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CMMSE: Analysis of order reduction when Lawson methods integrate nonlinear initial boundary value problems

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In this paper a thorough analysis is carried out of the type of order reduction that Lawson methods exhibit when used to integrate nonlinear initial boundary value problems. In particular, we focus on nonlinear reaction-diffusion problems, and therefore, this study is important in a large number of practical applications modeled by this type of nonlinear equations. A theoretical study of the local and global error of the total discretization of the problem is carried out, taking into account both, the error coming from the space discretization and that due to the integration in time. These results are also corroborated by the numerical experiments performed in this paper.

KEYWORDS
exponential methods, Lawson methods, nonlinear reaction-diffusion problems, order reduction

MSC CLASSIFICATION
65M12, 65M20

1 | INTRODUCTION

It is well known the advantages of exponential methods when integrating partial differential equations. The key is that this type of methods integrates the linear and stiff part of the differential equation exactly. In this way, it is possible to obtain exponential methods, which are explicit and linearly stable at the same time, which is not possible with classical methods. Moreover, the development of Krylov-type methods to calculate exponential-type functions over matrices which are applied over vectors has helped make exponential methods an efficient approach to integrate such problems in a stable way.

In this paper we are going to focus on Lawson methods, which are a particular type of exponential methods and which have the advantage of the possibility of being built with classical order as high as desired, associated to any given Runge–Kutta method of the same classical order. However, this advantage is affected by the severe order reduction that Lawson methods present when integrating partial differential problems subject to non-periodic and time-dependent boundary.

The analysis of such order reduction in the case that the partial differential problem to be integrated is linear is studied in Alonso–Mallo et al[1] and a technique to avoid such order reduction in this linear case is proposed in Alonso–Mallo et al.[2]

Although the linear case is also interesting, as it is well known, there are numerous practical applications where the mathematical model is based on a nonlinear partial differential problem. This is the case of applications that are described by nonlinear reaction-diffusion equations, which arise in several fields such as physical sciences, biological sciences, ecology, physiology, population dynamics, or finance among others.[3–8] Attention has also been paid in the literature to fractional reaction-diffusion equations,[9–11] although the analysis of models that include fractional derivatives is beyond the scope of this paper.

Therefore, in view of the practical interest of the nonlinear models, similarly as it has been done in Alonso–Mallo et al[1] and Alonso–Mallo et al[2] for the linear case, it is important to perform an analysis of the type of order reduction that Lawson methods present when integrating nonlinear initial boundary value problems, which is the purpose of this paper, and to develop a technique to avoid such order reduction, which is done in Cano and Reguera.[12]

More precisely, we are going to focus on nonlinear reaction-diffusion problems

$$\begin{aligned} u'(t) &= Au(t) + f(t, u(t)), \ 0 \le t \le T, \\ u(0) &= u_0 \in X, \\ \partial u(t) &= g(t) \in Y, \ 0 \le t \le T, \end{aligned} \tag{1}$$

where A is an elliptic differential operator, f is a smooth real function which acts as a reaction term, ∂ is a boundary operator, g is the boundary condition which in principle does not vanish and is time-dependent. Notice that, within this general formulation, very interesting equations for practical applications are included, such as the Fisher equation,[13] Turing-type models[14] such as Brusselator model[15] or the Gray-Scott model.[16]

The aim of this work is to analyze the different kinds of order reduction that Lawson methods suffer when integrating in time the nonlinear non homogeneous initial boundary value problem (1). The analysis we carry out in this work is not only interesting because the problem is more general than the one in Alonso–Mallo et al,[1] but also because now we are going to take the error coming from the space discretization into account, not only the error in time as it was done in Alonso–Mallo et al.[1]

2 | PRELIMINARIES

Let us consider the nonlinear abstract non homogeneous initial boundary value problem (1) where X and Y are Banach spaces and $A : D(A) \subset X \to X$ and $\partial : D(A) \to Y$ are linear operators, $f : [0, T] \times X \to X$ is a nonlinear function and $g : [0, T] \to Y$ is regular enough.

There are many evolution problems based on partial differential equations that are included in this abstract nonlinear formulation.[3,5,7,8,13–16]

For the theoretical study that we are performing in this work, we will consider the following hypotheses, similar to the ones in Alonso–Mallo et al[17] when analyzing the same kind of problems with exponential splitting methods.

(A1) The boundary operator $\partial : D(A) \subset X \to Y$ is onto and $g \in C^1([0, T], Y)$.

(A2) $\text{Ker}(\partial)$ is dense in X and $A_0 : D(A_0) = \ker(\partial) \subset X \to X$, the restriction of A to $\text{Ker}(\partial)$, is the infinitesimal generator of a C_0- semigroup $\{e^{tA_0}\}_{t\ge0}$ in X, which type ω is assumed to be negative.

(A3) If $z \in \mathbb{C}$ satisfies $\Re(z) > 0$ and $v \in Y$, then the steady state problem

$$Ax = zx, \tag{2}$$

$$\partial x = v, \tag{3}$$

possesses a unique solution denoted by $x = K(z)v$. Moreover, the linear operator $K(z) : Y \to D(A)$ satisfies

$$\|K(z)v\| \le C\|v\|, \tag{4}$$

where the constant C holds for any z such that $Re(z) \ge \omega_0 > \omega$.

(A4) The nonlinear source f belongs to $C^1([0, T] \times X, X)$.

