An inventory system with demand dependent on both time and price assuming backlogged shortages

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Abstract

In this work we analyze an inventory model for items whose demand is a bivariate function of price and time. It is supposed that the demand rate multiplicatively combines the effects of a time-power function and a price-logit function. The aim is to maximize the profit per time unit, assuming that the inventory cost per time unit is the sum of the holding, shortage, ordering and purchasing costs. An algorithm is developed to find the optimal price, the optimal lot size and the optimal replenishment cycle. Several numerical examples are introduced to illustrate the solution procedure.

Keywords: Inventory; Price and time-dependent demand; Backlogged demand; Profit optimization

1 Introduction

In the twenty-first century, with the globalization of the markets, there has been a considerable increase in trade throughout the world. Firms produce, maintain and distribute goods on all continents. Customers demand products that must be supplied quickly and efficiently. The coordination of the production, maintenance and distribution of the products to meet customer demand and not lose market share

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with respect to other firms, requires an adequate planning and administration of the inventories. Thus, Inventory Management has become a vital activity for companies to successfully compete in business.

Stock management models help to determine the optimal inventory policies that must be implemented to minimize the inherent costs associated with the maintenance and management of products. Some of the most common assumptions in the study of economic order quantity (EOQ) inventory models are to consider a constant demand rate (independent of time and the unit selling price) and to allow no shortages. However, in many real situations, the demand rate is not constant and may be dependent on time and/or the selling price. Stockouts may also occur and this must be permitted in the inventory model.

When demand is dependent on time, there are different ways by which products are withdrawn from stock during the inventory cycle. These shapes are defined as demand patterns. A demand pattern is known as a power pattern if the demand rate depends potentially on the quotient between time and the length of the inventory cycle. Some Inventory systems with a power demand pattern were developed by Naddor (1966). Later, Goel and Aggarwal (1981) and Datta and Pal (1988) studied inventory models with a power demand pattern for deteriorating items. Lee and Wu (2002); Dye (2004); Singh, Singh and Dutt (2009); Rajeswari and Vanjikkodi (2011) and Mishra and Singh (2013) developed inventory models for deteriorating items with a power demand pattern while also allowing shortages.

In all the above works, the length of the inventory cycle is always known and fixed. However, Sicilia, Febles-Acosta and Gonzalez-De la Rosa (2012) analyzed some inventory systems with power demand in which the length of the inventory cycle was not constant but a decision variable. They determined the optimal inventory policy for the system with backlogged shortages and for the system with lost sales.

Chen and Simchi-Levi (2012) described several price-dependent demand functions which may be used in the study of inventory systems. An interesting review of demand functions in decision modeling is published by Huang, Leng and Parlar (2013). They presented and commented several price-dependent demand functions that have appeared in the literature.

There are several papers on inventory models where demand is a price-dependent function. Thus, Smith, Martinez-Flores and Cardenas-Barron (2007) analyzed an EOQ inventory system with selling price-dependent demand rate. They developed the optimal policy for three specific demand functions. Kocabıyıkoğlu and Popescu (2011) studied the newsvendor problem with price-sensitive demand. Soni

(2013) analyzed an inventory model where the demand rate was additive with respect to the stock level and the unit selling price. Wu, Skouri, Teng and Ouyang (2014) corrected some deficiencies of Soni's model.

Some researchers have analyzed inventory systems where demand depends on time and price. The demand rate is usually a separable function of time and unit selling price. Thus, Avinadav, Herbon and Spiegel (2014) studied two inventory models with price and time-dependent demand, but without shortages (one with multiplicative influence of price and time, and the other with additive effect).

In this paper, we assume that shortages are allowed and completely backlogged. This assumption is also considered in other papers on inventory systems. Thus, San-Jose and Garcia-Laguna (2009) presented a composite lot size model with discounts in all units (constant demand) assuming full backlogging. Birbil, Bulbul, Frenk and Mulder (2015) studied EOQ models with constant demand and purchase-price and transportation cost functions, considering backlogged shortages. Jaksic and Fransoo (2015) developed a dynamic programming model for a finite horizon stochastic capacitated inventory system where shortages are fully backlogged. Mishra, Gupta, Yadav and Rawat (2015) presented an EOQ inventory model with full backlogging and deterioration in a fuzzy environment.

Economic order quantity replenishment models focus considerable attention in Inventory Control nowadays. Thus, some recent papers on this topic are the following: Muriana (2016), Bakal, Bayındır and Emer (2017), Demirag, Kumar and Rao (2017), Dobson, Pinker and Yildiz (2017), Herbon and Khmelnitsky (2017), and San-José, Sicilia, González-De-la-Rosa and Febles-Acosta (2017).

To the best of the authors' knowledge, there is no published model developing the optimal policy for an inventory system with full backlogging, where the inventory cycle is a decision variable and demand multiplicatively combines the effects of selling price and a power demand pattern, assuming that the demand rate is the product of a price-logit function and a power-time function.

The remainder of the paper is organized as follows. Section 2 presents the notation and the assumptions related to the inventory system here studied. In the next section, the development of the inventory model and the formulation of the optimization problem is shown. Then, we prove several results that derive to an algorithmic approach to optimally solve the inventory problem. Several numerical examples are discussed to illustrate the procedure for solving the inventory problem. Next, a sensitivity analysis on some input parameters associated with the demand rate of the inventory model is presented. Finally, the conclusions of the work are presented and future research areas are suggested.

2 Assumptions and notation

The notations used in this work are shown in Table 1.

Table 1. List of notations

τ_1	Time period where the net stock is positive (≥ 0).
τ_2	Time period where the net stock is less than or equal to zero (≥ 0).
T	Length of the inventory cycle, that is, $T = \tau_1 + \tau_2$ (> 0, *decision variable*).
M	Maximum level of the stock (≥ 0, *decision variable*).
b	Maximum backlogged quantity per cycle (≥ 0).
Q	Lot size per cycle, that is $Q = M + b$ (> 0).
p	Unit purchasing cost (> 0).
s	Unit selling price ($s \geq p$, *decision variable*).
K	Ordering cost per replenishment (> 0).
h	Holding cost per unit and per unit time (> 0).
ω	Shortage cost per backordered unit and per unit time (> 0).
$D(s,t)$	Demand rate at time t when the selling price is s, with $0 < t < T$.
$I(s,t)$	Inventory level at time t when the selling price is s, with $0 \leq t < T$.
n	Demand pattern index (> 0).
$B(s,M,T)$	Total profit per unit time.

