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EXPONENTIAL ORDERING FOR NEUTRAL FUNCTIONAL DIFFERENTIAL EQUATIONS WITH NON-AUTONOMOUS LINEAR D-OPERATOR

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Dedicated to Professor Russell A. Johnson on the occasion of his 60th birthday

Abstract. We study neutral functional differential equations with stable linear non-autonomous D-operator. The operator of convolution $\widehat{D}$ transforms BU into BU. We show that, if D is stable, then $\widehat{D}$ is invertible and, besides, $\widehat{D}$ and $\widehat{D}^{-1}$ are uniformly continuous for the compact-open topology on bounded sets. We introduce a new transformed exponential order and, under convenient assumptions, we deduce the 1-covering property of minimal sets. These conclusions are applied to describe the amount of material in a class of compartmental systems extensively studied in the literature.

1. Introduction

The present work is one of a series of papers devoted to the study of the long term behavior of the solutions of non-autonomous monotone differential equations and to provide a local or global qualitative description of the corresponding phase spaces (see Shen and Yi [24], Jiang and Zhao [12], and Novo, Obaya and Sanz [18] among others). This paper continues the theory initiated by Muñoz-Villarragut, Novo and Obaya [16], and Novo, Obaya and Villarragut [19], where a dynamical theory for non-autonomous functional differential equations (FDEs for short) with infinite delay and for non-autonomous neutral functional differential equations (NFDEs for short) with infinite delay and stable autonomous linear D-operator was developed. We now consider non-autonomous NFDEs with non-autonomous linear D-operator. In this situation, the main conclusions of the previous papers do not remain valid and, thus, the extension of the theory requires to solve new and challenging problems, which becomes the main objective of this paper. We also introduce an alternative definition of the exponential ordering which can be applied in the present context with more generality preserving the dynamical behavior of the previous theory.

We assume some recurrence properties on the temporal variation of the NFDEs; thus, its solutions induce a skew-product semiflow with a minimal flow $(\Omega,\sigma,\mathbb{R})$ on the base. In particular, the uniform almost periodic and almost automorphic cases are included in this formulation. The skew-product formalism permits the study of the trajectories using methods of ergodic theory and topological dynamics. We

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introduce the phase space $BU \subset C((-\infty, 0], \mathbb{R}^m)$ of bounded and uniformly continuous functions with the supremum norm, where the standard theory provides existence, uniqueness and continuous dependence of the solutions. For each $r > 0$, we denote $B_r = \{x \in BU : \|x\|_\infty \le r\}$. In our present setting, every bounded trajectory is relatively compact for the compact-open metric topology, the restriction of the semiflow to its omega-limit sets is continuous for the metric and it admits a flow extension. In fact, the main conclusions of this theory are obtained by the application of the metric topology on compact invariant sets. We introduce a non-autonomous operator $D : \Omega \times BU \to \mathbb{R}^m$ which is linear and continuous for the norm in its second variable, continuous for the metric topology on the bounded subsets of BU, and atomic at zero. We obtain an integral representation $D(\omega, x) = \int_{-\infty}^0 [d\mu(\omega)]x$, $(\omega, x) \in \Omega \times BU$, where $\mu(\omega)$ is an $m \times m$ matrix of real Borel regular measures with finite total variation and $\det(\mu(\omega)(\{0\})) \neq 0$ for every $\omega \in \Omega$. We can define $\widehat{D} : \Omega \times BU \to \Omega \times BU$ by $\widehat{D}(\omega, x) = (\omega, \widehat{D}_2(\omega, x))$ with $\widehat{D}_2(\omega, x)(t) = \int_{-\infty}^0 [d\mu(\omega{\cdot}t)(s)]x(t+s)$ for all $t \le 0$. Then $\widehat{D}_2$ is well-defined, it is linear and continuous for the norm in its second variable for all $\omega \in \Omega$ and it is uniformly continuous on $\Omega \times B_r$ for every $r > 0$ when we consider the restriction of the compact-open topology in the second factor. We prove that, if D is stable in the sense of Hale [6], and Hale and Verduyn-Lunel [7], then $\widehat{D}$ is invertible, $(\widehat{D}^{-1})_2(\omega, \cdot)$ is linear and continuous on BU and $\widehat{D}^{-1}$ is uniformly continuous on $\Omega \times B_r$ for every $r > 0$ when we take the restriction of the compact-open topology in the second factor. This behavior characterizes the stability of the operator. In fact D is stable if and only if $\widehat{D}$ is invertible and $\widehat{D}^{-1}$ is continuous for the product metric topology on bounded subsets. In addition, if D is stable and the flow $(\Omega, \sigma, \mathbb{R})$ is almost periodic, then $\widehat{D}$ and $\widehat{D}^{-1}$ are uniformly continuous on $\Omega \times B_r$ for all $r > 0$ when we take the norm in the second factor. In this almost periodic context, our theory is completely analogous to the one obtained in [16] for the autonomous operators D and $\widehat{D}$.

The exponential order was extensively studied by Smith and Thieme [25], [26] for autonomous functional differential equations with finite delay. Assuming that the vector field satisfies a strong version of a non-standard monotonicity condition and a supplementary irreducibility assumption, they proved that the induced semiflow is eventually strongly monotone in the phase space of Lipschitz functions and strongly order preserving in the phase space of continuous functions, from which they deduced the quasiconvergence of the trajectories. Krisztin and Wu [15] studied the dynamical properties of scalar neutral functional differential equations with finite delay which induce a monotone semiflow for the exponential ordering, and Wu and Zhao [30] analyzed the same question in a class of evolutionary equations with applications to reaction-diffusion models. Skew-product semiflows generated by families of non-autonomous FDEs with infinite delay and NFDEs with infinite delay and autonomous stable D-operator which are monotone for the exponential order were studied in [19]. Assuming some adequate hypotheses including the boundedness, relative compactness and uniform stability for the order on bounded subsets of the trajectories, they deduced that the omega-limit set of either each trajectory or at least each trajectory with Lipschitz continuous initial datum is a copy of the base, i.e. it reproduces exactly the dynamics exhibited by the time variation of the equation. It is important to point out that no irreducibility conditions are required, and hence the conclusion can be applied to general problems under natural physical conditions.

In the setting considered in this paper, it is obvious that the invertibility of the operator $\widehat{D}$ allows us to transform the original NFDEs with a time dependent D-operator into FDEs to which the conclusions in [19] can be applied. Thus we can define a *transformed exponential order* on our original phase space: we say that a semiflow is monotone for the transformed exponential order when the transformed semiflow is monotone for the exponential order introduced in [19]. This transformed exponential order has the same dynamical properties as the direct exponential order; under natural conditions of relative compactness and stability, the omega-limit sets are copies of the base. Two different approaches may be taken; on the one hand, we can use the exponential order for the transformed FDE on a compact interval of $(-\infty, 0]$ and the standard ordering on its complementary set; on the other hand, we can consider the exponential order for the transformed FDE on the complete interval $(-\infty, 0]$. Each approach yields different dynamical consequences.

The foregoing conclusions are applied to the study the evolution of the amount of material within the compartments of some non-autonomous neutral compartmental systems, extending previous results of the literature. Compartmental systems have been used as mathematical models for the study of the dynamical behavior in biological and physical sciences which depend on local mass balance conditions (see Jacquez [9], Jacquez and Simon [10], [11], Haddad, Chellaboina and Hiu [5], and the references therein). The papers [15], [16], [19], Arino and Bourad [1], and Wu and Freedman [29] apply dynamical methods in order to study monotone neutral compartmental systems. In this work, we consider a general class of compartmental systems and solve, under convenient hypotheses, the same kind of problems by using the transformed exponential order. Assuming the monotonicity and the existence of an adequate bounded semitrajectory, we conclude that either every trajectory or at least every trajectory with Lipschitz continuous transformed initial datum is bounded and uniformly stable for the transformed order on bounded sets.

We analyze in detail some specific closed neutral compartmental systems whose explicit expression ensures the stability of D. Now the total mass of the system is invariant and we obtain precise conditions guaranteeing the monotonicity for the transformed exponential order of the semiflow. Under these conditions, the omega-limit set of either every trajectory or at least every trajectory with Lipschitz continuous transformed initial datum is a copy of the base. We also give some alternative conditions leading to the monotonicity of the semiflow for the direct exponential order, which requires the differentiability of the coefficients of D rather than just their continuity, used for the transformed exponential order. Thus its applicability becomes much more restrictive. In addition the transformed exponential order is also more advantageous when dealing with rapidly oscillating coefficients of D; as a consequence, this is the natural exponential order when dealing with NFDEs with recurrent linear D-operator.

We now briefly describe the structure of the paper. Some basic notions on topological dynamics used throughout the rest of the paper are stated in Section 2. In Section 3, we study the invertibility of $\widehat{D}$ and show that, when D is stable, the theories of $\widehat{D}$ and $\widehat{D}^{-1}$ are symmetric. In Section 4, we introduce the transformed exponential order and prove that the main conclusions of Sections 4 and 5 in [19] remain valid for this order. Section 5 studies the boundedness and stability for the order of the trajectories of a quite general class of non-autonomous compartmental systems. Finally, in Section 6 we analyze some specific closed neutral compartmental systems, we characterize the relatively compact trajectories and prove that the minimal sets are copies of the base.

2. Some preliminaries

Let (Ω, d) be a compact metric space. A real *continuous flow* $(\Omega, \sigma, \mathbb{R})$ is defined by a continuous mapping $\sigma : \mathbb{R} \times \Omega \to \Omega,\ (t, \omega) \mapsto \sigma(t, \omega)$ satisfying

(i) $\sigma_0 = \mathrm{Id}$,
(ii) $\sigma_{t+s} = \sigma_t \circ \sigma_s$ for each $s,\ t \in \mathbb{R}$,

where $\sigma_t(\omega) = \sigma(t, \omega)$ for all $\omega \in \Omega$ and $t \in \mathbb{R}$. The set $\{\sigma_t(\omega) : t \in \mathbb{R}\}$ is called the *orbit* or the *trajectory* of the point ω. We say that a subset $\Omega_1 \subset \Omega$ is σ*-invariant* if $\sigma_t(\Omega_1) = \Omega_1$ for every $t \in \mathbb{R}$. A subset $\Omega_1 \subset \Omega$ is called *minimal* if it is compact, σ-invariant and its only nonempty compact σ-invariant subset is itself. Every compact and σ-invariant set contains a minimal subset; in particular it is easy to prove that a compact σ-invariant subset is minimal if and only if every trajectory is dense. We say that the continuous flow $(\Omega, \sigma, \mathbb{R})$ is *recurrent* or *minimal* if Ω is minimal.

The flow $(\Omega, \sigma, \mathbb{R})$ is *distal* if for any two distinct points $\omega_1,\ \omega_2 \in \Omega$ the orbits keep at a positive distance, that is, $\inf_{t\in\mathbb{R}} d(\sigma(t, \omega_1), \sigma(t, \omega_2)) > 0$. The flow $(\Omega, \sigma, \mathbb{R})$ is *almost periodic* when for every $\varepsilon > 0$ there is a $\delta > 0$ such that, if $\omega_1,\ \omega_2 \in \Omega$ with $d(\omega_1, \omega_2) < \delta$, then $d(\sigma(t, \omega_1), \sigma(t, \omega_2)) < \varepsilon$ for every $t \in \mathbb{R}$. Equivalently, the flow $(\Omega, \sigma, \mathbb{R})$ is almost periodic if the family $\{\sigma_t\}_{t\in\mathbb{R}}$ is equicontinuous. If $(\Omega, \sigma, \mathbb{R})$ is almost periodic, it is distal. The converse is not true; even if $(\Omega, \sigma, \mathbb{R})$ is minimal and distal, it does not need to be almost periodic. For the main properties of almost periodic and distal flows we refer the reader to Ellis [3], Sacker and Sell [20], and Sell [22], [23]. The reference Fink [4] describes the basic theory of almost periodic functions and almost periodic ordinary differential equations.

A *flow homomorphism* from another continuous flow $(Y, \Psi, \mathbb{R})$ to $(\Omega, \sigma, \mathbb{R})$ is a continuous map $\pi\colon Y \to \Omega$ such that $\pi(\Psi(t, y)) = \sigma(t, \pi(y))$ for every $y \in Y$ and $t \in \mathbb{R}$. If π is also bijective, it is called a *flow isomorphism*. Let $\pi : Y \to \Omega$ be a surjective flow homomorphism and suppose $(Y, \Psi, \mathbb{R})$ is minimal (then, so is $(\Omega, \sigma, \mathbb{R})$). $(Y, \Psi, \mathbb{R})$ is said to be an *almost automorphic extension* of $(\Omega, \sigma, \mathbb{R})$ if there is $\omega \in \Omega$ such that $\mathrm{card}(\pi^{-1}(\omega)) = 1$. Then, actually $\mathrm{card}(\pi^{-1}(\omega)) = 1$ for ω in a residual subset $\Omega_0 \subseteq \Omega$; in the nontrivial case $\Omega_0 \subsetneq \Omega$ the dynamics can be very complicated. A minimal flow $(Y, \Psi, \mathbb{R})$ is *almost automorphic* if it is an almost automorphic extension of an almost periodic minimal flow $(\Omega, \sigma, \mathbb{R})$. Johnson [13], [14] contain examples of almost periodic differential equations with almost automorphic solutions which are not almost periodic. We refer the reader to the work in [24] for a survey of almost periodic and almost automorphic dynamics.

Let E be a complete metric space and $\mathbb{R}^+ = \{t \in \mathbb{R} : t \geq 0\}$. A *semiflow* $(E, \Phi, \mathbb{R}^+)$ is determined by a continuous map $\Phi : \mathbb{R}^+ \times E \to E,\ (t, x) \mapsto \Phi(t, x)$ which satisfies

(i) $\Phi_0 = \mathrm{Id}$,
(ii) $\Phi_{t+s} = \Phi_t \circ \Phi_s$ for all $t,\ s \in \mathbb{R}^+$,

where $\Phi_t(x) = \Phi(t, x)$ for each $x \in E$ and $t \in \mathbb{R}^+$. The set $\{\Phi_t(x) : t \geq 0\}$ is the *semiorbit* of the point x. A subset E_1 of E is *positively invariant* (or just Φ*-invariant*) if $\Phi_t(E_1) \subset E_1$ for all $t \geq 0$. A semiflow $(E, \Phi, \mathbb{R}^+)$ admits a *flow* *extension* if there exists a continuous flow $(E,\widetilde{\Phi},\mathbb{R})$ such that $\widetilde{\Phi}(t,x)=\Phi(t,x)$ for all $x\in E$ and $t\in\mathbb{R}^+$. A compact and positively invariant subset admits a flow extension if the semiflow restricted to it admits one.

Write $\mathbb{R}^-=\{t\in\mathbb{R}\,|\,t\le 0\}$. A *backward orbit* of a point $x\in E$ in the semiflow $(E,\Phi,\mathbb{R}^+)$ is a continuous map $\psi:\mathbb{R}^-\to E$ such that $\psi(0)=x$ and for each $s\le 0$ it holds that $\Phi(t,\psi(s))=\psi(s+t)$ whenever $0\le t\le -s$. If for $x\in E$ the semiorbit $\{\Phi(t,x):t\ge 0\}$ is relatively compact, we can consider the *omega-limit set* of x,

$$\mathcal{O}(x)=\bigcap_{s\ge 0}\operatorname{cls}\{\Phi(t+s,x):t\ge 0\}\,,$$

which is a nonempty compact connected and Φ-invariant set. Namely, it consists of the points $y\in E$ such that $y=\lim_{n\to\infty}\Phi(t_n,x)$ for some sequence $t_n\uparrow\infty$. It is well-known that every $y\in\mathcal{O}(x)$ admits a backward orbit inside this set. Actually, a compact positively invariant set M admits a flow extension if every point in M admits a unique backward orbit which remains inside the set M (see [24], part II).

A compact positively invariant set M for the semiflow $(E,\Phi,\mathbb{R}^+)$ is *minimal* if it does not contain any other nonempty compact positively invariant set than itself. If E is minimal, we say that the semiflow is minimal.

A semiflow is *of skew-product type* when it is defined on a vector bundle and has a triangular structure; more precisely, a semiflow $(\Omega\times X,\tau,\mathbb{R}^+)$ is a *skew-product* semiflow over the product space $\Omega\times X$, for a compact metric space (Ω,d) and a complete metric space (X,d), if the continuous map τ is as follows:

$$\begin{array}{rccl}\tau: & \mathbb{R}^+\times\Omega\times X & \longrightarrow & \Omega\times X\\ & (t,\omega,x) & \mapsto & (\omega{\cdot}t,u(t,\omega,x))\,,\end{array}\tag{2.1}$$

where $(\Omega,\sigma,\mathbb{R})$ is a real continuous flow $\sigma:\mathbb{R}\times\Omega\to\Omega,\ (t,\omega)\mapsto\omega{\cdot}t$, called the *base flow*. The skew-product semiflow (2.1) is *linear* if $u(t,\omega,x)$ is linear in x for each $(t,\omega)\in\mathbb{R}^+\times\Omega$. The definitions of stability, uniform stability and asymptotic stability for non-autonomous differential equations used along the paper can be found in [22], [23] (see also Conley and Miller [2] for an interesting remark).

3. Non-Autonomous stable linear D-operators

Given an $m\times m$ matrix $\mu=[\mu_{ij}]_{ij}$ of measures with finite total variation on a measurable space (Y,ζ) and a measurable subset of Y, $E\in\zeta$, $|\mu_{ij}|(E)$ will denote the total variation of μ_{ij} over E; besides, the maximum norm of the $m\times m$ matrix $[|\mu_{ij}|(E)]_{ij}$ will be denoted by $\|\mu\|_\infty(E)$ and the $m\times m$ matrix of positive measures $[|\mu_{ij}|]_{ij}$ will be denoted by $|\mu|$.

