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Clifford elements in Lie algebras

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Abstract. Let L be a Lie algebra over a field $\mathbb{F}$ of characteristic zero or $p > 3$. An element $c \in L$ is called Clifford if $\mathrm{ad}_c^3 = 0$ and its associated Jordan algebra L_c is the Jordan algebra $\mathbb{F} \oplus X$ defined by a symmetric bilinear form on a vector space X over $\mathbb{F}$. In this paper we prove the following result: Let R be a centrally closed prime ring R of characteristic zero or $p > 3$ with involution $*$ and let $c \in \mathrm{Skew}(R, *)$ be such that $c^3 = 0$, $c^2 \neq 0$ and $c^2kc = ckc^2$ for all $k \in \mathrm{Skew}(R, *)$. Then c is a Clifford element of the Lie algebra $\mathrm{Skew}(R, *)$.
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1. Introduction

Let L be a Lie algebra over a field $\mathbb{F}$ of characteristic not 2 or 3. An element $a \in L$ is called a *Jordan element* if $\mathrm{ad}_a^3 L = 0$. In [10], a Jordan algebra was attached to any Jordan element $a \in L$. This Jordan algebra, denoted by L_a, inherits most of the properties of the Lie algebra L and in addition the nature of the Jordan element in question is reflected in the structure of the attached Jordan algebra. For instance, if L is nondegenerate ($\mathrm{ad}_x^2 L = 0 \Rightarrow x = 0$) so is the Jordan algebra L_a and, in this case, L_a is unital if and only if a is *von Neumann regular* ($a \in \mathrm{ad}_a^2 L$). Jordan techniques have proved to be very useful in some questions of Lie theory. Examples of the use of the Jordan-Lie connection can be found in the papers [3], [7], [11], [12] and [13].

By a *Clifford element* of L we mean a Jordan element $c \in L$ such that L_c is the Jordan algebra $J := \mathbb{F} \oplus X$ defined by a symmetric bilinear form on a vector space X over $\mathbb{F}$ (we do not discard the case $X = 0$, i.e., $J = \mathbb{F}$). Suppose now that L is nondegenerate, $\mathrm{char}(\mathbb{F}) = 0$ or $p > 5$ and c is a Clifford element of L. Since L_c is then unital, c is von Neumann regular, and hence, by the Jacobson-Morozov Lemma (see [6, Proposition 1.18]), L has a 5-grading $L = L_{-2} \oplus L_{-1} \oplus L_0 \oplus L_1 \oplus L_2$ such that the Jordan pair $V := (L_{-2}, L_2)$ is isomorphic to the Clifford Jordan pair defined by the Jordan algebra L_c, whose Tits-Kantor-Koecher algebra $TKK(V)$

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is a finitary orthogonal Lie algebra (see [8, 5.11]), that is, $TKK(V) \cong \mathrm{Skew}(R,*)$, where R is a simple ring coinciding with its socle containing at least three nonzero orthogonal idempotents, and $*$ is an involution of *orthogonal type*, i.e., the adjoint involution associated to a nondegenerate symmetric bilinear form. Thus every Clifford element c actually lives in a ring, and in this associative context verifies $c^3 = 0$ and $c^2 \neq 0$ (see [9, Lemma 3.7(ii)]). In this paper we prove the following converse of the above result:

Let R be a centrally closed prime ring of characteristic not 2 or 3, let $*$ be an involution of R and let c be a Jordan element of the Lie algebra $K := \mathrm{Skew}(R,*)$ such that $c^3 = 0$ and $c^2 \neq 0$. Then R has nonzero socle and contains at least three orthogonal idempotents, $*$ is of orthogonal type and c is a Clifford element of K.