In this work, an economic order quantity model is developed under the following assumptions:

1. The inventory system considers a single product.
2. The planning horizon is infinite and the replenishment is instantaneous.
3. The lead time is zero or negligible.
4. The demand rate $D(s,t)$ is a bivariate function of price and time. We suppose that $D(s,t) = d_1(s)d_2(t)$, where $d_1(s)$ is a known logit-funtion of price and $d_2(t)$ is a power time-dependent function. That is, the demand rate multiplicatively combines the effects of selling price and a power demand pattern.
5. The order cost is fixed regardless of the lot size.
6. The holding cost per unit is a linear function of time in storage.
7. The system allows shortages, which are completely backlogged.

8. There is single procurement of size Q units to the start of inventory cycle and is equal to the total demand throughout the inventory cycle.

3 Model development

In this work, a continuous review inventory system over an infinite-horizon with deterministic demand is analyzed. It is assumed that shortages are completely backlogged.

At the beginning of the inventory cycle there are M units in the stock. That amount meets demand during the time period $(0, \tau_1]$. Thus, we have

$$M = \int_0^{\tau_1} D(s,u)du.$$

Next, the inventory falls into shortage because there is not enough stock to meet demand. During the time period (τ_1, T), shortages are accumulated and fully backlogged. Thus, from $t = 0$ to T time units, the inventory level decreases due to demand. So, the net stock level $I(s,t)$ is a $T-$periodic function defined on the interval $[0, \infty)$. The net stock level at time t is given by

$$I(s,t) = M - \int_0^t D(s,u)du = \int_t^{\tau_1} D(s,u)du = d_1(s) \int_t^{\tau_1} d_2(u)du.$$

We suppose that $d_1(s)$ is the logit function given by

$$d_1(s) = \frac{\alpha e^{-\beta s}}{1 + e^{-\beta s}}, \text{ with } \alpha > 0 \text{ and } \beta > 0.$$

The parameter α represents the market size and the parameter β is a coefficient of the price sensitivity. The function $d_2(t)$ is a power time-dependent function given by

$$d_2(t) = \tfrac{1}{n} \left(\frac{t}{T}\right)^{(1-n)/n}, \text{ with } n > 0.$$

A justification of the practical utility of these functions $d_1(s)$ and $d_2(t)$ to describe the demand for certain products can be seen, respectively, in Sudhir (2001) and San-José, Sicilia, González-De-la-Rosa and Febles-Acosta (2017).

Therefore, the net stock level at time t is

$$I(s,t) = \frac{\alpha e^{-\beta s}}{1 + e^{-\beta s}} T \left[\left(\frac{\tau_1}{T}\right)^{1/n} - \left(\frac{t}{T}\right)^{1/n} \right] = M - \frac{\alpha e^{-\beta s}}{1 + e^{-\beta s}} T \left(\frac{t}{T}\right)^{1/n}. \quad (1)$$

Thus, the maximum positive stock level is

$$M = \frac{\alpha e^{-\beta s}}{1 + e^{-\beta s}} T \left(\frac{\tau_1}{T}\right)^{1/n}.$$

The totally backordered shortage amount during the inventory cycle is given by

$$b = \int_{\tau_1}^{T} D(s,u)du = \frac{\alpha e^{-\beta s}}{1+e^{-\beta s}} T - M.$$

The lot size Q is

$$Q = M + b = \frac{\alpha e^{-\beta s}}{1+e^{-\beta s}} T.$$

For a fixed value of s, Figures 1 to 3 illustrate the behavior of the inventory system for different demand pattern indexes.

Taking into account the above assumptions, the total profit per cycle of the inventory system is obtained as the difference between the revenue per cycle and the sum of the ordering cost, the purchasing cost, the inventory holding cost and the backordering cost per cycle. Thus, the revenue per cycle is sQ, the ordering cost is K, the purchasing cost is pQ, the holding cost is

$$h\int_0^{\tau_1} I(s,t)dt = \frac{\alpha h}{(n+1)\left(1+e^{\beta s}\right)} T^2 \left(\frac{\tau_1}{T}\right)^{1+1/n} = \frac{h}{n+1} TM \left(\frac{\left(1+e^{\beta s}\right)M}{\alpha T}\right)^n$$

and the backordering cost is given by

$$\omega \int_{\tau_1}^{T} \left[-I(s,t)\right]dt = \omega \left[\frac{\alpha n}{(n+1)\left(1+e^{\beta s}\right)} T^2 - MT + \frac{1}{n+1} TM \left(\frac{\left(1+e^{\beta s}\right)M}{\alpha T}\right)^n\right].$$

Consequently, the total profit per unit time is

$$B(s,M,T) = \frac{1}{T}\left[(s-p)Q - K - h\int_0^{\tau_1} I(s,t)dt + \omega \int_{\tau_1}^{T} I(s,t)dt\right] \quad (2)$$

$$= (s-p)\frac{\alpha}{1+e^{\beta s}} - \frac{K}{T} - \frac{h+\omega}{n+1} M \left(\frac{\left(1+e^{\beta s}\right)M}{\alpha T}\right)^n - \frac{\alpha \omega n}{(n+1)\left(1+e^{\beta s}\right)} T + \omega M \quad (3)$$

Thus, the optimization problem addressed in this work is given by

$$\max_{(s,M,T)\in\Omega} B(s,M,T), \quad (4)$$

where $\Omega = \left\{(s,M,T) : T > 0, 0 < M \le \alpha T / \left(1+e^{\beta s}\right) \text{ and } p \le s\right\}$.

