INNER IDEALS OF LIE ALGEBRAS OF SKEW ELEMENTS OF PRIME RINGS WITH INVOLUTION

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Dedicated to Professor W.S. Martindale, 3rd

Abstract. In this note we extend the Lie inner ideal structure of simple Artinian rings with involution, initiated by Benkart and completed by Benkart and Fernández López, to centrally closed prime rings with involution of characteristic not 2, 3 or 5. New Lie inner ideals (which we call special) occur when making this extension. We also give a purely algebraic description of the so-called Clifford inner ideals, which had only been described in geometric terms.

1. Introduction

Inner ideals of Lie algebras are the analogues of one-sided ideals in associative rings and of inner ideals in Jordan algebras ([9], [15]). They are Φ-submodules B of a Lie algebra L (over a ring of scalars Φ) such that $[[B, L], B] \subseteq B$. An abelian inner ideal of L is an inner ideal B which is also an abelian subalgebra, i.e., such that $[B, B] = 0$. Since their introduction over 40 years ago ([7],[4]), abelian inner ideals have proved to be a useful tool for classifying both finite-dimensional and infinite-dimensional simple Lie algebras. Premet ([16],[17]) showed that every finite-dimensional simple Lie algebra over an algebraically closed field of characteristic not 2 or 3 must contain one-dimensional inner ideals. Moreover, it follows from ([4],[18]) (see also [6]) that when the field is algebraically closed of characteristic $p > 5$, the classical Lie algebras (modular versions of the complex finite-dimensional simple Lie algebras) can be characterized as the finite-dimensional simple Lie algebras satisfying the following two equivalent conditions:

(i) They are generated by one-dimensional inner ideals.

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(ii) They are nondegenerate, that is, they have no nonzero absolute zero divisors (where by an absolute zero divisor or sandwich element we mean an element x such that $[x,[x,L]]=0$).

Further evidence of the usefulness of inner ideals comes from [11], where it is shown that an abelian inner ideal B of finite length in a nondegenerate Lie algebra L over a ring of scalars Φ such that 2 and 3 are invertible gives rise to a finite $\mathbb{Z}$-grading $L = L_{-n} \oplus \cdots \oplus L_0 \oplus \cdots \oplus L_n$ with $B = L_n$. Zelmanov in [21] described the simple Lie algebras over fields of characteristic 0 or $p > 4n+1$ with such gradings in terms of finite $\mathbb{Z}$-gradings of simple associative rings with involution. A description of these associative rings and their gradings was later provided by Smirnov in [19],[20]. As a result, any nondegenerate simple Lie algebra with a nonzero abelian inner ideal of finite length comes either from a simple associative ring with a finite $\mathbb{Z}$-grading by taking the Lie commutator, from the skew-symmetric elements of such a simple associative ring with involution or from the Tits-Kantor-Koecher construction of a Jordan algebra of Clifford type, or it is of exceptional type E_6, E_7, E_8, F_4 or G_2.

In this paper we extend the Lie inner ideal structure of simple Artinian rings with involution ([3], [5]) to centrally closed prime rings with involution.

Let R be a centrally closed prime ring of characteristic not 2, 3 or 5 with an involution $*$, and let K be the Lie algebra of the skew elements of R. We prove that there are at most four types of abelian inner ideals in K. Abelian inner ideals of the first type, called *isotropic inner ideals*, consist only of elements of zero square and are the cornerstone of the classification. Abelian inner ideals of the second type, called *standard inner ideals*, are of the form $V \oplus \mathrm{Skew}(Z(R),*)$, where V is an isotropic inner ideal of K and $\mathrm{Skew}(Z(R),*)$ is the set of skew elements of the centre of R (notice that $\mathrm{Skew}(Z(R),*) \neq 0$ if and only if R is unital and the involution $*$ is of the second kind). The third type of abelian inner ideals is the most exotic. These abelian inner ideals, called *special inner ideals*, only occur when R is unital, the involution $*$ is of the second kind and K contains elements of zero square which are not von Neumann regular (this is the reason why these inner ideals do not appear in the Artinian case). Finally, abelian inner ideals of the fourth type, *Clifford inner ideals*, only occur in prime rings with nonzero socle and involution of orthogonal type.

2. Preliminaries

Throughout this section we will be dealing with a non-necessarily unital associative algebra A with product xy, a Lie algebra L with $[x,y]$ denoting the Lie bracket and ad_x

the adjoint map determined by x, and a Jordan triple system T ([13]) with quadratic Jordan operator P_x and triple product $\{x, y, z\}$, all over a (commutative and unital) ring of scalars Φ. Since $2P_x y = \{x, y, x\}$, if $\frac{1}{2} \in \Phi$ then only the triple product is required to define a Jordan triple system over Φ. Eventually we also consider *linear Jordan pairs* (see [13, Notes in page 55] for the definition); in this case, $\frac{1}{6} \in \Phi$ will be required.

2.1. Assume that A has a Φ-linear involution $*$ and denote by $\mathrm{Skew}(A, *)$ the set of skew elements of A. Then:

(i) $\mathrm{Skew}(A, *)$ is a Lie algebra (over Φ) with Lie bracket given by $[x, y] := xy - yx$.