The proof is rather constructive. We start by showing some elementary associative properties of the Clifford element c and its square c^2. In particular, c^2 is von Neumann regular and can be paired with an element d that shares its properties; moreover, c is also von Neumann regular and can be paired with the element $\sqrt{d} := cd + dc$, which is also Clifford and will play the role of identity element in the Jordan algebra K_c. The element d helps to build a 3-grading of K in which $c \in K_{-1}$. We show that this component is actually independent of the choice of d, since it can be expressed just in terms of c in different ways, all important for our purposes. We also prove that $c^2Rc^2 = \mathcal{C}c^2$ and $cKc = \mathcal{C}c$ (with $\mathcal{C}$ being the extended centroid of R), facts that serve to build a linear form and a bilinear symmetric form over K, which in turn help to prove the main result about the structure of K_c.

2. Preliminaries

Throughout this section Φ will denote a ring of scalars, i.e., a commutative ring with 1, and $\mathbb{F}$ will stand for a field. An *algebra over* Φ (in short, a Φ-algebra) is a Φ-module A endowed with a product (bilinear operation). Thus no associativity condition is assumed; neither it is supposed the existence of a unit element in A. According to this definition, a ring is an associative $\mathbb{Z}$-algebra.

Jordan algebras and Lie algebras.

Suppose that 2 is invertible in Φ. A (linear) *Jordan algebra* is a Φ-algebra J whose product, denoted by $\bullet$, is commutative and satisfies the identity $x^2 \bullet (y \bullet x) = (x^2 \bullet y) \bullet x$ for all $x, y \in J$, where $x^2 := x \bullet x$. For each $x \in J$, the U-operator $U_x : J \to J$ defined by $U_x y := 2x \bullet (x \bullet y) - x^2 \bullet y$, $y \in J$, satisfies the identity $U_{U_x y} = U_x U_y U_x$ for all $x, y \in J$. A Jordan algebra is said to be *nondegenerate* if $U_x = 0$ implies $x = 0$.

Suppose that 2 is invertible in Φ and A is an associative Φ-algebra, whose product is denoted by juxtaposition. In the Φ-module A we define a new product by $x \circ y := xy + yx$. The resulting algebra is a Jordan algebra denoted by A^+, with $U_x y = 4xyx$. Note that A is semiprime if and only if A^+ is nondegenerate. A Jordan algebra J is called *special* if it is isomorphic to a subalgebra of A^+ for some associative algebra A. As usual, we denote by A^- the Lie algebra defined in the Φ-module A by the product $[x, y] := xy - yx$.

Let $\mathbb{F}$ be a field of characteristic not 2 and let X be an $\mathbb{F}$-vector space with a symmetric bilinear form $\langle\,,\rangle$. Then the vector space $\mathbb{F}\oplus X$ is endowed with a structure of Jordan algebra by defining

$$(\alpha, x)\bullet(\beta, y) := (\alpha\beta + \langle x, y\rangle,\ \beta x + \alpha y)$$

for $\alpha, \beta \in \mathbb{F}$ and $x, y \in X$. This Jordan algebra is unital, with $(1,0)$ as unit element, and special; in fact, it is isomorphic to a Jordan subalgebra of the Clifford (associative) algebra defined by $\langle\,,\rangle$ (see [14, II.3]). For this reason, $\mathbb{F}\oplus X$ is sometimes called a *Clifford* Jordan algebra.

Let L be a Lie Φ-algebra, with $[x, y]$ denoting the product and ad_x the adjoint map determined by x. Sometimes we will use capital letters instead, i.e., X for ad_x. An *inner ideal* of L is a Φ-submodule B of L such that $[[B, L], B]\subseteq B$. An *abelian inner ideal* is an inner ideal B which is also an abelian subalgebra, i.e., such that $[B, B] = 0$. For example, if $L = \bigoplus_{-n\le i\le n} L_i$ is a finite $\mathbb{Z}$-grading, then L_{-n} and L_n are easily checked to be abelian inner ideals of L. An element $a\in L$ is said to be a *Jordan element* whenever $ad_a^3 L = 0$. Every element in an abelian inner ideal is easily shown to be a Jordan element, and conversely, if L is 3-torsion free and $a\in L$ is Jordan, then $\Phi a + \mathrm{ad}_a^2 L$ is an abelian inner ideal of L (see [2, Lemma 1.8]).

