $t \geq 0$. Such a number exists because A is a Lyapunov attractor. If $z_* \in D$, then $\omega(z_*) \subset A$. Hence there exists a time $t_* = t_*(z_*) > 0$ such that, if $t \geq t_*$, then $H(\tilde{\tau}_t(z_*), A) < \delta$. There is a neighborhood $N_* = N_*(z_*)$ of z_* in W such that, if $z_1 \in N_*$, then $H(\tilde{\tau}_t(z_1), A) < \delta$, and by compactness of D we can find a fixed time t_f such that, if $z \in D$, then for some time $t_z \in (0, t_f]$ there holds $H(\tilde{\tau}_{t_z}(z), A) < \delta$. But this implies that, if $t \geq t_f$, then $H(\tilde{\tau}_t(z), A) \leq \varepsilon/2$. This is inconsistent with the condition (*). So indeed A is a pullback attractor.

Now let us suppose that the pullback condition is satisfied by A. Let $W \subset \mathfrak{P} \times \mathbb{R}^d$ be a neighborhood of A for which the pullback condition holds. We claim that, if $z = (p, x) \in W$, then $\omega(z) \subset A$. For suppose not. Then there exist $\varepsilon > 0$ and a sequence $t_n \to \infty$ such that

$$H(\tilde{\tau}_{t_n}(z), A) \geq \varepsilon. \tag{**}$$

Write $z_n = (p_n, x_n) = \tilde{\tau}_{t_n}(z)$, so that $\tilde{\tau}_{-t_n}(z_n) = z$. Then $x \in D_p$ for some compact set $D \subset W$ such that $D_{p_1} \neq \emptyset$ for all $p_1 \in \mathfrak{P}$ (one can take, for example, $D = A \cup \{z\}$). But then (**) contradicts the pullback condition, so indeed $\omega(z) \subset A$.

We now claim that the Lyapunov stability condition holds. For suppose not; then there is a number $\varepsilon > 0$ with the following property: there are sequences $\{\delta_n\} \subset (0, \infty)$, $\{z_n\} \subset Z$, and $\{t_n\} \subset (0, \infty)$ such that $\delta_n \to 0$, $H(z_n, A) \leq \delta_n$, and

$$H(\tilde{\tau}_{t_n}(z_n), A) \geq \varepsilon. \tag{***}$$

There is no loss of generality in assuming that $\tilde{\tau}_{t_n}(z_n) \in W$.

Now, however, $z_n = \tilde{\tau}_{-t_n}(\tilde{\tau}_{t_n}(z_n))$. Fix a compact neighborhood D of A in W. Then z_n lies in D for all large n. But then (***) violates the pullback condition. This shows that A is indeed a Lyapunov attractor, and completes the proof of Proposition 2.3. □

Next we consider the following situation. Let $V \subset \mathbb{R}^d$ be an open set. Let $\mathcal{C}_V = \{c : \mathfrak{P} \to V : c \text{ is bounded and continuous}\}$ with the usual metric

$$\rho(c_1, c_2) = \sup_{p \in \mathfrak{P}} \{|c_1(p) - c_2(p)|\}.$$

Then $(\mathcal{C}_V, \rho)$ is a complete metric space. Suppose that, for some $t_0 > 0$, there holds $\tilde{\tau}_{t_0}(\mathfrak{P} \times V) \subset \mathfrak{P} \times V$. In this case, one can define a map $\xi : \mathcal{C}_V \to \mathcal{C}_V$ as follows:

$$(p, \xi(c)(p)) = \tilde{\tau}_{t_0}(\tau_{-t_0}(p), c(\tau_{-t_0}(p))). \tag{2.2}$$

We will suppose that the map ξ is a *contraction* with respect to ρ. That is, there exists a constant $\alpha < 1$ with the property that

$$\rho(\xi(c_1), \xi(c_2)) \leq \alpha \rho(c_1, c_2)$$

for all $c_1, c_2 \in \mathcal{C}_V$.

Under these conditions, ξ admits a unique fixed point $a \in \mathcal{C}_V$: $\xi(a) = a$. Let

$$A = \{(p, a(p)) : p \in \mathfrak{P}\}.$$

Then $A \subset \mathfrak{P} \times V \subset Z$ is a compact set, and it is natural to conjecture that A is a Lyapunov attractor for $\tilde{\tau}$. We will verify that this is indeed the case.

Proposition 2.4. *Let $V \subset \mathbb{R}^d$ be an open set, and suppose that $t_0 > 0$ is a number such that $\tilde{\tau}_{t_0}(\mathfrak{P} \times V) \subset \mathfrak{P} \times V$. Suppose that the map $\xi : \mathcal{C}_V \to \mathcal{C}_V$ defined by (2.2) is a contraction in $\mathcal{C}_V$ with contraction constant $\alpha < 1$. Then the fixed point $a \in \mathcal{C}_V$ of ξ has the property that*

$$A = \{(p, a(p)) : p \in \mathfrak{P}\}$$

is a Lyapunov attractor for $\tilde{\tau}$.

Proof. We first show that A is $\tilde{\tau}$-invariant. This is not immediately clear because $\tilde{\tau}$ is only assumed to be a local flow. Let $U \subset \mathbb{R}^d$ be an open set such that, if $z = (p, x) \in \mathfrak{P} \times V$, then $\tilde{\tau}_t(z) \in \mathfrak{P} \times U$ for all $t \geq 0$. Define $\mathcal{C}_U$ in the same way in which $\mathcal{C}_V$ was defined; then $\mathcal{C}_V \subset \mathcal{C}_U$. Fix a pair of integers $n \geq 1$, $0 \leq r \leq n$, and set $u = rt_0/n$. Define a map $\xi_1 : \mathcal{C}_V \to \mathcal{C}_U$ via the relation

$$(p, \xi_1(c)(p)) = \tilde{\tau}_u(\tau_{-u}(p), c(\tau_{-u}(p))),\ c \in \mathcal{C}_V.$$

Then the iterate $\xi_1^{(k)}$ is defined for each $k \geq 1$. Note that $\rho(\xi_1(a), a) = \rho(\xi_1(\xi_1^{(n)}(a)), \xi_1^{(n)}(a)) \leq \alpha^r \rho(\xi_1(a), a)$, so a is a fixed point of ξ_1. This means that $\tilde{\tau}_u$ leaves A invariant for all u as above. By continuity of $\tilde{\tau}$, one has that $\tilde{\tau}_t(A) \subset A$ for all $t \in [0, t_0]$, and it follows that A is positively $\tilde{\tau}$-invariant.

We can now globally invert the local flow $\tilde{\tau}$ on A by setting $\tilde{\tilde{\tau}}_{-t}(p, a(p)) = (\tau_{-t}(p), a(\tau_{-t}(p)))$, $t \geq 0, p \in \mathfrak{P}$. We omit the proof that $\tilde{\tilde{\tau}}$ coincides with $\tilde{\tau}$ on A.