(ii) $\mathrm{Skew}(A, *)$ is also a Jordan triple system (over Φ) with $P_x y := xyx$ and $\{x, y, z\} := xyz + zyx$.

2.2. Let R be an (associative) ring such that $aR = 0 \ \Rightarrow a = 0$, $a \in R$ (e.g., R is a semiprime ring). Then its centroid $\Gamma := \Gamma(R)$ is a commutative unital ring and R can be regarded as an associative Γ-algebra. Moreover, any involution $*$ of R induces an involution in Γ, also denoted by $*$, and $K := \mathrm{Skew}(R, *)$ is then both a Lie algebra and a Jordan triple system over the ring of scalars $\mathrm{Sym}(\Gamma, *)$, the symmetric elements of Γ.

2.3. Suppose that $\frac{1}{2} \in \Phi$. An *inner ideal* of a Jordan triple system T is a Φ-submodule B of T such that $\{B, T, B\} \subseteq B$. Similarly, an *inner ideal* of a Lie algebra L is a Φ-submodule B of L such that $[[B, L], B] \subseteq B$. An *abelian inner ideal* of L is an inner ideal B which is also an abelian subalgebra, i.e., such that $[B, B] = 0$.

2.4. To avoid confusion, an *abelian* inner ideal of the Lie algebra K will be called a *Lie inner ideal* of K, while an inner ideal of the Jordan triple system K will be called a *Jordan inner ideal* of K.

2.5. Let B be an abelian inner ideal of a Lie algebra L over a ring of scalars Φ in which 6 is invertible.

(i) The *kernel* of B is the set $\ker_L B := \{a \in L : [B, [B, a]] = 0\}$.

(ii) The pair of Φ-modules $\mathrm{Sub}_L B := (B, L/\ker_L B)$ with the triple products given by

$$\begin{aligned} \{b, \overline{x}, c\} &:= [[b, x], c] \quad \text{for every } b, c \in B \text{ and } x \in L, \\ \{\overline{x}, b, \overline{y}\} &:= \overline{[[x, b], y]} \quad \text{for every } b \in B \text{ and } x, y \in L, \end{aligned}$$

where $\overline{x}$ denotes the coset of x relative to the submodule $\ker_L B$, is a Jordan pair called the *subquotient of* B ([11, Lemma 3.2]). Due to this notion, we can define a relation between abelian inner ideals of Lie algebras: if B and B' are abelian inner ideals of Lie algebras L and L' respectively, then B and B' are said to be *Jordan-isomorphic* if $\mathrm{Sub}_L B$ and $\mathrm{Sub}_{L'} B'$ are isomorphic as Jordan pairs.

3. Standard inner ideals

Throughout this section R will denote a semiprime ring with an involution $*$ and with $2\mathrm{id}_R$ being invertible in the centroid $\Gamma := \Gamma(R)$. Since we do not assume R to be unital, its centre $Z := Z(R)$ may be equal to zero. As noted previously, Γ is a commutative unital ring with involution, also denoted by $*$, and $K := \mathrm{Skew}(R, *)$ is both a Lie algebra and a Jordan triple system over the ring of scalars $\Phi := \mathrm{Sym}(\Gamma, *)$.

3.1. Let V be Φ-submodule of K such that $VV = 0$. Then V is a Jordan inner ideal of K if and only if it is a Lie inner ideal: V is clearly abelian and for $u, v \in V$ and $x \in K$, $VV = 0$ implies $[[u, x], v] = uxv + vxu = \{u, x, v\}$. In this case, V will be called an *isotropic inner ideal* of K.

Note that any Lie inner ideal B of K such that $b^2 = 0$ for all $b \in B$ is isotropic: for any $b, c \in B$, $0 = (b + c)^2 = 2bc$ implies $bc = 0$, since $\frac{1}{2} \in \Phi$.

It is also clear that if V is an isotropic inner ideal of K and Ω is a Φ-submodule of $\mathrm{Skew}(Z, *)$, then $B = V + \Omega$ is a Lie inner ideal of K. Moreover, the sum $V + \Omega$ is direct since Z does not contain nonzero nilpotent elements by semiprimeness of R.

Definition 3.2. A Lie inner ideal B of K will be called *standard* if $B = V \oplus \Omega$, where V is an isotropic inner ideal of K and Ω is a Φ-submodule of $\mathrm{Skew}(Z, *)$.

3.3. Given a Lie inner ideal B of K, we denote by V_B the subset of all zero square elements of the abelian subalgebra $B + \mathrm{Skew}(Z, *)$ of K.