The following identities (see [2, Lemma 1.7]) will be used in what follows. Let L be a 3-torsion free Lie algebra and let $a, x\in L$ with a being a Jordan element. Then

(JE1) $A^2XA = AXA^2$

(JE2) $\mathrm{ad}^2_{A^2x} = A^2X^2A^2$

where according to our notational convention A denotes the adjoint map ad_a and similarly X stands for ad_x.

Suppose that 2 and 3 are invertible in Φ. Let L be a Lie Φ-algebra and let $a\in L$ be a Jordan element. In the Φ-module L a new product is defined by $x\bullet y := [[x, a], y]$, $x, y\in L$. Denote by $L^{(a)}$ the resulting algebra. Then $\mathrm{Ker}(a) := \{x\in L : \mathrm{ad}_a^2 x = 0\}$ is an ideal of $L^{(a)}$ and the quotient algebra $L_a := L^{(a)}/\mathrm{Ker}(a)$ is a Jordan algebra (with product $\bar{x}\bullet\bar{y} := \overline{[[x, a], y]}$, where $\bar{x}$ stands for the coset of x for any $x\in L$), called the *Jordan algebra of L at a* (see [10, Theorem 2.4]).

Definition 2.1. If a is *von Neumann regular*, i.e., if a is Jordan and $a\in \mathrm{ad}_a^2 L$, then (1) L_a is unital with $\bar{b}$ as unit element for any $b\in L$ such that $a = [[a, b], a]$. In this case, (2) L_a is isomorphic to the Jordan algebra $J(a, b)$ defined in the Φ-module $\mathrm{ad}_a^2 L$ by the product $x\bullet y := [[x, b], y]]$ for all $x, y\in \mathrm{ad}_a^2 L$. We provide here a proof of these results under conditions less restrictive than those required in [10].

Proof. (1) $a = [[a, b], a]$ implies $A = \mathrm{ad}_{[[a,b],a]} = [[A, B], A] = 2ABA - A^2B - BA^2$. Multiplying both members of this equation on the left by A and using (JE1) we get $A^2 = 2A^2BA - ABA^2 = A^2BA$ (since $A^3 = 0$), which proves that L_a is unital with $\bar{b}$ as unit element.

(2) The map $\varphi : L_a \to J(a,b)$ defined by $\varphi(\bar{x}) := -A^2x$ is an algebra isomorphism. Clearly φ is a linear isomorphism, and since both algebras are commutative and $\frac{1}{2} \in \Phi$, it suffices to check that $\varphi(\bar{x})^2 = \varphi(\bar{x}^2)$:

$$\varphi(\bar{x})^2 = [[A^2x, b], A^2x] = -\mathrm{ad}^2_{A^2x}b = -A^2X^2A^2b = A^2X^2a = -A^2XAx = \varphi(\bar{x}^2),$$

where we have used (JE2), $A^2b = [a,[a,b]] = -ABa = -a$ and $XAx = -X^2a$. ∎

Involutions.

If R is a ring, an involution on R is an additive map $* : R \to R$ such that $*^2 = \mathrm{Id}_R$ and $(ab)^* = b^*a^*$ for all $a, b \in R$. If A is an algebra over a ring of scalars with involution $(\Phi, ^-)$, then an involution $*$ on A is an involution on the underlying ring of A which in addition satisfies $(\lambda a)^* = \bar{\lambda}a^*$ for every $\lambda \in \Phi$ and $a \in A$. If $^-$ is trivial (i.e., if it is the identity map) then $*$ is just an involution of A as a ring which is also a linear map.