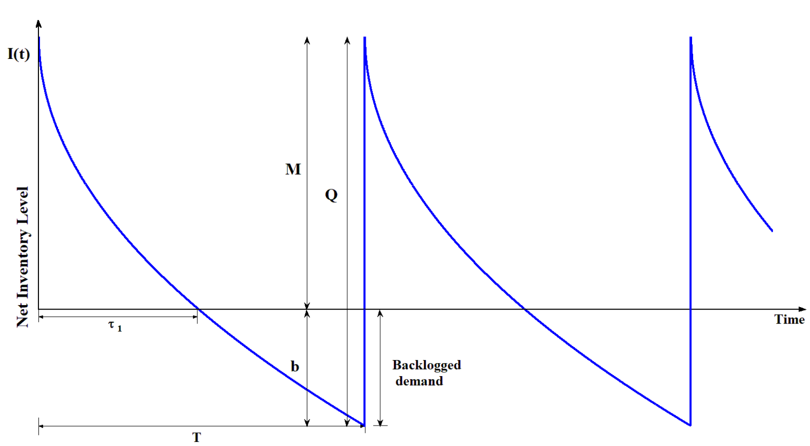

Fig. 1. Net stock level when $n > 1$

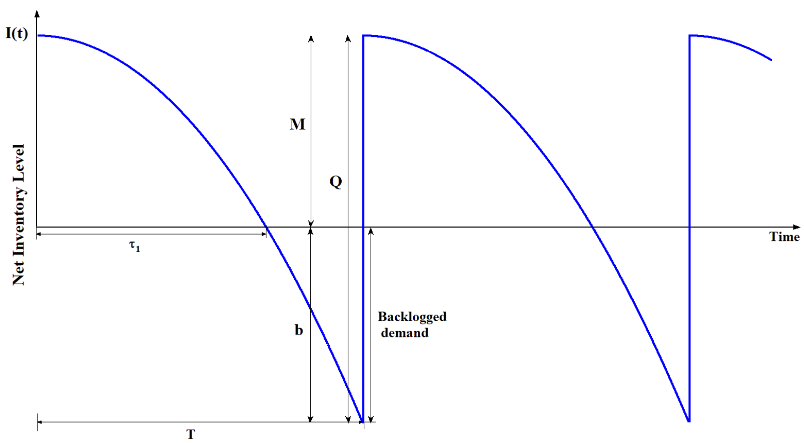

Fig. 2. Net stock level when $n < 1$

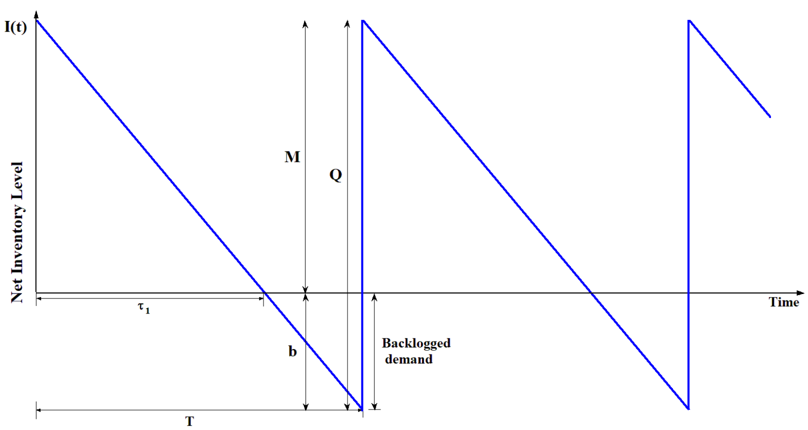

Fig. 3. Net stock level when $n = 1$

4 Analysis and solution of the problem

For a fixed value of s, the bivariate function $B_s(M,T) = B(s,M,T)$ is strictly concave and attains its maximum value at the point $(M^*(s), T^*(s))$, solving the simultaneous equations $\frac{\partial}{\partial M}B_s(M,T) = 0$ and $\frac{\partial}{\partial T}B_s(M,T) = 0$ (see Lemmas 1 and 2 in the Appendix). That is, the maximum point is given by

$$T^*(s) = \sqrt{\frac{(n+1)K\left(1+e^{\beta s}\right)}{n\alpha\omega\left(1-\left(\frac{\omega}{h+\omega}\right)^{1/n}\right)}} \tag{5}$$

$$M^*(s) = \frac{\alpha}{1+e^{\beta s}}\left(\frac{\omega}{h+\omega}\right)^{1/n} T^*(s) = \frac{\alpha}{(1+e^{\beta s})}\left(\frac{\omega}{h+\omega}\right)^{1/n}\sqrt{\frac{(n+1)K\left(1+e^{\beta s}\right)}{n\alpha\omega\left(1-\left(\frac{\omega}{h+\omega}\right)^{1/n}\right)}} \tag{6}$$

Hence, for a fixed selling price s, Eq. (5) provides the optimal inventory cycle and Eq. (6) gives the optimal inventory level at the beginning of the inventory cycle.

By evaluating the function $B_s(M,T)$ at the point $(M^*(s), T^*(s))$, we find the univariate function

$$P(s) = B_s(M^*(s), T^*(s)) = (s-p)\frac{\alpha}{1+e^{\beta s}} - 2\xi\sqrt{\frac{\alpha}{1+e^{\beta s}}}, \tag{7}$$

where, for simplicity, the parameter ξ is

$$\xi = \sqrt{\frac{1+e^{\beta s}}{\alpha}}\frac{K}{T^*(s)} = \sqrt{\frac{n}{n+1}K\omega\left(1-\left(\frac{\omega}{h+\omega}\right)^{1/n}\right)}. \tag{8}$$

Thus, we have reduced the three-variable optimization problem (4) to the single optimization problem

$$\max_{s\geq p} P(s). \tag{9}$$

Next, we shall show some interesting properties of the function $P(s)$.

Proposition 1 *Let $P(s)$ be given by (7). Then:*

1. *We have $P(p) < 0$ and $\lim_{s\to\infty} P(s) = 0$.*
2. *The function $P(s)$ is continuously differentiable on the interval (p, ∞) and sign$(P'(s))$ =sign$(f(s))$, where $f(s)$ is a strictly convex function defined on the set $\mathbb{R}$ of real numbers and is given by*

$$f(s) = 1 + e^{-\beta s} - \beta(s-p) + \beta\xi\sqrt{\frac{1+e^{\beta s}}{\alpha}}. \tag{10}$$

3. *The function $P(s)$ is strictly increasing on the interval (p, ∞) in the cases: (i) $f(s_o) \geq 0$ and (ii) $p \geq s_o$, where*

$$s_o = \arg_{s\in\mathbb{R}}\left\{f'(s) = 0\right\}. \tag{11}$$

4. *If $p \geq s_o$ and $f(s_o) < 0$, then the function $P(s)$ has a local maximum at the point*

$$s_1 = \arg_{s\in(p,s_o)}\{f(s)\}. \tag{12}$$

Proof. Please, see Appendix. ■

Figure 1 depicts the three possible behaviors of the function $P(s)$.