(A5) The solution u of (1) satisfies $u \in C^1([0, T], X)$, $u(t) \in D(A)$ for all $t \in [0, T]$ and $Au \in C([0, T], X)$.

We will suppose that (A1)–(A5) are satisfied from now on.

We will denote by $\{\varphi_j\}$ the standard functions which are used in exponential methods[18] and which are defined by

$$\varphi_j(tA_0) = \frac{1}{t^j}\int_0^t e^{(t-\tau)A_0}\frac{\tau^{j-1}}{(j-1)!}d\tau,\ j \geq 1. \tag{5}$$

Notice that, because of hypothesis (A2), $\{\varphi_j(tA_0)\}_{j=1}^3$ are bounded operators for $t > 0$. The following recursive formulas are used for computational purposes

$$\varphi_{j+1}(z) = \frac{\varphi_j(z) - 1/j!}{z},\ z \neq 0, \qquad \varphi_{j+1}(0) = \frac{1}{(j+1)!}, \qquad \varphi_0(z) = e^z. \tag{6}$$

We are going to consider Lawson methods for the time integration of problem (1). Lawson methods are exponential methods which are determined by an explicit Runge–Kutta tableau. When integrating in time a finite-dimensional nonlinear problem like

$$U'(t) = MU(t) + F(t, U(t)), \tag{7}$$

where M is a matrix, the numerical solution at each step is given by the formulas

$$K_{n,i} = e^{c_i kM}U_n + k\sum_{j=1}^{i-1} a_{ij}e^{(c_i-c_j)kM}F(t_n + c_jk, K_{n,j}),\ i = 1, \dots, s, \tag{8}$$

$$U_{n+1} = e^{kM}U_n + k\sum_{i=1}^{s} b_i e^{(1-c_i)kM}F(t_n + c_ik, K_{n,i}), \tag{9}$$

where $k > 0$ is the time stepsize, $t_n = t_0 + nk$ and the coefficients in the formulas are those of the associated Runge–Kutta tableau.

Following Alonso–Mallo et al,[17] we consider $X = C(\overline{\Omega})$ for a certain bounded domain $\Omega \in \mathbb{R}^d$ and we denote by Ω_h the grid of Ω considered for the spatial discretization of the problem. Then, if N is the number of nodes in the grid, the numerical approximation $u_h = [u_1, \dots, u_N]^T$ belongs to C^N and its maximum norm will be $\|u_h\|_h = \max_{1\leq i\leq N}|u_i|$.

We also consider P_h, the projection operator

$$P_h : X \to \mathbb{C}^N, \tag{10}$$

which takes a function to its values over the grid Ω_h.

Then, when the operator A is applied over functions satisfying a certain condition on the boundary $\partial u = g$, it is discretized by

$$A_{h,0}U_h + C_h g,$$

where $A_{h,0}$ is the discretization matrix of A_0 and $C_h : Y \to \mathbb{C}^N$ is an operator which contains the information on the boundary.

We also assume that

$$P_h f(t, u) = f(t, P_h u). \tag{11}$$

On the other hand, following Alonso–Mallo et al,[17] we consider the hypotheses:

(H1) The matrix $A_{h,0}$ satisfies

a. $\|e^{tA_{h,0}}\|_h \leq 1$,
b. $A_{h,0}$ is invertible and $\|A_{h,0}^{-1}\|_h \leq C$ for some constant C which does not depend on h.

(H2) We define the elliptic projection $R_h : D(A) \to \mathbb{C}^N$ as the solution of

$$A_{h,0}R_h u + C_h \partial u = P_h A u. \tag{12}$$

We assume that there exists a subspace $Z \subset D(A)$, such that, for $u \in Z$,

a. $A_0^{-1}u \in Z$ and $e^{tA_0}u, f(t, u) \in Z$, for $t \in [0, T]$.

b. for some ε_h and η_h which are both small with h,

$$\|A_{h,0}(P_h u - R_h u)\| \le \varepsilon_h \|u\|_Z, \quad \|P_h u - R_h u\| \le \eta_h \|u\|_Z. \tag{13}$$

(Although obviously, because of (H1b), η_h could be taken as $C\varepsilon_h$, for some discretizations η_h can decrease more quickly with h than ε_h and that leads to better error bounds in the following sections.)

c. $\|A_{h,0}^{-1} C_h\|_h \le C''$ for some constant C'' which does not depend on h. This resembles the continuous maximum principle which is satisfied because of (4) when $z = 0$.

(H3) The nonlinear source f belongs to $C^1([0,T] \times \mathbb{C}^N, \mathbb{C}^N)$ and the derivative with respect to the variable in $\mathbb{C}^N$ is uniformly bounded in a neighborhood of the solution where the numerical approximation stays.

As in Alonso–Mallo et al,[17] hypothesis (H1a) can be deduced in our numerical experiments by using the logarithmic norm of matrix $A_{h,0}$.