Let us now show that A is a pullback attractor. In fact we will show that $W = \mathfrak{P} \times V$ satisfies the condition of Definition 2.2. For this, let $D \subset W$ be a compact set such that $D_p \neq \emptyset$ for each $p \in \mathfrak{P}$. We must show that

$$\limsup_{t \to \infty} \sup_{p \in \mathfrak{P}} H(\tilde{\tau}_t(D(\tau_{-t}(p))), A) = 0.$$

Suppose for contradiction that there exists $\varepsilon > 0$ with the following property: there are sequences $t_n \to \infty$, $p_n \in \mathfrak{P}$, and $x_n \in D(\tau_{-t_n}(p_n))$ such that

$$H(\tilde{\tau}_{t_n}(\tau_{-t_n}(p_n), x_n), A) \geq \varepsilon. \tag{2.3}$$

For each $n \in \mathbb{N}$, there is a continuous map $c_n : \mathfrak{P} \to V$ such that $c_n(\tau_{-t_n}(p_n)) = x_n$. For each n, let k_n be the largest integer such that $k_n t_0 \leq t_n$. We have

$$\rho(\xi^{(k_n)}(c_n), a) = \rho(\xi^{(k_n)}(c_n), \xi^{(k_n)}(a)) \leq \alpha^{k_n} \rho(c_n, a),$$

and the right-hand side tends to zero as $n \to \infty$. Taking account of the definition of ξ and of the continuity of $\tilde{\tau}$, we see that (2.3) is violated for sufficiently large n. This completes the proof of Proposition 2.4. □

3. Asymptotic phase and reduction principle

In this section, we state and prove the principle of asymptotic phase and the Pliss reduction principle in a form appropriate for ordinary differential systems with rapidly varying coefficients. We first discuss the integral manifold theory for such a system as it is presented in [FJP].

Let $I \subset \mathbb{R}$ be a compact interval containing $\varepsilon = 0$ in its interior. Let $f : \mathbb{R}\times\mathbb{R}^d\times I \to \mathbb{R}^d$ be a function with regularity and recurrence properties which will be specified presently. Consider the non-autonomous differential system

$$x' = |\varepsilon| f(t, x, \varepsilon), \ t \in \mathbb{R}, \ x \in \mathbb{R}^d, \ \varepsilon \in I \tag{3.1}$$

where ε is small. We write $|\varepsilon|$ instead of ε before f because one might want, for instance, to study the change of stability of an equilibrium point of (3.1) when ε passes through zero.

The integral manifold theory for (3.1) can be formulated in an elegant way when f depends on t in a "uniquely ergodic" manner. We pause to explain what this means.

Let $\mathfrak{P}$ be a compact metric space and let $\{\tau_t\}$ be a flow on $\mathfrak{P}$. We review some basic notions of ergodic theory. Let μ be a regular Borel probability measure on $\mathfrak{P}$. Then μ is said to be *invariant* if, for each Borel set $B \subset \mathfrak{P}$ and each $t \in \mathbb{R}$, there holds $\mu(\tau_t(B)) = \mu(B)$. An invariant measure μ is said to be *ergodic* if it satisfies the following indecomposibility condition: for each Borel set $B \subset \mathfrak{P}$, the property

$$\mu(\tau_t(B) \triangle B) = 0 \text{ for all } t \in \mathbb{R} \ (\triangle = \text{symmetric difference})$$

implies that either $\mu(B) = 0$ or $\mu(B) = 1$. It is known [NS] that there exists at least one regular Borel measure μ on $\mathfrak{P}$ which is ergodic (with respect to $\{\tau_t\}$).

Let us now impose the following hypotheses, which will be valid in all that follows. First, we assume that the flow $(\mathfrak{P}, \{\tau_t\})$ admits a *unique* invariant measure μ, which is then necessarily ergodic. Second, we assume that there exist: (i) a continuous map $\tilde{f} : \mathfrak{P} \times \mathbb{R}^d \times I \to \mathbb{R}^d$ and (ii) a point $\tilde{p} \in \mathfrak{P}$ such that

$$f(t, x, \varepsilon) = \tilde{f}(\tau_t(\tilde{p}), x, \varepsilon)$$

for all $t \in \mathbb{R}$, $x \in \mathbb{R}^d$, and $\varepsilon \in I$. This means in effect that $f(\cdot, x, \varepsilon)$ has recurrence properties which are reflected in the structure of the flow $(\mathfrak{P}, \{\tau_t\})$. Third, we assume that $\tilde{f}(p, 0, \varepsilon) = 0$ for all $p \in \mathfrak{P}$, $\varepsilon \in I$.

There are well-known conditions on f which ensure the existence of objects $\mathfrak{P}, \{\tau_t\}, \mu$, and $\tilde{f}$ which satisfy the above conditions. For example, if f is almost periodic in t, uniformly on each set of the form $K \times I$ where $K \subset \mathbb{R}^d$ is compact, then the above conditions are fulfilled. See [FJP] (also [Se] and many others references) for a discussion of this matter. We will say no more about it; rather, we let $\mathfrak{P}, \{\tau_t\}, \mu$, and $\tilde{f}$ be as above, then write f instead of $\tilde{f}$, and consider the family of differential systems

$$x' = |\varepsilon| f(\tau_t(p), x, \varepsilon) \tag*{$(3.2)_p$}$$

which includes the ε-dependent equation (3.1). We generally will not indicate explicitly the dependence of the family $(3.2)_p$ on $\varepsilon \in I$.

We proceed to outline the integral manifold theory for a "uniquely ergodic family" $(3.2)_p$ as it is worked out in [FJP]. We will assume throughout that, for some $r \geq 1$, the function $x \mapsto f(p, x, \varepsilon) : \mathbb{R}^d \to \mathbb{R}^d$ is of class C^r for each $p \in \mathfrak{P}$ and $\varepsilon \in I$. We further require that the Fréchet derivatives $\partial_x^k f : \mathfrak{P} \times \mathbb{R}^d \times I \to L^k(\mathbb{R}^d, \mathbb{R}^d)$ are continuous, $0 \leq k \leq r$. Here $L^k(\mathbb{R}^d, \mathbb{R}^d)$ is the usual space of $\mathbb{R}^d$-valued, k-linear maps defined on $\mathbb{R}^d \times \cdots \times \mathbb{R}^d$ (k times).

We introduce the average

$$\bar{f}(x, \varepsilon) = \int_{\mathfrak{P}} f(p, x, \varepsilon) d\mu(p) = \lim_{|t|\to\infty} \frac{1}{t} \int_0^t f(\tau_s(p), x, \varepsilon) ds.$$

Because of the unique ergodicity of $(\mathfrak{P}, \{\tau_t\})$, the limit on the right is uniform on $\mathfrak{P}\times K\times I$ for each compact subset $K \subset \mathbb{R}^d$. Note that $\bar{f}(0,\varepsilon) = 0$ for all $\varepsilon \in I$. Let us write

$$\bar{f}(x,\varepsilon) = \bar{l}_\varepsilon x + \bar{n}_\varepsilon(x) \tag{3.3}$$

where $\bar{n}_\varepsilon(x) = o(|x|)$ as $x \to 0$, uniformly in $\varepsilon \in I$. Set $\varepsilon = 0$; we impose a condition on the matrix $\bar{l}_0$.

Hypothesis 3.1. The set of eigenvalues of $\bar{l}_0$ is the union of two disjoint nonempty subsets, namely

$$\Sigma^{(0)} = \{\lambda \in \mathbb{C} : \lambda \text{ is an eigenvalue of } \bar{l}_0 \text{ and } \mathrm{Re}(\lambda) = 0\},$$

$$\Sigma^{(-)} = \{\lambda \in \mathbb{C} : \lambda \text{ is an eigenvalue of } \bar{l}_0 \text{ and } \mathrm{Re}(\lambda) < 0\}.$$

Let $\beta > 0$ be a number such that $\mathrm{Re}(\lambda) < -\beta$ for all $\lambda \in \Sigma^{(-)}$. Let $L^{(-)} \subset \mathbb{R}^d$ be the intersection with $\mathbb{R}^d$ of the sum of the generalized eigenspaces of $\bar{l}_0$ which correspond to eigenvalues in $\Sigma^{(-)}$. Define $L^{(0)}$ in the analogous way. Let $Q_0 : \mathbb{R}^d \to \mathbb{R}^d$ be the projection with image $L^{(-)}$ and kernel $L^{(0)}$.