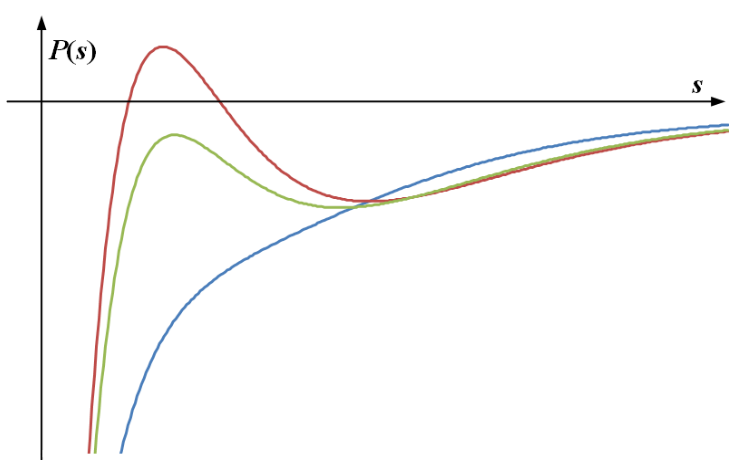

Figure 1. Graphs of the function $P(s)$

We can now provide a criterion for determining the optimal selling price s^*, which is an important consequence of the above proposition.

Theorem 2 *Let $P(s)$, $f(s)$, s_o and s_1 be given, respectively, by (7), (10), (11) and (12)*

1. *If $p \geq s_o$, then $s^* = \infty$ and $B^* = P(s^*) = 0$.*
2. *If $p < s_o$ and $f(s_o) \geq 0$, then $s^* = \infty$ and $B^* = P(s^*) = 0$.*
3. *If $p < s_o$ and $f(s_o) < 0$, then:*
   (a) *$s^* = \infty$ and $B^* = 0$, when $P(s_1) < 0$.*
   (b) *$s^* = s_1$ and $B^* = P(s_1)$, otherwise.*

Proof. Please, see Appendix. ■

Remark 1 *Note that the inventory system is profitable only in the case (3.b) of the previous theorem.*

Next, we formulate some of the conditions of the above theorem as a function of the input parameters of the inventory system.

Proposition 3 *Let ξ, s_o and s_1 be given, respectively, by (8), (11) and (12).*

1. *The case (2) of Theorem 2 is satisfied if*

$$s_o - \frac{2 + 3e^{\beta s_o}}{2\sqrt{\alpha\left(1 + e^{\beta s_o}\right)}}\xi \leq p$$

2. *The case (3.a) of Theorem 2 is satisfied if*

$$s_1 \geq \frac{1}{\beta} \ln\left(\frac{\alpha + \sqrt{\alpha^2 + 4\alpha\beta^2\xi^2}}{2\beta^2\xi^2}\right)$$

Proof. Please, see Appendix. ■

As a consequence of the previous results, we provide a procedure to solve the problem analyzed in this paper. The following algorithm develops the optimal inventory policy.

Algorithm 1 Step 1 *Calculate $s_o = \arg_{s\in\mathbb{R}}\left\{f'(s) = 0\right\}$.*

Step 2 *If $s_o \leq p$, then go to* Step 7*. Otherwise, go to* Step 3.

Step 3 *If $s_o - \frac{2+3e^{\beta s_o}}{2\sqrt{\alpha(1+e^{\beta s_o})}}\xi \leq p$, then go to* Step 7*. Otherwise, go to* Step 4.

Step 4 *Calculate $s_1 = \arg_{s\in[p,s_o]}\{f(s)\}$.*

Step 5 *If* $s_1 \geq \frac{1}{\beta} \ln \left(\frac{\alpha + \sqrt{\alpha^2 + 4\alpha\beta^2\xi^2}}{2\beta^2\xi^2} \right)$*, then go to* Step 7. *Otherwise, go to* Step 6.

Step 6 *Take* $s^* = s_1$.

From (5), calculate $T^* = T^*(s_1)$.

From (6), calculate $M^* = M^*(s_1)$.

From (7), calculate $B^* = P(s_1)$. *Stop.*

Step 7 *Consider* $s^* = \infty$. *Put* $B^* = 0$, $M^* = 0$ *and* $T^* = \infty$. *Stop.*

5 Numerical examples

In this section, we present several numerical examples to show how the algorithm proposed in the previous section can be applied to obtain the optimal inventory policy.

Example 1 Let us consider the inventory system with the following parameters: $p = 8$, $K = 500$, $h = 2$, $\omega = 3.2$, $\alpha = 2500$, $\beta = 0.2$ and $n = 2.5$. We have $\xi = 14.2030$ and $s_o = 35.6236$. As $s_o \geq p$ and $s_o - (2 + 3e^{\beta s_1})\xi / \left(2\sqrt{\alpha\,(1 + e^{\beta s_1})}\right) = 20.6035 > p$, we calculate $s_1 = 14.5202$. Taking into account that $\ln\left[\left(\alpha + \sqrt{\alpha^2 + 4\alpha\beta^2\xi^2}\right) / \left(2\beta^2\xi^2\right)\right] / \beta = 28.6961 > s_1$, we conclude that $s^* = s_1$. From (5), we obtain the optimal inventory cycle $T^* = T^*(s_1) = 3.08895$ and, from (6), the maximum level of the stock is $M^* = 330.390$. Consequently, the maximum profit per unit time is $B^* = 523.144$ and the economic order quantity is $Q^* = 401.207$.

Example 2 Suppose the same parameters as in Example 1, but change the value of β to $\beta = 0.4$. Now $s_o = 14.3640 > p$ and $s_o - (2 + 3e^{\beta s_o})\xi / \left(2\sqrt{\alpha\,(1 + e^{\beta s_o})}\right) = 6.82397 < p$. Therefore, we fall into the case described by step 3 of Algorithm 1. Therefore, the inventory system is non-profitable for any unit selling price.