3 | ANALYSIS OF THE FULL DISCRETIZATION

In this section, we are going to make an analysis of the local and global errors of the full discretization when considering the method of lines to integrate the nonlinear non homogeneous initial boundary value problem (1). As a difference with the analysis carried out in Alonso–Mallo et al[1] for linear problems, now we are going to take also the error coming from the space discretization into account, not only the error in time as it was done in Alonso–Mallo et al.[1]

Notice that, after discretizing (1) in space, we obtain the following semidiscrete problem

$$\begin{aligned} U_h'(t) &= A_{h,0} U_h(t) + C_h g(t) + f(t, U_h(t)), \\ U_h(t_0) &= P_h u(t_0). \end{aligned} \tag{14}$$

Then, applying Lawson method (8) and (9) to this semidiscrete problem, we obtain the formulas which allow us to obtain U_h^{n+1} from U_h^n:

$$\begin{aligned} K_{h,i}^n &= e^{c_i k A_{h,0}} U_h^n + k \sum_{j=1}^{i-1} a_{ij} e^{(c_i - c_j) k A_{h,0}} [C_h g(t_n + c_j k) + f(t_n + c_j k, K_{h,j}^n)], \quad i = 1, \dots, s, \\ U_h^{n+1} &= e^{k A_{h,0}} U_h^n + k \sum_{i=1}^{s} b_i e^{(1 - c_i) k A_{h,0}} [C_h g(t_n + c_i k) + f(t_n + c_i k, K_{h,i}^n)]. \end{aligned} \tag{15}$$

We define the local error as $\rho_{h,n+1} = \bar{U}_h^{n+1} - P_h u(t_{n+1})$, where $u(t)$ is the solution of (1) and $\bar{U}_h^{n+1}$ is deduced as U_h^{n+1} but starting from $P_h u(t_n)$ instead of U_h^n. Then,

$$\bar{U}_h^{n+1} = e^{k A_{h,0}} P_h u(t_n) + k \sum_{i=1}^{s} b_i e^{(1 - c_i) k A_{h,0}} [C_h g(t_n + c_i k) + f(t_n + c_i k, \bar{K}_{h,i}^n)], \tag{16}$$

where, for $i = 1, \dots, s$,

$$\bar{K}_{h,i}^n = e^{c_i k A_{h,0}} P_h u(t_n) + k \sum_{j=1}^{i-1} a_{ij} e^{(c_i - c_j) k A_{h,0}} [C_h g(t_n + c_j k) + f(t_n + c_j k, \bar{K}_{h,j}^n)]. \tag{17}$$

Our objective is to study the local and global order of the total discretization, taking into account both the errors coming from the integration in time and those caused by the spatial discretization. For this purpose, we will distinguish two cases. First, we assume that the boundary conditions vanish ($g(t) \equiv 0$), which is the most favorable case in terms of order reduction. Then, we will analyze the more general and interesting case in which the boundary conditions do not cancel and are time dependent ($g(t) \neq 0$).

3.1 | Local and global error when $g(t) \equiv 0$

In this subsection, we will analyze the error under the assumption that the boundary conditions vanish, which is the only case which has been studied for other standard exponential methods when integrating nonlinear problems and where just the error coming from the time integration has been analyzed.[19] As distinct, here we also consider the error coming from the space discretization.

Theorem 1. *Under hypotheses (A1)–(A5) and (H1)–(H3), whenever $u \in C([0,T],Z)$ and $\partial u \equiv 0$,*

$$\rho_{h,n} = \bar{U}_h^{n+1} - P_h u(t_{n+1}) = O(k),$$

where the constant in Landau notation is independent of k and h. Moreover, if $\sum_{i=1}^{s} b_i = 1$ and $u \in C^2([0,T],X)$, it happens that

$$A_{h,0}^{-1}\rho_{h,n} = O(\eta_h k + k^2).$$

Proof. First, we notice that, for $i = 1, \dots, s$, $\bar{K}_{h,i}^n$ in (17) are uniformly bounded on h when $g(t) \equiv 0$, which can be proved by induction on i by using (H1a), (A4) and (A5). Then, by using (6),

$$\begin{aligned}\bar{U}_h^{n+1} &= P_h u(t_n) + kA_{h,0}\varphi_1(kA_{h,0})P_h u(t_n) + k\sum_{i=1}^{s} b_i e^{(1-c_i)kA_{h,0}} f(t_n + c_i k, \bar{K}_{h,i}^n)\\ &= P_h u(t_n) + O(k),\end{aligned}$$

where we have used that $A_{h,0}\varphi_1(kA_{h,0})P_h u(t_n)$ is uniformly bounded on h because

$$\begin{aligned}A_{h,0}\varphi_1(kA_{h,0})P_h u(t_n) &= \varphi_1(kA_{h,0})A_{h,0}R_h u(t_n) + \varphi_1(kA_{h,0})A_{h,0}(P_h - R_h)u(t_n)\\ &= \varphi_1(kA_{h,0})P_h A u(t_n) + O(\varepsilon_h).\end{aligned} \tag{18}$$

(Here, the second equality comes from the fact that $A_{h,0}R_h u(t) = P_h A u(t)$ due to (12) with $\partial u = 0$, and also to (13) considering that $u \in C([0,T],Z)$.)

As for the second result, notice that, by using (6) again and an argument similar to (18), $\bar{K}_{h,i}^n$ can also be written as

$$\bar{K}_{h,i}^n = P_h u(t_n) + c_i k A_{h,0}\varphi_1(c_i k A_{h,0})P_h u(t_n) + O(k) = P_h u(t_n) + O(k). \tag{19}$$

