Note

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Diagonal dominance and invertibility of matrices

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Abstract: A weakly diagonally dominant matrix may or may not be invertible. We characterize, in terms of combinatorial structure and sign pattern when such a matrix is invertible, which is the common case. Examples are given.

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A matrix $A = (a_{ij}) \in \mathcal{M}_n(\mathbb{R})$ is (row) *diagonally dominant* (DD) if

$$|a_{ii}| \geq \sum_{j \neq i} |a_{ij}|, \quad i = 1, \ldots, n.$$

If each inequality is strict, A is called *strictly* (row) *DD*, in which case A is invertible. However, less is needed for invertibility. If A is irreducible, and at least one inequality is strict, A is called *irreducibly* (row) DD, and, again, A is invertible. If none of the inequalities is strict (all are equalities), we call A *weakly* (row) DD. In this case A need not be invertible, *e.g.*,

$$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}.$$

But, A might also be invertible, *e.g.*,

$$\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}.$$

Of course, by transposition, there is a parallel situation for columns. The standard theory of diagonal dominance and invertibility may be found, for example, in [1], where there is no general discussion of when a weakly DD matrix is invertible.

This raises the natural question of when a weakly DD matrix is invertible? We realized that this is an important question in the preparation of [2]. Let $|A|$ denote the entry-wise absolute value of A, $|A| = (|a_{ij}|)$, and $r(A) = |A|e$, in which e, as usual, is the vector of all 1's. If A is to be invertible, it can have no 0 rows, so we assume, without loss of generality, that $r(A)$ is entry-wise positive, and let D_r be the invertible diagonal matrix for which $D_r e = r(A)$, *i.e.*, $D_r = \mathrm{Diag}(r(A))$. Now, A is invertible if and only if $D_r^{-1}A$ is invertible, so that we may assume that $|A|$ is row stochastic. Given a row stochastic, weakly (row) DD matrix T, we consider

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$$\Omega(T) = \{A \in \mathcal{M}_n(\mathbb{R}) \ : \ |A| = T\},$$

and ask two questions: (1) when is T invertible?; (2) which $A \in \Omega(T)$ are invertible? We assume without loss of generality that T is irreducible, because, if not, we may consider its irreducible blocks individually.

By definition

$$T = \frac{1}{2}I + \frac{1}{2}R,$$

in which R is row stochastic and hollow (0 diagonal). Since T is irreducible, R is, as well. Then T is invertible, unless -1 is an eigenvalue of R (which is equivalent to the singularity of T). Since R is row stochastic, 1 is the spectral radius of R and is an eigenvalue of R. Of course, if R were primitive, then -1 is not an eigenvalue of R. But, even if R were non-primitive, -1 might not be an eigenvalue. It depends upon the "index of imprimitivity" [1, Ch 8]. The greatest common divisor (gcd) of the cycle lengths of R determines how many eigenvalues, equally spaced around the unit circle may tie for spectral radius. The value -1 is not an eigenvalue of R if and only if this gcd is odd (the case in which the gcd is 1 is primitivity). If this gcd $=d$, then, up to permutation similarity, which does not change the problem,

$$R = \begin{pmatrix} 0 & R_1 & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \ddots & R_{d-1} \\ R_d & 0 & \cdots & 0 & 0 \end{pmatrix},$$

in which each R_i, $i = 1, \ldots, d$, is non-negative with row sum 1. If d is odd, -1 is not an eigenvalue of R, and conversely. Our answer to question (1) is then

Theorem 1. *The row stochastic irreducible, weakly row DD matrix $A \in \mathcal{M}_n(\mathbb{R})$ is invertible if and only if the gcd of the cycle lengths of $A - \frac{1}{2}I$ is odd.*

Since gcd = 1 is generic, we conclude that an irreducible, row stochastic, weakly row DD matrix is almost always invertible. Of course it is not always invertible, as indicated by simple examples that are characterized by Theorem 1. Note that in the example from the introduction

$$\frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix},$$

$R = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, which is a 2-cycle.