Example 3 Assume the same parameters as in Example 2, but modify the value of α to $\alpha = 5000$. We have $s_o = 16.0849$, $s_o - (2 + 3e^{\beta s_o})\xi / \left(2\sqrt{\alpha\,(1 + e^{\beta s_o})}\right) = 8.56486 > p$ and $s_1 = 13.5167 > \ln\left[\left(\alpha + \sqrt{\alpha^2 + 4\alpha\beta^2\xi^2}\right) / \left(2\beta^2\xi^2\right)\right] / \beta = 12.6232$. So, we fall into the case described by step 5 of Algorithm 1. As in the previous example, the inventory system is always non-profitable.

Example 4 Now, we consider the same parameters as in Example 2, but change the values of α, p and n to $\alpha = 1250$, $p = 12$ and $n = 0.5$, respectively. We obtain $\xi = 18.2033$ and $s_o = 11.4431 < p$. Thus, we fall into the case described by step 2 of Algorithm 1. Consequently, we obtain the same conclusion as in

Example 2.

5.1 Sensitivity analysis

Let us suppose the following input data of the inventory system: $p = 8$, $K = 500$, $h = 2$ and $\omega = 3.2$.

To study the impact of the parameters associated with the demand rate α, β and n, we provide four tables that show the behavior of s^*, T^*, M^* and B^*as functions of α, β and n. Tables 2 to 5 display computational results when $\alpha \in \{1875, 2500, 3125, 3750, 4375, 5000\}$, $\beta \in \{0.05, 0.08, 0.12, 0.16, 0.2, 0.24, 0.28, 0.32\}$ and $n \in \{0.25, 0.5, 1, 2.5\}$. These tables provide certain insights into the model studied. Some issues are the following:

1. Fixed α and n, if the value of β is increasing, then there is a point, say $\widetilde{\beta}$, such that $s^*(\beta)$ is finite for all $\beta \leq \widetilde{\beta}$ and $s^*(\beta) = \infty$ if $\beta > \widetilde{\beta}$ and, moreover, $P(s^*(\widetilde{\beta})) = 0$. For example, if $\alpha = 1875$ and $n = 2.5$, then $\widetilde{\beta} = 0.31563877$. When $\beta \leq \widetilde{\beta}$, the maximum level of the stock M^* and the maximum profit B^* are strictly decreasing as β increases, while the optimal inventory cycle T^* is strictly increasing. However, the optimal selling price s^* begins by decreasing and then increases as the parameter β increases.

2. Fixed β and n, if the value of α is decreasing, then there is a point, say $\widetilde{\alpha}$, such that $s^*(\alpha)$ is finite for all $\alpha \geq \widetilde{\alpha}$ and $s^*(\alpha) = \infty$ if $\alpha < \widetilde{\alpha}$ and, moreover, $P(s^*(\widetilde{\alpha})) = 0$. When $\alpha \geq \widetilde{\alpha}$, the optimal selling price s^* and the optimal inventory cycle T^* are strictly increasing as the parameter α decreases, while the maximum level of the stock M^* and the maximum profit B^* are strictly decreasing.

3. Fixed α and β, the maximum stock level M^* increases as the demand pattern index n increases. The optimal inventory cycle T^* and the optimal profit B^* start decreasing and then increase, while the optimal selling price s^* begins by growing and then decreases.

4. As conclusions, we have:

   (i) In general, the optimal policy and the maximum profit are more sensitive to changes in the parameter β than to changes in the parameter α. Moreover, the sensitivities of these parameters are greater when the value n is small.

   (ii) The optimal profit B^* is not very sensitive to changes in the pattern demand index n. The same occurs with the optimal inventory policy.

Table 2. Effects of α and β on optimal policy when $n = 0.25$

α		$\beta = 0.05$	$\beta = 0.08$	$\beta = 0.12$	$\beta = 0.16$	$\beta = 0.2$	$\beta = 0.24$	$\beta = 0.28$	$\beta = 0.32$
1875	s^*	32.8240	23.4626	18.4616	16.1655	15.0053	14.5087	14.6136	∞
	T^*	1.73120	1.91434	2.22363	2.63584	3.20419	4.03839	5.44044	∞
	M^*	75.5541	68.3260	58.8223	49.6234	40.8212	32.3889	24.0420	0
	B^*	6976.70	3325.87	1479.99	692.543	310.217	116.371	19.9850	0
2500	s^*	32.7160	23.3373	18.3066	15.9688	14.7461	14.1442	14.0234	∞
	T^*	1.49588	1.65068	1.90965	2.24957	2.70733	3.35226	4.34431	∞
	M^*	87.4398	79.2395	68.4938	58.1440	48.3130	39.0182	30.1082	0
	B^*	9405.57	4528.01	2054.00	991.663	470.072	200.362	60.9003	0
3125	s^*	32.6426	23.2525	18.2024	15.8382	14.5773	13.9150	13.6815	13.9267
	T^*	1.33590	1.47209	1.69847	1.99268	2.38301	2.91967	3.70761	5.04449
	M^*	97.9109	88.8528	77.0099	65.6397	54.8883	44.7993	35.2785	25.9291
	B^*	11845.3	5740.08	2636.80	1298.53	636.730	290.352	107.240	14.1856
3750	s^*	32.5886	23.1903	18.1264	15.7437	14.4565	13.7544	13.4522	13.5414
	T^*	1.21813	1.34094	1.54415	1.80636	2.15061	2.61617	3.27995	4.33287
	M^*	107.377	97.5431	84.7064	72.4102	60.8195	49.9964	39.8784	30.1877
	B^*	14292.6	6959.19	3225.84	1610.89	808.204	384.571	157.342	38.4116
4375	s^*	32.5468	23.1422	18.0678	15.6712	14.3648	13.6342	13.2851	13.2798
	T^*	1.12678	1.23940	1.42510	1.66342	1.97385	2.38865	2.96806	3.84921
	M^*	116.082	105.534	91.7826	78.6327	66.2659	54.7586	44.0689	33.9807
	B^*	16745.8	8183.65	3819.62	1927.41	983.322	481.978	210.257	65.2128
5000	s^*	32.5131	23.1036	18.0209	15.6134	14.2921	13.5399	13.1567	13.0873
	T^*	1.05327	1.15781	1.32969	1.54935	1.83372	2.21018	2.72809	3.49298
	M^*	124.184	112.971	98.3678	84.4220	71.3300	59.1804	47.9453	37.4462
	B^*	19203.6	9412.34	4417.14	2247.22	1161.32	581.903	265.384	94.0026