Then, using now (6) to expand $e^{kA_{h,0}}$ till $\varphi_2(kA_{h,0})$ and $e^{(1-c_i)kA_{h,0}}$ till $\varphi_1((1-c_i)kA_{h,0})$ and again an argument similar to (18) for $A_{h,0}\varphi_2(kA_{h,0})P_h u(t_n)$,

$$\begin{aligned}A_{h,0}^{-1}\rho_{h,n+1} = A_{h,0}^{-1}\Big[&P_h u(t_n) + kA_{h,0}P_h u(t_n) + k^2 A_{h,0}^2\varphi_2(kA_{h,0})P_h u(t_n)\\ &+ k\sum_{i=1}^{s} b_i[f(t_n + c_i k, \bar{K}_{h,i}^n) + (1-c_i)kA_{h,0}\varphi_1((1-c_i)kA_{h,0})f(t_n + c_i k, \bar{K}_{h,i}^n)]\\ &- P_h u(t_n) - kP_h\dot{u}(t_n) + O(k^2)\Big]\\ = k[&P_h u(t_n) - A_{h,0}^{-1}P_h\dot{u}(t_n) + A_{h,0}^{-1}f(t_n, P_h u(t_n))] + O(k^2),\end{aligned} \tag{20}$$

where (H1b) and (19) have been used as well as the fact that $\sum_{i=1}^{s} b_i = 1$. Using now that

$$A_{h,0}^{-1}P_h\dot{u}(t) = A_{h,0}^{-1}P_h[Au(t) + f(t,u(t))] = R_h u(t) + A_{h,0}^{-1}P_h f(t,u(t)),$$

the bracket in (20) is $O(\eta_h)$ according to (13) and using (11), from what the result follows. □

Using the classical argument for the global error, the first result of the previous theorem would not lead to convergence. However, by using the second result and a few more assumptions, a summation-by-parts argument leads to the following result.

Theorem 2. *Under the hypotheses of Theorem 1, and assuming also that* $u \in C^3([0,T],X) \cap C^1([0,T],Z)$*, with* $\dot{u}(t) \in D(A)$ *for* $t \in [0,T]$*,* $\dot{u} \in C([0,T],Z)$*,* $A\dot{u} \in C([0,T],X)$ *and*

$$\left\| kA_{h,0} \sum_{r=1}^{n-1} e^{rkA_{h,0}} \right\|_h \leq C, \; 0 \leq nk \leq T, \tag{21}$$

it happens that

$$e_{h,n} = U_h^n - P_h u(t_n) = O(\eta_h + k).$$

3.2 | Local and global error when $g(t) \neq 0$

In this subsection we study the more general case in which $g(t) \neq 0$, that is, when the boundary conditions do not vanish and are time dependent. For the sake of brevity, we will consider just the case in which all the nodes c_i are strictly increasing with the index i, but we will distinguish between the case in which $c_i \neq 1$ for all $i = 1, \ldots, s$ and the case in which some of those c_i is equal to 1. One of the objectives is to explain the differences between both cases.

Theorem 3. *Under hypotheses (A1)–(A5) and (H1)–(H3), assuming also that* $\partial u \neq 0$*,* $c_1 < c_2 < \ldots < c_s$*,* $c_i \neq 1$ *for* $i = 1, \ldots, s$ *and that there exists* C'*,* h_0 *such that*

$$\|\tau A_{h,0} e^{\tau A_{h,0}}\|_h \leq C', \; \tau \geq 0, \; h \leq h_0, \tag{22}$$

it happens that

$$\rho_{h,n} = O(1), \; A_{h,0}^{-1}\rho_{h,n} = O(k),$$

where the constant in Landau notation is independent of k and h.

Proof. First, we notice that now the stages $\bar{K}_{h,i}^n$ in (17) are also uniformly bounded because, as the nodes are different,

$$k e^{(c_i - c_j)kA_{h,0}} C_h g(t_n + c_j k) = \frac{1}{(c_i - c_j)} [(c_i - c_j)kA_{h,0} e^{(c_i - c_j)kA_{h,0}}] A_{h,0}^{-1} C_h g(t_n + c_j k).$$

Then, the bracket is bounded because of (22) and the last factor because of (H2c). With the same argument, as $c_i \neq 1$, $\bar{U}_h^{n+1}$ in (16) is also uniformly bounded, which implies that $\rho_{h,n}$ is just $O(1)$. On the other hand, by using (6) in the first term of (16),

$$\begin{aligned} A_{h,0}^{-1}\rho_{h,n+1} = A_{h,0}^{-1} \Bigg[& P_h u(t_n) + kA_{h,0}\varphi_1(kA_{h,0}) P_h u(t_n) \\ & + k \sum_{i=1}^{s} b_i e^{(1-c_i)kA_{h,0}} [f(t_n + c_i k, \bar{K}_{h,i}^n) + C_h g(t_n + c_i k)] - P_h u(t_n) + O(k) \Bigg], \end{aligned}$$

which again is $O(k)$ because of (H2c). □

Now, a summation-by-parts argument allow us to obtain the following result for the global error.

Theorem 4. *Under the same hypotheses of Theorem 3, if* $u \in C^2([0,T],X)$ *and (21) holds, then* $e_{h,n} = U_h^n - P_h u(t_n) = O(1)$.

Let us now consider the case in which some $c_i = 1$.

Theorem 5. *Under hypotheses (A1)–(A5) and (H1)–(H3), assuming also that* $\partial u \neq 0$*,* $c_1 < c_2 < \ldots < c_s$*,* $c_i = 1$ *for some* $i \in \{1, \ldots, s\}$ *and (22), it happens that*

$$\rho_{h,n} = O(1 + k\|C_h\|), \; A_{h,0}^{-1}\rho_{h,n} = O(k),$$

where the constant in Landau notation is independent of k and h.