Let (Ω,d) be a compact metric space and let $\sigma:\mathbb{R}\times\Omega\to\Omega$ be a continuous real flow on Ω. We will denote $\omega{\cdot}t=\sigma(\omega,t)$, $t\in\mathbb{R}$, $\omega\in\Omega$. We will assume in the remainder of the paper that the flow σ is minimal.

Let $X=C((-\infty,0],\mathbb{R}^m)$, which is a Fréchet space when endowed with the compact-open topology, i.e. the topology of uniform convergence over compact subsets. This topology happens to be metric for the distance

$$\mathsf{d}(x,y)=\sum_{n=1}^{\infty}\frac{1}{2^n}\frac{\|x-y\|_n}{1+\|x-y\|_n}\,,\quad x,y\in X\,,$$

where $\|x\|_n = \sup_{s\in[-n,0]} \|x(s)\|$, and $\|\cdot\|$ denotes the maximum norm in $\mathbb{R}^m$. Let $BU \subset X$ be the Banach space

$$BU = \{x \in X : x \text{ is bounded and uniformly continuous}\}$$

with the supremum norm $\|x\|_\infty = \sup_{s\in(-\infty,0]} \|x(s)\|$. Given $r > 0$, we will denote

$$B_r = \{x \in BU : \|x\|_\infty \leq r\}.$$

As usual, given $I = (-\infty, a] \subset \mathbb{R}$, $t \in I$ and a continuous function $x : I \to \mathbb{R}^m$, x_t will denote the element of X defined by $x_t(s) = x(t + s)$ for $s \in (-\infty, 0]$.

Let $D : \Omega \times BU \to \mathbb{R}^m$ be an operator satisfying the following hypotheses:

(D1) D is linear and continuous in its second variable and the map $\Omega \to \mathcal{L}(BU, \mathbb{R}^m)$, $\omega \mapsto D(\omega, \cdot)$ is continuous;

(D2) for each $r > 0$, $D\colon \Omega \times B_r \to \mathbb{R}^m$ is continuous when we take the restriction of the compact-open topology to B_r, i.e. if $\omega_n \to \omega$ and $x_n \xrightarrow{\mathsf{d}} x$ as $n \to \infty$ with (ω_n, x_n), $(\omega, x) \in \Omega \times B_r$, then $\lim_{n\to\infty} D(\omega_n, x_n) = D(\omega, x)$.

Lemma 3.1. *For each $\omega \in \Omega$, there exists an $m \times m$ matrix $\mu(\omega) = [\mu_{ij}(\omega)]_{ij}$ of real Borel regular measures with finite total variation such that*

$$D(\omega, x) = \int_{-\infty}^{0} [d\mu(\omega)]x, \quad (\omega, x) \in \Omega \times BU.$$

Proof. This result follows by applying Proposition 3.1 in [16] to each $\omega \in \Omega$. □

For each $\omega \in \Omega$, let $B(\omega) = \mu(\omega)(\{0\})$. Now let $\nu(\omega) = B(\omega)\delta_0 - \mu(\omega)$, $\omega \in \Omega$, where δ_0 is the Dirac measure at 0, that is, $\int_{-\infty}^{0}[d\delta_0]x = x(0)$ for all $x \in BU$. It is clear that $|\nu_{ij}(\omega)|(\{0\}) = 0$ for all $i, j \in \{1, \ldots, m\}$ and all $\omega \in \Omega$. Besides, from the dominated convergence theorem, it follows that

$$\lim_{\rho\to 0^+} |\nu_{ij}(\omega)|([-\rho, 0]) = 0 \text{ and } \lim_{\rho\to\infty} |\nu_{ij}(\omega)|((-\infty, -\rho]) = 0$$

for each $\omega \in \Omega$.

Proposition 3.2. *$B : \Omega \to \mathbb{M}_m(\mathbb{R})$, $\omega \mapsto B(\omega)$ is a continuous map.*

Proof. For each $\rho > 0$, let $\varphi_\rho : (-\infty, 0] \to \mathbb{R}$ be the function given for $s \leq 0$ by

$$\varphi_\rho(s) = \begin{cases} 0 & \text{if } s \leq -2\rho, \\ \rho^{-1}s + 2 & \text{if } -2\rho < s \leq -\rho, \\ 1 & \text{if } -\rho < s \leq 0. \end{cases} \tag{3.1}$$

Let $i \in \{1, \ldots, m\}$ and let $\{\omega_n\}_n \subset \Omega$ be a sequence converging to some $\omega_0 \in \Omega$. On the one hand, a straightforward application of the dominated convergence theorem yields

$$\lim_{\rho\to 0^+} \int_{-\infty}^{0} [d\mu(\omega_n)]\varphi_\rho e_i = B(\omega_n)e_i \text{ for all } n \in \mathbb{N} \text{ and } \lim_{\rho\to 0^+} D(\omega_0, \varphi_\rho e_i) = B(\omega_0)e_i.$$

On the other hand, from (D1) we deduce that

$$\lim_{n\to\infty} \int_{-\infty}^{0} [d\mu(\omega_n)]\varphi_\rho e_i = D(\omega_0, \varphi_\rho e_i)$$

uniformly for $\rho > 0$, and the result follows immediately. □

Proposition 3.3. *Let $L : \Omega \times BU \to \mathbb{R}^m$, $(\omega, x) \mapsto B(\omega)x(0) - D(\omega, x)$. Then the mapping $\Omega \to \mathcal{L}(BU, \mathbb{R}^m)$, $\omega \mapsto L(\omega, \cdot)$ is continuous. Equivalently, for every sequence $\{\omega_n\}_n \subset \Omega$ converging to $\omega_0 \in \Omega$ and all i, $j \in \{1, \ldots, m\}$, we have that $\lim_{n\to\infty} |\nu_{ij}(\omega_n) - \nu_{ij}(\omega_0)|((-\infty, 0]) = 0$.*

Proof. Let ω_1, $\omega_2 \in \Omega$ and $x \in BU$ with $\|x\|_\infty \le 1$; then

$$\|L(\omega_1, x) - L(\omega_2, x)\| \le \|B(\omega_1) - B(\omega_2)\| + \|D(\omega_1, x) - D(\omega_2, x)\|.$$

The result follows from (D1) and Proposition 3.2. □

Corollary 3.4. *Under hypotheses* (D1)–(D2)*, the following statements hold:*

(i) $\lim_{\rho\to 0^+} \|\nu(\omega)\|_\infty([-\rho, 0]) = 0$ *uniformly for $\omega \in \Omega$;*
(ii) $\lim_{\rho\to\infty} \|\nu(\omega)\|_\infty((-\infty, -\rho]) = 0$ *uniformly for $\omega \in \Omega$.*

Proof. Let us prove (i). Note that, for each ω_1, $\omega_2 \in \Omega$ and each i, $j \in \{1, \ldots, m\}$, we have that

$$||\nu_{ij}(\omega_1)|([-\rho, 0]) - |\nu_{ij}(\omega_2)|([-\rho, 0])| \le |\nu_{ij}(\omega_1) - \nu_{ij}(\omega_2)|((-\infty, 0]).$$

Moreover, $\lim_{\rho\to 0^+} \|\nu(\omega)\|_\infty([-\rho, 0]) = 0$ for all $\omega \in \Omega$. This is a family of continuous functions decreasing to 0. As a result, Dini's theorem implies that this family converges to 0 uniformly for $\omega \in \Omega$. The proof of (ii) is analogous. □

Let us assume one more hypothesis on the operator D, which is a natural generalization of the atomic character as seen in [6] and [7]:

(D3) $B(\omega)$ is a regular matrix for all $\omega \in \Omega$.

Theorem 3.5. *For all $h \in C([0, \infty), \mathbb{R}^m)$ and all $(\omega, \varphi) \in \Omega \times BU$ with $D(\omega, \varphi) = h(0)$, there exists $x \in C(\mathbb{R}, \mathbb{R}^m)$ such that*

$$\begin{cases} D(\omega{\cdot}t, x_t) = h(t), & t \ge 0, \\ x_0 = \varphi. \end{cases} \tag{3.2$_\omega$}$$

Moreover, given $\rho > 0$, there are positive constants k^1_ρ, k^2_ρ such that for each $t \in [0, \rho]$

$$\|x_t\|_\infty \le k^1_\rho \sup_{0\le u\le t} \|h(u)\| + k^2_\rho \|\varphi\|_\infty\,.$$

This bound for the solution leads us to its uniqueness. Namely, if x^1, x^2 are solutions of the equation, then for all $t \ge 0$, we have

$$\begin{cases} D(\omega{\cdot}t, x^1_t - x^2_t) = 0\,, & t \ge 0\,, \\ x^1_0 - x^2_0 = 0, \end{cases}$$

Thus, given $t > 0$, $\|x^1(t) - x^2(t)\| \le k^1_t\, 0 + k^2_t\, 0 = 0$.

Theorem 3.6. *Let us define the map*

$$\begin{array}{rccc} \widehat{D}: & \Omega \times BU & \longrightarrow & \Omega \times BU \\ & (\omega, x) & \mapsto & (\omega, \widehat{D}_2(\omega, x)) \end{array} \tag{3.3}$$

where $\widehat{D}_2(\omega, x) : (-\infty, 0] \to \mathbb{R}^m$, $s \mapsto D(\omega{\cdot}s, x_s)$. Then $\widehat{D}$ is well defined, $\widehat{D}_2$ is linear and continuous for the norm in its second variable for all $\omega \in \Omega$ and, for all $r > 0$, $\widehat{D}$ is uniformly continuous on $\Omega \times B_r$ when we take the restriction of the compact-open topology to B_r. In addition, if the flow $(\Omega, \sigma, \mathbb{R})$ is almost periodic, then $\widehat{D}$ is uniformly continuous on $\Omega \times B_r$ for all $r > 0$ when we take the norm on B_r.

Proof. Let us check that $\widehat{D}$ is well defined. Let $(\omega, x) \in \Omega \times BU$ and let $h = \widehat{D}_2(\omega, x) : (-\infty, 0] \to \mathbb{R}^m$. From (D1) and the uniform continuity of σ on, say, $[0,1] \times \Omega$, it follows that, for all $\varepsilon > 0$, there exists $\delta \in (0,1)$ such that, if t, $s \le 0$ and $|t-s| < \delta$ then

$$\|D(\omega{\cdot}t,\cdot) - D(\omega{\cdot}s,\cdot)\|\,\|x\|_\infty \le \frac{\varepsilon}{2} \quad\text{and}\quad \sup_{\omega_1\in\Omega}\|D(\omega_1,\cdot)\|\,\|x_t - x_s\|_\infty \le \frac{\varepsilon}{2},$$

whence

$$\|D(\omega{\cdot}t, x_t) - D(\omega{\cdot}s, x_s)\| \le \varepsilon.$$

Clearly, $\|h\|_\infty \le \sup_{\omega_1\in\Omega}\|D(\omega_1,\cdot)\|\,\|x\|_\infty$ and, consequently, $h \in BU$. This way, $\widehat{D}$ is well defined.

The linearity of $\widehat{D}_2$ in its second variable is clear. Besides, for all $\omega \in \Omega$, the continuity of $\widehat{D}_2(\omega,\cdot)$ for the norm on BU is a straightforward consequence of (D1).

Let us check the uniform continuity of $\widehat{D}$ on $\Omega \times B_r$, $r \ge 0$, when we take the restriction of the compact-open topology to B_r. In order to do so, let us fix $\rho > 0$ and $\varepsilon > 0$; there is a $\delta > 0$ such that, for all ω^1, $\omega^2 \in \Omega$ with $d(\omega^1, \omega^2) < \delta$ and all $s \in [-\rho, 0]$, it holds that $\|D(\omega^1{\cdot}s,\cdot) - D(\omega^2{\cdot}s,\cdot)\| < \varepsilon/(2r)$. Thanks to Corollary 3.4, there exists $\rho_0 > 0$ such that $\sup_{\omega\in\Omega}\|\mu(\omega)\|_\infty((-\infty,-\rho_0]) < \varepsilon/(8r)$. Now, let (ω^1, x^1), $(\omega^2, x^2) \in \Omega \times B_r$ such that $d(\omega^1,\omega^2) < \delta$ and satisfying $\sup_{\omega\in\Omega}\|\mu(\omega)\|_\infty((-\infty,0])\|x^1 - x^2\|_{[-\rho-\rho_0,0]} < \varepsilon/4$. If $s \in [-\rho, 0]$, then

$$\begin{aligned}\|D(\omega^1{\cdot}s, x^1_s) - D(\omega^2{\cdot}s, x^2_s)\| &\le \|D(\omega^1{\cdot}s,\cdot) - D(\omega^2{\cdot}s,\cdot)\|\,r + \|D(\omega^2{\cdot}s,(x^1-x^2)_s)\|\\ &\le \frac{\varepsilon}{2r}r + \left\|\int_{-\infty}^{-\rho_0}[d\mu(\omega^2{\cdot}s)](x^1-x^2)_s\right\| + \left\|\int_{-\rho_0}^{0}[d\mu(\omega^2{\cdot}s)](x^1-x^2)_s\right\|\\ &\le \frac{\varepsilon}{2} + \frac{\varepsilon}{8r}2r + \frac{\varepsilon}{4} = \varepsilon.\end{aligned}$$

This inequality yields the expected result.

Finally, assume that $(\Omega, \sigma, \mathbb{R})$ is almost periodic. Let us prove that $\widehat{D}_2$ is uniformly continuous on $\Omega \times B_r$, $r > 0$, when we take the norm on B_r. From (D1) and the almost periodicity of $(\Omega,\sigma,\mathbb{R})$, it follows that, for all $\varepsilon > 0$, there exists $\delta > 0$ such that, if ω^1, $\omega^2 \in \Omega$, $s \le 0$ and $d(\omega^1,\omega^2) < \delta$, then $\|D(\omega^1{\cdot}s,\cdot) - D(\omega^2{\cdot}s,\cdot)\| < \varepsilon/(2r)$. Taking (ω^1, x^1), $(\omega^2, x^2) \in \Omega \times B_r$ with $d(\omega^1,\omega^2) < \delta$ and such that $\sup_{\omega\in\Omega}\|\mu(\omega)\|_\infty((-\infty,0])\|x^1-x^2\|_\infty < \varepsilon/2$, an argument similar to the previous one yields the desired property. □

Definition 3.7. The mapping D is said to be *stable* if there is a continuous function $c \in C([0,\infty),\mathbb{R})$ with $\lim_{t\to\infty} c(t) = 0$ such that, for each $(\omega,\varphi) \in \Omega \times BU$ with $D(\omega,\varphi) = 0$, the solution of the homogeneous problem

$$\begin{cases} D(\omega{\cdot}t, x_t) = 0\,, & t \ge 0\\ x_0 = \varphi\,, \end{cases}$$

satisfies $\|x(t)\| \le c(t)\,\|\varphi\|_\infty$ for each $t \ge 0$.

The following statement, whose proof is similar to the one in [6], provides a nonhomogeneous version of the concept of stability for a D-operator.

Theorem 3.8. *Let us assume that D is stable. Then there are a continuous function $c \in C([0,\infty),\mathbb{R})$ with $\lim_{t\to\infty} c(t) = 0$ and a positive constant $k > 0$ such that the* *solution of the equation*

$$\begin{cases} D(\omega{\cdot}t,x_t)=h(t)\,, & t\geq 0\,,\\ x_0=\varphi\,, \end{cases}$$

where $h\in C([0,\infty),\mathbb{R}^m)$*,* $(\omega,\varphi)\in\Omega\times BU$ *and* $D(\omega,\varphi)=h(0)$*, satisfies*

$$\|x(t)\|\leq c(t)\,\|\varphi\|_\infty+k\sup_{0\leq u\leq t}\|h(u)\|$$

for each $t\geq 0$*.*

It is easy to see that $\widehat{D}_2$ has the following integral representation:

$$\widehat{D}_2(\omega,x)(s)=B(\omega{\cdot}s)x(s)-\int_{-\infty}^0[d\nu(\omega{\cdot}s)]\,x_s$$

for each $(\omega,x)\in\Omega\times BU$ and $s\in(-\infty,0]$.

We are now in a position to state the main theorem of this section which assures the invertibility of $\widehat{D}$ and specifies the regularity of its inverse when the linear operator D is stable.

Theorem 3.9. *Under hypotheses* (D1)–(D3)*, the following statements hold:*

- (i) *let us assume that D is stable; then there is a positive constant $k>0$ such that $\|x^h\|_\infty\leq k\,\|h\|_\infty$ for all $h\in BU$, $\omega\in\Omega$ and $x^h\in BU$ satisfying $D(\omega{\cdot}s,x^h_s)=h(s)$ for $s\leq 0$;*
- (ii) *if D is stable, then $\widehat{D}$ is invertible, $\widehat{D}_2^{-1}(\omega,\cdot)$ is linear and continuous on BU for all $\omega\in\Omega$ when we consider the norm on BU, and $\widehat{D}^{-1}$ is uniformly continuous on $\Omega\times B_r$ for all $r>0$ when we take the restriction of the compact-open topology to B_r. In addition, if the flow $(\Omega,\sigma,\mathbb{R})$ is almost periodic, $\widehat{D}^{-1}$ is uniformly continuous on $\Omega\times B_r$ for all $r>0$ when we take the norm on B_r;*
- (iii) *let us assume that, for each $r>0$ and each sequence $\{(\omega_n,x_n)\}_n\subset\Omega\times BU$ such that $\|\widehat{D}_2(\omega_n,x_n)\|_\infty\leq r$, $\omega_n\to\omega\in\Omega$ and $\widehat{D}_2(\omega_n,x_n)\xrightarrow{\mathsf{d}}0$ as $n\to\infty$, it holds that $x_n(0)\to 0$ as $n\to\infty$; then D is stable.*

Proof. The proof of (i) is analogous to the one of Proposition 3.7 in [16] due to the uniform character over Ω of the function c and the constant k of Theorem 3.8.