Table 3. Effects of α and β on optimal policy when $n = 0.5$

α		$\beta = 0.05$	$\beta = 0.08$	$\beta = 0.12$	$\beta = 0.16$	$\beta = 0.2$	$\beta = 0.24$	$\beta = 0.28$	$\beta = 0.32$
1875	s^*	32.9046	23.5564	18.5787	16.3160	15.2078	14.8059	15.1570	∞
	T^*	1.57721	1.74680	2.03529	2.42434	2.97109	3.80244	5.33309	∞
	M^*	181.151	163.564	140.380	117.852	96.1647	75.1397	53.5738	0
	B^*	6919.29	3273.98	1435.39	655.009	279.464	92.1512	2.34286	0
2500	s^*	32.7854	23.4177	18.4059	16.0946	14.9111	14.3743	14.3881	∞
	T^*	1.36250	1.50551	1.74621	2.06522	2.50129	3.13154	4.15428	∞
	M^*	209.699	189.779	163.620	138.345	114.227	91.2375	68.7759	0
	B^*	9339.12	4467.82	2002.04	947.643	433.610	171.073	38.5520	0
3125	s^*	32.7044	23.3239	18.2901	15.9481	14.7192	14.1072	13.9670	∞
	T^*	1.21659	1.34220	1.55211	1.82721	2.19677	2.71536	3.50678	∞
	M^*	234.848	212.870	184.081	156.367	130.061	105.221	81.4748	0
	B^*	11770.8	5672.58	2578.36	1248.80	595.260	256.651	80.9174	0
3750	s^*	32.6449	23.2552	18.2057	15.8423	14.5825	13.9219	13.6916	13.9446
	T^*	1.10921	1.22234	1.41044	1.65496	1.97951	2.42607	3.08259	4.20012
	M^*	257.583	233.743	202.571	172.641	144.336	117.768	92.6865	68.0252
	B^*	14211.0	6885.08	3161.55	1556.01	762.219	346.907	127.495	16.1464
4375	s^*	32.5987	23.2020	18.1407	15.7613	14.4790	13.7841	13.4939	13.6091
	T^*	1.02594	1.12959	1.30124	1.52303	1.81477	2.21051	2.77770	3.68770
	M^*	278.490	252.937	219.570	187.596	157.438	129.253	102.860	77.4776
	B^*	16657.6	8103.46	3749.94	1867.80	933.192	440.685	177.207	40.0123
5000	s^*	32.5616	23.1593	18.0886	15.6968	14.3972	13.6765	13.3435	13.3696
	T^*	.958934	1.05507	1.21379	1.41788	1.68446	2.04216	2.54537	3.32153
	M^*	297.950	270.800	235.390	201.508	169.617	139.908	112.249	86.0188
	B^*	19109.2	9326.49	4342.45	2183.21	1107.33	537.242	229.373	66.1337

Table 4. Effects of α and β on optimal policy when $n = 1$

α		$\beta = 0.05$	$\beta = 0.08$	$\beta = 0.12$	$\beta = 0.16$	$\beta = 0.2$	$\beta = 0.24$	$\beta = 0.28$	$\beta = 0.32$
1875	s^*	32.8722	23.5186	18.5314	16.2550	15.1253	14.6833	14.9243	∞
	T^*	1.63564	1.81036	2.10672	2.50446	3.05907	3.88998	5.35955	∞
	M^*	305.691	276.188	237.336	199.644	163.448	128.535	93.2913	0
	B^*	6942.36	3294.82	1453.29	670.048	291.756	101.787	9.27609	0
2500	s^*	32.7575	23.3853	18.3658	16.0437	14.8440	14.2800	14.2357	∞
	T^*	1.41311	1.56059	1.80820	2.13510	2.57921	3.21434	4.22171	∞
	M^*	353.830	320.392	276.518	234.182	193.858	155.553	118.435	0
	B^*	9365.82	4492.00	2022.89	965.290	448.200	182.753	47.4015	0
3125	s^*	32.6796	23.2952	18.2548	15.9037	14.6616	14.0287	13.8488	14.2337
	T^*	1.26186	1.39148	1.60763	1.88994	2.26727	2.79231	3.58064	4.99825
	M^*	396.240	359.329	311.016	264.558	220.530	179.063	139.640	100.035
	B^*	11800.7	5699.69	2601.82	1268.74	611.864	270.109	91.3756	2.66748
3750	s^*	32.6223	23.2291	18.1737	15.8024	14.5314	13.8537	13.5930	13.7739
	T^*	1.15054	1.26734	1.46116	1.71237	2.04432	2.49782	3.15605	4.24264
	M^*	434.579	394.527	342.194	291.993	244.580	200.175	158.426	117.851
	B^*	14243.8	6914.86	3187.36	1578.02	780.639	361.960	139.376	24.9264
4375	s^*	32.5778	23.1779	18.1113	15.7249	14.4327	13.7231	13.4083	13.4713
	T^*	1.06420	1.17125	1.34823	1.57627	1.87505	2.27783	2.84897	3.74454
	M^*	469.835	426.893	370.857	317.204	266.660	219.507	175.502	133.528
	B^*	16693.0	8135.68	3777.92	1891.71	953.279	457.200	190.380	49.9855
5000	s^*	32.5421	23.1369	18.0613	15.6631	14.3546	13.6210	13.2670	13.2523
	T^*	.994727	1.09405	1.25776	1.46774	1.74103	2.10570	2.61400	3.38371
	M^*	502.650	457.016	397.531	340.659	287.187	237.450	191.278	147.767
	B^*	19147.1	9360.98	4372.44	2208.89	1128.97	555.111	243.739	77.1881