Lemma 3.4. *Let B be a Lie inner ideal of K such that $B \subseteq V_B \oplus \mathrm{Skew}(Z, *)$. Then:*

(i) *V_B is an isotropic inner ideal of K satisfying $[[B, K], B] \subseteq \{V_B, K, V_B\} \subseteq B$.*

(ii) *If in addition $V_B \subseteq B$, then B is standard. In particular, this is so if $\mathrm{Skew}(Z, *) \subseteq B$ or if every $v \in V_B$ is von Neumann regular.*

Proof. (i) Since $B + \mathrm{Skew}(Z, *)$ is a Φ-submodule of K,

$$V_B + V_B \subseteq B + \mathrm{Skew}(Z, *) \subseteq V_B \oplus \mathrm{Skew}(Z, *).$$

Thus for any $u, v \in V_B$ there exists $w \in V_B$ and $z \in \mathrm{Skew}(Z, *)$ such that $u + v = w + z$, with $z = 0$ since $u + v - w$ is a nilpotent central element and R is semiprime. This proves that $V_B + V_B \subseteq V_B$, and since V_B is clearly invariant under Φ, V_B is a Φ-submodule of K. Then, for any $u, v \in V_B$, $2uv = (u+v)^2 = 0$ implies $uv = 0$. Hence, for any $x \in K$, $\{u, x, v\} = uxv + vxu = [[u, x], v] \in [[B + Z, K], B + Z] = [[B, K], B] \subseteq B$, with $\{u, x, v\}^2 = 0$. This proves that V_B is an isotropic inner ideal of K satisfying $\{V_B, K, V_B\} \subseteq B$. Note also that $[[B, K], B] \subseteq \{V_B, K, V_B\} \subseteq V_B$.

(ii) Suppose in addition that $V_B \subseteq B$. By the Modular Law we have

$$B = B \cap (V_B \oplus \mathrm{Skew}(Z, *)) = V_B \oplus (\mathrm{Skew}(Z, *) \cap B),$$

so B is standard. Note finally that if $\mathrm{Skew}(Z, *) \subseteq B$, then $V_B \subseteq B + \mathrm{Skew}(Z, *) \subseteq B$; the same holds if every $v \in V_B$ is von Neumann regular, since then $V_B = \{V_B, K, V_B\} \subseteq B$ by (i). This completes the proof. □

Theorem 3.5. *A Lie inner ideal B of K is standard if and only if the following condition holds:*

$$V_B \subseteq B \subseteq V_B \oplus \mathrm{Skew}(Z, *). \tag{ST}$$

Proof. By Lemma 3.4, condition (ST) is sufficient for B to be standard. Suppose then that B is standard, i.e., $B = V \oplus \Omega$, where V is an isotropic inner ideal of K and Ω is a Φ-submodule of $\mathrm{Skew}(Z, *)$. Clearly $V \subseteq V_B$ and, by the Modular Law,

$$V_B = V_B \cap (B + \mathrm{Skew}(Z, *)) = V_B \cap (V \oplus \mathrm{Skew}(Z, *)) = V,$$

since Z does not contain nonzero nilpotent elements. Thus $B = V_B \oplus \Omega$ and therefore it satisfies (ST). □

4. Special inner ideals

Throughout this section R will denote a unital semiprime ring with an involution $*$ which does not act as the identity on the centre $Z := Z(R)$ of R. We will also assume that Z is a field of characteristic not 2. Then $K := \mathrm{Skew}(R, *)$ is both a Lie algebra and a Jordan triple system over the field $\mathbb{F} := \mathrm{Sym}(Z, *)$.

Theorem 4.1. *Let V be a nonzero isotropic inner ideal of K and let $f : V \to \mathrm{Skew}(Z, *)$ be a nonzero $\mathbb{F}$-linear map with $[[V, K], V] \subseteq \ker f$. Then $\mathrm{Inn}(V, f) := \{v + f(v) : v \in V\}$ is a Lie inner ideal of K which is not standard.*

Proof. Set $B := \mathrm{Inn}(V, f)$. Then:

(1) B is a Lie inner ideal of K.

Indeed, $[[B, K], B] = [[V, K], V] \subseteq \ker f \subseteq B$ since $v = v + f(v) \in B$ for every $v \in \ker f$, and $[B, B] = [V, V] = 0$.

(2) $V \cap B = \ker f$.

As noted in (1), $\ker f \subseteq V \cap B$. Conversely, let $v \in V \cap B$. Then $v = u + f(u)$ for some $u \in V$. But then $v - u = f(u) \in V \cap Z = 0$, so $v = u \in \ker f$.

(3) $V_B = V$.

By definition of B, $V \subseteq B + \mathrm{Skew}(Z, *)$, and since $VV = 0$, $V \subseteq V_B$. Conversely, let $x = b + z \in V_B$, with $b = v + f(v)$ for some $v \in V$ and $z \in \mathrm{Skew}(Z, *)$. Then

$$0 = x^2 = (b + z)^2 = (v + (f(v) + z))^2 = 2(f(v) + z)v + (f(v) + z)^2$$

implies $f(v) + z = 0$ since the sum $V + Z$ is direct; so $x = v \in V$.

(4) B is not standard.