Now, we will prove statement (ii). $\widehat{D}$ is injective because, if we have (ω^1,x^1), $(\omega^2,x^2)\in\Omega\times BU$ with $\widehat{D}(\omega^1,x^1)=\widehat{D}(\omega^2,x^2)$, then $\omega^1=\omega^2$ and, from (i) and the fact that $D(\omega^1{\cdot}s,x^1_s-x^2_s)=0$ for $s\leq 0$, we get $x^1=x^2$.

In order to show that $\widehat{D}$ is surjective, let $(\omega,h)\in\Omega\times BU$ and $\{h_n\}_n\subseteq B_r$, for some $r>0$, be a sequence of continuous functions whose components are of compact support such that $h_n\xrightarrow{\mathsf{d}}h$ as $n\to\infty$. Moreover, it is easy to choose them with the same modulus of uniform continuity as h. Let us check that, for each $n\in\mathbb{N}$, there is an $x^n\in BU$ such that $\widehat{D}_2(\omega,x^n)=h_n$, that is, $D(\omega{\cdot}s,x^n_s)=h_n(s)$ for $s\leq 0$ and $n\in\mathbb{N}$. Fix $n\in\mathbb{N}$ and $\rho_n>0$ such that $\mathrm{supp}(h_n)\subset[-\rho_n,0]$. Let $\widetilde{h}_n:[0,\infty)\to\mathbb{R}^m$ be the function defined for $t\geq 0$ by

$$\widetilde{h}_n(t)=\begin{cases} h_n(t-\rho_n) & \text{if } t\in[0,\rho_n],\\ h_n(0) & \text{if } t\geq\rho_n. \end{cases}$$

Since $\widetilde{h}_n(0)=0$, by Theorem 3.5, we have that there exists $\widetilde{x}^n\in C(\mathbb{R},\mathbb{R}^m)$ such that $D(\omega{\cdot}(t-\rho_n),\widetilde{x}^n_t)=\widetilde{h}_n(t)$ for $t\geq 0$ and $\widetilde{x}^n_0=0$. Let $x^n:(-\infty,0]\to\mathbb{R}^m$

be the function defined by $x^n(s) = \widetilde{x}^n(s+\rho_n)$ for $s \le 0$. Clearly, the function x^n is continuous and of compact support. Now, if $s \in [-\rho_n, 0]$, then $D(\omega{\cdot}s, x^n_s) = D(\omega{\cdot}(-\rho_n + (s+\rho_n)), \widetilde{x}^n_{s+\rho_n}) = \widetilde{h}_n(s+\rho_n) = h_n(s)$ and, if $s \le -\rho_n$, then $D(\omega{\cdot}s, x^n_s) = D(\omega{\cdot}(-\rho_n + (s+\rho_n)), \widetilde{x}^n_{s+\rho_n}) = D(\omega{\cdot}s, 0) = h_n(s) = 0$ as wanted.

From (i), there exists $k > 0$ such that $\|x^n\|_\infty \le k\|h_n\|_\infty \le kr$. Let us fix $\varepsilon > 0$; since the restriction of σ to $[-1,0]\times\Omega$ is uniformly continuous, we can fix $\delta > 0$ such that, for all $\tau \in [-\delta, 0]$ and all $s \le 0$, $\|h_n - (h_n)_\tau\|_\infty < \varepsilon/(2k)$ and $\|D(\omega{\cdot}(s+\tau), \cdot) - D(\omega{\cdot}s, \cdot)\| < \varepsilon/(2k^2r)$. For each $n \in \mathbb{N}$ and each $\tau \in [-\delta, 0]$, let $g^\tau_n : (-\infty, 0] \to \mathbb{R}^m$, $s \mapsto D(\omega{\cdot}s, (x^n - x^n_\tau)_s)$. Then, for all $s \le 0$, all $\tau \in [-\delta, 0]$ and all $n \in \mathbb{N}$,

$$\begin{aligned}\|g^\tau_n(s)\| \le & \|D(\omega{\cdot}s, x^n_s) - D(\omega{\cdot}(s+\tau), x^n_{s+\tau})\| + \|D(\omega{\cdot}(s+\tau), x^n_{s+\tau}) - D(\omega{\cdot}s, x^n_{s+\tau})\| \\ \le & \frac{\varepsilon}{2k} + \frac{\varepsilon}{2k^2r}\|x^n\|_\infty \le \frac{\varepsilon}{k}.\end{aligned}$$

From (i), we deduce again that

$$\|x^n - x^n_\tau\|_\infty \le k\,\|g^\tau_n\|_\infty \le k\,\frac{\varepsilon}{k} = \varepsilon$$

for all $n \in \mathbb{N}$ and all $\tau \in [-\delta, 0]$. Thus $\{x^n\}_n$ is equicontinuous and, consequently, relatively compact for the compact-open topology. Hence, there is a convergent subsequence of $\{x^n\}_n$, let us assume the whole sequence, i.e. there is a continuous function x such that $x^n \xrightarrow{\mathsf{d}} x$ as $n \to \infty$. Therefore, we have that $\|x\|_\infty \le k\,r$ and $\|x^n(s) - x^n(s+t)\| \to \|x(s) - x(s+t)\|$ as $n \to \infty$ for all $s, t \le 0$, which implies that $x \in BU$. From this, $x^n_s \xrightarrow{\mathsf{d}} x_s$ for each $s \le 0$ and the expression of D yields $D(\omega{\cdot}s, x^n_s) = h_n(s) \to D(\omega{\cdot}s, x_s)$, i.e. $D(\omega{\cdot}s, x_s) = h(s)$ for $s \le 0$ and $\widehat{D}_2(\omega, x) = h$. Then $\widehat{D}$ is surjective, as claimed.

Let us check that $\widehat{D}^{-1}$ is uniformly continuous on $\Omega \times B_r$, $r > 0$, when we take the restriction of the compact-open topology to B_r. Fix $\varepsilon > 0$ and $\rho > 0$; using Theorem 3.8, it is clear that we can find a $\rho_0 > 0$ such that $c(t) < \varepsilon/(4kr)$ for all $t \ge \rho_0$. Besides, there is a $\delta > 0$ such that, if ω_1, $\omega_2 \in \Omega$ and $d(\omega_1, \omega_2) < \delta$, then $\|D(\omega_1{\cdot}s, x) - D(\omega_2{\cdot}s, x)\| < \varepsilon/(4k)$ for all $s \in [-\rho_0 - \rho, 0]$ and all $x \in B_{kr}$, thanks to the uniform continuity of σ on $[-\rho_0 - \rho, 0]\times\Omega$ and (D1). Let (ω_1, h_1), $(\omega_2, h_2) \in \Omega \times B_r$ such that $d(\omega_1, \omega_2) < \delta$ and $\|h_1 - h_2\|_{[-\rho_0-\rho, 0]} \le \varepsilon/(4k)$; $x_1 = \widehat{D}_2^{-1}(\omega_1, h_1)$, $x_2 = \widehat{D}_2^{-1}(\omega_2, h_2) \in B_{kr}$ thanks to (i). Then, for all $t \in [0, \rho_0 + \rho]$,

$$D(\omega_i{\cdot}(-\rho_0 - \rho + t), (x_i)_{-\rho_0-\rho+t}) = h_i(-\rho_0 - \rho + t), \quad i = 1, 2,$$

whence it follows that, for all $t \in [0, \rho_0 + \rho]$,

$$\begin{aligned}&\|D(\omega_1{\cdot}(-\rho_0 - \rho + t), (x_1)_{-\rho_0-\rho+t}) - D(\omega_1{\cdot}(-\rho_0 - \rho + t), (x_2)_{-\rho_0-\rho+t})\| \le \\ &\quad \le \|D(\omega_1{\cdot}(-\rho_0 - \rho + t), (x_1)_{-\rho_0-\rho+t}) - D(\omega_2{\cdot}(-\rho_0 - \rho + t), (x_2)_{-\rho_0-\rho+t})\| \\ &\qquad + \|D(\omega_2{\cdot}(-\rho_0 - \rho + t), (x_2)_{-\rho_0-\rho+t}) - D(\omega_1{\cdot}(-\rho_0 - \rho + t), (x_2)_{-\rho_0-\rho+t})\| \\ &\quad \le \|h_1(-\rho_0 - \rho + t) - h_2(-\rho_0 - \rho + t)\| + \frac{\varepsilon}{4k} \le \frac{\varepsilon}{4k} + \frac{\varepsilon}{4k} = \frac{\varepsilon}{2k}.\end{aligned}$$

Now, the stability of D and Theorem 3.8 yield that, for all $t \in [\rho_0, \rho_0 + \rho]$,

$$\|x_1(-\rho_0 - \rho + t) - x_2(-\rho_0 - \rho + t)\| \le \sup_{t \in [\rho_0, \rho_0 + \rho]} c(t)\|x_1 - x_2\|_\infty + k\frac{\varepsilon}{2k} \le \frac{\varepsilon}{4kr}2kr + \frac{\varepsilon}{2} = \varepsilon,$$

that is, $\|x_1 - x_2\|_{[-\rho, 0]} \le \varepsilon$ and the expected result holds.

Finally, we prove that, provided that $(\Omega,\sigma,\mathbb{R})$ is almost periodic, $\widehat{D}^{-1}$ is uniformly continuous on $\Omega\times B_r$, $r>0$, when the norm is considered on B_r. Let $\varepsilon>0$; from Theorem 3.6 and the almost periodicity of $(\Omega,\sigma,\mathbb{R})$, it follows that there exists $\delta>0$ such that, for all $\omega_1,\omega_2\in\Omega$ with $d(\omega_1,\omega_2)<\delta$ and all $x\in B_{kr}$, $\|\widehat{D}_2(\omega_1,x)-\widehat{D}_2(\omega_2,x)\|_\infty<\varepsilon/(2k)$. Now, fix (ω_1,h_1), $(\omega_2,h_2)\in\Omega\times B_r$ with $d(\omega_1,\omega_2)<\delta$ and $\|h_1-h_2\|_\infty\le\varepsilon/(2k)$. From (i), we have that $x_1=(\widehat{D}^{-1})_2(\omega_1,h_1)$, $x_2=(\widehat{D}^{-1})_2(\omega_2,h_2)\in B_{kr}$. Let $y=(\widehat{D}^{-1})_2(\omega_1,h_2)$; using (i),

$$\frac{1}{k}\|y-x_2\|_\infty\le\|\widehat{D}_2(\omega_1,y)-\widehat{D}_2(\omega_1,x_2)\|_\infty=\|\widehat{D}_2(\omega_2,x_2)-\widehat{D}_2(\omega_1,x_2)\|_\infty\le\frac{\varepsilon}{2k},$$

which, together with (i) again, yields

$$\|x_1-x_2\|_\infty\le\|x_1-(\widehat{D}^{-1})_2(\omega_1,h_2)\|_\infty+\|(\widehat{D}^{-1})_2(\omega_1,h_2)-x_2\|_\infty\le k\frac{\varepsilon}{2k}+\frac{\varepsilon}{2}=\varepsilon,$$

as desired.

Now we prove statement (iii). Let $C_D=\{(\omega,\varphi)\in\Omega\times BU:D(\omega,\varphi)=0\}$. For each $\rho>0$, we define $\mathcal{L}_\rho:C_D\to\mathbb{R}^m$, $(\omega,\varphi)\mapsto x(\rho)$, where x is the solution of

$$\begin{cases} D(\omega{\cdot}t,x_t)=0\,, & t\ge 0\,,\\ x_0=\varphi\,. \end{cases}$$

Let us observe that $C_D=\{(\omega,\varphi):\omega\in\Omega,\ \varphi\in C_D(\omega)\}$, where $C_D(\omega)=\{\varphi\in BU: D(\omega,\varphi)=0\}$ is a vector space for each $\omega\in\Omega$.

From the uniqueness of the solution of $(3.2)_\omega$, it is easy to check that $\mathcal{L}_\rho$ is well defined and linear in its second variable. In addition, from Theorem 3.5, we deduce that $\|\mathcal{L}_\rho(\omega,\varphi)\|=\|x(\rho)\|\le k_\rho^2\,\|\varphi\|_\infty$ for all $(\omega,\varphi)\in C_D$, whence $\|\mathcal{L}_\rho(\omega,\cdot)\|\le k_\rho^2$ for all $\omega\in\Omega$.

Next, we check that $\sup_{\omega\in\Omega}\|\mathcal{L}_\rho(\omega,\cdot)\|_\infty\to 0$ as $\rho\to\infty$, which shows the stability of D because $\|x(\rho)\|\le c(\rho)\,\|\varphi\|_\infty$ for all $(\omega,\varphi)\in C_D$, where $c(\rho)=\sup_{\omega\in\Omega}\|\mathcal{L}_\rho(\omega,\cdot)\|_\infty$. Let us assume, on the contrary, that there exist $\delta>0$, a sequence $\rho_n\uparrow\infty$ and a sequence $\{\varphi_n\}_n$ with $\varphi_n\in C_D(\omega_n)$, $\|\varphi_n\|_\infty\le 1$ and such that $\|\mathcal{L}_{\rho_n}(\omega_n,\varphi_n)\|\ge\delta$ for each $n\in\mathbb{N}$. That is, $\|x^n(\rho_n)\|\ge\delta$ where x^n is the solution of

$$\begin{cases} D(\omega_n{\cdot}t,x^n_t)=0\,, & t\ge 0\,,\\ x^n_0=\varphi_n\,. \end{cases}$$

Therefore,

$$\begin{cases} D(\omega_n{\cdot}(\rho_n+s),(x^n_{\rho_n})_s)=D(\omega_n{\cdot}(\rho_n+s),x^n_{\rho_n+s})=0 & \text{if } s\in[-\rho_n,0]\,,\\ D(\omega_n{\cdot}(\rho_n+s),(x^n_{\rho_n})_s)=D(\omega_n{\cdot}(\rho_n+s),(\varphi_n)_{\rho_n+s}) & \text{if } s\le-\rho_n\,, \end{cases}$$

and taking $r=\sup_{\omega\in\Omega}\|D(\omega,\cdot)\|$, the sequence $\{x^n_{\rho_n}\}_{n\in\mathbb{N}}\subset BU$ satisfies that $\|\widehat{D}(\omega_n{\cdot}\rho_n,x^n_{\rho_n})\|_\infty\le r$ and $\widehat{D}(\omega_n{\cdot}\rho_n,x^n_{\rho_n})\xrightarrow{\mathsf{d}}0$ as $n\to\infty$. We can assume without loss of generality that $\omega_n{\cdot}\rho_n\to\omega\in\Omega$. Consequently, $x^n_{\rho_n}(0)=x^n(\rho_n)\to 0$ as $n\to\infty$, which contradicts the fact that $\|x^n(\rho_n)\|\ge\delta$, and finishes the proof. □

As a consequence, the operator D is stable if and only if $\widehat{D}$ is invertible and $\widehat{D}^{-1}$ is uniformly continuous on $\Omega\times B_r$ for the compact-open topology for all $r>0$.

The following statement provides a symmetric theory for the operators $\widehat{D}$ and $\widehat{D}^{-1}$. In particular, $\widehat{D}^{-1}$ is generated by a linear operator D^* which satisfies (D1)–(D3). We omit the proof, which follows the arguments of Proposition 2.10 in [16].

Proposition 3.10. *Suppose that D is stable and define*

$$\begin{array}{rccc} D^*: & \Omega \times BU & \longrightarrow & \mathbb{R}^m \\ & (\omega, x) & \mapsto & (\widehat{D}^{-1})_2(\omega, x)(0). \end{array}$$

Then D^* also satisfies (D1)–(D3) *and is stable. Moreover, for all $s \le 0$ and all $(\omega, x) \in \Omega \times BU$, it holds that $(\widehat{D}^{-1})_2(\omega, x)(s) = D^*(\omega{\cdot}s, x_s)$. In particular, for all $\omega \in \Omega$, there is an $m \times m$ matrix $\mu^*(\omega) = [\mu^*_{ij}(\omega)]_{ij}$ of real Borel regular measures with finite total variation such that*

$$(\widehat{D}^{-1})_2(\omega, x)(s) = \int_{-\infty}^{0} [d\mu^*(\omega{\cdot}s)]x_s, \quad (\omega, x) \in \Omega \times BU,\ s \le 0.$$

4. Transformed exponential order and structure of omega-limit sets

Let $F : \Omega \times BU \to \mathbb{R}^m$, let (Ω, d) be a compact metric space and let $\mathbb{R} \times \Omega \to \Omega$, $\omega \mapsto \omega{\cdot}t$ be a minimal real flow on Ω. Let $D : \Omega \times BU \to \mathbb{R}^m$ be a stable operator satisfying hypotheses (D1)–(D3).