Table 5. Effects of α and β on optimal policy when $n = 2.5$

α		$\beta = 0.05$	$\beta = 0.08$	$\beta = 0.12$	$\beta = 0.16$	$\beta = 0.2$	$\beta = 0.24$	$\beta = 0.28$	$\beta = 0.32$
1875	s^*	32.7094	23.3297	18.2972	15.9571	14.7308	14.1232	13.9912	∞
	T^*	2.01320	2.22126	2.56912	3.02532	3.63881	4.50118	5.82170	∞
	M^*	506.935	459.451	397.241	337.340	280.466	226.732	175.303	0
	B^*	7058.91	3400.29	1544.20	746.890	355.167	152.382	47.3056	0
2500	s^*	32.6173	23.2233	18.1667	15.7937	14.5202	13.8388	13.5717	13.7379
	T^*	1.74012	1.91660	2.20931	2.58845	3.08895	3.77179	4.75993	6.38101
	M^*	586.488	532.486	461.936	394.274	330.390	270.585	214.407	159.937
	B^*	9500.69	4614.29	2128.71	1055.26	523.144	243.532	94.6796	17.9303
3125	s^*	32.5546	23.1512	18.0788	15.6847	14.3818	13.6565	13.3158	13.3268
	T^*	1.55438	1.70999	1.96674	2.29658	2.72686	3.30306	4.11092	5.34858
	M^*	656.573	596.824	518.911	444.383	374.263	308.975	248.256	190.810
	B^*	11951.7	5836.80	2720.76	1370.26	696.935	339.779	146.572	43.8000
3750	s^*	32.5085	23.0983	18.0145	15.6055	14.2822	13.5273	13.1396	13.0622
	T^*	1.41758	1.55814	1.78918	2.08424	2.46592	2.97058	3.66348	4.68300
	M^*	719.933	654.985	570.408	489.656	413.866	343.556	278.577	217.929
	B^*	14409.4	7065.36	3318.17	1689.98	874.846	439.627	201.631	72.5292
4375	s^*	32.4727	23.0573	17.9649	15.5447	14.2062	13.4298	13.0092	12.8739
	T^*	1.31144	1.44052	1.65204	1.92098	2.26666	2.71944	3.33187	4.20891
	M^*	778.198	708.467	617.757	531.271	450.249	375.283	306.302	242.476
	B^*	16872.0	8298.50	3919.63	2013.26	1055.88	542.197	259.073	103.378
5000	s^*	32.4439	23.0244	17.9252	15.4961	14.1459	13.3530	12.9078	12.7313
	T^*	1.22600	1.34595	1.54204	1.79047	2.10820	2.52136	3.07388	3.84973
	M^*	832.429	758.246	661.824	569.995	484.090	404.766	332.010	265.099
	B^*	19338.6	9535.27	4524.31	2339.38	1239.37	646.917	318.391	135.883

6 Conclusions

We have analyzed an economic order quantity inventory model where demand is a bivariate function dependent on price and time. More concretely, the demand rate is the product of a price-logit function and a power-time function. Thus, the demand rate multiplicatively combines the effects of selling price and a power demand pattern. The replenishing of the inventory is instantaneously assumed and the lead time is zero or negligible. Shortages are allowed and fully backlogged.

The objective is to maximize the profit per unit time, considering the sales revenue and assuming the sum of the following costs: ordering cost, purchasing cost, holding cost and backordering cost.

We have developed several properties with the aim of characterizing the optimal inventory policy.

In this inventory system, it is not possible to obtain optimal policies in a closed form, but the optimal solutions can be determined by using some classic numerical procedures, such as the Newton-Raphson method or the bisection method.

We have provided an algorithmic approach to determine the optimal inventory policy. It can be obtained by using an efficient algorithm which finds the optimal selling price, the maximum level of the stock, the optimal inventory cycle and the maximum profit per unit time.

Several numerical examples are introduced to illustrate the solution procedure. Also, to study the impact on the optimal solution of some parameters associated with demand rate, we provide computational results which permits a sensitivity analysis of the inventory policy to be established.

Some directions for future research are as follows: to include in the demand rate other functions that depend on the selling price, to assume partial backlogging, to consider the possibility that the item deteriorates over time or to suppose non-linear holding cost in the model.

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Appendix

Lemma 1 *For a fixed value of s, the bivariate function $B_s(M,T) = B(s,M,T)$ given by (3) is strictly concave on the set $\Omega_s = \left\{(M,T) : T > 0, 0 < M \leq \alpha T / \left(1 + e^{\beta s}\right)\right\}$.*

Proof. Since the function $B_s(M,T)$ is twice-differentiable, we only need to prove that the Hessian matrix is negative definite on the set Ω_s.

The first partial derivatives of $B_s(M,T)$ are

$$\frac{\partial B_s(M,T)}{\partial M} = \omega - (h+\omega)\left(\frac{\left(1+e^{\beta s}\right)M}{\alpha T}\right)^n \tag{13}$$

$$\frac{\partial B_s(M,T)}{\partial T} = \frac{K}{T^2} + \frac{n\,(h+\omega)}{n+1}\frac{M}{T}\left(\frac{\left(1+e^{\beta s}\right)M}{\alpha T}\right)^n - \frac{\alpha\omega n}{(n+1)\left(1+e^{\beta s}\right)} \tag{14}$$

Therefore, the second partial derivatives are given by

$$\begin{aligned}
\frac{\partial^2 B_s(M,T)}{\partial M^2} &= -\frac{n(h+\omega)\left(1+e^{\beta s}\right)}{\alpha T}\left(\frac{\left(1+e^{\beta s}\right)M}{\alpha T}\right)^{n-1} \\
\frac{\partial^2 B_s(M,T)}{\partial T^2} &= -\frac{2K}{T^3} - \frac{n\,(h+\omega)\,M}{T^2}\left(\frac{\left(1+e^{\beta s}\right)M}{\alpha T}\right)^n \\
\frac{\partial^2 B_s(M,T)}{\partial M \partial T} &= \frac{(h+\omega)n}{T}\left(\frac{\left(1+e^{\beta s}\right)M}{\alpha T}\right)^n
\end{aligned}$$

and the Hessian matrix is

$$H = \begin{pmatrix} -\frac{n(h+\omega)\left(1+e^{\beta s}\right)}{\alpha T}\left(\frac{\left(1+e^{\beta s}\right)M}{\alpha T}\right)^{n-1} & \frac{(h+\omega)n}{T}\left(\frac{\left(1+e^{\beta s}\right)M}{\alpha T}\right)^n \\ \frac{(h+\omega)n}{T}\left(\frac{\left(1+e^{\beta s}\right)M}{\alpha T}\right)^n & -\frac{2K}{T^3} - \frac{n(h+\omega)M}{T^2}\left(\frac{\left(1+e^{\beta s}\right)M}{\alpha T}\right)^n \end{pmatrix}$$