By Theorem 3.5 and (3), it is enough to see that V is not contained in B. Suppose otherwise that $V \subseteq B$. Then, by (2), $V = V \cap B = \ker f$ yields a contradiction. □

Definition 4.2. A Lie inner ideal B of K is called *special* if $B = \mathrm{Inn}(V, f)$ for some isotropic inner ideal V of K and some nonzero $\mathbb{F}$-linear map $f : V \to \mathrm{Skew}(Z, *)$ as in the theorem above.

Proposition 4.3. *K contains a special inner ideal if and only if there exists $x \in K$ such that $x^2 = 0$ and x is not von Neumann regular.*

Proof. Let $B = \mathrm{Inn}(V, f)$ be a special inner ideal of K. It follows from (3) of the proof of Theorem 4.1 and Lemma 3.4(i) that $\{V, K, V\} = [[V, K], V] \subseteq \ker f$. Hence V contains an element which is not von Neumann regular, since otherwise $V = \{V, K, V\} = \ker f$, which is a contradiction. Note also that any $v \in V$ is of zero square.

Suppose conversely that there exists $x \in K$ such that $x^2 = 0$ and x is not von Neumann regular. Then the sum $\mathbb{F}x + xKx$ is direct and it is easily checked that $V := \mathbb{F}x \oplus xKx$ is an isotropic inner ideal of K. Given a nonzero skew-symmetric element $z \in Z$ (which exists because $*$ does not act as the identity in Z by assumption), define a $\mathbb{F}$-linear map $f : V \to \mathrm{Skew}(Z, *)$ by $f(xKx) = 0$ and $f(x) = z$. Then $\mathrm{Inn}(V, f)$ is a special inner ideal of K since $[[V, K], V] = xKx = \ker f$. □

In spite of what we have proved in Theorem 4.1, isotropic inner ideals and special inner ideals are the same kind of thing from the Jordan point of view:

Proposition 4.4. *Let V be a nonzero isotropic inner ideal of K and let $f : V \to$ $\mathrm{Skew}(Z, *)$ be a nonzero $\mathbb{F}$-linear map such that $[[V, K], V] \subseteq \ker f$. Then the inner ideals V and $\mathrm{Inn}(V, f)$ are Jordan-isomorphic.*

Proof. Clearly, $\ker_L B = \ker_L V$. Then $\overline{K} := K/\ker_L B = K/\ker_L V$, $\mathrm{Sub}_K V = (V, \overline{K})$ and $\mathrm{Sub}_K B = (B, \overline{K})$. We claim that the pair of linear maps $(\varphi, \mathrm{id}_{\overline{K}}) : \mathrm{Sub}_K V \to \mathrm{Sub}_K B$ is an isomorphism of Jordan pairs, where $\varphi(v) = v + f(v)$ and $\mathrm{id}_{\overline{K}}$ is the identity on $\overline{K}$. Clearly $\varphi : V \to B$ is a linear isomorphism, and for $u, v \in V$ and $x, y \in K$, we have

$$\varphi(\{u, \overline{x}, v\}) = [[u, x], v] + f([[u, x], v]) = [[u, x], v] = [[u + f(u), x], v + f(v)] = \{\varphi(u), \overline{x}, \varphi(v)\}$$

since $[[V, K], V] \subseteq \ker f$ and $f(V) \subseteq \mathrm{Skew}(Z, *)$; and

$$\{\overline{x}, v, \overline{y}\} = \overline{[[x, v], y]} = \overline{[[x, \varphi(v)], y]} = \{\overline{x}, \varphi(v), \overline{y}\},$$

which completes the proof. □

5. Clifford inner ideals

Throughout this section $\mathbb{F}$ will denote a field of characteristic not 2 and X a vector space of dimension greater than 2 over $\mathbb{F}$ with a nondegenerate symmetric bilinear form denoted by $\langle \cdot\,, \cdot \rangle$.

5.1. Denote by $\mathcal{L}_X(X)$ the associative $\mathbb{F}$-algebra of linear maps $a : X \to X$ having a (unique) adjoint $a^* : X \to X$, that is, such that $\langle ax, y \rangle = \langle x, a^*y \rangle$ for all $x, y \in X$. Then:

(i) $\mathcal{L}_X(X)$ is a prime (in fact primitive) algebra ([2, Theorem 4.3.8(ii)]) with involution $*$ (the adjoint), whose socle is the ideal $\mathcal{F}_X(X)$ of all $a \in \mathcal{L}_X(X)$ having finite rank ([2, Theorem 4.3.8(iv)]).
(ii) $\mathrm{Skew}(\mathcal{L}_X(X), *)$ is the *orthogonal algebra* $\mathfrak{o}(X)$ and $\mathrm{Skew}(\mathcal{F}_X(X), *)$ is the *finitary orthogonal algebra* $\mathfrak{fo}(X)$ ([1]).
(iii) $\langle bx, x \rangle = 0$ for every $b \in \mathfrak{o}(X)$ and $x \in X$.