Next write

$$f(p,x,\varepsilon) = l_\varepsilon(p)x + n_\varepsilon(p,x)$$

for $p \in \mathfrak{P}, x \in \mathbb{R}^d, \varepsilon \in I$. Consider the family of linear systems

$$x' = |\varepsilon| l_\varepsilon(\tau_t(p))x. \tag*{$(3.4)_p$}$$

If $\varepsilon \neq 0$, the change of variables $s = |\varepsilon| t$ transforms $(3.4)_p$ into

$$\frac{dx}{ds} = l_\varepsilon(\tau_{s/|\varepsilon|}(p))x. \tag*{$(3.5)_p$}$$

Moreover, $(3.2)_p$ is transformed into

$$\frac{dx}{ds} = l_\varepsilon(\tau_{s/|\varepsilon|}(p))x + n_\varepsilon(\tau_{s/|\varepsilon|}(p), x). \tag*{$(3.6)_p$}$$

We recall a basic definition [Co], [SS].

Definition 3.2. The family of equations $(3.5)_p$ is said to have an *exponential dichotomy* (*ED*) over $\mathfrak{P}$ if the following conditions hold. Let $\Phi_p(s)$ be the fundamental matrix solution of $(3.5)_p$; then there are constants $k > 0, \sigma > 0$ together with a continuous, projection-valued function $p \mapsto Q_p : \mathbb{R}^d \to \mathbb{R}^d$, $Q_p^2 = Q_p$, such that the following estimates hold:

$$|\Phi_p(u) Q_p \Phi_p(s)^{-1}| \leq k e^{-\sigma(u-s)},\ u \geq s,$$

$$|\Phi_p(u)(I - Q_p)\Phi_p(s)^{-1}| \leq k e^{\sigma(u-s)},\ u \leq s.$$

We introduce the *dynamical spectrum* of the family $(3.5)_p$. For this, let $\lambda \in \mathbb{R}$ and consider the translated family

$$\frac{dx}{ds} = [-\lambda I + l_\varepsilon(\tau_{s/|\varepsilon|}(p))]x. \tag*{$(3.7)_p$}$$

Then the dynamical spectrum $\Sigma(\varepsilon)$ of the family $(3.5)_p$ is

$$\Sigma(\varepsilon) = \{\lambda \in \mathbb{R} : \text{the family } (3.7)_p \text{ does } \textit{not} \text{ admit ED over } \mathfrak{P}\}.$$

We can use a basic perturbation theorem of Sacker and Sell [SS] to prove the following result.

Theorem 3.3. *Let β be as above, and let $\alpha > 0$ be a real number. There exists $\varepsilon_1 > 0$ such that, if $0 < |\varepsilon| \leq \varepsilon_1$, then $\Sigma(\varepsilon) = \Sigma^{(0)}(\varepsilon) \cup \Sigma^{(-)}(\varepsilon)$ where $\Sigma^{(0)}(\varepsilon) \subset \{\lambda \in \mathbb{R} : |\lambda| < \alpha\}$ and $\Sigma^{(-)}(\varepsilon) \subset \{\lambda \in \mathbb{R} : \lambda < -\beta\}$.*

In the following developments, we will assume that $\alpha < \beta$ and that $I \subset [-\varepsilon_1, \varepsilon_1]$. We then have that $\Sigma^{(0)}(\varepsilon) \cap \Sigma^{(-)}(\varepsilon) = \emptyset$ for all $\varepsilon \in I$.

Next, fix $\lambda_* \in (-\beta, -\alpha)$, so that the family of translated equations

$$\frac{dx}{ds} = [-\lambda_* I + l_\varepsilon(\tau_{s/|\varepsilon|}(p))]x$$

admits an ED over $\mathfrak{P}$. This family is of course $(3.7)_p$ with $\lambda = \lambda_*$. Let us write $Q_{p,\varepsilon}$ for the dichotomy projection of this family of equations; see Definition 3.2. It turns out that, if $\varepsilon \in I$, then $Q_{p,\varepsilon}$ does not depend on the choice of $\lambda_* \in (-\beta, -\alpha)$ [SS]. Furthermore, one has the following important continuity result; see [Co], [SS].

Theorem 3.4. *The mapping*

$$(p, \varepsilon) \mapsto \begin{cases} Q_{p,\varepsilon} & p \in \mathfrak{P},\ 0 \neq \varepsilon \in I, \\ Q_0 & p \in \mathfrak{P},\ \varepsilon = 0, \end{cases}$$

is continuous on $\mathfrak{P} \times I$.

Let us write $Q_{p,0} = Q_0$ for all $p \in \mathfrak{P}$.

We can now describe the integral manifold theory for the nonlinear family $(3.6)_p$. Let us write $L^{(0)}(p, \varepsilon) = \mathrm{Ker}(Q_{p,\varepsilon}) \subset \mathbb{R}^d$ and $L^{(-)}(p, \varepsilon) = \mathrm{Im}(Q_{p,\varepsilon}) \subset \mathbb{R}^d$. Let $e = \dim L^{(0)}$ so that $d - e = \dim L^{(-)}$. It follows from Theorem 3.4 that $\dim L^{(0)}(p, \varepsilon) = e$ and $\dim L^{(-)}(p, \varepsilon) = d - e$ for all $(p, \varepsilon) \in \mathfrak{P} \times I$. If $\delta > 0$ is a real number, let $B_\delta = \{x \in \mathbb{R}^d : |x| \leq \delta\}$, then set $L^{(0)}(\delta, p, \varepsilon) = L^{(0)}(p, \varepsilon) \cap B_\delta$, $L^{(-)}(\delta, p, \varepsilon) = L^{(-)}(p, \varepsilon) \cap B_\delta$ for each $(p, \varepsilon) \in \mathfrak{P} \times I$. Let us note that, for each $0 \neq \varepsilon \in I$, the family of nonlinear equations $(3.6)_p$ induces a skew-product local flow on $\mathfrak{P} \times \mathbb{R}^d$. Precisely, if $p \in \mathfrak{P}$ and $x_0 \in \mathbb{R}^d$, set $\tilde{\tau}_2^\varepsilon(p, x_0, s) = \varphi(s)$ where $\varphi(s)$ is the maximal solution of $(3.6)_p$ satisfying $\varphi(0) = x_0$. Then $\tilde{\tau}^\varepsilon$ is a local skew-product flow on $\mathfrak{P} \times \mathbb{R}^d$ which covers the "fast flow" τ^ε on $\mathfrak{P}$ defined by $\tau_s^\varepsilon(p) = \tau_{s/|\varepsilon|}(p)$. Say that a subset $Y \subset \mathfrak{P} \times \mathbb{R}^d$ is *locally invariant* with respect to $\tilde{\tau}^\varepsilon$ if, for each $y = (p, x) \in Y$, there exists $s_0 > 0$ such that, if $|s| < s_0$, then $\tilde{\tau}^\varepsilon(y, s) \in Y$.