Since $H_{11} = \partial^2 B_s(M,T)/\partial M^2 < 0$ for all $(M,T) \in \Omega_s$, if we prove that the determinant of the Hessian matrix is positive, the assertion follows. Indeed,

$$\det(H) = \frac{2(h+\omega)Kn}{MT^3}\left(\frac{\left(1+e^{\beta s}\right)M}{\alpha T}\right)^n > 0$$

for all $(M,T) \in \Omega_s$. ■

Lemma 2 *For a fixed value of s, the function $B_s(M,T)$ attains its maximum value at the point $(M^*(s), T^*(s))$ given by*

$$\begin{aligned}
M^*(s) &= \frac{\alpha}{\left(1+e^{\beta s}\right)}\left(\frac{\omega}{h+\omega}\right)^{1/n}\sqrt{\frac{(n+1)K\left(1+e^{\beta s}\right)}{n\alpha\omega\left(1-\left(\frac{\omega}{h+\omega}\right)^{1/n}\right)}} \\
T^*(s) &= \sqrt{\frac{(n+1)K\left(1+e^{\beta s}\right)}{n\alpha\omega\left(1-\left(\frac{\omega}{h+\omega}\right)^{1/n}\right)}}
\end{aligned}$$

Proof. By the previous lemma, it is sufficient to show that the point $(M^*(s), T^*(s)) \in \Omega_s$ and that the gradient at that point $\nabla B_s(M^*(s), T^*(s)) = 0$, which is easy to check. ■

Proof of Proposition 1.

1. It is immediate.

2. Indeed, taking the first derivative of the function $P(s)$, we have

$$\begin{aligned} P'(s) &= \left[1 - \frac{\beta e^{\beta s}}{1+e^{\beta s}}\left(s - p - \sqrt{\frac{1+e^{\beta s}}{\alpha}}\xi\right)\right]\frac{\alpha}{1+e^{\beta s}} \\ &= \left[e^{-\beta s} + 1 - \beta(s-p) + \beta\xi\sqrt{\frac{1+e^{\beta s}}{\alpha}}\right]\frac{\alpha e^{\beta s}}{(1+e^{\beta s})^2} \\ &= f(s)\frac{\alpha e^{\beta s}}{(1+e^{\beta s})^2}, \text{where } f(s) = 1 + e^{-\beta s} - \beta(s-p) + \beta\xi\sqrt{\frac{1+e^{\beta s}}{\alpha}}. \end{aligned}$$

   From this, we conclude that sign$(P'(s))$ =sign$(f(s))$.

   On the other hand, it is easy to check that the two first derivatives of the function $f(s)$ are

$$\begin{aligned} f'(s) &= -\beta\left(e^{-\beta s} + 1\right) + \frac{\beta^2 e^{\beta s}}{2\sqrt{\alpha}\sqrt{1+e^{\beta s}}}\xi \\ f''(s) &= \beta^2 e^{-\beta s} + \frac{\beta^3 e^{\beta s}(2+e^{\beta s})}{4\sqrt{\alpha}\sqrt{(1+e^{\beta s})^3}}\xi. \end{aligned} \tag{15}$$

   Therefore, $f''(s) > 0$, which proves the convexity of $f(s)$.

3. Since $\lim_{s\to\infty} f(s) = \infty$, $\lim_{s\to-\infty} f(s) = \infty$ and $f(s)$ is a strictly convex function, there exists a real point s_o in which f attains its minimum value. Moreover, as f is a differentiable function on $\mathbb{R}$, this point s_o can be calculated by (11), that is, solving the equation $f'(s) = 0$.

   The case (i) is obvious, because $f(s) > f(s_o) \geq 0$ for all $s \neq s_o$ and, according to the previous property, we have $P'(s) > 0$ for all $s \geq p$. Hence $P(s)$ is strictly increasing in (p, ∞). To prove the other case, consider $s > p$. Since $p \geq s_o$, we have $0 = f'(s_o) \leq f'(p) < f'(s)$, which implies that the function $f(s)$ is strictly increasing in the interval (p, ∞). Therefore, $f(s) > f(p) > 0$. The rest of the proof runs as before.

4. We can assert the existence of only one root s_1 of the function $f(s)$ in the interval (p, s_o) because $f(s)$ is strictly decreasing in such interval and, moreover, $f(p)f(s_o) < 0$. Thus, we have $f(s) > 0$ for $s \in (p, s_1)$ and $f(s) < 0$ for $s \in (s_1, s_o)$, which implies that $P(s)$ has a relative maximum at s_1.

Proof of Theorem 2.

It is obvious by Proposition 1.

Proof of Proposition 3.

1. From (15), $f'(s_o) = 0$ implies that $1 + e^{-\beta s_o} = \beta\xi e^{\beta s_o} / \left(2\sqrt{\alpha\,(1+e^{\beta s_o})}\right)$. Substituting the left-hand side into (10), yields

$$\begin{aligned} f(s_o) &= \frac{\beta\xi e^{\beta s_o}}{2\sqrt{\alpha\,(1+e^{\beta s_o})}} - \beta(s_o - p) + \beta\xi\sqrt{\frac{1+e^{\beta s_o}}{\alpha}} \\ &= \beta\left(p - s_o + \frac{2+3e^{\beta s_o}}{2\sqrt{\alpha\,(1+e^{\beta so})}}\xi\right). \end{aligned}$$

   The rest is immediate.

2. From (10) and (12), we have $p = s_1 - (1+e^{-\beta s_1})/\beta - \xi\sqrt{(1+e^{\beta s_1})/\alpha}$. Substituting this value into (7), an easy computation shows that

$$P(s_1) = \frac{\alpha e^{-\beta s_1}}{\beta} - \sqrt{\frac{\alpha}{1+e^{\beta s_1}}}\xi.$$

   Hence $P(s_1) < 0$ if $\alpha/(\beta\xi)^2 < e^{2\beta s_1}/(1+e^{\beta s_1})$ and so $\beta^2\xi^2(e^{\beta s_1})^2 - \alpha e^{\beta s_1} - \alpha > 0$. The rest is straightforward. ■

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