Let us consider the family of equations

$$\frac{d}{dt} D(\omega{\cdot}t, z_t) = F(\omega{\cdot}t, z_t), \quad t \ge 0,\ \omega \in \Omega. \qquad (4.1)_\omega$$

We take the componentwise partial order relation on $\mathbb{R}^m$, that is, if v, $w \in \mathbb{R}^m$, then

$$\begin{aligned} v \le w \quad &\Longleftrightarrow \quad v_j \le w_j \quad \text{for } j = 1, \dots, m\,, \\ v < w \quad &\Longleftrightarrow \quad v \le w \quad \text{and} \quad v_j < w_j \quad \text{for some } j \in \{1, \dots, m\}\,, \end{aligned}$$

We write $A \le B$ for $m \times m$ matrices $A = [a_{ij}]_{ij}$ and $B = [b_{ij}]_{ij}$ whenever $a_{ij} \le b_{ij}$ for all i, j. Let A be an $m \times m$ quasipositive matrix, i.e. a matrix such that there exists $\lambda > 0$ with $A + \lambda I \ge 0$. Let $\rho > 0$; let us recall the definitions of exponential ordering on BU given in [19]. If x, $y \in BU$, then

$$\begin{aligned} x \le_{A,\rho} y &\iff x \le y \ \text{ and } \ y(t) - x(t) \ge e^{A(t-s)}(y(s) - x(s))\,, -\rho \le s \le t \le 0\,, \\ x <_{A,\rho} y &\iff x \le_{A,\rho} y \quad \text{and} \quad x \ne y\,, \text{ and} \end{aligned}$$

$$\begin{aligned} x \le_{A,\infty} y &\iff x \le y \ \text{ and } \ y(t) - x(t) \ge e^{A(t-s)}(y(s) - x(s))\,, -\infty < s \le t \le 0\,, \\ x <_{A,\infty} y &\iff x \le_{A,\infty} y \ \text{ and } \quad x \ne y\,. \end{aligned}$$

In what follows, $\le_A$ will denote any of the order relations $\le_{A,\rho}$ and $\le_{A,\infty}$. However, in the case of $\le_{A,\infty}$, we will assume without further notice that all the eigenvalues of A have strictly negative real parts. The theory will provide different dynamical conclusions for each choice. The aforementioned relations define positive cones in BU, $BU_A^+ = \{x \in BU : x \ge_A 0\}$, in the sense that they are closed subsets of BU and satisfy $BU_A^+ + BU_A^+ \subset BU_A^+$, $\mathbb{R}^+ BU_A^+ \subset BU_A^+$ and $BU_A^+ \cap (-BU_A^+) = \{0\}$. Note that, if $\le_A = \le_{A,\rho}$, then a smooth function (resp. a Lipschitz continuous function) x belongs to BU_A^+ if and only if $x \ge 0$ and $x'(s) \ge Ax(s)$ for each (resp. a.e.) $s \in [-\rho, 0]$, and, if $\le_A = \le_{A,\infty}$, then it belongs to BU_A^+ if and only if $x \ge 0$ and $x'(s) \ge Ax(s)$ for each (resp. a.e.) $s \in (-\infty, 0]$.

On each fiber of the product $\Omega \times BU$, we define the following *transformed exponential order* relation: if (ω, x), $(\omega, y) \in \Omega \times BU$, then

$$(\omega, x) \le_{D,A} (\omega, y) \iff \widehat{D}_2(\omega, x) \le_A \widehat{D}_2(\omega, y).$$

Let us assume the following hypothesis:

(F1) $F : \Omega \times BU \to \mathbb{R}^m$ is continuous on $\Omega \times BU$ and its restriction to $\Omega \times B_r$ is Lipschitz continuous in its second variable when the norm is considered on B_r for all $r > 0$.

As seen in Wang and Wu [27] and Wu [28], for each $\omega \in \Omega$, the local existence and uniqueness of the solutions of equation $(4.1)_\omega$ follow from (F1). Moreover, given $(\omega, x) \in \Omega \times BU$, if $z(\cdot, \omega, x)$ represents the solution of equation $(4.1)_\omega$ with initial datum x, then the mapping $u(t, \omega, x) : (-\infty, 0] \to \mathbb{R}^m$, $s \mapsto z(t + s, \omega, x)$ is an element of BU for all $t \geq 0$ where $z(\cdot, \omega, x)$ is defined.

Therefore, a local skew-product semiflow on $\Omega \times BU$ can be defined as follows:

$$\begin{array}{rccc} \tau : & \mathcal{U} \subset \mathbb{R}^+ \times \Omega \times BU & \longrightarrow & \Omega \times BU \\ & (t, \omega, x) & \mapsto & (\omega{\cdot}t, u(t, \omega, x)). \end{array}$$

Let $(\omega, y) \in \Omega \times BU$. For each $t \geq 0$ where $u(t, \widehat{D}^{-1}(\omega, y))$ is defined, we define $\widehat{u}(t, \omega, y) = \widehat{D}_2(\omega{\cdot}t, u(t, \widehat{D}^{-1}(\omega, y)))$. Let us check that

$$\widehat{z}(\cdot, \omega, y) : t \mapsto \begin{cases} y(t) & \text{if } t \leq 0, \\ \widehat{u}(t, \omega, y)(0) & \text{if } t \geq 0, \end{cases}$$

is the solution of

$$\widehat{z}\,'(t) = G(\omega{\cdot}t, \widehat{z}_t),\ t \geq 0,\ \omega \in \Omega \tag{4.2$_\omega$}$$

through (ω, y), where $G = F \circ \widehat{D}^{-1}$. Let $x = (\widehat{D}^{-1})_2(\omega, y)$; if $t \geq 0$, then

$$\begin{aligned} \frac{d}{dt}\widehat{z}(t, \omega, y) &= \frac{d}{dt}\Big[\widehat{D}_2(\omega{\cdot}t, u(t, \omega, x))(0)\Big] = \frac{d}{dt}D(\omega{\cdot}t, u(t, \omega, x)) = F(\omega{\cdot}t, u(t, \omega, x)) \\ &= F \circ \widehat{D}^{-1}(\widehat{D}(\omega{\cdot}t, u(t, \omega, x))) = F \circ \widehat{D}^{-1}(\omega{\cdot}t, \widehat{u}(t, \omega, x)). \end{aligned}$$

It only remains to observe that a simple calculation yields $\widehat{z}(\cdot, \omega, y)_t = \widehat{u}(t, \omega, y)$ for all $t \geq 0$. Let us assume some more hypotheses concerning the map F.

(F2) $F(\Omega \times B_r)$ is a bounded subset of $\mathbb{R}^m$ for all $r > 0$.

(F3) The restriction of F to $\Omega \times B_r$ is continuous when the compact-open topology is considered on B_r, for $r > 0$.

(F4) If $(\omega, x), (\omega, y) \in \Omega \times BU$ and $(\omega, x) \leq_{D,A} (\omega, y)$, then $F(\omega, y) - F(\omega, x) \geq A(D(\omega, y) - D(\omega, x))$.

Proposition 4.1. *Under hypotheses* (F1)–(F4)*, the following assertions hold:*

(i) *G is continuous on $\Omega \times BU$ and its restriction to $\Omega \times B_r$ is Lipschitz continuous in its second variable when the norm is considered on B_r for all $r > 0$;*

(ii) *$G(\Omega \times B_r)$ is a bounded subset of $\mathbb{R}^m$ for all $r > 0$;*

(iii) *the restriction of G to $\Omega \times B_r$ is continuous when the compact-open topology is considered on B_r, for $r > 0$;*

(iv) *if $(\omega, x), (\omega, y) \in \Omega \times BU$ and $x \leq_A y$, then $G(\omega, y) - G(\omega, x) \geq A(y(0) - x(0))$.*

Proof. First, let us check (i). Let $\{(\omega_n, x_n)\}_n \subset \Omega \times BU$ be a sequence with $\omega_n \to \omega$ and $\|x_n - x\|_\infty \to 0$ as $n \to \infty$ for some $(\omega, x) \in \Omega \times BU$. Then $x_n \overset{\mathsf{d}}{\to} x$ as $n \to \infty$ and there is an $r > 0$ such that $x_n, x \in B_r$ for all $n \in \mathbb{N}$. Consequently, $\widehat{D}^{-1}(\omega_n, x_n) \overset{\mathsf{d}}{\to} \widehat{D}^{-1}(\omega, x)$ as $n \to \infty$ and, from Theorem 3.9, $\widehat{D}^{-1}(\omega_n, x_n), \widehat{D}^{-1}(\omega, x) \in B_{kr}$ for all $n \in \mathbb{N}$. Thanks to (F3), $G(\omega_n, x_n) \to G(\omega, x)$ as $n \to \infty$. As for the Lipschitz continuity, let $r > 0$ and fix $(\omega, y_1), (\omega, y_2) \in \Omega \times B_r$;

denote by $x_i = (\widehat{D}^{-1})_2(\omega, y_i)$, $i = 1, 2$. From Theorem 3.9, it follows that $\|x_i\|_\infty \leq k\|y_i\|_\infty \leq kr$, $i = 1, 2$. Let $L > 0$ be the Lipschitz constant of F on $\Omega \times B_{kr}$. Again Theorem 3.9 yields

$$\|G(\omega, y_1) - G(\omega, y_2)\| = \|F(\omega, x_1) - F(\omega, x_2)\| \leq L\|x_1 - x_2\|_\infty \leq Lk\|y_1 - y_2\|_\infty,$$

and the result is proved.
As for (ii), let $r > 0$; from Theorem 3.9, it follows that

$$G(\Omega \times B_r) = F(\widehat{D}^{-1}(\Omega \times B_r)) \subset F(\Omega \times B_{kr})$$

and the latter is bounded thanks to (F2).
Let us focus on (iii). Let $r > 0$; once more, Theorem 3.9 implies that $\widehat{D}^{-1}(\Omega \times B_r) \subset \Omega \times B_{kr}$, and F is continuous there when we consider the compact-open topology. Finally, (iv) is a straightforward consequence of (F4). □

We may now define another local skew-product semiflow on $\Omega \times BU$ from the solutions of the equations of the family $(4.2)_\omega$ (see Hino, Murakami and Naito [8]) in the following manner:

$$\begin{array}{rccl} \widehat{\tau}: & \widehat{\mathcal{U}} \subset \mathbb{R}^+ \times \Omega \times BU & \longrightarrow & \Omega \times BU \\ & (t, \omega, x) & \mapsto & (\omega{\cdot}t, \widehat{u}(t, \omega, x)). \end{array}$$

Given $r > 0$, a forward orbit $\{\widehat{\tau}(t, \omega_0, x_0) : t \geq 0\}$ of the transformed skew-product semiflow $\widehat{\tau}$ is said to be *uniformly stable for the order* $\leq_A$ *in* B_r if, for every $\varepsilon > 0$, there is a $\delta > 0$, called the *modulus of uniform stability*, such that, if $s \geq 0$ and $\mathsf{d}(\widehat{u}(s, \omega_0, x_0), x) \leq \delta$ for certain $x \in B_r$ with $x \leq_A \widehat{u}(s, \omega_0, x_0)$ or $\widehat{u}(s, \omega_0, x_0) \leq_A x$, then for each $t \geq 0$,

$$\mathsf{d}(\widehat{u}(t + s, \omega_0, x_0), \widehat{u}(t, \omega_0{\cdot}s, x)) = \mathsf{d}(\widehat{u}(t, \omega_0{\cdot}s, \widehat{u}(s, \omega_0, x_0)), \widehat{u}(t, \omega_0{\cdot}s, x)) \leq \varepsilon.$$

If this happens for each $r > 0$, the forward orbit is said to be *uniformly stable for the order* $\leq_A$ *in bounded sets.*

Notice that an argument similar to the one given in Proposition 4.1 in [18] and Proposition 4.2 in [16] ensures that, for all $(\omega_0, x_0) \in \Omega \times BU$ giving rise to a bounded solution, $\mathrm{cls}_X\{u(t, \omega_0, x_0) : t \geq 0\}$ is a compact subset of BU and the omega-limit set of (ω_0, x_0) can be defined as

$$\mathcal{O}(\omega_0, x_0) = \{(\omega, x) \in \Omega \times BU : \exists t_n \uparrow \infty \text{ with } \omega_0{\cdot}t_n \to \omega,\ u(t_n, \omega_0, x_0) \xrightarrow{\mathsf{d}} x\}.$$

Moreover, the restriction of τ to $\mathcal{O}(\omega_0, x_0)$ is continuous when the compact-open topology is considered on BU, and $\mathcal{O}(\omega_0, x_0)$ admits a flow extension. The main objective of this section is to transfer the dynamical structure of the semiflow $(\Omega \times BU, \widehat{\tau}, \mathbb{R}^+)$ to $(\Omega \times BU, \tau, \mathbb{R}^+)$.

Theorem 4.2. *Fix* (ω, x), $(\omega, y) \in \Omega \times BU$ *such that* $(\omega, x) \leq_{D,A} (\omega, y)$*. Then*

$$\tau(t, \omega, x) \leq_{D,A} \tau(t, \omega, y)$$

for all $t \geq 0$ *where they are defined.*

Proof. It is clear that $\widehat{D}_2(\omega, x) \leq_A \widehat{D}_2(\omega, y)$. Now, from Theorem 3.5 in [19] and Proposition 4.1, it follows that $\widehat{u}(t, \widehat{D}(\omega, x)) \leq_A \widehat{u}(t, \widehat{D}(\omega, y))$ or, equivalently, $\widehat{D}_2(\omega{\cdot}t, u(t, \omega, x)) \leq_A \widehat{D}_2(\omega{\cdot}t, u(t, \omega, y))$ whenever they are defined. Therefore, we have $\tau(t, \omega, x) \leq_{D,A} \tau(t, \omega, y)$ for all $t \geq 0$ where they are defined, as wanted. □

Let us assume two more hypotheses concerning F and the semiflow $\widehat{\tau}$. The fact that we are imposing a condition on the semiflow $\widehat{\tau}$ seems to suggest that such condition should be difficult to verify when studying specific systems of equations. As it will be shown later on, this kind of condition arises naturally in some systems and is easier to check.

(F5) There exists $r_0 > 0$ such that all the trajectories for $\widehat{\tau}$ with a Lipschitz continuous initial datum within $B_{\widehat{r}_0}$ are relatively compact for the product metric topology and uniformly stable for the order $\leq_A$ in bounded sets, where

$$\widehat{r}_0 = \|A^{-1}\|(\sup\{\|F(\omega,x)\| : (\omega,x) \in \widehat{D}^{-1}(\Omega \times B_{r_0})\} + \|A\|r_0).$$

(F6) If (ω,x), $(\omega,y) \in \Omega \times BU$ admit a backward orbit extension for the semiflow τ, $(\omega,x) \leq_{D,A} (\omega,y)$ and there exists $J \subset \{1,\dots,m\}$ such that

$$\widehat{D}_2(\omega,x)_i = \widehat{D}_2(\omega,y)_i \text{ for all } i \notin J \text{ and}$$
$$\widehat{D}_2(\omega,x)_i(s) < \widehat{D}_2(\omega,y)_i(s) \text{ for all } i \in J \text{ and all } s \leq 0,$$

then $F_i(\omega,y) - F_i(\omega,x) - [A(D(\omega,y) - D(\omega,x))]_i > 0$ for all $i \in J$.

The next result is an immediate consequence of these two properties, and so we give it without a proof.

Proposition 4.3. *Under hypotheses* (F5) *and* (F6)*, the following assertions hold:*

(i) *there exists $r_0 > 0$ such that all the trajectories for $\widehat{\tau}$ with a Lipschitz continuous initial datum within $B_{\widehat{r}_0}$ are relatively compact for the product metric topology and uniformly stable for the order $\leq_A$ in bounded sets, where*

$$\widehat{r}_0 = \|A^{-1}\|(\sup\{\|G(\omega,x)\| : (\omega,x) \in \Omega \times B_{r_0}\} + \|A\|r_0). \tag{4.3}$$

(ii) *if (ω,x), $(\omega,y) \in \Omega \times BU$ admit a backward orbit extension for the semiflow $\widehat{\tau}$, $x \leq_A y$ and there exists $J \subset \{1,\dots,m\}$ such that*

$$x_i = y_i \text{ for all } i \notin J \text{ and}$$
$$x_i(s) < y_i(s) \text{ for all } i \in J \text{ and all } s \leq 0,$$

then $G_i(\omega,y) - G_i(\omega,x) - [A(y(0) - x(0))]_i > 0$ for all $i \in J$.

Relation (4.3) is an improved version of formula (5.1) in [19]. Following this paper, we come now to the main result of this section, which establishes the 1-covering property of the omega-limit sets.

Theorem 4.4. *Assume that conditions* (D1)–(D3) *are satisfied and that D is stable; furthermore, assume conditions* (F1)–(F6)*. Fix $(\omega_0,x_0) \in \widehat{D}^{-1}(\Omega \times B_{r_0})$ such that $\{\widehat{\tau}(t,\widehat{D}(\omega_0,x_0)) : t \geq 0\}$ is relatively compact for the product metric topology and uniformly stable for $\leq_A$ in bounded sets, and such that $K = \mathcal{O}(\omega_0,x_0) \subset \widehat{D}^{-1}(\Omega \times B_{r_0})$. If $\leq_A = \leq_{A,\infty}$, then we will further assume that $\widehat{D}_2(\omega_0,x_0)$ is Lipschitz continuous. Then $K = \{(\omega,c(\omega)) : \omega \in \Omega\}$ and*

$$\lim_{t\to\infty} \mathsf{d}(u(t,\omega_0,x_0), c(\omega_0{\cdot}t)) = 0,$$

where $c : \Omega \to BU$ is a continuous equilibrium, i.e. $c(\omega{\cdot}t) = u(t,\omega,c(\omega))$ for all $t \geq 0$ and all $\omega \in \Omega$, considering on BU the compact-open topology.

Proof. As seen in Theorem 5.6 in [19], from Propositions 4.1 and 4.3, it follows that $\mathcal{O}(\widehat{D}(\omega_0, x_0)) = \{(\omega, \widehat{c}(\omega)) : \omega \in \Omega\}$ and

$$\lim_{t\to\infty} \mathsf{d}(\widehat{u}(t, \omega_0, x_0), \widehat{c}(\omega_0{\cdot}t)) = 0,$$

where $\widehat{c} : \Omega \to BU$ is a continuous equilibrium considering on BU the compact-open topology. We observe that

$$K = \widehat{D}^{-1}(\mathcal{O}(\widehat{D}(\omega_0, x_0))) = \widehat{D}^{-1}(\{(\omega,\widehat{c}(\omega)) : \omega \in \Omega\}) = \{(\omega, (\widehat{D}^{-1})_2(\omega,\widehat{c}(\omega))) : \omega \in \Omega\}.$$

Let us define $c : \Omega \to BU$, $\omega \mapsto (\widehat{D}^{-1})_2(\omega, \widehat{c}(\omega))$. The continuity of c when we consider the compact-open topology on BU is a consequence of Theorem 3.9 and the fact that $\mathcal{O}(\widehat{D}(\omega_0, x_0)) \subset \Omega \times B_{r_0}$. Moreover, c is an equilibrium, for its graph defines an omega-limit set. Finally, from Theorem 3.9 again and the boundedness of the trajectory and of $\widehat{c}(\Omega)$, we conclude that

$$\lim_{t\to\infty} \mathsf{d}(u(t, \omega_0, x_0), c(\omega_0{\cdot}t)) = 0,$$

and the proof is finished. □

5. Neutral compartmental systems

Let (Ω, d) be a compact metric space and let $\mathbb{R} \times \Omega \to \Omega$, $\omega \mapsto \omega{\cdot}t$ be a minimal real flow on Ω. In this section, we apply the foregoing results to the study of compartmental models, used to describe processes in which the transport of material among some compartments takes a non-negligible length of time, and each compartment produces or swallows material.