Proof. The proof is the same as that of Theorem 3, with the difference that now one of the terms in (16) can just be bounded by $k\|C_h\|$ because $c_i = 1$ for some i. However, the result for $A_{h,0}^{-1}\rho_{h,n}$ is the same as in that theorem because of (H2c). □

As we will see in the numerical experiments, this explains that, in this case, the local error behaves very badly, because it grows when h diminishes. However, in spite of that, for fixed but small h, it behaves with order 1 in k because the term in $k\|C_h\|$ dominates. The same happens with the global error, as the following theorem states by using again a summation-by-parts argument.

Theorem 6. *Under the same hypotheses of Theorem 5, if* $u \in C^2([0,T],X)$ *and (21) holds, then* $e_{h,n} = U_h^n - P_h u(t_n) = O(1 + k\|C_h\|)$.

Notice that, from the previous results, it follows that the behavior of the local and global errors will be different depending on whether all coefficients c_i for $i = 1, \ldots, s$ are different from 1 (Theorems 3, 4), or there is a coefficient c_i equal to 1 (Theorems 5, 6). This different behavior will be further confirmed by the numerical experiments.

4 | NUMERICAL EXPERIMENTS

In order to corroborate the previous theoretical results, we are going to show in this section some numerical experiments. For that, we are going to consider the following test problems:

- Dirichlet problem with vanishing boundary conditions:

$$\begin{cases} u_t(t,x) &= u_{xx}(t,x) + u^2(t,x) + h_1(t,x), \;\; x \in [1,2], \;\; t \in [0,1], \\ u(0,x) &= (x-1)(x-2)\sin(x), \\ u(t,1) &= 0 \\ u(t,2) &= 0 \end{cases} \tag{23}$$

  with $h_1(t,x) = (8 - 7x + x^2)\cos(t+x) + \sin(t+x)((x-3)x - (2 - 3x + x^2)^2 \sin(t+x))$. The exact solution of this problem is $u(t,x) = (x-1)(x-2)\sin(x+t)$.
- Dirichlet problem with non-vanishing boundary conditions:

$$\begin{cases} u_t(t,x) &= u_{xx}(t,x) + u^2(t,x) + h_2(t,x), \;\; x \in [1,2], \;\; t \in [0,1], \\ u(0,x) &= \sin(x), \\ u(t,1) &= \sin(t+1) \\ u(t,2) &= \sin(t+2) \end{cases} \tag{24}$$

  with $h_2(t,x) = \cos(t+x) + \sin(t+x) - \sin{(t+x)}^2$. The exact solution of this problem is $u(t,x) = \sin(x+t)$.
- mixed (Dirichlet/Neumann) problem with non-vanishing boundary conditions:

$$\begin{cases} u_t(t,x) &= u_{xx}(t,x) + u^2(t,x) + h_2(t,x), \;\; x \in [1,2], \;\; t \in [0,1], \\ u(0,x) &= \sin(x), \\ u(t,1) &= \sin(t+1) \\ u_x(t,2) &= \cos(t+2) \end{cases} \tag{25}$$

  whose exact solution is $u(t,x) = \sin(x+t)$.

First, we carry out the space discretization considering the symmetric second-order difference scheme for which, in the Dirichlet case (23) and (24),

$$A_{h,0} = \text{tridiag}(1,-2,1)/h^2, \;\; C_h[g_0(t), g_1(t)]^T = [g_0(t), 0, \ldots, 0, g_1(t)]^T/h^2, \tag{26}$$

TABLE 1 Local error and local order when integrating Dirichlet problem with vanishing boundary conditions (23) with the Lawson method associated to the second-order RK (28)

h	**k = 1e−3**	**k = 5e−4**	**k = 2.5e−4**	**k = 1.25e−4**
h = 1e−3 Local error	4.6602e−4	2.2810e−4	1.1051e−4	5.2781e−5
h = 1e−3 Local order		1.03	1.05	1.07
h = 5e−4 Local error	4.7880e−4	2.3705e−4	1.1673e−4	5.7082e−5
h = 5e−4 Local order		1.014	1.02	1.03
h = 2.5e−4 Local error	4.8528e−4	2.4162e−4	1.1994e−4	5.9322e−5
h = 2.5e−4 Local order		1.01	1.01	1.02

TABLE 2 Global error and global order when integrating Dirichlet problem with vanishing boundary conditions (23) with the Lawson method associated to the second-order RK (28)

h	**k = 1e−3**	**k = 5e−4**	**k = 2.5e−4**	**k = 1.25e−4**
h = 1e−3 Global error	1.2578e−03	6.1523e−04	2.9815e−04	1.4257e−04
h = 1e−3 Global order		1.03	1.05	1.06
h = 5e−4 Global error	1.2913e−03	6.3865e−04	3.1443e−04	1.5380e−04
h = 5e−4 Globalcal order		1.02	1.02	1.03
h = 2.5e−4 Global error	1.3083e−03	6.5060e−04	3.2281e−04	1.5966e−04
h = 2.5e−4 Global order		1.01	1.01	1.02

and, in the Dirichlet/Neumann case (25),

$$A_{h,0} = \begin{bmatrix} -2 & 1 & 0 & \dots & 0 \\ 1 & -2 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & 1 & -2 & 1 \\ & & 0 & 2 & -2 \end{bmatrix}, \quad C_h \begin{bmatrix} g_0(t) \\ g_1(t) \end{bmatrix} = \begin{bmatrix} \frac{1}{h^2} g_0(t) \\ 0 \\ \vdots \\ 0 \\ \frac{2}{h} g_1(t) \end{bmatrix}. \tag{27}$$

All differential problems (23)–(25) satisfy hypotheses (A1)–(A5) with $X = C([0,1])$ and the space discretizations (26) and (27) satisfy hypotheses (H1)–(H3), as it was justified in Alonso–Mallo et al[17] for $Z = C^4([0,1])$, ε_h, η_h being $O(h^2)$ for (26) and $\varepsilon_h = O(h)$, $\eta_h = O(h^2)$ for (27). Moreover the considered solutions and f are smooth enough so that all conditions of regularity needed in the paper are satisfied. On the other hand, although we do not provide a proof for the condition (22), it can be numerically verified that this condition holds uniformly on h for $A_{h,0}$ in (26) and (27).