5.2. Given $x, y \in X$, write y^*x to denote the linear map on X defined by $y^*x(x') := \langle x', y \rangle x$, $x' \in X$. We also set $[x, y] := x^*y - y^*x$ for all $x, y \in X$. The following identities (which can be easily checked) will be used in what follows without further mention:

(i) $(y^*x)^* = x^*y$ and therefore $y^*x \in \mathcal{F}_X(X)$. In fact, $\mathcal{F}_X(X)$ is the additive span of these rank-one linear maps ([2, Theorem 4.3.2]).
(ii) $a(y^*x) = y^*ax$ and $(y^*x)b = (b^*y)^*x$ for all $x, y \in X$, any linear map a on X and any $b \in \mathcal{L}_X(X)$.
(iii) $(y^*x)(z^*v) = \langle v, y \rangle z^*x$, $x, y, z, v \in X$.
(iv) $[x, y] \in \mathfrak{fo}(X)$, $x, y \in X$. In fact, $\mathfrak{fo}(X) = [X, X]$, the additive span of all $[x, y]$.

5.3. By a *hyperbolic pair* we mean a pair (x, y) of isotropic vectors of X such that $\langle x, y\rangle = 1$, i.e., such that $H = \mathbb{F}x \oplus \mathbb{F}y$ is a hyperbolic plane of X. The following assertions are immediate:

(i) Any nonzero isotropic vector $x \in X$ can be extended to a *unique* hyperbolic pair (x, y).

(ii) $X = H \oplus H^\perp$ for any hyperbolic plane of X, where $H^\perp = \{z \in X : \langle z, H\rangle = 0\}$.

5.4. An idempotent $e \in \mathcal{L}_X(X)$ is said to be $*$-*orthogonal* if $ee^* = 0 = e^*e$. Note that e is a rank-one $*$-orthogonal idempotent if and only if $e = x^*y$, where (x, y) is a hyperbolic pair.

Proposition 5.5. *Set $B := [x, H^\perp] = \{[x, z] : z \in H^\perp\}$, where H is a hyperbolic plane of X and x is a nonzero isotropic vector of H. Then:*

(i) B is a Lie inner ideal of $\mathfrak{o}(X)$ contained in $\mathfrak{fo}(X)$, with $b^3 = 0$ for every $b \in B$ and $b_0^2 \neq 0$ for some $b_0 \in B$. Hence B is neither standard nor special.

(ii) B coincides with its centralizer in $\mathfrak{o}(X)$ and hence it is a maximal Lie inner ideal of $\mathfrak{o}(X)$.

Proof. (i) By [10, Lemma 3.7(i)], B is a Lie inner ideal of $\mathfrak{o}(X)$ contained in $\mathfrak{fo}(X)$, and by [10, Lemma 3.7(ii)], $b^3 = 0$ for every $b = [x, z] \in B$, with $b^2 = 0$ if and only if $z \in H^\perp$ is isotropic. Since $\dim_{\mathbb{F}} X > 2$, $H^\perp$ must contain some anisotropic vector, so there exists $b_0 \in B$ such that $b_0^2 \neq 0$. This implies that B is not standard. Since the adjoint involution $*$ of $\mathcal{L}_X(X)$ is of the first kind, B is not special either.

(ii) Let $a \in \mathfrak{o}(X)$ be such that

$$a(x^*z - z^*x) = (x^*z - z^*x)a, \;\; z \in H^\perp. \tag{1}$$

The proof will be complete if we prove that, for the isotropic vector $y \in H$ such that $\langle x, y\rangle = 1$, we have $ay \in H^\perp$ and $a = [x, ay]$.

Since $a^* = -a$, equation (1) can be written as

$$x^*az - z^*ax = (az)^*x - (ax)^*z, \;\; z \in H^\perp, \tag{2}$$

which evaluated in y yields

$$az = \langle y, az\rangle x - \langle y, ax\rangle z, \;\; z \in H^\perp. \tag{3}$$

Take $z \in H^\perp$ anisotropic in (3), which is possible because $\dim_{\mathbb{F}} X > 2$, and consider $\langle z, az\rangle$. Since $\langle az, z\rangle = 0$ because $a = -a^*$, we get $\langle y, ax\rangle = 0$. Thus

$$az = \langle y, az\rangle x, \;\; z \in H^\perp. \tag{4}$$

Evaluating (2) in z and applying (4), we get that for any $z \in H^\perp$,

$$-\langle z, z\rangle ax = -\langle z, ax\rangle z = \langle az, x\rangle z = \langle\langle y, az\rangle x, x\rangle z = 0.$$

Taking z anisotropic we get

$$ax = 0. \tag{5}$$

Then $\langle ay, x\rangle = -\langle y, ax\rangle = 0$, and since $\langle ay, y\rangle = 0$, we get that $ay \in H^\perp$. Using the decomposition $X = H \oplus H^\perp = \mathbb{F}x \oplus \mathbb{F}y \oplus H^\perp$ we will prove that $a = [x, ay]$ to complete the proof.

(i) $[x, ay]x = \langle x, x\rangle ay - \langle x, ay\rangle x = 0 = ax$ by (5),
(ii) $[x, ay]y = \langle y, x\rangle ay - \langle y, ay\rangle x = ay$, and for $z \in H^\perp$,
(iii) $[x, ay]z = \langle z, x\rangle ay - \langle z, ay\rangle x = -\langle z, ay\rangle x = \langle az, y\rangle x = az$, by (4).