Proposition 3.5 ([FJP])**.** *There exist positive numbers $\varepsilon_2 \leq \varepsilon_1$ and $\delta \in \mathbb{R}$ together with a family of maps*

$$h_{p,\varepsilon} : L^{(0)}(\delta, p, \varepsilon) \longrightarrow L^{(-)}(\delta, p, \varepsilon),\ p \in \mathfrak{P},\ 0 < |\varepsilon| \leq \varepsilon_2$$

such that the following conditions hold.

(a) *If $0 \neq |\varepsilon| \leq \varepsilon_2$, then the set*

$$M_\varepsilon = \bigcup_{p \in \mathfrak{P}} \{(p, x) : x = u + h_{p,\varepsilon}(u),\ u \in L^{(0)}(\delta, p, \varepsilon)\}$$

is a locally invariant subset of $\mathfrak{P} \times \mathbb{R}^d$ with respect to the local flow $\tilde{\tau}^\varepsilon$. We call M_ε an integral manifold *of the family* $(3.6)_p$.

(b) *Introduce the "center bundle" $E_\delta^{(0)} = \{(p, u, \varepsilon) : u \in L^{(0)}(\delta, p, \varepsilon), p \in \mathfrak{P}, 0 \neq |\varepsilon| \leq \varepsilon_2\}$ and the "stable bundle" $E_\delta^{(-)} = \{(p, u, \varepsilon) : u \in L^{(-)}(\delta, p, \varepsilon), p \in \mathfrak{P}, 0 \neq |\varepsilon| \leq \varepsilon_2\}$. Then the map $(p, u, \varepsilon) \mapsto (p, h_{p,\varepsilon}(u), \varepsilon) : E_\delta^{(0)} \to E_\delta^{(-)}$ is continuous. Moreover, for each $p \in \mathfrak{P}$ and $0 \neq |\varepsilon| \leq \varepsilon_2$, the map $u \mapsto h_{p,\varepsilon}(u) : L^{(0)}(\delta, p, \varepsilon) \to \mathbb{R}^d$ is of class C^r. For each $k = 0, 1, \ldots, r$, the Fréchet derivatives $\partial_u^k h_{p,\varepsilon}$, $p \in \mathfrak{P}$, $0 \neq |\varepsilon| \leq \varepsilon_2$, are well-defined. The collection of maps $\{h_{p,\varepsilon}\}$ is F^r in the sense of Foster* [Fo] *for each ε with $0 \neq |\varepsilon| \leq \varepsilon_2$. This means that, for each $k = 1, \ldots, r$ one has the following statement. If $(p_n, u_n, \varepsilon_n) \to (p, u, \varepsilon)$*

in $E^{(0)}_\delta$, *and if* $x^{(1)}_n \to x^{(1)}, \dots, x^{(k)}_n \to x^{(k)}$ *are convergent sequences in* $\mathbb{R}^d$, *then* $\partial^k_{u_n} h_{p_n,\varepsilon_n}(x^{(1)}_n, \dots, x^{(k)}_n) \to \partial^k_u h_{p,\varepsilon}(x^{(1)}, \dots, x^{(k)})$.

(c) *For each* $p \in \mathfrak{P}$ *and each* $0 \neq |\varepsilon| \leq \varepsilon_2$, *one has* $h_{p,\varepsilon}(0) = \partial_x h_{p,\varepsilon}(0) = 0$. *Thus the manifold* $M_\varepsilon \cap (\{p\} \times \mathbb{R}^d)$ *is tangent to* $L^{(0)}(p, \varepsilon)$ *at the origin* $x = 0$.

One actually has that the family $\{h_{p,\varepsilon}\}$ extends in a C^r-way to $\varepsilon = 0$. It is fairly clear what this means; in any case see [FJP].

From now on, we assume that $I \subset [-\varepsilon_2, \varepsilon_2]$. Our next goal is the following. Fix $\varepsilon \in I$ with $\varepsilon \neq 0$. We want to introduce (time-dependent) coordinates $u \in \mathbb{R}^e$, $v \in \mathbb{R}^{d-e}$ in $\mathbb{R}^d$ such that, for each $p \in \mathfrak{P}$, the family of linear systems $(3.5)_p$ assumes the block-diagonal form $\frac{du}{ds} = au$, $\frac{dv}{ds} = bv$. This is because we wish to apply Palmer's methods [Pa2] to the study of the asymptotic phase and the Pliss reduction principle relative to the integral manifold M_ε. It turns out, however, that in general we cannot achieve a block-diagonalization *over* $\mathfrak{P}$. Instead we must introduce an appropriate "extension" of the flow $(\mathfrak{P}, \{\tau^\varepsilon_s\})$ to do so. We now discuss this question in more detail.

It is convenient to assume that the original flow $(\mathfrak{P}, \{\tau_t\})$ is *minimal*, i.e. that the orbit $\{\tau_t(p) : t \in \mathbb{R}\}$ is dense in $\mathfrak{P}$ for each $p \in \mathfrak{P}$. (This implies that, for each $\varepsilon \neq 0$ in I, the fast flow $(\mathfrak{P}, \{\tau^\varepsilon_s\})$ is minimal.) See [E] for a study of minimal flows. The condition of minimality entails little loss of generality for the following reason. The flow $(\mathfrak{P}, \{\tau_t\})$ admits by assumption just one invariant measure μ, which is therefore ergodic. Let $\mathfrak{P}_\mu$ be the topological support of μ; that is, $\mathfrak{P}_\mu$ is the complement in $\mathfrak{P}$ of the union over all open sets $O \subset \mathfrak{P}$ satisfying $\mu(O) = 0$. Then $\mathfrak{P}_\mu$ is invariant and $(\mathfrak{P}_\mu, \{\tau_t\})$ is minimal.

We assume form now on that $(\mathfrak{P}, \{\tau_t\})$ is minimal and uniquely ergodic (one says that the flow is *strictly ergodic*). This implies that $(\mathfrak{P}, \{\tau^\varepsilon_s\})$ is strictly ergodic for each $0 \neq \varepsilon \in I$.

An *extension* $(\mathfrak{Q}, \{T_t\})$ of $(\mathfrak{P}, \{\tau_t\})$ consists of a compact metric space $\mathfrak{Q}$, a flow $\{T_t\}$ on $\mathfrak{Q}$, and a continuous surjective map $\pi : \mathfrak{Q} \to \mathfrak{P}$ such that

$$\tau_t(\pi(q)) = \pi(T_t(q)),\ t \in \mathbb{R},\ q \in \mathfrak{Q}.$$

One says that π is a *homomorphism* of the flows $(\mathfrak{Q}, \{T_t\})$ and $(\mathfrak{P}, \{\tau_t\})$. We will show that, if $0 \neq \varepsilon$ is sufficiently small, then there is an extension of $(\mathfrak{P}, \{\tau^\varepsilon_s\})$ with respect to which one can block-diagonalize the family $(3.5)_p$ after it has been "lifted" to $\mathfrak{Q}$ in an appropriate way. We discuss the appropriate concept of lifting. To lighten the notation, we will write $\{\tau_s\}$ instead of $\{\tau^\varepsilon_s\}$ when $\varepsilon \in I$ has been fixed.