Let us suppose that we have a system formed by m compartments $C_1, \dots, C_m$, and denote by $z_i(t)$ the amount of material within compartment C_i at time t for each $i \in \{1, \dots, m\}$. Material flows from compartment C_j into compartment C_i through a pipe having a transit time distribution given by a positive regular Borel measure μ_{ij} with total variation $\mu_{ij}((-\infty, 0]) = 1$, for each i, $j \in \{1, \dots, m\}$, whereas the outcome of material from C_i to C_j is assumed to be instantaneous. Let $g_{ij} : \Omega \times \mathbb{R} \to \mathbb{R}$ be the so-called *transport function* determining the volume of material flowing from C_j to C_i given in terms of the time and the amount of material within C_j for i, $j \in \{1, \dots, m\}$. For each $i \in \{1, \dots, m\}$, there is an inflow of material from the environment given by $I_i : \Omega \to \mathbb{R}$, an outflow of material toward the environment given by $g_{0i} : \Omega \times \mathbb{R} \to \mathbb{R}$, and the compartment C_i produces or swallows material itself at a rate given by some regular Borel measures $\nu_{ij}(\omega)$, $\omega \in \Omega$, and some functions $b_{ij} : \Omega \to \mathbb{R}$, $j \in \{1, \dots, m\}$.

Once the destruction and creation of material is taken into account, the change of the amount of material of any compartment C_i, $1 \le i \le m$, equals the difference between the amount of total inflow into and total outflow out of C_i, and we obtain a model governed by the following system of NFDEs:

$$\begin{aligned} \frac{d}{dt} \sum_{j=1}^{m} \left[b_{ij}(\omega{\cdot}t) z_j(t) - \int_{-\infty}^{0} z_j(t+s)\, d\nu_{ij}(\omega{\cdot}t)(s) \right] &= -g_{0i}(\omega{\cdot}t, z_i(t)) \\ - \sum_{j=1}^{m} g_{ji}(\omega{\cdot}t, z_i(t)) + \sum_{j=1}^{m} \int_{-\infty}^{0} g_{ij}(\omega{\cdot}(t+s), z_j(t+s))\, d\mu_{ij}(s) &+ I_i(\omega{\cdot}t), \end{aligned} \tag{5.1$_\omega$}$$

for $t \ge 0$, $i = 1, \dots, m$ and $\omega \in \Omega$, where $g_{ij} : \Omega \times \mathbb{R} \to \mathbb{R}$, $I_i : \Omega \to \mathbb{R}$ and μ_{ij}, $\nu_{ij}(\omega)$ are regular Borel measures with finite total variation for all $\omega \in \Omega$.

For the sake of simplicity, let us denote $g_{i0}:\Omega\times\mathbb{R}\to\mathbb{R}$, $(\omega,v)\mapsto I_i(\omega)$ and let $g=(g_{ij})_{ij}:\Omega\times\mathbb{R}\to\mathbb{R}^{m^2+2m}$. We denote by $B(\omega)$ and $\nu(\omega)$ the matrices $[b_{ij}(\omega)]_{ij}$ and $[\nu_{ij}(\omega)]_{ij}$, $\omega\in\Omega$, respectively.

Let $F:\Omega\times BU\to\mathbb{R}^m$ be the map defined for $(\omega,x)\in\Omega\times BU$ by

$$F_i(\omega,x)=-\sum_{j=0}^m g_{ji}(\omega,x_i(0))+\sum_{j=1}^m\int_{-\infty}^0 g_{ij}(\omega{\cdot}s,x_j(s))\,d\mu_{ij}(s)+I_i(\omega),$$

$i=1,\dots,m$, and let $D:\Omega\times BU\to\mathbb{R}^m$ be the map defined for $(\omega,x)\in\Omega\times BU$ by

$$D(\omega,x)=\left(\sum_{j=1}^m\left[b_{ij}(\omega)x_j(0)-\int_{-\infty}^0 x_j\,d\nu_{ij}(\omega)\right]\right)_{i=1}^m=B(\omega)x(0)-\int_{-\infty}^0[d\nu(\omega)]x.$$

With this notation, the family of equations $(5.1)_\omega$ can be written as

$$\frac{d}{dt}D(\omega{\cdot}t,z_t)=F(\omega{\cdot}t,z_t),\quad t\ge 0,\,\omega\in\Omega. \tag{5.2$_\omega$}$$

We will assume the following hypotheses on the family $(5.1)_\omega$.

(C1) g_{ij} is C^1 and nondecreasing in its second variable, and $g_{ij}(\omega,0)=0$ for each $\omega\in\Omega$, $i\in\{0,\dots,m\}$ and $j\in\{1,\dots,m\}$.

(C2) $\mu_{ij}((-\infty,0])=1$ and $\int_{-\infty}^0|s|\,d\mu_{ij}(s)<\infty$ for $i,j=1,\dots,m$.

(C3) $\nu_{ij}(\omega)(\{0\})=0$ for all $\omega\in\Omega$ and the mapping $\nu:\Omega\to\mathcal{M}$, $\omega\mapsto\nu(\omega)$ is continuous, where $\mathcal{M}$ is the Banach space of $m\times m$ matrices of Borel measures on $(-\infty,0]$ with the supremum norm defined from the total variation of the measures.

(C4) $B(\omega)$ is a regular matrix for all $\omega\in\Omega$ and $B:\Omega\to\mathbb{M}_m(\mathbb{R})$, $\omega\mapsto B(\omega)$ is continuous; moreover, $B(\omega)^{-1}\ge 0$, $B(\omega)^{-1}\nu(\omega)$ is a matrix of positive measures for all $\omega\in\Omega$, and

$$\|B(\omega)^{-1}\nu(\omega)\|_\infty((-\infty,0])<1.$$

First, in this section we will prove the stability of the operator D. Later, we will deduce that at least the trajectories transformed by the operator $\widehat{D}$ with Lipschitz continuous initial data are uniformly stable for $\le_A$ on bounded sets.

Conditions (C1)–(C4) yield the following result.

Proposition 5.1. *The mapping D satisfies* (D1)–(D3) *and the mapping F satisfies* (F1)–(F3) *respectively.*

We define the map $\widehat{D}:\Omega\times BU\to\Omega\times BU$ as in Theorem 3.6. For each $\omega\in\Omega$, let us define $\widehat{B}_\omega:BU\to BU$, $\widehat{B}_\omega(x)(s)=B(\omega{\cdot}s)x(s)$. It is easy to check that $\widehat{B}_\omega$ is a linear isomorphism of BU and it is continuous for the norm; in addition $(\omega,x)\mapsto\widehat{B}_\omega(x)$ is continuous on each set of the form $\Omega\times B_r$, $r>0$, when the compact-open topology is considered. We can denote $(\widehat{B}_\omega)^{-1}=(\widehat{B^{-1}})_\omega$, $\omega\in\Omega$.

Theorem 5.2. *For each $\omega\in\Omega$, consider the continuous linear operator $\widehat{L}_\omega:BU\to BU$ defined for $x\in BU$ and $s\le 0$ by*

$$\widehat{L}_\omega(x)(s)=B(\omega{\cdot}s)^{-1}\int_{-\infty}^0[d\nu(\omega{\cdot}s)]x_s.$$

Then the following statements hold:

(i) $\sup_{\omega\in\Omega}\|\widehat{L}_\omega\| < 1$ *and* $\widehat{D}_2(\omega,x) = [\widehat{B}_\omega \circ (I - \widehat{L}_\omega)](x)$ *for each* $(\omega,x) \in \Omega \times BU$*;*

(ii) $\widehat{D}$ *is invertible and*

$$(\widehat{D}^{-1})_2(\omega,x) = \sum_{n=0}^{\infty}(\widehat{L}_\omega^n \circ (\widehat{B}_\omega)^{-1})(x)$$

for every $(\omega,x) \in \Omega \times BU$*;*

(iii) $(\widehat{D}^{-1})_2(\omega,x) \geq 0$ *for every* $(\omega,x) \in \Omega \times BU$ *with* $x \geq 0$*;*

(iv) *the map* $\Omega \times B_r \to B_r$*,* $(\omega,x) \mapsto \widehat{L}_\omega(x)$ *is uniformly continuous for the compact-open topology for each* $r > 0$*;*

(v) D *is stable.*

Proof. From condition (C4), we conclude that $\sup_{\omega\in\Omega}\|\widehat{L}_\omega\| < 1$. It is immediate to check that $\widehat{D}_2(\omega,x) = [\widehat{B}_\omega \circ (I - \widehat{L}_\omega)](x)$ for each $(\omega,x) \in \Omega \times BU$. This way, (i) is proved, whence $\widehat{D}$ is invertible and

$$(\widehat{D}^{-1})_2(\omega,x) = \sum_{n=0}^{\infty}(\widehat{L}_\omega^n \circ (\widehat{B}_\omega)^{-1})(x)$$

for every $(\omega,x) \in \Omega \times BU$, which proves (ii). Now, (iii) follows from (ii) and hypothesis (C4).

In addition, for all $r_1 > 0$ and all $(\omega,x) \in \Omega \times B_{r_1}$, it is clear that $(\widehat{B}_\omega)^{-1}(x) \in \Omega\times B_{r_2}$ and $\widehat{L}_\omega(x) \in \Omega\times B_{r_1}$, where $r_2 = \sup_{\omega_1\in\Omega}\|B(\omega_1)^{-1}\|r_1$; besides, if (ω_1,x_1), $(\omega_2,x_2) \in \Omega \times B_{r_1}$ and $s \in [-\rho,0]$ for some $\rho > 0$, then

$$\begin{aligned}
\|\widehat{L}_{\omega_1}(x_1)(s)-\widehat{L}_{\omega_2}(x_2)(s)\| &\leq \|\widehat{L}_{\omega_1}(x_1)(s)-\widehat{L}_{\omega_1}(x_2)(s)\|+\|\widehat{L}_{\omega_1}(x_2)(s) - \widehat{L}_{\omega_2}(x_2)(s)\| \\
&\leq \sup_{\omega\in\Omega}\|B(\omega)^{-1}\| \sup_{\omega\in\Omega}\|\nu(\omega)\|_\infty((-\infty,-\rho])2r_1 + \|x_1 - x_2\|_{[-2\rho,0]} \\
&\quad + \|B(\omega_1{\cdot}s)^{-1} - B(\omega_2{\cdot}s)^{-1}\| \sup_{\omega\in\Omega}\|\nu(\omega)\|_\infty((-\infty,0])r_1 \\
&\quad + \sup_{\omega\in\Omega}\|B(\omega)^{-1}\|\|\nu(\omega_1{\cdot}s) - \nu(\omega_2{\cdot}s)\|_\infty((-\infty,0])r_1.
\end{aligned}$$

This proves (iv). Now it is immediate to deduce that, given $r > 0$, $\widehat{D}^{-1}$ is uniformly continuous on $\Omega \times B_r$ for the compact-open topology. Let us fix $\varepsilon > 0$ and $\rho > 0$. There is an $n_0 \in \mathbb{N}$ such that $\sum_{n=n_0}^{\infty}\|\widehat{L}_\omega^n \circ (\widehat{B}_\omega)^{-1}\| < \varepsilon/(3r)$. From (iv) and the continuity of $(\omega,x) \mapsto \widehat{B}_\omega(x)$ for the product metric topology on each ball, it follows that there exist $\rho_0 > 0$ and $\delta > 0$ such that, if (ω_1,x_1), $(\omega_2,x_2) \in \Omega \times B_r$ with $d(\omega_1,\omega_2) < \delta$ and $\|x_1(s) - x_2(s)\| < \delta$ for all $s \in [-\rho_0,0]$, then $\|\widehat{L}_{\omega_1}^j \circ (\widehat{B}_{\omega_1})^{-1}(x_1)(s) - \widehat{L}_{\omega_2}^j \circ (\widehat{B}_{\omega_2})^{-1}(x_2)(s)\| \leq \varepsilon/(3n_0)$, $j \in \{0,\ldots,n_0-1\}$, $s \in [-\rho,0]$. As a consequence, $\left\|\sum_{n=0}^{\infty}\widehat{L}_{\omega_1}^n \circ (\widehat{B}_{\omega_1})^{-1}(x_1)(s) - \sum_{n=0}^{\infty}\widehat{L}_{\omega_2}^n \circ (\widehat{B}_{\omega_2})^{-1}(x_2)(s)\right\| \leq \varepsilon$ for every $s \in [-\rho,0]$, which proves, according to Theorem 3.9, that D is stable. This completes the proof. □

We will assume now that (F4) is satisfied. Some sufficient conditions for (F4) to hold will be studied in the next section.

Consider the *total mass* of the system $(5.1)_\omega$, $M : \Omega \times BU \to \mathbb{R}$, defined for $(\omega,x) \in \Omega \times BU$ by

$$M(\omega,x) = \sum_{i=1}^{m} D_i(\omega,x) + \sum_{i=1}^{m}\sum_{j=1}^{m}\int_{-\infty}^{0}\left(\int_{s}^{0} g_{ji}(\omega{\cdot}\tau,x_i(\tau))d\tau\right)d\mu_{ji}(s).$$

M is well defined due to (C2) and because, if $i,j\in\{1,\dots,m\}$ and $(\omega,x)\in\Omega\times BU$, then $\left|\int_s^0 g_{ji}(\omega{\cdot}\tau,x_i(\tau))d\tau\right|\le c_1|s|$, where c_1 is a bound of g_{ji} on $\Omega\times[-\|x\|_\infty,\|x\|_\infty]$.

The next result is in the line of some results found in [16], [19] and [29], and states an important equality satisfied by the total mass of a compartmental system.

Proposition 5.3. *M is uniformly continuous on $\Omega\times B_r$ for all $r>0$ for the product metric topology. Moreover,*

$$M(\tau(t,\omega,x))=M(\omega,x)+\sum_{i=1}^m\int_0^t\left(I_i(\omega{\cdot}s)-g_{0i}(\omega{\cdot}s,z_i(s,\omega,x))\right)ds$$

for all $t\ge 0$ where the solution is defined and all $(\omega,x)\in\Omega\times BU$.

Lemma 5.4. *Fix (ω,x), $(\omega,y)\in\Omega\times BU$ with $(\omega,x)\le_{D,A}(\omega,y)$. Then*

$$0\le D_i(\tau(t,\omega,y))-D_i(\tau(t,\omega,x))\le M(\omega,y)-M(\omega,x)$$

for each $i=1,\dots,m$ and whenever $z(t,\omega,x)$ and $z(t,\omega,y)$ are defined.

Proof. It follows from (F4) and Theorem 4.2 that the skew-product semiflow induced by $(5.1)_\omega$ is monotone. Hence, if $(\omega,x)\le_{D,A}(\omega,y)$ then $\tau(t,\omega,x)\le_{D,A}\tau(t,\omega,y)$ whenever they are defined. From this and Theorem 5.2, we also deduce that $x\le y$ and $u(t,\omega,x)\le u(t,\omega,y)$. Therefore, $D_i(\tau(t,\omega,x))\le D_i(\tau(t,\omega,y))$ and $z_i(t,\omega,x)\le z_i(t,\omega,y)$ for each $i=1,\dots,m$. In addition, the monotonicity of transport functions yields $g_{ij}(\omega{\cdot}t,z_j(t,\omega,x))\le g_{ij}(\omega{\cdot}t,z_j(t,\omega,y))$. From all these inequalities, the definition of total mass and Proposition 5.3, we deduce that

$$\begin{aligned}
0&\le D_i(\tau(t,\omega,y))-D_i(\tau(t,\omega,x))\le\sum_{i=1}^m\left[D_i(\tau(t,\omega,y))-D_i(\tau(t,\omega,x))\right]\\
&\le M(\tau(t,\omega,y))-M(\tau(t,\omega,x))\\
&=M(\omega,y)-M(\omega,x)+\sum_{i=1}^m\int_0^t\left(g_{0i}(\omega{\cdot}s,z_i(s,\omega,x))-g_{0i}(\omega{\cdot}s,z_i(s,\omega,y))\right)ds\\
&\le M(\omega,y)-M(\omega,x)\,,
\end{aligned}$$

as stated. □

Proposition 5.5. *Fix $r>0$. Given $\varepsilon>0$ there exists $\delta>0$ such that, if (ω,x), $(\omega,y)\in\Omega\times B_r$ with $\mathsf{d}(x,y)<\delta$ and $x\le_A y$, then $\|\widehat{z}(t,\omega,x)-\widehat{z}(t,\omega,y)\|\le\varepsilon$ whenever they are defined.*

Proof. Let $r_1=r\sup_{\omega\in\Omega}\|(\widehat{D}^{-1})_2(\omega,\cdot)\|$. Fix $\varepsilon>0$; it follows from Proposition 5.3 that there is a $\delta_1>0$ such that, if (ω,x), $(\omega,y)\in\Omega\times B_{r_1}$ with $\mathsf{d}(x,y)<\delta_1$, then $|M(\omega,y)-M(\omega,x)|\le\varepsilon$. Now, thanks to Theorem 5.2, there is a $\delta>0$ such that, if (ω,x), $(\omega,y)\in\Omega\times B_r$ with $\mathsf{d}(x,y)<\delta$, then

$$\mathsf{d}((\widehat{D}^{-1})_2(\omega,x),(\widehat{D}^{-1})_2(\omega,y))<\delta_1.$$