4.1 | Second-order method

Let us first show the numerical results which are obtained when integrating problem (23), that is, the Dirichlet problem with vanishing boundary conditions, with the Lawson method which is built from the second-order RK tableau

$$\begin{array}{c|cc} 0 & 0 & \\ 1 & 1 & 0 \\ \hline & \frac{1}{2} & \frac{1}{2} \end{array}. \tag{28}$$

The scheme (15) shows local and global order 1 in time, as shown in Figures 1 and 2, as well as in Tables 1 and 2 for different values of the spatial parameter h, small enough so that the error in space is negligible (This corroborates Theorems 1 and 2).

The behavior is very different if we consider problem (24), that is, also a Dirichlet problem but now with a solution that does not vanish at the boundary. In this case, it is important to notice that, for the method we are considering, one of the coefficients c_i is 1. We can observe in Figures 3 and 4, as well as in Tables 3 and 4 that, although the local and global orders are still 1, the errors are very big and grow when h diminishes. This is justified in Theorems 5 and 6 which respectively predicted $\rho_{h,n} = O(1 + k\|C_h\|)$ for the local error and $e_{h,n} = O(1 + k\|C_h\|)$ for the global error. Notice that, in both results, the timestep k is multiplied by the norm of the boundary term C_h, which grows when h diminishes. This explains that both, the local and the global error behave very badly, because they grow when h diminishes. However, in spite of that, for fixed but small h, it behaves with order 1 in k because the term in k, $k\|C_h\|$ dominates.

4.2 | Third-order method

In this subsection we show the results which are obtained when considering the problem (25), with mixed Dirichlet/Neumann boundary conditions and whose solution does not vanish at the boundary and integrating it in time

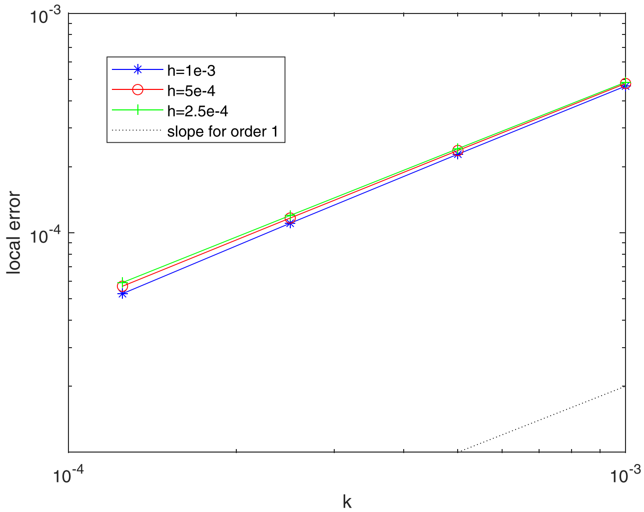

FIGURE 1 Local error when integrating Dirichlet problem with vanishing boundary conditions (23) with the Lawson method associated to the second-order RK (28) [Colour figure can be viewed at wileyonlinelibrary.com]

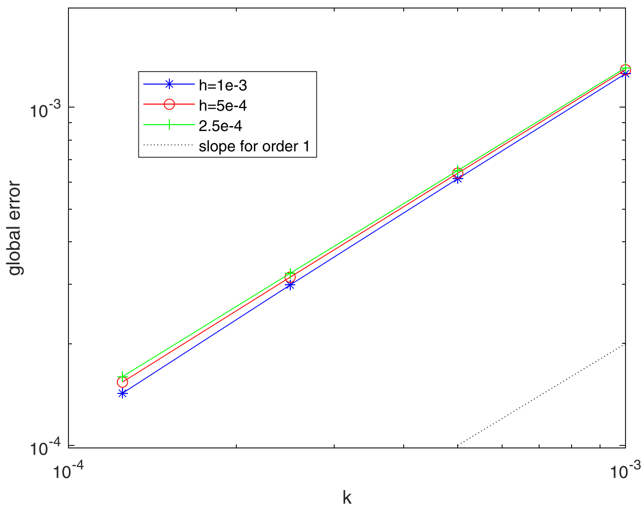

FIGURE 2 Global error when integrating Dirichlet problem with vanishing boundary conditions (23) with the Lawson method associated to the second-order RK (28) [Colour figure can be viewed at wileyonlinelibrary.com]

TABLE 3 Local error and local order when integrating Dirichlet problem with non-vanishing boundary conditions (24) with the Lawson method associated to the second-order RK (28)

h	$k = 1e{-}3$	$k = 5e{-}4$	$k = 2.5e{-}4$	$k = 1.25e{-}4$
$h = 1e{-}3$ Local error	4.5355e+02	2.2639e+02	1.1278e+02	5.5976e+01
$h = 1e{-}3$ Local order		1.00	1.01	1.01
$h = 5e{-}4$ Local error	1.8169e+03	9.0819e+02	4.5371e+02	2.2643e+02
$h = 5e{-}4$ Local order		1.00	1.00	1.00
$h = 2.5e{-}4$ Local error	7.2701e+03	3.6355e+03	1.8175e+03	9.0835e+02
$h = 2.5e{-}4$ Local order		1.00	1.00	1.00