Suppose that $0 \neq \varepsilon \in I$, and that $(\mathfrak{Q}, \{T_s\})$ is an extension of $(\mathfrak{P}, \{\tau_t\})$ with flow homomorphism $\pi : \mathfrak{Q} \to \mathfrak{P}$. Set $l(q) = l_\varepsilon(\pi(q))$ and $n(q, x) = n_\varepsilon(\pi(q), x)$. Consider the family of linear equations

$$x' = l(T_s(q))x \qquad (3.8)_q$$

and the family of nonlinear equations

$$x' = l(T_s(q))x + n(T_s(q), x). \qquad (3.9)_q$$

It is natural to view these families as lifts of the families $(3.5)_p$ and $(3.6)_p$. Clearly the statements of Theorems 3.3, 3.4 and 3.5 can be modified as to be valid for the lifted families $(3.8)_q$ and $(3.9)_q$. We will not write out these modified versions of Theorems 3.3-3.5.

For each integer $n \geq 1$, let $\mathbb{M}_n$ be the set of $n \times n$ real matrices with the usual operator norm. If $\varphi : \mathfrak{Q} \to \mathbb{M}_n$ is a map, let $|\varphi|_0 = \sup\{|\varphi(q)| : q \in \mathfrak{Q}\}$.

Theorem 3.6. *There is a positive number* $\varepsilon_3 \leq \varepsilon_2$ *such that, if* $0 \neq |\varepsilon| \leq \varepsilon_3$, *then there exist a minimal extension* $(\mathfrak{Q}, \{T_s\})$ *of* $(\mathfrak{P}, \{\tau_s\})$ *together with a continuous function*

$\sigma : \mathfrak{Q} \to \mathbb{M}_d$ with the following properties. First, the map $\sigma^{-1} : \mathfrak{Q} \to \mathbb{M}_d$ is well-defined and continuous. Second, the map $\sigma' : \mathfrak{Q} \to \mathbb{M}_d : \sigma'(q) = \frac{d}{ds}\sigma(T_s(q))\big|_{s=0}$ is well-defined and continuous. Third, for each $q \in \mathfrak{Q}$ the change of variables

$$x = \sigma(T_s(q))y$$

transforms equation $(3.8)_q$ into the block form

$$y' = \begin{pmatrix} a(T_s(q)) & 0 \\ 0 & b(T_s(q)) \end{pmatrix} y, \tag{3.10_q}$$

where $a : \mathfrak{Q} \to \mathbb{M}_e$ and $b : \mathfrak{Q} \to \mathbb{M}_{d-e}$ are continuous. Fourth, $|a|_0 \le |l|_0$ and $|b|_0 \le |l|_0$.

Proof. The first step involves an application of some results presented in Coppel's lecture notes ([Co, pp. 37-41]). See also Daletskii-Krein [DK].

Let us note that, if $\lambda \in (-\beta, -\alpha)$ is fixed, and if $0 \ne \varepsilon \in I$, then the family $(3.7)_p$ admits an ED over $\mathfrak{P}$ with a dichotomy constant k which does not depend on ε. Moreover, we can use Theorem 3.4 to determine a positive number $\varepsilon_3 \le \varepsilon_2$ such that, if $0 < |\varepsilon| \le \varepsilon_3$, then for every $p \in \mathfrak{P}$, the angle [DK] between $\mathrm{Im}(Q_{p,\varepsilon})$ and $\mathrm{Im}(Q_0)$ is less than $\pi/6$, and the angle between $\mathrm{Ker}(Q_{p,\varepsilon})$ and $\mathrm{Ker}(Q_0)$ is less than $\pi/6$.

Taking account of these facts, we can apply Lemma 3 on p. 41 of [Co] to determine a constant $\theta \ge 0$, which is independent of $p \in \mathfrak{P}$ and $0 \ne |\varepsilon| \le \varepsilon_3$, for which the following statements can be verified.

Let $\lambda \in (-\beta, -\alpha)$, $\bar{p} \in \mathfrak{P}$, and $0 \ne \bar{\varepsilon} \in I$ be fixed. Let us write $l(s) = -\lambda I + l_{\bar{\varepsilon}}(\tau_s^{\bar{\varepsilon}}(\bar{p}))$. Consider the linear system

$$\frac{dx}{ds} = l(s)x. \tag{3.11}$$

There is a continuously differentiable function $\sigma : \mathbb{R} \to \mathbb{M}_d$, with continuously differentiable inverse σ^{-1}, with the following properties.

(i) The change of variables

$$x = \sigma(s)y$$

transforms equation (3.11) into the block-form

$$y' = m(s)y,\ m(s) = \begin{pmatrix} a(s) & 0 \\ 0 & b(s) \end{pmatrix},\ s \in \mathbb{R}$$

where $a(\cdot) \in \mathbb{M}_e$ and $b(\cdot) \in \mathbb{M}_{d-e}$.

(ii) $|m(s)| \le |l(s)|$ for all $s \in \mathbb{R}$.

(iii) One has $\sup_{s\in\mathbb{R}} |\sigma(s)| \le \theta$, $\sup_{s\in\mathbb{R}} |\sigma(s)^{-1}| \le \theta$.

(iv) $\frac{d\sigma}{ds} = l(s)\sigma(s) - \sigma(s)l(s)$, $s \in \mathbb{R}$.

It follows from (iii) and (iv) that $\frac{d\sigma}{ds}$ is uniformly bounded, so σ and σ^{-1} are uniformly continuous functions. It then follows from (iv) that $\frac{d\sigma}{ds}$ is (bounded and) uniformly continuous.

The second step involves a Bebutov-type flow and basic methods of topological dynamics. We view σ as an element of the space $\mathcal{C} = \{c : \mathbb{R} \to \mathbb{M}_d : c \text{ is bounded and continuous}\}$. We equip $\mathcal{C}$ with the topology of uniform convergence on compact sets. There is a flow $\{T_s\}$ on $\mathcal{C}$ induced by the translations: thus $T_s(c)(u) = c(s+u)$ for $c \in \mathcal{C}$ and s, $u \in \mathbb{R}$. This is a flow of Bebutov-type [Be]. Since σ is uniformly continuous, the closure $\mathcal{C}_\sigma = \mathrm{cls}\{T_t(\sigma) : t \in \mathbb{R}\}$ is compact; clearly $\mathcal{C}_\sigma$ is $\{T_t\}$-invariant.

If $c \in \mathcal{C}_\sigma$, set $C(c) = c(0)$. Then $C : \mathcal{C}_\sigma \to \mathbb{M}_d$ is continuous. One can verify that $C^{-1} : \mathcal{C}_\sigma \to \mathbb{M}_d : C^{-1}(c) = c(0)^{-1}$ is well-defined and continuous. Moreover, a basic result of Analysis can be used to show that the map C' defined by $C'(c) = \frac{dc}{ds}(0)$ is well-defined and continuous.

Let $p \in \mathfrak{P}$ be a general point, and let $\{t_n\} \subset \mathbb{R}$ be a sequence such that $\tau_{t_n}(\bar{p}) \to p$. There is a subsequence $\{t_k\} \subset \{t_n\}$ such that $T_{t_k}(\sigma)$ converges in $\mathcal{C}$, say to σ_p. Let $l_p(s) = -\lambda I + l_{\bar{\varepsilon}}(\tau_s^{\bar{\varepsilon}}(p))$. One can check that, if l_p is substituted for l and σ_p is substituted for σ, then all statements (i)-(iv) above are valid when $m(s)$ is substituted by

$$m_p(s) = \sigma_p(s)^{-1} l_p(s) \sigma_p(s) - \sigma_p(s)^{-1} \frac{d\sigma_p}{ds}(s).$$

Caution: the function m_p is not *uniquely* determined by p, since different subsequences $\{t_k\}$ may give rise to different limit functions σ_p.