Next, note that in $\Omega(T)$ there are always singular matrices. Consider $A \in \Omega(T)$ with all off-diagonal entries non-positive. Its row sums are 0, so that e lies in the null space of A, which then must be singular. There are also always invertible elements of $\Omega(T)$, in fact many. Now, we say that the irreducible matrix $A(R) = 2T - I$ is *odd type*, if the gcd of its cycle lengths is odd, and $A(R) = 2T - I$ is of *even type* if the gcd of its cycle lengths is even. Note that Theorem 1 in this language is: the row stochastic irreducible, weakly row DD matrix $A \in \mathcal{M}_n(\mathbb{R})$ is invertible if and only if $A(R)$ is odd type. Recall that a signature matrix is a diagonal matrix of ± 1's, and a signature similarity of A is a matrix SAS, in which S is a signature matrix. The effect of a signature similarity is to resign some off-diagonal entries, without changing the eigenvalues. No matter which type R is, if it is signature similar to an entry-wise non-positive matrix, -1 will be an eigenvalue of R, and $A \in \Omega(T)$ will be non-invertible. On the other hand, if R is of odd type, this is the only signature similarity class for which -1 is an eigenvalue.

If A is of even type, then $|A|$ has -1 as an eigenvalue and any signature similarity class of $|A|$ does as well. No other signature similarity class, except that of $-|A|$, can have -1.

Theorem 2. *Suppose that A is weakly row DD and that $|A|$ is row stochastic and irreducible. Let $R = |2A| - I$.*
(1) *If A is of odd type, a resigning A' of $|A|$ that does not change the diagonal of $|A|$ is non-singular if and only if the off-diagonal part of A' is not signature similar to the off-diagonal part of $-|A|$.*
(2) *If A is of even type, a resigning A' of $|A|$ is non-singular if and only if the off-diagonal part of A' is neither signature similar to that of $|A|$ nor that of $-|A|$.*

Example 3. Matrix

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

is of even type and, thus, not invertible. Matrix

$$A' = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$$

is invertible, as it is a resigning that falls in neither category of Theorem 2, part 2. Matrix

$$A'' = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$$

is a resigning that falls in both categories and thus is also not invertible.

Example 4. Matrix

$$\begin{pmatrix} 1/2 & 1/4 & 1/4 \\ 1/4 & 1/2 & 1/4 \\ 1/4 & 1/4 & 1/2 \end{pmatrix}$$

is of odd type and, thus, invertible. The resignings that are signature similar to

$$\begin{pmatrix} 1/2 & -1/4 & -1/4 \\ -1/4 & 1/2 & -1/4 \\ -1/4 & -1/4 & 1/2 \end{pmatrix}$$

are all not invertible. They are

$$\begin{pmatrix} 1/2 & -1/4 & 1/4 \\ -1/4 & 1/2 & 1/4 \\ 1/4 & 1/4 & 1/2 \end{pmatrix} \begin{pmatrix} 1/2 & 1/4 & -1/4 \\ 1/4 & 1/2 & 1/4 \\ -1/4 & 1/4 & 1/2 \end{pmatrix} \begin{pmatrix} 1/2 & 1/4 & 1/4 \\ 1/4 & 1/2 & -1/4 \\ 1/4 & -1/4 & 1/2 \end{pmatrix}.$$

All other resignings of the off-diagonal part are invertible.

The same remarks are valid for any matrix of the form

$$\begin{pmatrix} 1 & a & 1-a \\ 1-b & 1 & b \\ c & 1-c & 1 \end{pmatrix}$$

for $a, b, c \in (0, 1)$.

Example 5. Matrix

$$A = \frac{1}{2}\begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \end{pmatrix}$$

is of even type and, thus, not invertible. If any even number of signs of the off-diagonal are changed, the result is still not invertible, but all are signature similar. All other off-diagonal resignings are invertible.

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References

[1] R. Horn and C. R. Johnson, *Matrix Analysis*, Cambridge University Press, New York, 2013.
[2] C. R. Johnson, C. Marijuán, M. Pisonero, and I. Spitkovsky, Diagonal Entries of Inverses of Diagonally Dominant Matrices. Linear Algebra Appl. submitted.