TABLE 4 Global error and global order when integrating Dirichlet problem with non-vanishing boundary conditions (24) with the Lawson method associated to the second-order RK (28)

h	$k = 1e{-}3$	$k = 5e{-}4$	$k = 2.5e{-}4$	$k = 1.25e{-}4$
$h = 1e{-}3$ Global error	4.5376e+02	2.2645e+02	1.1280e+02	5.5982e+01
$h = 1e{-}3$ Global order		1.00	1.01	1.01
$h = 5e{-}4$ Global error	1.8177e+03	9.0840e+02	4.5376e+02	2.2645e+02
$h = 5e{-}4$ Global order		1.00	1.00	1.00
$h = 2.5e{-}4$ Global error	7.2735e+03	3.6363e+03	1.8177e+03	9.0840e+02
$h = 2.5e{-}4$ Global order		1.00	1.00	1.00

with the Lawson method associated to the third-order Heun method

$$\begin{array}{c|ccc} 0 & & & \\ \frac{1}{3} & \frac{1}{3} & & \\ \frac{2}{3} & 0 & \frac{2}{3} & \\ \hline & \frac{1}{4} & 0 & \frac{3}{4} \end{array} . \tag{29}$$

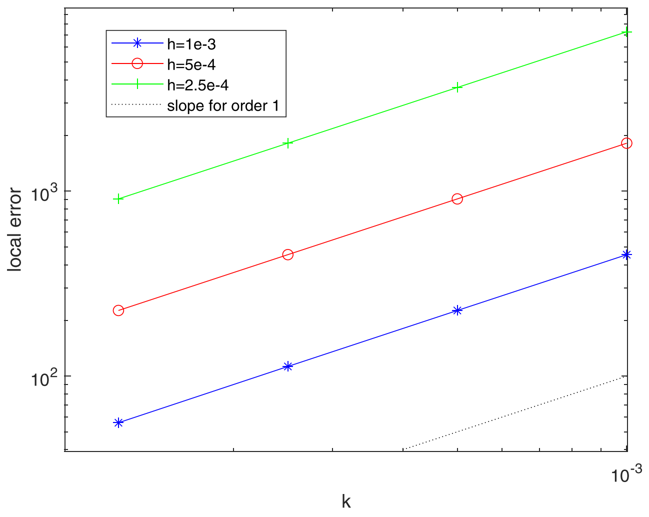

FIGURE 3 Local error when integrating Dirichlet problem with non-vanishing boundary conditions (24) with the Lawson method associated to the second-order RK (28) [Colour figure can be viewed at wileyonlinelibrary.com]

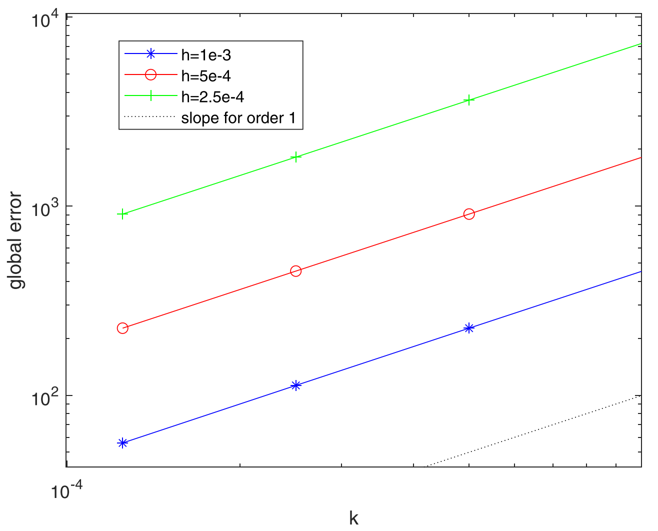

FIGURE 4 Global error when integrating Dirichlet problem with non-vanishing boundary conditions (24) with the Lawson method associated to the second-order RK (28) [Colour figure can be viewed at wileyonlinelibrary.com]

TABLE 5 Local error and local order when integrating mixed D/N problem with non-vanishing boundary conditions (25) with the Lawson method associated to the third-order RK (29)

h	k = 2e−01	1e−01	5e−02	2.5e−02
h = 1e−3 Local error	9.2858e−01	8.8642e−01	8.6077e−01	8.4539e−01
h = 1e−3 Local order		0.07	0.04	0.03
h = 5e−4 Local error	9.3031e−01	8.8882e−01	8.6410e−01	8.5005e−01
h = 5e−4 Local order		0.07	0.04	0.02
h = 2.5e−4 Local error	9.3117e−01	8.9001e−01	8.6576e−01	8.5238e−01
h=2.5e−4 Local order		0.07	0.04	0.02