Now let $\mathfrak{Q} \subset \mathcal{C}_\sigma$ be an invariant set such that $(\mathfrak{Q}, \{T_s\})$ is minimal (see e.g. [E]). There is a natural projection $\pi : \mathfrak{Q} \to \mathbb{M}_d : \sigma(q) = C(q)$. By stepping through the definition, one can now verify all the assertions of Theorem 3.6. This completes the proof. □

From now on, we suppose that $I \subset [-\varepsilon_3, \varepsilon_3]$. We note an important corollary of the proof of Theorem 3.6.

Corollary 3.7. *Let $0 \neq \varepsilon \in I$, and let $(\mathfrak{Q}, \{T_s\})$ and σ be as in the statement of Theorem 3.6.*

(i) *There is a constant $\theta \geq 0$, which does not depend on $q \in \mathfrak{Q}$ and $0 \neq \varepsilon \in I$, such that $|\sigma(q)| \leq \theta$ and $|\sigma^{-1}(q)| \leq \theta$.*

(ii) *Let $\Phi_q^{(a)}(s)$ be the fundamental matrix solution of the system*

$$\frac{du}{ds} = a(T_s(q))u,$$

and let $\Phi_q^{(b)}(s)$ be the fundamental matrix solution of the system

$$\frac{dv}{ds} = b(T_s(q))v.$$

There is a constant $k_1 > 0$, which does not depend on $q \in \mathfrak{Q}$ and $0 \neq \varepsilon \in I$, such that

$$|\Phi_q^{(a)}(s)| \leq k_1 e^{\alpha|s|},\ s \in \mathbb{R},$$
$$|\Phi_q^{(b)}(s)| \leq k_1 e^{-\beta s},\ s \geq 0.$$

The number k_1 is only distantly related to the dichotomy constant k of the family $(3.7)_p$ introduced in the proof of Theorem 3.6, and may be much larger. Nevertheless, we will indicate k_1 by k in the succeeding developments.

Next let $0 \neq \varepsilon \in I$. Introduce a minimal flow $(\mathfrak{Q}, \{T_s\})$ which satisfies the conditions of Theorem 3.6. Let σ, a, and b be the functions of that theorem. Consider the nonlinear family $(3.9)_q$ obtained by lifting the family $(3.6)_p$ to $\mathfrak{Q}$. For each $q \in \mathfrak{Q}$, introduce the change of variables $x = \sigma(T_s(q))y$ in equation $(3.9)_q$. Set $y = \begin{pmatrix} u \\ v \end{pmatrix}$ where $u \in \mathbb{R}^e$, $v \in \mathbb{R}^{d-e}$, and $\mathbb{R}^d = \mathbb{R}^e \oplus \mathbb{R}^{d-e}$. Further set

$$g(q, u, v, \varepsilon) = \sigma^{-1}(q) n(q, \sigma(q)y),$$

then put $g = \begin{pmatrix} g_1 \\ g_2 \end{pmatrix}$ where $g_1 \in \mathbb{R}^e$, $g_2 \in \mathbb{R}^{d-e}$. Then equations $(3.9)_q$ take the form

$$\begin{aligned} \frac{du}{ds} &= a(T_s(q))u + g_1(T_s(q), u, v, \varepsilon), \\ \frac{dv}{ds} &= b(T_s(q))v + g_2(T_s(q), u, v, \varepsilon). \end{aligned} \qquad (3.12)_q$$

Fix $\varepsilon \in I$. Return to the spaces $L^{(0)}(p,\varepsilon)$, $L^{(-)}(p,\varepsilon)$, $L^{(0)}(\delta,p,\varepsilon)$, and $L^{(-)}(\delta,p,\varepsilon)$ of Proposition 3.5. These spaces can be lifted to $\mathfrak{Q}$ by setting $L^{(0)}(q,\varepsilon) = L^{(0)}(\pi(q),\varepsilon)$, etc.; we have made an obvious abuse of notation. The functions $h_{p,\varepsilon}$ lift naturally to functions $h_{q,\varepsilon} : L^{(0)}(\delta,q,\varepsilon) \to L^{(-)}(\delta,q,\varepsilon)$ where again we have abused notation. For a given number $\delta > 0$, let $\hat{\delta} = \delta\theta^{-1}$ where θ is the constant of Corollary 3.7. Let us write

$$\mathbb{R}^e_{\hat{\delta}} = \mathbb{R}^e \cap \{y \in \mathbb{R}^d : |y| \le \hat{\delta}\}.$$

We see that $\sigma(q)\mathbb{R}^e \subset L^{(0)}(q,\varepsilon)$, $\sigma(q)\mathbb{R}^{d-e} \subset L^{(-)}(q,\varepsilon)$, and $\sigma(q)\mathbb{R}^e_{\hat{\delta}} \subset L^{(0)}(\delta,q,\varepsilon)$ for all $q \in \mathfrak{Q}$.

Note that the solutions of the family $(3.12)_q$ of non-autonomous differential systems determine a local skew-product flow $\tilde{T}^\varepsilon$ on $\mathfrak{Q} \times \mathbb{R}^d$.

Consider the functions

$$\hat{h}_{q,\varepsilon} = \sigma(q)^{-1} h_{q,\varepsilon} \sigma(q) : \mathbb{R}^e_{\hat{\delta}} \longrightarrow \mathbb{R}^{d-e},$$

and set

$$\hat{M}_\varepsilon = \bigcup_{q \in \mathfrak{Q}} \{(q,y) : y = u + \hat{h}_{q,\varepsilon}(u),\ u \in \mathbb{R}^e_{\hat{\delta}}\}.$$

Then $\hat{M}_\varepsilon$ is a locally invariant subset of $\mathfrak{Q} \times \mathbb{R}^d$ with respect to the local flow $\tilde{T}^\varepsilon$. Moreover, the functions $\hat{h}_{q,\varepsilon}$ satisfy conditions analogous to those stated in parts (b) and (c) of Proposition 3.5. In particular, the collection of maps $\{\hat{h}_{q,\varepsilon} : q \in \mathfrak{Q}\}$ is F^r in the sense of Foster, and $\hat{h}_{q,\varepsilon}(0) = \partial_u \hat{h}_{q,\varepsilon}(0) = 0$. We abuse notation still again, and write δ for $\hat{\delta}$, $h_{q,\varepsilon}$ for $\hat{h}_{q,\varepsilon}$, and M_ε for $\hat{M}_\varepsilon$.

We will now discuss results concerning the asymptotic phase and the Pliss reduction principle which make reference to the set M_ε defined by the family $(3.12)_q$. It will be clear that these results imply corresponding statements which make reference to the set M_ε as originally defined for the family $(3.5)_p$.

The preceding constructions allow us to follow Palmer's arguments [Pa2] in a fairly straightforward way. He works in the context of autonomous differential systems, however. For the reader's convenience, we sketch how his statements and proofs can be modified so as to apply to our family of non-autonomous equations $(3.12)_q$.