□

Definition 5.6. Let L be a subalgebra of $\mathfrak{o}(X)$ containing $\mathfrak{fo}(X)$. An abelian inner ideal B of L is called *Clifford* if $B = [x, H^\perp]$, where H is a hyperbolic plane of X and $x \in H$ is a nonzero isotropic vector. This terminology is motivated by the fact that the *subquotient* of B is the *Clifford Jordan pair* $(H^\perp, H^\perp)$ (see [5, Proposition 4.4(i)]).

The following proposition is the converse of the statement (i) of Proposition 5.5.

Proposition 5.7. *Let L be a subalgebra of $\mathfrak{o}(X)$ containing $\mathfrak{fo}(X)$ and let B be an abelian inner ideal of L. If B contains an element b such that $b^3 = 0$ and b^2 has rank one, then B is Clifford.*

Proof. Since b^2 is symmetric and of rank one, we have that $b^2 = \alpha x^* x$, where both $\alpha \in \mathbb{F}$ and $x \in X$ are nonzero. Now $b^3 = 0$ implies by 5.2(iii) that $0 = b^2 b^2 = \alpha^2\langle x, x\rangle x^* x$, so x is isotropic. Extend x to the hyperbolic pair (x, y) and set $H := \mathbb{F}x \oplus \mathbb{F}y$. We have the following identities:

(i) $b^2 y = (\alpha x^* x)y = \alpha\langle y, x\rangle x = \alpha x$,
(ii) $\langle by, by\rangle = -\langle y, b^2 y\rangle = -\alpha$, so by is anisotropic,
(iii) $by \in H^\perp$, since $\langle by, y\rangle = 0$ and $\langle by, x\rangle = \langle by, \alpha^{-1} b^2 y\rangle = \langle b^3 y, \alpha^{-1} y\rangle = 0$.

Let $w \in H^\perp$ and set $a := [y, w]$. Then:

(iv) $ax = \langle x, y\rangle w - \langle x, w\rangle y = w$,
(v) $b^2 a = \alpha(x^* x)a = -\alpha(ax)^* x = -\alpha w^* x$,
(vi) $ab^2 = \alpha a(x^* x) = \alpha x^* ax = \alpha x^* w$,
(vii) $bab = b(y^* w - w^* y)b = (bw)^* by - (by)^* bw = [bw, by]$, and

(viii) $\mathrm{ad}_b^2 a = b^2a + ab^2 - 2bab = \alpha[x,w] - 2[bw, by]$.

Taking $w = by$ in (viii), we get by (i) that

$$\mathrm{ad}_b^2[y, by] = \alpha[x, by] - 2[b^2y, by] = \alpha[x, by],$$

so $[x, by] \in B$. Since by is anisotropic, by [10, Lemma 3.7(iii)] we have $[x, H^\perp] = \mathrm{ad}^2_{[x,by]}\mathfrak{fo}(X) \subseteq B$, and hence $B = [x, H^\perp]$ since $[x, H^\perp]$ is maximal by Proposition 5.5(ii). This proves that B is Clifford. $\square$

We describe Clifford inner ideals in algebraic terms. To this end we introduce the following notation, which makes sense for any subset S of a ring R with involution: $\kappa(S) := \{a - a^* : a \in S\}$.

Proposition 5.8. *Let L be a subalgebra of $\mathfrak{o}(X)$ containing $\mathfrak{fo}(X)$. An abelian inner ideal B of L is Clifford if and only if $B = \kappa((1-e)Re)$, where R is any $*$-subalgebra of $\mathcal{L}_X(X)$ containing L such that $\mathcal{F}_X(X) \subseteq R$, and e is a rank-one $*$-orthogonal idempotent.*

Proof. As previously noted in (5.4), $e = x^*y$, where (x, y) is a hyperbolic pair. Let $H = \mathbb{F}x \oplus \mathbb{F}y$ be the associated hyperbolic plane and set $f := e + e^*$. Then

$$Re = R(x^*y) = x^*Ry = x^*X, \tag{1}$$

and since $1 - f$ is the orthogonal projection on $H^\perp$, we have by (1) that

$$(1-f)Re = (1-f)x^*X = x^*(1-f)X = x^*H^\perp. \tag{2}$$

Since $\langle by, y\rangle = 0$ for every $b \in \mathfrak{o}(X)$,

$$e^*be = (x^*y)^*b(x^*y) = (y^*x)b(x^*y) = (y^*x)(x^*by) = \langle by, y\rangle x^*x = 0. \tag{3}$$

Hence, for every $a \in R$,

$$\kappa((1-f)ae) = \kappa((1-e)ae - e^*ae) = \kappa((1-e)ae) - e^*\kappa(a)e = \kappa((1-e)ae). \tag{4}$$

Then, by (2) and (4), $[x, H^\perp] = \kappa(x^*H^\perp) = \kappa((1-f)Re) = \kappa((1-e)Re)$. $\square$

6. Centrally closed prime rings with involution

In this section R will be a prime ring. The extended centroid $\mathcal{C} := \mathcal{C}(R)$ of R (see [2, 2.3] or [12, 14C] for the definition and basic results) is a field containing the centroid $\Gamma := \Gamma(R)$ of R, and the *central closure* $\mathcal{C}R$ of R is a prime associative algebra over $\mathcal{C}$. A prime ring R is said to be *centrally closed* if it is its own central closure, equivalently, if $\mathcal{C}(R) = \Gamma(R)$. The central closure of a prime ring is centrally closed and so is any simple ring. Moreover, it follows from [2, Theorem 4.3.7(ix)] that rings of the form $\mathcal{L}_X(X)$ as in (5.1) are also centrally closed.