TABLE 6 Global error and Global order when integrating mixed D/N problem with non-vanishing boundary conditions (25) with the Lawson method associated to the third-order RK (29)

h	k = 2e−01	1e−01	5e−02	2.5e−02
h = 1e−3 Global error	9.0587e−01	9.0438e−01	9.0231e−01	8.9938e−01
h = 1e−3 Global order		0.00	0.00	0.00
h = 5e−4 Global error	9.0759e−01	9.0684e−01	9.0580e−01	9.0434e−011
h = 5e−4 Global order		0.00	0.00	0.00
h = 2.5e−4 Global error	9.0844e−01	9.0807e−01	9.0755e−01	9.0682e−01
h = 2.5e−4 Global order		0.00	0.00	0.00

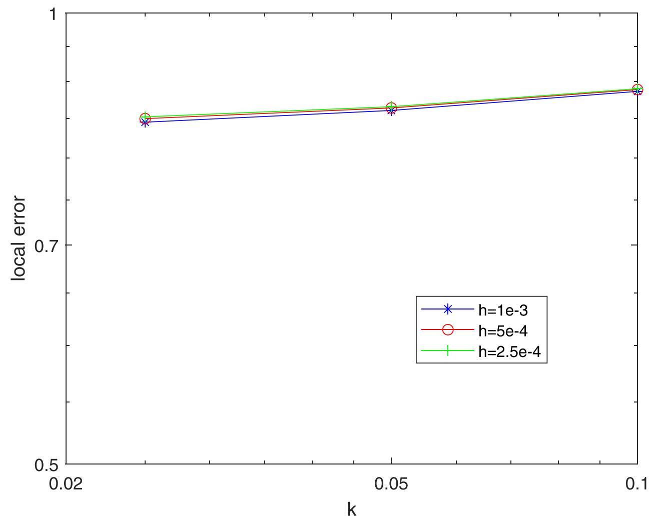

FIGURE 5 Local error when integrating mixed D/N problem with non-vanishing boundary conditions (25) with the Lawson method associated to the third-order RK (29) [Colour figure can be viewed at wileyonlinelibrary.com]

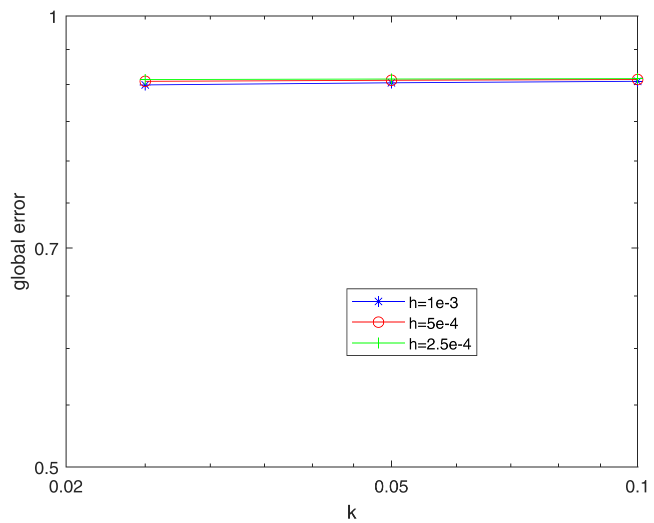

FIGURE 6 Global error when integrating mixed D/N problem with non-vanishing boundary conditions (25) with the Lawson method associated to the third-order RK (29) [Colour figure can be viewed at wileyonlinelibrary.com]

The approach (15) shows no convergence either on the local or global error where the timestepsize diminishes, as it is justified in Theorems 3 and 4, and it is shown in Figures 5 and 6, as well as in Tables 5 and 6. Notice the different behavior with respect to the numerical experiment in Figures 3 and 4 and Tables 3 and 4. Here the errors do not diminish with k but, they neither grow when h diminishes as it happens in those tables. This is due to the fact that now every c_i is different from 1.

5 | CONCLUSIONS

In this paper we have analyzed the different types of order reduction that Lawson methods suffer when integrating in time nonlinear reaction-diffusion problems. For this purpose, we have taken into account both the error coming from the spatial discretization and the error due to the time integration. An analysis of both, the local and global errors, of the full discretization when considering the method of lines to integrate the nonlinear non homogeneous initial value problem (1) is carried out in this work. In order to perform this analysis, we distinguish two cases: when the boundary conditions vanish ($g(t) \equiv 0$), which is the most favorable case in regard of order reduction, and the more interesting and general case, which is when the boundary conditions do not vanish ($g(t) \neq 0$). In all cases the theoretical results that we proof are corroborated by the numerical experiments presented in this work.

In the first case (vanishing boundary conditions), we proof that under certain hypothesis of regularity, local order one and global order one in time is achieved: $\rho_{h,n} = O(k)$, $e_{h,n} = O(\eta_h + k)$.

In the case of non-vanishing boundary conditions, the behavior is different depending on whether all the coefficients c_i of the Runge–Kutta associated to the Lawson method are different from 1. In such a case, we prove that there is not even convergence either on the local or global error as the timestesize diminishes: $\rho_{h,n} = O(1)$, $e_{h,n} = O(1)$.

However, the behavior is very different when, considering also non vanishing boundary conditions, at least one of the coefficients c_i of the method is equal to 1. In this case, in the numerical experiments it is observed that, although the local and global orders are one, the errors in practice are very large and grow when the spatial parameter h decreases. This behavior is consistent with the theoretical study carried out in this work, since in this case we prove $\rho_{h,n} = O(1 + k\|C_h\|)$ for the local error and $e_{h,n} = O(1 + k\|C_h\|)$ for the global error, which explains the bad behavior of the errors when h decreases and also that for fixed and small h, local and global order one in time is observed.

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CONFLICT OF INTEREST

This work does not have any conflicts of interest.

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