Altogether, using Lemma 5.4, if (ω,x), $(\omega,y)\in\Omega\times B_r$ with $\mathsf{d}(x,y)<\delta$ and $x\le_A y$, then

$$0\le D_i(\tau(t,\widehat{D}^{-1}(\omega,y)))-D_i(\tau(t,\widehat{D}^{-1}(\omega,x)))\le M(\widehat{D}^{-1}(\omega,y))-M(\widehat{D}^{-1}(\omega,x))\le\varepsilon,$$

whence

$$0\le\widehat{z}_i(t,\omega,y)-\widehat{z}_i(t,\omega,x)\le\varepsilon$$

for all $t\ge 0$ where they are defined and all $i\in\{1,\dots,m\}$. The result is proved. □

Proposition 5.6. *Assume hypotheses* (C1)–(C4) *together with* (F4)*. Suppose that there exists* $(\omega_0, x_0) \in \Omega \times BU$ *such that* $\widehat{\tau}(\cdot, \omega_0, y_0)$ *is bounded, where* $y_0 = \widehat{D}_2(\omega_0, x_0)$*. The following statements hold:*

(i) *when the order* $\leq_A$ *associated to* $\leq_{A,\rho}$ *is considered, then it holds that, for all* $(\omega, x) \in \Omega \times BU$*,* $\widehat{\tau}(\cdot, \omega, y)$ *is bounded and uniformly stable for* $\leq_A$ *in bounded subsets, where* $y = \widehat{D}_2(\omega, x)$*;*

(ii) *when the order* $\leq_A$ *associated to* $\leq_{A,\infty}$ *is considered, if* y_0 *is Lipschitz continuous, then it holds that, for all* $(\omega, x) \in \Omega \times BU$ *such that* $y = \widehat{D}_2(\omega, x)$ *is Lipschitz continuous,* $\widehat{\tau}(\cdot, \omega, y)$ *is bounded and uniformly stable for* $\leq_A$ *in bounded subsets.*

Proof. To prove (i), let us define $\widetilde{y} \in BU$ as the solution of

$$\begin{cases} y'(t) = Ay(t) + \mathbf{1}, & t \in [-\rho, 0], \\ y_{-\rho} \equiv \mathbf{1}, \end{cases} \tag{5.3}$$

where $\mathbf{1} = (1, \ldots, 1) \in \mathbb{R}^m$. Since the matrix A is quasipositive, the system of ordinary differential equations (5.3) is cooperative. The standard comparison theory for its solutions allows us to conclude that there exists $k_0 > 0$ such that $\widetilde{y}(t) \geq k_0\mathbf{1}$ for all $t \leq 0$ and $\widetilde{y} \geq_{A,\rho} 0$. Let us fix $z \in BU$ and suppose that z is Lipschitz continuous on $[-\rho, 0]$; let us check that there is a $\lambda_0 > 0$ such that, for all $\lambda \geq \lambda_0$, $-\lambda\widetilde{y} \leq_{A,\rho} z \leq_{A,\rho} \lambda\widetilde{y}$. Since $\widetilde{y}(t) \geq k_0\mathbf{1}$ for all $t \leq 0$, it is clear that, if $\lambda \geq \lambda_0$, then $-\lambda\widetilde{y} \leq z \leq \lambda\widetilde{y}$ holds for all big enough λ_0. In addition, $(\lambda\widetilde{y} - z)' - A(\lambda\widetilde{y} - z) = \lambda\mathbf{1} - (z' - Az)$ is greater or equal to 0 a.e. in $[-\rho, 0]$ for all big enough λ_0 and we are done.

Now, let K be the omega-limit set of (ω_0, y_0) for the semiflow $\widehat{\tau}$ and let $r_1 > 0$ be such that $K \subset \Omega \times B_{r_1}$. We will prove that $\widehat{z}(\cdot, \omega, \lambda\widetilde{y})$ and $\widehat{z}(\cdot, \omega, -\lambda\widetilde{y})$ are bounded for all $\omega \in \Omega$ and all sufficiently big λ. In order to do this, fix $\omega \in \Omega$ and $z \in K_\omega$. We know that there is a $\lambda_0 > 0$ (irrespective of ω) such that, for all $\lambda \geq \lambda_0$, $-\lambda\widetilde{y} \leq_{A,\rho} z \leq_{A,\rho} \lambda\widetilde{y}$. For each $s \in [0, 1]$, let $y_s = (1 - s)z + s\lambda\widetilde{y}$; clearly, $y_s \leq_{A,\rho} y_t$ for all $0 \leq s \leq t \leq 1$. Besides, there exists $r > 0$ such that $\{y_s\}_{s\in[0,1]} \subset B_r$. An application of Proposition 5.5 for $\varepsilon = 1$, implies that there are a $\delta > 0$ and a partition $0 = s_0 \leq s_1 \leq \cdots \leq s_n = 1$ of $[0, 1]$ such that $\mathsf{d}(y_{s_j}, y_{s_{j+1}}) < \delta$ for all $j \in \{1, \ldots, n-1\}$, and therefore $\|\widehat{z}(t, \omega, y_{s_j}) - \widehat{z}(t, \omega, y_{s_{j+1}})\| \leq 1$ for all $t \geq 0$ where they are defined. Consequently, for each $j \in \{0, \ldots, n\}$, the solution $\widehat{z}(\cdot, \omega, y_{s_j})$ is globally defined and $\|\widehat{z}(t, \omega, z) - \widehat{z}(t, \omega, \lambda\widetilde{y})\| \leq n$ for all $t \geq 0$, which implies that $\widehat{z}(\cdot, \omega, \lambda\widetilde{y})$ is bounded. Analogously, $\widehat{z}(\cdot, \omega, -\lambda\widetilde{y})$ is bounded as well.

Finally, let $(\omega, x) \in \Omega \times BU$ and $y = \widehat{D}_2(\omega, x)$. Let $z = \widehat{u}(\rho, \omega, y)$; since z is Lipschitz continuous on $[-\rho, 0]$, there exists $\lambda \geq \lambda_0$ such that $-\lambda\widetilde{y} \leq_{A,\rho} z \leq_{A,\rho} \lambda\widetilde{y}$, which implies that $\widehat{z}(\cdot, \omega{\cdot}\rho, z)$ is bounded thanks to the fact that $\widehat{z}(\cdot, \omega, \lambda\widetilde{y})$ and $\widehat{z}(\cdot, \omega, -\lambda\widetilde{y})$ are bounded and to the monotonicity of $\widehat{\tau}$. Consequently, the trajectory through (ω, y) for $\widehat{\tau}$ is bounded. The remainder of the proof follows from Proposition 5.5.

Now we deal with statement (ii). Notice that the fact that all the eigenvalues of A have a negative real part implies that A is a hyperbolic matrix. Consider the cooperative system of ordinary differential equations $y' = Ay + \mathbf{1}$. It is well-known that there exists a unique solution of the aforementioned system that is bounded and exponentially stable when $t \to \infty$, namely $\widetilde{y} \equiv -A^{-1}\mathbf{1}$. Since 0 is a strong subequilibrium of the system, as seen in Novo, Núñez and Obaya [17], there exists $k_0 > 0$ such that $\widetilde{y} \geq k_0\mathbf{1}$. Denote again by $\widetilde{y}$ its restriction to $(-\infty, 0]$. The rest of the proof is analogous to the one of statement (i). □

This proposition proves condition (F5) stated in Section 4. The following result is a direct consequence of Theorem 4.4.

Theorem 5.7. *Under the hypotheses of* Proposition 5.6*, for all* $(\omega, x) \in \Omega \times BU$ *satisfying the conditions of statements* (i) *or* (ii) *respectively, we have that* $\mathcal{O}(\omega, x)$ *is a copy of the base whenever* (F6) *holds as well.*

Sufficient conditions under which (F6) holds will be given in the next section together with those guaranteeing (F4).

6. Some specific neutral compartmental systems

Let (Ω, d) be a compact metric space and let $\mathbb{R} \times \Omega \to \Omega$, $\omega \mapsto \omega{\cdot}t$ be a minimal real flow on Ω. Let us focus on the study of the following family of compartmental systems:

$$\frac{d}{dt}[z_i(t)-c_i(\omega{\cdot}t)z_i(t-\alpha_i)] = -\sum_{j=1}^{m} g_{ji}(\omega{\cdot}t, z_i(t)) + \sum_{j=1}^{m} g_{ij}(\omega{\cdot}(t-\rho_{ij}), z_j(t-\rho_{ij})) \quad (6.1)_\omega$$

for $i \in \{1, \ldots, m\}$, $t \geq 0$ and $\omega \in \Omega$, where $g_{ij} : \Omega \times \mathbb{R} \to \mathbb{R}$ and $c_i : \Omega \to \mathbb{R}$ are continuous functions and α_i, $\rho_{ij} \in \mathbb{R}$ for i, $j \in \{1, \ldots, m\}$. Throughout this section, we will use the notation introduced in Section 5. Let us assume the following conditions on equation $(6.1)_\omega$:

- (G1) g_{ij} is C^1 and nondecreasing in its second variable, and $g_{ij}(\omega, 0) = 0$ for each $\omega \in \Omega$, i, $j \in \{1, \ldots, m\}$;
- (G2) $\alpha_i > 0$, $\rho_{ij} \geq 0$ and $0 \leq c_i(\omega) < 1$ for all $i, j \in \{1, \ldots, m\}$ and all $\omega \in \Omega$.

Notice that the family of equations $(6.1)_\omega$ corresponds to a *closed* compartmental system, that is, a system where there is no incoming material from the environment and there is no outgoing material toward the environment either. As a result, the total mass of the system is invariant along the trajectories and 0 is a constant bounded solution of all the equations of the family.

Definition 6.1. Let us consider a function $\mathfrak{c} : \Omega \to \mathbb{R}^m$.

- (i) $\mathfrak{c}$ is said to be *Lipschitz continuous along the flow* if, for some $\omega \in \Omega$, the mapping $\mathbb{R} \to \mathbb{R}^m$, $t \mapsto \mathfrak{c}(\omega{\cdot}t)$ is Lipschitz continuous;
- (ii) $\mathfrak{c}$ is *continuously differentiable along the flow* if, for all $\omega \in \Omega$, the mapping
$$\begin{array}{ccc} \Omega & \longrightarrow & \mathbb{R}^m \\ \omega & \mapsto & \frac{d}{dt}\mathfrak{c}(\omega{\cdot}t)\big|_{t=0} \end{array}$$
is well-defined and continuous. We will refer to this mapping as the derivative of $\mathfrak{c}$.

It is easy to check that, due to the density of all trajectories within Ω, if $\mathfrak{c} : \Omega \to \mathbb{R}^m$ is Lipschitz continuous along the flow, then the mappings $\mathbb{R} \to \mathbb{R}^m$, $t \mapsto \mathfrak{c}(\widetilde{\omega}{\cdot}t)$, $\widetilde{\omega} \in \Omega$, are all Lipschitz continuous with the same Lipschitz constant. From now on, $c : \Omega \to \mathbb{R}^m$ will denote $c = (c_i)_{i=1}^m : (\omega, x) \mapsto (c_i(\omega, x))_{i=1}^m$. It is noteworthy that, given $\omega \in \Omega$, if c is a Lipschitz continuous function, then $x \in BU$ is Lipschitz continuous if and only if $\widehat{D}_2(\omega, x)$ is Lipschitz continuous.

For each $i \in \{1, \ldots, m\}$, let $c_i^{[0]}(\omega) = 1$ and, for each $n \in \mathbb{N}$, define

$$c_i^{[n]}(\omega) = \prod_{j=0}^{n-1} c_i(\omega{\cdot}(-j\alpha_i)), \ \omega \in \Omega.$$

Proposition 6.2. *Assume that c is continuously differentiable along the flow. Suppose that $(\omega,x)\in\Omega\times BU$ admits a backward orbit extension and that there is an $r_1>0$ such that $u(t,\omega,x)\in B_{r_1}$ for each $t\in\mathbb{R}$. Then $z=z(\cdot,\omega,x)$, the solution of $(6.1)_\omega$ with initial value x, belongs to $C^1(\mathbb{R},\mathbb{R}^m)$.*

Proof. Let $\widehat{z}=\widehat{z}(\cdot,\widehat{D}(\omega,x))$. It is clear that $\widehat{z}$ is of class C^1 and it is bounded by $r_1\sup_{\omega_1\in\Omega}\|D(\omega_1,\cdot)\|$. Then, for all $t\in\mathbb{R}$ and all $i\in\{1,\dots,m\}$, it holds that

$$z_i(t)=\sum_{n=0}^{\infty}c_i^{[n]}(\omega{\cdot}t)\widehat{z}_i(t-n\alpha_i).$$

From (G2), it follows that this series converges uniformly on $\mathbb{R}$. Analogously, the formal derivative of the former series, namely

$$\sum_{n=0}^{\infty}c_i^{[n]}(\omega{\cdot}t)\widehat{z}_i{}'(t-n\alpha_i)+\sum_{n=0}^{\infty}\frac{d}{ds}c_i^{[n]}(\omega{\cdot}(t+s))\bigg|_{s=0}\widehat{z}_i(t-n\alpha_i), \tag{6.2}$$

converges uniformly on $\mathbb{R}$ thanks to (G2). Consequently, z_i is continuously differentiable on $\mathbb{R}$. □

Note that, in the conditions of the previous proposition, the derivative of z is given by (6.2). The following result is a straightforward consequence of hypotheses (G1)–(G2).

Proposition 6.3. *Under hypotheses* (G1)–(G2)*, the family of equations* $(6.1)_\omega$ *satisfies conditions* (C1)–(C4)*.*

We next analyze some situations where the previous theory can be applied. They are chosen to describe different types of conditions which assure the monotonicity of the semiflow for some transformed exponential order leading to different dynamical implications. In the next statement, we take $\rho_{ii}=2\alpha_i$, $i=1,\dots,m$. Note that condition (G3.2), independently of the matrix A, is always required. The conclusion of the theorem also guarantees that all the trajectories are relatively compact.

For each $\omega\in\Omega$ and each i, $j\in\{1,\dots,m\}$, let $l_{ii}^-(\omega)=\inf_{v\in\mathbb{R}}\frac{\partial g_{ii}}{\partial v}(\omega,v)$, $l_{ij}^+(\omega)=\sup_{v\in\mathbb{R}}\frac{\partial g_{ij}}{\partial v}(\omega,v)$ and $L_i^+(\omega)=\sum_{j=1}^m l_{ji}^+(\omega)$. In this section, we will assume that $L_i^+(\omega)<\infty$ for all $\omega\in\Omega$.

Theorem 6.4. *Let us assume hypotheses* (G1)–(G2) *together with*

(G3) *for each $i\in\{1,\dots,m\}$, if $c_i\not\equiv 0$, then $\rho_{ii}=2\alpha_i$ and there exists $a_i\in(-\infty,0]$ such that, for all $\omega\in\Omega$, the following conditions hold:*
- (G3.1) $(-a_i-L_i^+(\omega))e^{a_i\alpha_i}-L_i^+(\omega)c_i(\omega)\geq 0$,
- (G3.2) $l_{ii}^-(\omega{\cdot}(-\rho_{ii}))-L_i^+(\omega)c_i^{[2]}(\omega)\geq 0$,

where at least one of the inequalities is strict. Then all the trajectories of the family $(6.1)_\omega$ are bounded and their omega-limit sets are copies of the base.

Proof. Proposition 6.3 guarantees that conditions (C1)–(C4) are satisfied. For each $i\in\{1,\dots,m\}$ such that $c_i\equiv 0$, let $a_i=-\sup_{\omega\in\Omega}L_i^+(\omega)-1$. Let A be the $m\times m$ diagonal matrix with diagonal elements $a_1,\dots,a_m$. We consider the order $\leq_{D,A}$ associated to $\leq_{A,\rho}$ for $\rho=\max\{\rho_{11},\dots,\rho_{mm}\}$. Let us check that the family of equations $(6.1)_\omega$ satisfies conditions (F4) and (F6). First, let us focus on condition (F4). If $c_i\equiv 0$, then $[AD(\omega,z)]_i=a_iz_i(0)\leq -L_i^+(\omega)z_i(0)\leq F_i(\omega,y)-F_i(\omega,x)$ and we are done. Let us suppose that $c_i\not\equiv 0$. Fix $i\in\{1,\dots,m\}$. Let (ω,x), $(\omega,y)\in$

$\Omega\times BU$ with $(\omega,x)\le_{D,A}(\omega,y)$ and denote $z=y-x$. Then we have that $D_i(\omega,z)\ge e^{a_i\alpha_i}D_i(\omega{\cdot}(-\alpha_i),z_{-\alpha_i})$, whence

$$z_i(0)-c_i(\omega)z_i(-\alpha_i)\ge e^{a_i\alpha_i}(z_i(-\alpha_i)-c_i(\omega{\cdot}(-\alpha_i))z_i(-\rho_{ii})). \tag{6.3}$$

From Theorem 5.2, it follows that $z\ge 0$. Thus

$$\begin{aligned}F_i(\omega,y)-F_i(\omega,x)&\ge -L_i^+(\omega)z_i(0)+l_{ii}^-(\omega{\cdot}(-\rho_{ii}))z_i(-\rho_{ii})\\&+\sum_{j\neq i}l_{ij}^-(\omega{\cdot}(-\rho_{ij}))z_j(-\rho_{ij})\ge -L_i^+(\omega)z_i(0)+l_{ii}^-(\omega{\cdot}(-\rho_{ii}))z_i(-\rho_{ii}).\end{aligned}$$

Note that (G3.1) implies that $-a_i-L_i^+(\omega)\ge 0$ for all $\omega\in\Omega$. Then, from (6.3), it follows that

$$\begin{aligned}F_i(\omega,y)-F_i(\omega,x)-[AD(\omega,z)]_i&\ge(-L_i^+(\omega)-a_i)z_i(0)+a_ic_i(w)z_i(-\alpha_i)\\&+l_{ii}^-(\omega{\cdot}(-\rho_{ii}))z_i(-\rho_{ii})\ge[(-L_i^+(\omega)-a_i)e^{a_i\alpha_i}-L_i^+(\omega)c_i(\omega)]z_i(-\alpha_i)\\&+[l_{ii}^-(\omega{\cdot}(-\rho_{ii}))+(L_i^+(\omega)+a_i)c_i(\omega{\cdot}(-\alpha_i))e^{a_i\alpha_i}]z_i(-\rho_{ii}).\end{aligned}$$

On the other hand, $D_i(\omega{\cdot}(-\alpha_i),z_{-\alpha_i})\ge 0$, which in turn implies that $z_i(-\alpha_i)-c_i(\omega{\cdot}(-\alpha_i))z_i(-\rho_{ii})\ge 0$. Consequently, thanks to (G3.1) and (G3.2),

$$\begin{aligned}&F_i(\omega,y)-F_i(\omega,x)-[AD(\omega,z)]_i\ge\\&\quad\ge[l_{ii}^-(\omega{\cdot}(-\rho_{ii}))-L_i^+(\omega)\,c_i(\omega)\,c_i(\omega{\cdot}(-\alpha_i))]z_i(-\rho_{ii})\ge 0.\end{aligned} \tag{6.4}$$

As a result, (G3.2) is a sufficient condition for (F4) to hold.