Any involution $*$ of R induces an involution in $\mathcal{C}$, also denoted by $*$, and therefore it can be extended to an involution of $\mathcal{C}R$. The involution $*$ of R is said to be of *the first kind* if it acts as the identity on $\mathcal{C}$, otherwise $*$ is said to be of *the second kind.*

We set $\tilde{R} := \overline{\mathcal{C}} \otimes_{\mathcal{C}} \mathcal{C}R$, with $\overline{\mathcal{C}}$ denoting the algebraic closure of the field $\mathcal{C}$, and $K := \mathrm{Skew}(R, *)$. Recall that K is both a Lie algebra and a Jordan triple system over the ring of scalars $\mathrm{Sym}(\Gamma, *)$.

Proposition 6.1. *Let R be a prime ring of characteristic not 2 with an involution $*$ of the first kind and let $\langle K\rangle$ be the subring of R generated by K.*

(i) *If the Lie algebra K is not abelian, then $\langle K\rangle$ is prime with $\mathcal{C}(\langle K\rangle) = \mathcal{C}(R)$.*

(ii) *If K is abelian, then $K = 0$ or $\tilde{R} = \mathrm{M}_2(\overline{\mathcal{C}})$.*

Proof. (i) Taking $U = K$ in [2, Theorem 9.1.13(d)], we have that $[K, K] \neq 0$ implies $[K, K]^2 \neq 0$. Let I be the ideal of R generated by $[K, K]^2$. By [2, Lemma 9.1.4], $0 \neq I \subseteq \langle K\rangle$ and hence it follows from [12, Theorem 14.14 and subsequent Remark] that $\langle K\rangle$ is prime with $\mathcal{C}(\langle K\rangle) = \mathcal{C}(R)$.

(ii) Take $U = K$ in [2, Theorem 9.1.13(a)]. $\square$

Proposition 6.2. *Let R be a centrally closed prime ring of characteristic not $2, 3$ or 5 with an involution $*$ of the first kind such that $[K, K] \neq 0$. If B is a Lie inner ideal of K, then* (i) *$b^3 = 0$ for every $b \in B$. Moreover, if $b^2 \neq 0$ for some $b \in B$, then* (ii) *K is a subalgebra of $\mathfrak{o}(X)$ containing $\mathfrak{fo}(X)$, where X is a vector space of dimension greater than 2 with a nondegenerate symmetric bilinear form over the field $\mathcal{C}$, and* (iii) *B is a Clifford inner ideal of K.*

Proof. (i) By Proposition 6.1(i), $\langle K\rangle$ is a centrally closed prime ring of characteristic not $2, 3$ or 5 with $\mathcal{C}(\langle K\rangle) = \mathcal{C}(R)$. For any $b \in B$, $\mathrm{ad}_b^3 K \in [B, [B, [B, K]]] \subseteq [B, B] = 0$, and since $\langle K\rangle$ is spanned by the elements of K and their squares ([2, Lemma 9.1.5]), we have (using the Leibniz rule) that $\mathrm{ad}_b^5\langle K\rangle = 0$. Then it follows from [14, Corollary 1] that $(b - \alpha)^3 = 0$ for some $\alpha \in \mathcal{C}$ (the formula making sense in the unitization of $\langle K\rangle$). But as observed in the proof of [3, Lemma 4.22], the involution $*$ being of the first kind forces $\alpha = 0$. Thus $b^3 = 0$ for every $b \in B$.

(ii) Suppose that there exists $b \in B$ such that $b^2 \neq 0$. Then $\mathrm{ad}_b^3 K = 0$, and since $b^3 = 0$, we have $0 = \mathrm{ad}_b^4 K = 6b^2 K b^2$, which implies $b^2 K b^2 = 0$ because $\mathrm{char}(R) \neq 2, 3$. Put $c := b^2 \in \mathrm{Sym}(R, *)$. For every $x \in R$ we have $c(x - x^*)c = 0$, and hence, for all