Choose $\delta > 0$ so that $|\partial_u h_{q,\varepsilon}| \le 1$ for all $q \in \mathfrak{Q}$, $|u| \le \delta$, $0 \ne \varepsilon \in I$. Then $|h_{q,\varepsilon}(u)| \le |u|$ whenever $|u| \le \delta$. Fix $0 \ne \varepsilon \in I$. For each positive number $\Delta \le \delta$, let $\omega(\Delta)$ be the maximum, as q ranges over $\mathfrak{Q}$, of the upper bounds of the norms of the derivatives $\partial_u g_1$, $\partial_v g_1$, $\partial_u g_2$, $\partial_v g_2$, $\partial_u h_{q,\varepsilon}$ in the set $|u| \le \Delta$, $|v| \le \Delta$. Then $\omega(\Delta)$ decreases to zero as $\Delta \to 0^+$.

If $q \in \mathfrak{Q}$, consider the linearization of $(3.12)_q$:

$$\begin{pmatrix} \frac{du}{ds} \\ \frac{dv}{ds} \end{pmatrix} = \begin{pmatrix} a(T_s(q)) & 0 \\ 0 & b(T_s(q)) \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}. \tag{3.13$_q$}$$

It admits the fundamental matrix solution

$$\begin{pmatrix} \Phi^{(a)}_q(s) & 0 \\ 0 & \Phi^{(b)}_q(s) \end{pmatrix}.$$

By Corollary 3.7, there is a constant $k > 0$, which does not depend on $q \in \mathfrak{Q}$ and $0 \ne \varepsilon \in I$, such that

$$|\Phi^{(a)}_q(s)| \le k e^{\alpha|s|},\ s \in \mathbb{R},$$
$$|\Phi^{(b)}_q(s)| \le k e^{-\beta s},\ s \ge 0.$$

The conditions above are analogous to those stated on [Pa2, p. 274]. Caution: our numbers α resp. β play the roles of his β resp. α.

Lemma 3.8. *Let $0 \neq \varepsilon \in I$, $q \in \mathfrak{Q}$. Write $h(s,u) = h_{T_s(q),\varepsilon}(u)$, $a(s) = a(T_s(q))$, $b(s) = b(T_s(q))$, $g_i(s,u,v) = g_i(T_s(q),u,v)$, $1 \le i \le 2$, $s \in \mathbb{R}$, $u \in \mathbb{R}^e$, $v \in \mathbb{R}^{d-e}$. Let γ be a real number such that $0 < \gamma < \beta - \alpha$.*

There is a positive number $\Delta \le \delta/2$, which does not depend on the choice of ε and q, such that the following statements are valid. Let $S > 0$, and let $\begin{pmatrix} u(s) \\ v(s) \end{pmatrix}$ be a solution of $(3.12)_q$ which is defined on the interval $[0,S]$. Then the solution $\tilde{u}(s)$ of the system

$$\frac{du}{ds} = a(s)u + g_1(s,u,h(s,u)) \tag{3.14}$$

with $\tilde{u}(s) = u(s)$ is defined on $[0,S]$. Moreover, $|\tilde{u}(s)| \le 2\Delta$ and

$$|u(s) - \tilde{u}(s)| + |v(s) - h(s,\tilde{u}(s))| \le 2k|v(0) - h(0,u(0))|e^{-(\beta-\gamma)s}$$

for all $0 \le s \le S$.

Proof. Let us first show that $h(s,u)$ is a C^1 function of (s,u). This is not immediately obvious, because Proposition 3.5 states only that h is continuous in s. However, we will see that the local invariance of M_ε actually implies that h is C^1 in (s,u). Fix $\Delta < \delta$, and let $\mathbb{R}^e_\Delta = \{u \in \mathbb{R}^e : |u| \le \Delta\}$.

We will show that the partial derivatives $\partial_s h$, $\partial_u h$ exist at each point $(s_0,u_0) \in \mathbb{R} \times \mathbb{R}^e_\Delta$, and are jointly continuous on $\mathbb{R} \times \mathbb{R}^e_\Delta$. First note that, by Proposition 3.5, the partial Fréchet derivative $\partial_u h(s_0,u_0)$ is defined and continuous as (s_0,u_0) varies over $\mathbb{R} \times \mathbb{R}^e_\Delta$. Moreover one has

$$h(s_0,u_0+u) - h(s_0,u_0) - \partial_u h(s_0,u_0)u = o(u)$$

where $o(u)/|u| \to 0$ as $u \to 0$, uniformly in $s_0 \in \mathbb{R}$.

We show that the partial derivative $\partial_s h$ exists and is continuous on $\mathbb{R} \times \mathbb{R}^e_\Delta$. For this, fix $s_0 \in \mathbb{R}$ and $u_0 \in \mathbb{R}^e_\Delta$. Let $v_0 = h(s_0,u_0)$, and let $\begin{pmatrix} \bar{u}(\cdot) \\ \bar{v}(\cdot) \end{pmatrix}$ be the solution of $(3.12)_q$ satisfying $\bar{u}(s_0) = u_0$, $\bar{v}(s_0) = v_0$. We have

$$h(s_0+s,u_0) - h(s_0,u_0) = h(s_0+s,u_0) - h(s_0+s,\bar{u}(s_0+s)) + h(s_0+s,\bar{u}(s_0+s)) - h(s_0,u_0).$$

However $h(s_0+s,u_0) - h(s_0+s,\bar{u}(s_0+s)) = -\partial_u h(s_0+s,u_0)(\bar{u}(s_0+s)-u_0) + o(\bar{u}(s_0+s)-u_0)$ and $h(s_0+s,\bar{u}(s_0+s)) - h(s_0-u_0) = \frac{d\bar{v}}{ds}(s_0) + o(s)$ where we use the invariance of M_ε to obtain the second relation. Letting $s \to 0$ we obtain

$$\begin{aligned} \partial_s h(s_0,u_0) = &- \partial_u h(s_0,u_0)[a(s_0)u_0 + g_1(s_0,u_0,h(s_0,u_0))] \\ &+ [b(s_0)h(s_0,u_0) + g_2(s_0,u_0,h(s_0,u_0))]. \end{aligned} \tag{*}$$

The explicit expression for $\partial_s h$ in (*) shows that it is continuous in its arguments.

Now we follow the arguments of [Pa2]. Choose $\Delta > 0$ such that $\Delta \le \delta/2$ and $4k^2\omega(2\Delta) \le \min\{2\gamma, \beta - \alpha - \gamma, 4k^2\}$. Let $\begin{pmatrix} u(s) \\ v(s) \end{pmatrix}$ be the solution of $(3.12)_q$ referred to in the statement of the present lemma. Write $z(s) = v(s) - h(s,u(s))$. Then

$$\begin{aligned} \frac{dz}{ds} =& b(s)z(s) + g_2(s,u(s),v(s)) - g_2(s,u(s),h(s,u(s))) \\ &- \partial_u h(s,u(s))[g_1(s,u(s),v(s)) - g_1(s,u(s),h(s,u(s)))] \end{aligned}$$

where we used (*). One completes the proof if the lemma by mimicking the estimates in [Pa2]; we omit the details. □

Fix numbers $\gamma \in (0, \beta - \alpha)$ and $\Delta > 0$ which satisfy the conditions of Lemma 3.8. We retain the notation introduced in the statement of Lemma 3.8.

Corollary 3.9. *Fix $0 \neq \varepsilon \in I$ and $q \in \mathfrak{Q}$. Let $\begin{pmatrix} u(s) \\ v(s) \end{pmatrix}$ be a solution of $(3.12)_q$ such that $|u(s)| \leq \Delta$, $|v(s)| \leq \Delta$ for all $s \in \mathbb{R}$. Then $v(s) = h(s, u(s))$ for all $s \in \mathbb{R}$, and hence $\begin{pmatrix} u(s) \\ v(s) \end{pmatrix} \in M_\varepsilon$ for all $s \in \mathbb{R}$.*

This corollary may be proved by adapting the reasoning of [Pa2, Proposition 1]. The next statement is proved by appropriate modification of the arguments of [Pa2, Proposition 2].