As for (F6), if we had that $D_i(\omega{\cdot}s,x_s)<D_i(\omega{\cdot}s,y_s)$ for all $s\le 0$, then $z_i(s)>c_i(\omega{\cdot}s)z_i(s-\alpha_i)\ge 0$, $s\le 0$. It is clear that, if condition (G3.1) is strict, then the first inequality in (6.4) is strict; on the other hand, if condition (G3.2) is strict, then the second inequality in (6.4) is strict. This way, the fact that at least one of the inequalities in (G3) is strict together with the choice of a_i when $c_i\equiv 0$ and an argument similar to the foregoing one yield (F6), as expected. The remainder of the theorem follows from Theorem 5.7. □

Let us focus on a different approach to the study of family $(6.1)_\omega$; we will give another valid monotonicity condition for the transformed exponential order defined on the whole interval $(-\infty,0]$. For each $i\in\{1,\dots,m\}$, each $\omega\in\Omega$ and each $a\in(-\infty,0]$, we consider $q_{i0}(\omega,a)=-L_i^+(\omega)-a$ and

$$\begin{aligned}p_{in}(\omega,a)&=-\,L_i^+(\omega)\,c_i^{[n]}(\omega)+e^{a(\alpha_i-\rho_{ii})}l_{ii}^-(\omega{\cdot}(-\rho_{ii}))\,c_i^{[n-1]}(\omega{\cdot}(-\rho_{ii})),\\q_{in}(\omega,a)&=q_{in-1}(\omega,a)\,e^{a\alpha_i}+p_{in}(\omega,a),\ n\in\mathbb{N}.\end{aligned}$$

In the conditions of the next statement, an infinite sequence of consecutive inequalities is required. If c is Lipschitz continuous, then the conclusions hold when dealing with Lipschitz continuous initial data.

Theorem 6.5. *Assume hypotheses* (G1)–(G2) *together with the following one:*

(G4) *for each $i\in\{1,\dots,m\}$, if $c_{ii}\not\equiv 0$, then $\rho_{ii}\le\alpha_i$ and there exists $a_i\in(-\infty,0]$ such that, for all $\omega\in\Omega$, there is an $n_0(\omega)\in\mathbb{N}\cup\{0\}$ such that*

$$\begin{aligned}&q_{in}(\omega,a_i)\ge 0,\ n\in\{0,\dots,n_0(\omega)-1\}\ (\text{if } n_0(\omega)\ge 1),\\&q_{in_0(\omega)}(\omega,a_i)>0,\ \ \text{and}\\&p_{in}(\omega,a_i)\ge 0\ \ \text{for all } n>n_0(\omega).\end{aligned}$$

Then, for all $(\omega, x) \in \Omega \times BU$ such that $\widehat{D}_2(\omega, x)$ is Lipschitz continuous, the trajectory $\{\tau(t, \omega, x) : t \geq 0\}$ is bounded and its omega-limit set is a copy of the base.

Proof. First, Proposition 6.3 yields conditions (C1)–(C4). For each $i \in \{1, \ldots, m\}$ such that $c_i \equiv 0$, let $a_i = -\sup_{\omega\in\Omega} L_i^+(\omega) - 1$. Let A be the $m \times m$ diagonal matrix with diagonal elements $a_1, \ldots, a_m$ and consider the order $\leq_{D,A}$ associated to $\leq_{A,\infty}$. Let us check that the family of equations $(6.1)_\omega$ satisfies conditions (F4) and (F6). Let (ω, x), $(\omega, y) \in \Omega \times BU$ with $(\omega, x) \leq_{D,A} (\omega, y)$ and let $z = y - x$, $\widehat{z} = \widehat{D}_2(\omega, z)$. Fix $i \in \{1, \ldots, m\}$. As we know

$$z_i(s) = \sum_{n=0}^{\infty} c_i^{[n]}(\omega{\cdot}s)\widehat{z}_i(s - n\alpha_i).$$

Now, if $c_i \equiv 0$, then $F_i(\omega, y) - F_i(\omega, x) - [AD(\omega, z)]_i \geq (-L_i^+(\omega) - a_i)\widehat{z}_i(0)$ and (F4) and (F6) hold. Assume now that $c_i \not\equiv 0$; then we have that

$$\begin{aligned}
F_i(\omega, y) - F_i(\omega, x) - [AD(\omega, z)]_i &\geq -L_i^+(\omega)z_i(0) + l_{ii}^-(\omega{\cdot}(-\rho_{ii}))z_i(-\rho_{ii}) - a_i\widehat{z}_i(0) \\
= - L_i^+(\omega) &\sum_{n=0}^{\infty} c_i^{[n]}(\omega)\widehat{z}_i(-n\alpha_i) \\
+ l_{ii}^-(\omega{\cdot}(-\rho_{ii})) &\sum_{n=0}^{\infty} c_i^{[n]}(\omega{\cdot}(-\rho_{ii}))\widehat{z}_i(-\rho_{ii} - n\alpha_i) - a_i\widehat{z}_i(0).
\end{aligned}$$

Note that $\widehat{z} \geq_A 0$ and $\rho_{ii} \leq \alpha_i$; hence, $\widehat{z}_i(-\rho_{ii} - n\alpha_i) \geq e^{a_i(\alpha_i - \rho_{ii})}\widehat{z}_i(-(n+1)\alpha_i)$ and

$$\begin{aligned}
F_i(\omega, y) - F_i(\omega, x) - [AD(\omega, z)]_i &\geq \\
\geq &(-L_i^+(\omega) - a_i)\widehat{z}_i(0) - L_i^+(\omega)\sum_{n=1}^{\infty} c_i^{[n]}(\omega)\widehat{z}_i(-n\alpha_i) \\
&+ e^{a_i(\alpha_i - \rho_{ii})}l_{ii}^-(\omega{\cdot}(-\rho_{ii}))\sum_{n=1}^{\infty} c_i^{[n-1]}(\omega{\cdot}(-\rho_{ii}))\widehat{z}_i(-n\alpha_i) \\
= &\, q_{i0}(\omega, a_i)\widehat{z}_i(0) + \sum_{n=1}^{\infty} p_{in}(\omega, a_i)\widehat{z}_i(-n\alpha_i).
\end{aligned} \tag{6.5}$$

If $n_0(\omega) = 0$, we are done. Otherwise, we observe that, for all $n \in \mathbb{N} \cup \{0\}$, $\widehat{z}_i(-n\alpha_i) \geq e^{a_i\alpha_i}\widehat{z}_i(-(n+1)\alpha_i)$. Hence,

$$\begin{aligned}
F_i(\omega, y) - F_i(\omega, x) - [AD(\omega, z)]_i &\geq \\
&\geq (q_{i0}(\omega, a_i)e^{a_i\alpha_i} + p_{i1}(\omega, a_i))\widehat{z}_i(-\alpha_i) + \sum_{n=2}^{\infty} p_{in}(\omega, a_i)\widehat{z}_i(-n\alpha_i) \\
&= q_{i1}(\omega, a_i)\widehat{z}_i(-\alpha_i) + \sum_{n=2}^{\infty} p_{in}(\omega, a_i)\widehat{z}_i(-n\alpha_i) \\
&\geq \cdots \geq \; q_{in_0(\omega)}(\omega, a_i)\widehat{z}_i(-n_0(\omega)\alpha_i) + \sum_{n=n_0(\omega)+1}^{\infty} p_{in}(\omega, a_i)\widehat{z}_i(-n\alpha_i).
\end{aligned}$$

This way, (F4) and (F6) follow easily from (G4). The desired result is a consequence of Theorem 5.7. □

The scalar version of the family of functional differential equations $(6.1)_\omega$ with autonomous linear operator and $\rho = \alpha$ was studied in [1] using the standard ordering. In particular, in this paper it was established that all the minimal sets are copies of the base. Our next result obtains the same conclusion when the linear operator is non-autonomous and condition (G5) holds, by using the transformed exponential ordering.

Proposition 6.6. *Assume conditions* (G1) *and* (G2). *Consider the family* $(6.1)_\omega$ *and assume that, for all $i \in \{1, \ldots, m\}$, if $c_i \not\equiv 0$, then $\alpha_i = \rho_{ii}$ and the following assertion holds:*

(G5) $l_{ii}^-(\omega\cdot(-\rho_{ii})) - L_i^+(\omega)c_i(\omega) \geq 0.$

Then, for all $(\omega, x) \in \Omega \times BU$, the trajectory $\{\tau(t, \omega, x) : t \geq 0\}$ is bounded and its omega-limit set is a copy of the base.

Proof. Let $\rho > 0$ and take the order $\leq_{D,A}$ associated to $\leq_{A,\rho}$. For each $i \in \{1, \ldots, m\}$, let $a_i = -\sup_{\omega\in\Omega} L_i^+(\omega) - 1$. Let $n_0(\omega) = 0$; then $q_{i0}(\omega, a_i) > 0$ for all $\omega \in \Omega$ and $p_{in}(\omega, a) = c_i^{[n-1]}(\omega\cdot(-\alpha_i))(l_{ii}^-(\omega\cdot(-\rho_{ii})) - L_i^+(\omega)c_i(\omega)) \geq 0$ for all $a \leq 0$ and all $n \in \mathbb{N}$. Following the arguments of Theorem 6.5, we obtain again the relation (6.5). Now, it is immediate from (G5) that condition (G4) holds, which implies the monotonicity of the semiflow. The remainder of the proof follows from Theorem 5.7. □

Let us consider the following hypothesis:

(G6) $\alpha_i > 0$, $\rho_{ij} \geq 0$, $0 \leq c_i(\omega)$ for all $i, j \in \{1, \ldots, m\}$ and $\sum_{i=1}^m c_i(\omega) < 1$ for all $\omega \in \Omega$.

We will give an alternative condition which provides the monotonicity for the direct exponential order of the semiflow τ associated to the family $(6.1)_\omega$. The following result extends the conclusions in [15] for the scalar periodic case to the m-dimensional system of recurrent NFDEs $(6.1)_\omega$. Note that we provide precise conditions which assure the monotonicity of the semiflow on $\Omega \times BU$. It is important to mention that, in the present situation, the conclusions in [19] remain valid.

Theorem 6.7. *Assume that conditions* (G1) *and* (G6) *hold and, moreover, the following condition is satisfied:*

(G7) *c is continuously differentiable along the flow; let $\gamma : \Omega \to \mathbb{R}^m$ be its derivative. Besides, for each $i \in \{1, \ldots, m\}$, there exists $a_i \in (-\infty, 0]$ such that, if A is the $m \times m$ diagonal matrix with diagonal elements $a_1, \ldots, a_m$ and we consider the order $\leq_A = \leq_{A,\infty}$, then, for all $\omega \in \Omega$, the following inequalities hold:*

(G7.1) *if (ω, x), $(\omega, y) \in \Omega \times BU$ and $x \leq_A y$, then $F_i(\omega, y) - F_i(\omega, x) - a_i D_i(\omega, y - x) + \gamma_i(\omega)(y_i(-\alpha_i) - x_i(-\alpha_i)) \geq 0$.*

(G7.2) *if (ω, x), $(\omega, y) \in \Omega \times BU$ admit a backward orbit extension, $x \leq_A y$ and there exists $J \subset \{1, \ldots, m\}$ such that $x_i = y_i$ for all $i \notin J$ and $x_i(s) < y_i(s)$ for all $i \in J$ and all $s \leq 0$, then $F_i(\omega, y) - F_i(\omega, x) - a_i D_i(\omega, y - x) + \gamma_i(\omega)(y_i(-\alpha_i) - x_i(-\alpha_i)) > 0$ for all $i \in J$.*

Fix (ω, x), $(\omega, y) \in \Omega \times BU$ such that $x \leq_A y$. Then

$$u(t, \omega, x) \leq_A u(t, \omega, y)$$

for all $t \geq 0$ where they are defined. Moreover, all Lipschitz continuous initial data give rise to trajectories of the family $(6.1)_\omega$ which are bounded, and their omega-limit sets are copies of the base.

Proof. Proposition 6.3 guarantees that conditions (C1)–(C4) are satisfied. Fix (ω, x), $(\omega, y) \in \Omega \times BU$ such that $x \leq_A y$. Let $\rho > 0$ such that $u(t, \omega, x)$, $u(t, \omega, y)$ are defined on $[0, \rho]$. Let $\varepsilon > 0$ and denote by y^ε the solution of

$$\begin{cases} \frac{d}{dt} D(\omega{\cdot}t, z_t) = F(\omega{\cdot}t, z_t) + \varepsilon \mathbf{1}, \quad t \geq 0, \\ z_0 = y, \end{cases}$$

where, $\mathbf{1} = (1, \ldots, 1)^T \in \mathbb{R}^m$. There exists $\varepsilon_0 > 0$ such that, for all $\varepsilon \in [0, \varepsilon_0)$, $z = z(\cdot, \omega, x)$ and y^ε are defined on $[0, \rho]$. Let $z^\varepsilon = y^\varepsilon - z$ and denote by t_1 the greatest element of $[0, \rho]$ such that $z^\varepsilon_{t_1} \geq_A 0$. Suppose that $t_1 < \rho$. Since $z^\varepsilon_{t_1} \geq_A 0$, $z^\varepsilon_i(t_1) \geq e^{a_i \alpha_i} z^\varepsilon_i(t_1 - \alpha_i)$ for all $i \in \{1, \ldots, m\}$ and, from (G7.1), it follows that

$$\begin{aligned} \frac{d}{dt}(z^\varepsilon_i(t) &- c_i(\omega{\cdot}t)\, z^\varepsilon_i(t - \alpha_i))|_{t = t_1} - a_i(z^\varepsilon_i(t_1) - c_i(\omega{\cdot}t_1)\, z^\varepsilon_i(t_1 - \alpha_i)) \\ &+ \gamma_i(\omega{\cdot}t_1)\, z^\varepsilon_i(t_1 - \alpha_i) = F_i(\omega{\cdot}t_1, y^\varepsilon_{t_1}) - F_i(\omega{\cdot}t_1, z_{t_1}) \\ &- a_i(z^\varepsilon_i(t_1) - c_i(\omega{\cdot}t_1)\, z^\varepsilon_i(t_1 - \alpha_i)) + \gamma_i(\omega{\cdot}t_1)\, z^\varepsilon_i(t_1 - \alpha_i) + \varepsilon \geq \varepsilon. \end{aligned}$$

Hence, taking $\alpha = \min\{\alpha_1, \ldots, \alpha_m\}$, there exists $h \in (0, \alpha)$ such that, if $i \in \{1, \ldots, m\}$ and $t \in [t_1, t_1 + h]$, then

$$\frac{d}{dt}(z^\varepsilon_i(t) - c_i(\omega{\cdot}t) z^\varepsilon_i(t - \alpha_i)) - a_i(z^\varepsilon_i(t) - c_i(\omega{\cdot}t) z^\varepsilon_i(t - \alpha_i)) + \gamma_i(\omega{\cdot}t) z^\varepsilon_i(t - \alpha_i) \geq 0.$$

Let us fix $i \in \{1, \ldots, m\}$; for $t_1 \leq s \leq t \leq t_1 + h$, integrating between s and t, we have

$$\begin{aligned} z^\varepsilon_i(t) - c_i(\omega{\cdot}t) z^\varepsilon_i(t - \alpha_i) \geq & e^{a_i(t-s)}(z^\varepsilon_i(s) - c_i(\omega{\cdot}s) z^\varepsilon_i(s - \alpha_i)) \\ & - \int_s^t \gamma_i(\omega{\cdot}u) e^{a_i(t-u)} z^\varepsilon_i(u - \alpha_i)\, du. \end{aligned} \tag{6.6}$$

Fix $\eta > 0$; there exists an analytic function $\widetilde{c}_i : [t_1, t_1 + h] \to \mathbb{R}$ (for instance, a polynomial) such that

$$0 \leq \widetilde{c}_i(t) - c_i(\omega{\cdot}t) \leq \eta \ \text{ and } \ |\widetilde{c}_i\,'(t) - \gamma_i(\omega{\cdot}t)| \leq \eta$$

for every $t \in [t_1, t_1 + h]$. As a result, from (6.6), it follows that, for $t_1 \leq s \leq t \leq t_1 + h$,

$$\begin{aligned} z^\varepsilon_i(t) - \widetilde{c}_i(t)\, z^\varepsilon_i(t - \alpha_i) &\geq e^{a_i(t-s)}(z^\varepsilon_i(s) - \widetilde{c}_i(s)\, z^\varepsilon_i(s - \alpha_i)) \\ - \int_s^t \widetilde{c}_i\,'(u) e^{a_i(t-u)} z^\varepsilon_i(u - \alpha_i)\, du &- \eta\, \|z^\varepsilon_{t_1}\|_\infty (1 + e^{a_i(t-s)} + (t - s)). \end{aligned} \tag{6.7}$$

Now, as $\widetilde{c}_i$ is analytic, there exist $s = s_0 \leq s_1 \leq s_2 \leq \cdots \leq s_J = t$ such that, for all $j \in \{1, \ldots, J\}$, either $\widetilde{c}_i\,'(u) \geq 0$ or $\widetilde{c}_i\,'(u) < 0$ for all $u \in (s_{j-1}, s_j)$. Let $j \in \{0, \ldots, J - 1\}$ and fix $N \in \mathbb{N}$ such that $N \geq 3$; we define $s^N_{j0} = s_j$, $s^N_{j1} = (1 - 1/N) s_j + 1/N s_{j+1}$, $s^N_{j2} = 1/N s_j + (1 - 1/N) s_{j+1}$ and $s^N_{j3} = s_{j+1}$. Then we have

$$\begin{aligned}
\int_{s_j}^{s_{j+1}} \widetilde{c}_i\,'(u) e^{a_i(t-u)} z_i^\varepsilon(u-\alpha_i)\,du &= \sum_{l=1}^{3} \int_{s_{j\,l-1}^N}^{s_{jl}^N} \widetilde{c}_i\,'(u) e^{a_i(t-u)} z_i^\varepsilon(u-\alpha_i)\,du \\
&= \sum_{l=1}^{3} (\widetilde{c}_i(s_{jl}^N) - \widetilde{c}_i(s_{j\,l-1}^N)) e^{a_i(t-u_{jl}^N)} z_i^\varepsilon(u_{jl}^N - \alpha_i) \\
&= \widetilde{c}_i(s_{j+1}) e^{a_i(t-u_{j3}^N)} z_i^\varepsilon(u_{j3}^N - \alpha_i) - \widetilde{c}_i(s_j) e^{a_i(t-u_{j1}^N)} z_i^\varepsilon(u_{j1}^N - \alpha_i) \\
&\quad + \sum_{l=1}^{2} \widetilde{c}_i(s_{jl}^N) \left(e^{a_i(t-u_{jl}^N)} z_i^\varepsilon(u_{jl}^N - \alpha_i) - e^{a_i(t-u_{j\,l+1}^N)} z_i^\varepsilon(u_{j\,l+1}^N - \alpha_i) \right) \\
&\leq \widetilde{c}_i(s_{j+1}) e^{a_i(t-u_{j3}^N)} z_i^\varepsilon(u_{j3}^N - \alpha_i) - \widetilde{c}_i(s_j) e^{a_i(t-u_{j1}^N)} z_i^\varepsilon(u_{j1}^N - \alpha_i)
\end{aligned}$$

where the points $u_{jl}^N \in [s_{j\,l-1}^N, s_{jl}^N]$ for $l = 1, 2, 3$. As a consequence, $u_{jl}^N - \alpha_i \leq t_1$ for $l = 1, 2, 3$.