$x, y \in R$,

$$(cxc)yc = c(xcy)c = c(xcy)^*c = cy^*cx^*c = cycxc = cy(cxc).$$

Then we have by [2, Theorem 2.3.4] that for every $x \in R$ there exists $\lambda_x \in \mathcal{C}$ such that $cxc = \lambda_x c$, which proves that $cRc = \mathcal{C}c$, since $cRc \neq 0$ by primeness of R. This implies that cR is a minimal right ideal of R, so R has nonzero socle. Let $cR = eR$ where $e = ca$ is a minimal idempotent of R ([2, Proposition 4.3.3]). We have

$$eRe = cRe = (cRc)a = \mathcal{C}ca = \mathcal{C}e,$$

which proves that $\mathcal{C}$ itself is the division ring of R. Now it follows from Kaplansky's Theorem ([2, Theorem 4.6.8]) that the involution $*$ of R is either of transpose type or of symplectic type; but the latter cannot occur because c is a symmetric rank-one element, so $*$ is of transpose type. Since $*$ is of the first kind by hypothesis, we have (again by Kaplansky's Theorem) that it is actually of *orthogonal type*, i.e., $*$ is the adjoint involution coming from a nondegenerate symmetric bilinear form $\langle \cdot, \cdot \rangle$ of a vector space X over the field $\mathcal{C}$, R is a $\mathcal{C}$-subalgebra of $\mathcal{L}_X(X)$ containing $\mathcal{F}_X(X)$, and K is a subalgebra of $\mathfrak{o}(X)$ containing $\mathfrak{fo}(X)$, with $\dim_{\mathcal{C}} X > 2$ since $[K, K] \neq 0$.

(iii) It follows from Proposition 5.7. □

Theorem 6.3 (Main Theorem). *Let R be a centrally closed prime ring of characteristic not $2, 3$ or 5 with an involution $*$, let $\overline{\mathcal{C}}$ be the algebraic closure of the extended centroid $\mathcal{C}$ of R, and suppose that $\overline{\mathcal{C}} \otimes_{\mathcal{C}} R$ is not the full matrix algebra $M_2(\overline{\mathcal{C}})$ with the transpose involution. If B is a Lie inner ideal of K, then either*

(i) $B = V$ *is an isotropic inner ideal,*
(ii) $B = V \oplus \mathrm{Skew}(Z(R), *)$ *is a standard inner ideal,*
(iii) $B = \mathrm{Inn}(V, f)$ *is special, or*
(iv) $B = \kappa((1 - e)Re)$ *is Clifford.*

Moreover, in cases (ii) and (iii) R is unital and $*$ is of the second kind, while in case (iv) R has nonzero socle and $*$ is of orthogonal type.

Proof. Suppose first that $*$ is of the second kind and let ξ be a nonzero skew-symmetric element of $\mathcal{C}$. Then $R = K \oplus \xi K$. Set $C := B \oplus \xi B$. It is straightforward to see that C is an abelian inner ideal of the Lie algebra R^-. By [8, Theorem 5.4], either (i) $C = U$, where U is an inner ideal of R^- with $UU = 0$; or R is unital and either (ii) $C = U \oplus Z(R)$, where U is as in (i); or (iii) $C = \{u + g(u) : u \in U\}$, where U is as in (i) and $g : U \to Z(R)$ is a nonzero linear form such that $[[U, R], U] \subseteq \ker g$. If $C = U$ as in (i), then $B = \mathrm{Skew}(U, *)$ is an isotropic inner ideal of K (see. 3.1). Suppose then that C is as in (ii) or (iii). In both cases U is $*$-invariant: $U^* \subseteq C^* = C \subseteq U \oplus Z(R)$ and hence $[U^*, U] = 0$ since $UU = 0$. Thus for any $u \in U$, $u^* = v + z$ where $u, v \in U$ and $z \in Z(R)$. Since $u^* - v$ is nilpotent, $u^* - v = 0$, so $u^* = v \in U$ as claimed. If (ii), then $B = \mathrm{Skew}(U, *) \oplus \mathrm{Skew}(Z(R), *)$, with $\mathrm{Skew}(U, *)$ being an isotropic inner ideal of K; if (iii), then $B = \{v + f(v) : v \in V\}$, where $V = \mathrm{Skew}(U, *)$ is an isotropic inner ideal of K and $f : V \to \mathrm{Skew}(Z(R), *)$ is the restriction of g to V, which satisfies $[[V, K], V] \subseteq \ker f$.

Suppose now that the involution $*$ is of the first kind. If $b^2 = 0$ for every $b \in B$, then B is an isotropic inner ideal. Thus we may assume that $b_0^2 \neq 0$ for some $b_0 \in B$. Then we have by Proposition 6.2 that B is a Clifford inner ideal. □

Remarks 6.4. (1) The Lie algebra $\mathrm{Skew}(M_2(\mathbb{F}), *)$, where $\mathbb{F}$ is a field of characteristic not 2 and $*$ is the transpose involution, is an abelian inner ideal in itself which does not lie in any of the four cases of the theorem above. Thus the exception in the statement is not superfluous.

(2) Let B be a Lie inner ideal of $K = \mathrm{Skew}(R, *)$, where R is still prime but non-necessarily centrally closed. Then $\mathrm{Sym}(\mathcal{C}, *)B$ is a Lie inner ideal of $\mathrm{Skew}(\mathcal{C}R, *)$ and therefore one of those described in Theorem 6.3.

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