Theorem 3.10 (Asymptotic phase)**.** *Let $\begin{pmatrix} u(s) \\ v(s) \end{pmatrix}$ be a solution of $(3.12)_q$ which is defined and satisfies $|u(s)| \leq \Delta$, $|v(s)| \leq \Delta$ for all $s \geq 0$. Then there exists a solution $u_\infty(s)$ of (3.14) such that*

$$|u(s) - u_\infty(s)| + |v(s) - h(s, u_\infty(s))| \leq 2k|v(0) - h(0, u(0))|e^{-(\beta-\gamma)s}$$

for $s \geq 0$.

Observe that $\begin{pmatrix} u_\infty(s) \\ h(s, u_\infty(s)) \end{pmatrix} \in M_\varepsilon$ for all $s \geq 0$. Thus the solution $\begin{pmatrix} u(s) \\ v(s) \end{pmatrix}$ "tracks" a positive semiorbit in M_ε as $s \to \infty$.

Finally we have

Theorem 3.11 (Pliss Reduction Principle)**.** *Let $A \subset \mathfrak{Q} \times \mathbb{R}^d$ be a compact, $\tilde{T}^\varepsilon$-invariant set such that, if $(q, y) \in A$ and $y = \begin{pmatrix} u \\ v \end{pmatrix}$, then $|u| \leq \Delta/2$ and $|v| \leq \Delta/2$. Then $A \subset M_\varepsilon$. Moreover, if A is a Lyapunov attractor relative to M_ε, then it is a Lyapunov attractor relative to $\mathfrak{Q} \times \mathbb{R}^d$.*

Proof. It follows immediately from Corollary 3.9 that $A \subset M_\varepsilon$.

We indicate how the second statement can be proved. Introduce the family of equations

$$\frac{du}{ds} = a(T_s(q))u + g_1(T_s(q), u, h_{T_s(q),\varepsilon}(u)). \qquad (3.15)_q$$

Let $U = \{u \in \mathbb{R}^e : |u| < \delta\}$. The family $(3.15)_q$ gives rise to a local flow on $\mathfrak{Q} \times U$. The mapping $i : \mathfrak{Q} \times U \to M_\varepsilon : (q, u) \mapsto (q, h_{q,\varepsilon}(u))$ is a diffeomorphism onto its image, which maps trajectories of the local flow in $\mathfrak{Q} \times U$ onto trajectories of the local flow in M_ε. Let $A_0 \subset \mathfrak{Q} \times U$ be the compact invariant subset which is the preimage of A with respect to this diffeomorphism.

One now argues as in [Pa2, Proposition 3] to show that, if A_0 is a Lyapunov attractor relative to (the local flow on) $\mathfrak{Q} \times U$, then it is a Lyapunov attractor relative to $\mathfrak{Q} \times \mathbb{R}^d$. This clearly implies that the second statement of Theorem 3.11 is valid. ☐

References

[Be] M. Bebutov, Sur les systèmes dynamiques dans l'espace des fonctions continues, Bull. Math. Univ. Moscou 2 (1940).

[CRC] T. Caraballo, J. Real, I. D. Chueshov, Pullback attractors for stochastic heat equations in materials with memory, Discr. Contin. Dyn. Syst. Ser. B (2008), 525–540.

[Ca] J. Carr, Applications of Centre Manifold Theory, Springer-Verlag, Berlin (1981).

[CDG] D. Cheban, J. Duan, A. Gherco, Generalization of the second Bogoliubov's theorem for non-almost periodic systems, Nonlin. Anal.: Real World Appl. 4 (2003), 599–613.

[CKS] D. Cheban, P.E. Kloeden, B. Schmalfuß, The relationship between pullback, forward and global attractors of nonautonomous dynamical systems, Nonlinear Dyn. Syst. Theory, 2 (2002), 9-28.

[Co] W.A. Coppel, Dichotomies in Stability Theory, Lecture Notes in Math. 629, Springer-Verlag, Berlin, 1978.

[CP] W.A. Coppel, K.J. Palmer, Averaging and integral manifolds, Bull. Austral. Math. Soc. 2 (1970), 197-222.

[DK] L. Daletskii, M.G. Krein, Stability of Solutions of Differential Equations in Banach Space, Trans. Math. Monographs 43, Amer. Math. Soc., Providence (1974).

[E] R. Ellis, Lectures on Topological Dynamics, Benjamin, New York, 1969.

[FJM] R. Fabbri, R. Johnson, F. Mantellini, A nonautonomous saddle-node bifurcation pattern, Stoch. Dyn. 4(3) (2004), 335–350.

[FJP] R. Fabbri, R. Johnson, K.J. Palmer, Another look at averaging and integral manifolds, Jour. Difference Eqns. Appl. 13 (2007), 723–739.

[Fo] M.J. Foster, Fibre derivatives and stable manifolds: a note, Bull. London Math. Soc. 8 (1976), 286–288.

[Ja] K. Janglajew, On the reduction principle of difference equations, Dyn. Contin. Discr. Impuls. Syst. 6 (1999), 381-388.

[JM] R. Johnson, F. Mantellini, A nonautonomous transcritical bifurcation problem with an application to quasi-periodic bubbles, Discr. Contin. Dyn. Syst. 9 (2003), 209–224.

[KR] P.E. Kloeden, H. Rodrigues, Dynamics of a class of ODEs more general than almost periodic, manuscript (2008).

[Pa1] K.J. Palmer, Averaging and integral manifolds (II), Bull. Austral. Math. Soc. 2 (1970), 369–400.

[Pa2] K.J. Palmer, On the stability of center manifolds, Zeit. angew. Math. Phys. 38 (1987), 273–278.

[Pl] V. Pliss, Principle of reduction in the theory of stability of motion, Izv. Akad. Nauk SSSR, Ser. Mat. 28 (1964), 1297–1324 (Russian).

[NS] V. Nemytskii, V. Stepanoff, Qualitative Theory of Differential Equations, Princeton University Press, Princeton, NJ, 1960.

[Pö] C. Pötzsche, Stability of center fiber bundles for nonautonomous difference equations, Fields Inst. Commun. 42 (2004), 295–304.

[RJ] A. Reinfelds, K. Janglajew, Reduction principle in the theory of stability of difference equations, Discr. Contin. Dyn. Syst. - Supplement (2007), 864–874.

[SS] R.J. Sacker, G.R. Sell, A spectral theory for linear differential systems, Jour. Diff. Eqns. 27 (1978), 320–358.

[S-H] K.R. Schenk-Hoppé, Random attractors: general properties, existence and applications to stochastic bifurcation theory, Discr. Contin. Dyn. Syst. 4(1) (1998), 99–130.

[S] B. Schmalfuss, Backward cocycles and attractors of stochastic differential equations, in: V. Reitmann, T. Riedrich, N. Koksch (Eds.), International Seminar on Applied Mathematics-Nonlinear Dynamics: Attractor Approximation and Global Behaviour, TU Dresden (1992), 185-192.

[Se] G.R. Sell, Topological Dynamics and Ordinary Differential Equations, Van Nostrand-Reinhold, New York, 1971.

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