Taking limits when $N \to \infty$, we obtain

$$\begin{aligned}
\int_{s_j}^{s_{j+1}} \widetilde{c}_i\,'(u) e^{a_i(t-u)} z_i^\varepsilon(u-\alpha_i) du \leq{}& \widetilde{c}_i(s_{j+1}) e^{a_i(t-s_{j+1})}\, z_i^\varepsilon(s_{j+1} - \alpha_i) \\
&- \widetilde{c}_i(s_j) e^{a_i(t-s_j)} z_i^\varepsilon(s_j - \alpha_i).
\end{aligned}$$

Consequently,

$$\begin{aligned}
\int_s^t \widetilde{c}_i\,'(u) e^{a_i(t-u)} z_i^\varepsilon(u-\alpha_i)\,du &\leq \sum_{j=0}^{J-1} \int_{s_j}^{s_{j+1}} \widetilde{c}_i\,'(u) e^{a_i(t-u)} z_i^\varepsilon(u-\alpha_i)\,du \\
&\leq \widetilde{c}_i(t) z_i^\varepsilon(t-\alpha_i) - \widetilde{c}_i(s) e^{a_i(t-s)} z_i^\varepsilon(s-\alpha_i)
\end{aligned}$$

and, using (6.7), it yields

$$z_i^\varepsilon(t) - e^{a_i(t-s)} z_i^\varepsilon(s) \geq -\eta\, \|z_{t_1}^\varepsilon\|_\infty (1 + e^{a_i(t-s)} + (t-s)).$$

Letting $\eta \to 0$, we obtain $z_i^\varepsilon(t) - e^{a_i(t-s)} z_i^\varepsilon(s) \geq 0$ for all $i \in \{1, \dots, m\}$. This is not possible due to the choice of t_1. Thus $z_t^\varepsilon \geq_A 0$ for all $t \in [0, \rho]$ and, taking limits as $\varepsilon \to 0$, $u(t, \omega, x) \leq_A u(t, \omega, y)$ for all $t \in [0, \rho]$ and hence for all $t \geq 0$ where they are defined.

Finally, a generalization of the results in [19] provides the 1-covering property under these conditions. The theorem is proved. □

Note that this theorem requires hypothesis (G6), which is significantly stronger than (G2), which was required for the transformed exponential order. Besides, according to the previous theory, we should remark that the application of the direct exponential order requires the differentiability along the flow of the vector function c, instead of just the continuity of this map, only needed by the transformed exponential order. Thus the transformed exponential order becomes more natural in the study of NFDEs with non-autonomous linear D-operator.

Even in the periodic case, it is known that, given an open set $U \subset C(\Omega, \mathbb{R}^m)$, the subset of differentiable functions is dense, has empty interior and

$$\sup\{\|\mathfrak{c}'\|_\infty : \mathfrak{c} \in U \text{ and } \mathfrak{c} \text{ is differentiable along the flow}\} = \infty$$

(see Schwartzman [21]). As a consequence, the transformed exponential order is also more advantageous when dealing with rapidly oscillating differentiable coefficients c_i, $i \in \{1, \dots, m\}$. In practice, the conditions which allow us to apply the direct exponential order or the transformed exponential order are frequently quite different; the particular problem to be studied, will determine the advantages and disadvantages of each order.

We clarify the hypotheses of Theorem 6.7 in some specific situations. For each $t \in \mathbb{R}$, let $\mathfrak{n}(t) = \min\{t, 0\}$. Next, we give a condition on the coefficients of the equation implying condition (G7).

Proposition 6.8. *Assume that conditions* (G1) *and* (G6) *together with the following one are satisfied:*

(G8) *c is continuously differentiable along the flow; let $\gamma : \Omega \to \mathbb{R}^m$ be its derivative. Besides, for each $i \in \{1, \dots, m\}$, there exists $a_i \in (-\infty, 0]$ such that, for all $\omega \in \Omega$, the following inequality holds:*

$$-L_i^+(\omega) - a_i + \mathfrak{n}(a_i c_i(\omega) + \gamma_i(\omega)) e^{-a_i \alpha_i} > 0.$$

Then, for all $(\omega, x) \in \Omega \times BU$ such that x is Lipschitz continuous, the trajectory of (ω, x) for the family $(6.1)_\omega$ is bounded and its omega-limit set is a copy of the base.

Proof. Let A be the $m \times m$ diagonal matrix with diagonal elements $a_1, \dots, a_m$ and consider the order $\leq_A = \leq_{A,\infty}$. It is clear that, if (ω, x), $(\omega, y) \in \Omega \times BU$ and $x \leq_A y$, then

$$\begin{aligned} F_i(\omega, y) - F_i(\omega, x) - a_i(y_i(0) - x_i(0) - c_i(\omega)(y_i(-\alpha_i) - x_i(-\alpha_i))) \\ + \gamma_i(\omega)(y_i(-\alpha_i) - x_i(-\alpha_i)) \geq \\ \geq (-L_i^+(\omega) - a_i + \mathfrak{n}(a_i c_i(\omega) + \gamma_i(\omega)) e^{-a_i \alpha_i})(y_i(0) - x_i(0)). \end{aligned}$$

Consequently, property (G8) guarantees both (G7.1) and (G7.2) and Theorem 6.7 yields the expected result. □

We take $\rho_{ii} = 2\alpha_i$ under the assumptions of Proposition 6.8. Note that now condition (G3.2) is not required. However, if it holds, then (G3.1) is less restrictive than (G8) even in their autonomous versions.

Finally, we turn to the study of $(6.1)_\omega$ when another monotonicity condition is considered, which improves the conclusions of the previous statement.

Proposition 6.9. *Assume hypotheses* (G1) *and* (G6) *together with the following one:*

(G9) *c is continuously differentiable along the flow; let $\gamma : \Omega \to \mathbb{R}^m$ be its derivative. Besides, for each $i \in \{1, \dots, m\}$, if $c_i \not\equiv 0$, then $\rho_{ii} \leq \alpha_i$ and there exists $a_i \in (-\infty, 0]$ such that, for all $\omega \in \Omega$,*

(G9.1) $-a_i - L_i^+(\omega) \geq 0$,

(G9.2) $e^{a_i \rho_{ii}}(-a_i - L_{ii}^+(\omega)) + l_{ii}^-(\omega{\cdot}(-\rho_{ii})) + e^{a_i(\rho_{ii} - \alpha_i)} \mathfrak{n}(a_i c_i(\omega) + \gamma_i(\omega)) \geq 0$,

where at least one of the inequalities is strict. Then all Lipschitz continuous initial data for the family $(6.1)_\omega$ give rise to bounded trajectories and their omega-limit sets are copies of the base.

Proof. First, Proposition 6.3 yields conditions (C1)–(C4). For each $i \in \{1, \dots, m\}$ such that $c_i \equiv 0$, let $a_i = -\sup_{\omega \in \Omega} L_i^+(\omega) - 1$. Now, let A be the $m \times m$ diagonal matrix with diagonal elements $a_1, \dots, a_m$ and consider the order $\leq_A = \leq_{A,\infty}$. Let us check that the family of equations $(6.1)_\omega$ satisfies conditions (F4) and (F6). Let

(ω, x), $(\omega, y) \in \Omega \times BU$ with $x \leq_A y$ and let $z = y - x$. Fix $i \in \{1, \ldots, m\}$. If $c_i \equiv 0$, then $F_i(\omega, y) - F_i(\omega, x) - [AD(\omega, z)]_i + \gamma_i(\omega) z_i(-\alpha_i) \geq (-L_i^+(\omega) - a_i) z_i(0)$, whence (F4) holds and an argument similar to the one given in Proposition 6.4 yields (F6). Let us assume that $c_i \not\equiv 0$; in this case,

$$\begin{aligned} F_i(\omega, y) - F_i(\omega, x) - [AD(\omega, z)]_i + \gamma_i(\omega) z_i(-\alpha_i) &\geq -L_i^+(\omega) z_i(0) \\ &+ l_{ii}^-(\omega\cdot(-\rho_{ii})) z_i(\omega\cdot(-\rho_{ii})) - a_i(z_i(0) - c_i(\omega) z_i(-\alpha_i)) + \gamma_i(\omega) z_i(-\alpha_i) \\ &= (-a_i - L_i^+(\omega)) z_i(0) + l_{ii}^-(\omega\cdot(-\rho_{ii})) z_i(-\rho_{ii}) + \mathfrak{n}(a_i c_i(\omega) + \gamma_i(\omega)) z_i(-\alpha_i). \end{aligned}$$

Since $z \geq_A 0$ and $\rho_{ii} \leq \alpha_i$, we have that $z_i(-\rho_{ii}) \geq e^{a_i(\alpha_i - \rho_{ii})} z_i(-\alpha_i)$, $z_i(0) \geq e^{a_i \rho_{ii}} z_i(-\rho_{ii})$ and, thanks to (G9.1), it follows that

$$\begin{aligned} F_i(\omega, y) - F_i(\omega, x) - [AD(\omega, z)]_i + \gamma_i(\omega) z_i(-\alpha_i) &\geq [e^{a_i \rho_{ii}}(-a_i - L_i^+(\omega)) \\ &+ l_{ii}^-(\omega\cdot(-\rho_{ii})) + e^{a_i(\rho_{ii} - \alpha_i)} \mathfrak{n}(a_i c_i(\omega) + \gamma_i(\omega))] z_i(-\rho_{ii}). \end{aligned} \tag{6.8}$$

As a result, (G9.2) is a sufficient condition for (F4) to hold.

As for (F6), if we had that $D_i(\omega\cdot s, x_s) < D_i(\omega\cdot s, y_s)$ for all $s \leq 0$, then $z_i(s) > c_i(\omega\cdot s) z_i(s - \alpha_i) \geq 0$, $s \leq 0$. Clearly, if condition (G9.1) is strict, then the first inequality in (6.8) is strict. As a result, the fact that at least one of the inequalities in (G9) is strict together with an argument similar to the previous one yield (F6), as desired. The rest of the proof follows from Theorem 6.7. □

Observe that, under the hypotheses of Proposition 6.9, only two supplementary conditions are required, instead of the infinite sequence needed in Theorem 6.5 to apply the transformed exponential order.

References

[1] O. Arino, F. Bourad, *On the asymptotic behavior of the solutions of a class of scalar neutral equations generating a monotone semiflow*, J. Differential Equations **87** (1990), 84–95.

[2] C.C. Conley, R.A. Miller, *Asymptotic stability without uniform stability: almost periodic coefficients*, J. Differential Equations **1** (1965), 333–336.

[3] R. Ellis, *Lectures on Topological Dynamics*, Benjamin, New York, 1969.

[4] A.M. Fink, *Almost periodic differential equations*, Lecture Notes in Mathematics **377**, Springer-Verlag, Berlin, Heidelberg, New York, 1974.

[5] W.M. Haddad, V. Chellaboina, Q. Hui, *Nonnegative and Compartmental Dynamical Systems*, Princeton University Press, 2010.

[6] J.K. Hale, *Theory of Functional Differential Equations*, Applied Mathematical Sciences **3**, Springer-Verlag, Berlin, Heidelberg, New York 1977.

[7] J.K. Hale, S.M. Verduyn Lunel, *Introduction to Functional Differential Equations*, Applied Mathematical Sciences **99**, Springer-Verlag, Berlin, Heidelberg, New York 1993.

[8] Y. Hino, S. Murakami, T. Naito, *Functional Differential Equations with Infinite Delay*, Lecture Notes in Math. **1473**, Springer-Verlag, Berlin, Heidelberg, 1991.

[9] J.A. Jacquez, *Compartmental Analysis in Biology and Medicine*, Third Edition, Thomson-Shore Inc., Ann Arbor, Michigan, 1996.

[10] J.A. Jacquez, C.P. Simon, *Qualitative theory of compartmental systems*, SIAM Review **35** No.1 (1993), 43–79.

[11] J.A. Jacquez, C.P. Simon, *Qualitative theory of compartmental systems with lags*, Math. Biosci. **180** (2002), 329–362.

[12] J. Jiang, X.-Q. Zhao, *Convergence in monotone and uniformly stable skew-product semiflows with applications*, J. Reine Angew. Math **589** (2005), 21–55.

[13] R. Johnson, *A linear, almost periodic equation with an almost automorphic solution*, Proc. Amer. Math. Soc. **82** No.2 (1981), 199–205.

[14] R. Johnson, *On almost-periodic linear differential systems of Millionščikov and Vinograd*, J. Math. Anal. Appl. **85** (1982), 452–460.

[15] T. Krisztin, J. Wu, *Asymptotic Periodicity, Monotonicity, and Oscillation of Solutions of Scalar Neutral Functional Differential Equations*, J. Math. Anal. Appl. **199** (1996), 502–525.
[16] V. Muñoz-Villarragut, S. Novo, R. Obaya, *Neutral functional differential equations with applications to compartmental systems*, SIAM J. Math. Anal. **40** No. 3 (2008), 1003–1028.
[17] S. Novo, C. Núñez, R. Obaya, *Almost automorphic and almost periodic dynamics for quasi-monotone non-autonomous functional differential equations*, J. Dynamics Differential Equations **17** No.3 (2005), 589–619.
[18] S. Novo, R. Obaya, A.M. Sanz, *Stability and extensibility results abstract skew-product semiflows*, J. Differential Equations **235** No. 2 (2007), 623–646.
[19] S. Novo, R. Obaya, V.M. Villarragut, *Exponential ordering for nonautonomous neutral functional differential equations*, SIAM J. Math. Anal. **41** No. 3 (2009), 1025–1053.
[20] R.J. Sacker, G.R. Sell, *Lifting properties in skew-products flows with applications to differential equations*, Mem. Amer. Math. Soc. **190**, Amer. Math. Soc., Providence 1977.
[21] S. Schwartzman, *Asymptotic cycles*, Ann. Math. **66** No. 2 (1957), 270–284.
[22] G. Sell, *Nonautonomous differential equations and topological dynamics. I. The basic theory*, Trans. Amer. Math. Soc. **127** (1967), 241–262.
[23] G. Sell, *Nonautonomous differential equations and topological dynamics. II. Limiting equations*, Trans. Amer. Math. Soc. **127** (1967), 263–283.
[24] W. Shen, Y. Yi, *Almost Automorphic and Almost Periodic Dynamics in Skew-Product Semiflows*, Mem. Amer. Math. Soc. **647**, Amer. Math. Soc., Providence 1998.
[25] H.L. Smith, H.R. Thieme, *Monotone semiflows in scalar non-quasi monotone functional differential equations*, J. Math. Anal. Appl. **150** (1990), 289–306.
[26] H.L. Smith, H.R. Thieme, *Strongly order preserving semiflows generated by functional differential equations*, J. Differential Equations **93** (1991), 332–363.
[27] Z. Wang, J. Wu, *Neutral Functional Differential Equations with Infinite delay*, Funkcial. Ekvac. **28** (1985), 157–170.
[28] J. Wu, *Unified treatment of local theory of NFDEs with infinite delay*, Tamkang J. Math. No. 1 (1991), 51–72.
[29] J. Wu, H.I. Freedman, *Monotone semiflows generated by neutral functional differential equations with application to compartmental systems*, Can. J. Math. **43** No. 5 (1991), 1098–1120.
[30] J. Wu, X.-Q. Zhao, *Diffusive monotonicity and threshold dynamics of delayed reaction diffusion equations*, J. Differential Equations **186** (2002), 470–484.

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