Let A be an algebra with involution $*$ over $(\Phi, ^-)$. Denote by Γ the centroid of A as a ring. Denote by H (respectively by K) the set of the symmetric (respectively, skew-symmetric) elements of A, i.e., $H := \mathrm{Sym}(A,*) = \{x \in A : x = x^*\}$ and $K := \mathrm{Skew}(A,*) = \{x \in A : x = -x^*\}$. Then K is a subalgebra of the Lie algebra A^- restricted to $\mathrm{Sym}(\Phi, ^-)$ and, if $\frac{1}{2} \in \Gamma$, then H is a subalgebra of the Jordan algebra A^+ restricted to $\mathrm{Sym}(\Phi, ^-)$ (so it is a special Jordan algebra) and $A = H \oplus K$. Set $\kappa(x) := x - x^* \in K$ for every $x \in A$. Note that the mapping $x \mapsto \kappa(x)$ is linear and satisfies $\kappa(axa^*) = a\kappa(x)a^*$ for all $a, x \in A$. Note also that for $h \in H, k \in K$ we have

$$h \circ k = hk + kh = hk - (hk)^* = \kappa(hk) \in K,$$

a simple identity that will show up frequently.

If $\frac{1}{2} \in \Phi$ and M is a Φ-submodule of A which is $*$-*invariant*, i.e., such that $M^* = M$, then $\kappa(M) = \mathrm{Skew}(M,*)$, since if $k \in \mathrm{Skew}(M,*)$ then $k = \frac{1}{2}(k+k) = \frac{1}{2}(k - k^*) = \frac{1}{2}\kappa(k)$ and $\kappa(x) = x - x^* \in M \cap K = \mathrm{Skew}(M,*)$ for every $x \in M$. In particular $\kappa(A) = K$. If M is not $*$-invariant, then $\kappa(M) = \kappa(M^*)$ implies that $\kappa(M) = \kappa(M) + \kappa(M^*) = \kappa(M + M^*) = (M + M^*) \cap K$.

Let R be a ring with involution $*$. If $a \in R$ is von Neumann regular, i.e, if $a = axa$ for some $x \in R$, then by replacing x by $b := xax$ we obtain $a = aba$ and $b = bab$. If a is also symmetric and $\frac{1}{2} \in \Gamma$ then b can be chosen symmetric by replacing x by $\frac{1}{2}(x + x^*)$. The following lemma is a further step in the choice of b.

Lemma 2.2. *Let R be a ring and let $c \in R$ be a von Neumann regular element such that $c^2 = 0$. Then there exists $d \in R$ such that $c = cdc$, $d = dcd$ and $d^2 = 0$. Moreover, if R has involution, $\frac{1}{2} \in \Gamma$ and c is symmetric (skew-symmetric), then d can be chosen to be symmetric (respectively, skew-symmetric).*

Proof. Let c be a von Neumann regular element of R. By the argument above, there exists $b \in R$ such that $cbc = c$ and $b = bcb$. We claim that $d := b - b^2c$ satisfies $c = cdc$, $d = dcd$ and $d^2 = 0$. Indeed,

$$d^2 = (b-b^2c)(b-b^2c) = b^2 - b^3c - b(bcb) + b(bcb)bc = b^2 - b^3c - b^2 - b^3c = 0,$$
$$cdc = c(b-b^2c)c = cbc = c, \text{ and}$$
$$dcd = (b-b^2c)c(b-b^2c) = bc(b-b^2c) = bcb - (bcb)bc = b - b^2c = d.$$

Suppose now that c is symmetric. Since $\frac{1}{2} \in \Gamma$ we can take $b \in H$ such that $cbc = b$ and $b = bcb$. We claim that

$$d := b - \frac{1}{2}(cb^2 + b^2c) + \frac{1}{4}cb^3c$$

satisfies the required properties. It is clear that $d^* = d$. Moreover, we have:

$$d^2 = \left(b - \frac{1}{2}(cb^2 + b^2c) + \frac{1}{4}cb^3c\right)\left(b - \frac{1}{2}(cb^2 + b^2c) + \frac{1}{4}cb^3c\right) = b^2 - \frac{1}{2}(bcb)b$$
$$- \frac{1}{2}b^3c + \frac{1}{4}(bcb)b^2c - \frac{1}{2}cb^3 + \frac{1}{4}cb(bcb)b + \frac{1}{4}cb^4c - \frac{1}{8}cb(bcb)b^2c - \frac{1}{2}b(bcb)$$
$$+ \frac{1}{4}b(bcb)bc + \frac{1}{4}cb^2(bcb) - \frac{1}{8}cb^2(bcb)bc = b^2 - \frac{1}{2}b^2 - \frac{1}{2}b^3c + \frac{1}{4}b^3c - \frac{1}{2}cb^3$$
$$+ \frac{1}{4}cb^3 + \frac{1}{4}cb^4c - \frac{1}{8}cb^4c - \frac{1}{2}b^2 + \frac{1}{4}b^3c + \frac{1}{4}cb^3 - \frac{1}{8}cb^4c = 0,$$

$$cdc = c(b - \tfrac{1}{2}(cb^2 + b^2c))c = cbc = c, \text{ and}$$

$$dcd = \left(b - \frac{1}{2}(cb^2 + b^2c)\right)c\left(b - \frac{1}{2}(cb^2 + b^2c)\right)(b - \frac{1}{2}cb^2)c(b - \frac{1}{2}b^2c)$$
$$= bcb - \frac{1}{2}(bcb)bc - \frac{1}{2}cb(bcb) + \frac{1}{4}cb(bcb)bc = bcb - \frac{1}{2}b^2c - \frac{1}{2}cb^2 + \frac{1}{4}cb^3c = d.$$

If c is skew-symmetric, then the same d works taking $b \in K$. ■

Prime rings.

Let R be a prime ring. The *extended centroid* $\mathcal{C}$ of R (see ([1, Section 2.3]) is a field containing the centroid Γ, and the *central closure* $\mathcal{C}R$ of R is a prime associative algebra over the field $\mathcal{C}$. A prime ring R is *centrally closed* if it coincides with its central closure. The following lemma (see [4, Theorem A.7]) plays a fundamental role in our work.

Lemma 2.3 (Martindale). *Let R be a prime ring with extended centroid $\mathcal{C}$. Let $a_i, b_i \in R$ with $b_1 \neq 0$ be such that $\sum_{i=1}^n a_i x b_i = 0$ for every $x \in R$. Then $a_1 \in \sum_{i=2}^n \mathcal{C}a_i$.*

Let R be a centrally closed prime ring with involution $*$. Then $*$ naturally extends to an involution of the extended centroid $\mathcal{C}$ of R, also denoted by $*$, so that R is an algebra with involution over $(\mathcal{C}, *)$. If $*$ acts trivially on $\mathcal{C}$ then it is called *of the first kind*. In this case K can be regarded as a Lie algebra over $\mathcal{C}$.

3. Clifford elements of a prime ring with involution

Throughout this section R will denote a centrally closed prime ring of characteristic not 2 or 3 which is endowed with an involution $*$. Then K, the set of skew-symmetric elements of R, is a Lie algebra over the field $\mathrm{Sym}(\mathcal{C}, *)$. It follows from [5, Propostion 6.2] (here characteristic greater than 5 is required) that if K is not abelian and $*$ is of the first kind, then for any Jordan element $a \in K$ we have $a^3 = 0$. This leads us to the following:

Definition 3.1. By a *Clifford element* of R we mean an element $c \in K$ such that $c^3 = 0$, $c^2 \neq 0$ and c is a Jordan element of the Lie algebra K: $\mathrm{ad}_c^3 k = c^3k - 3c^2kc + 3ckc^2 - kc^3 = 0$ for all $k \in K$.

The square of a Clifford element

Proposition 3.2. *Let $c \in K$ be a Clifford element of R. Then:*

1. $c^2kc = ckc^2$ *for all* $k \in K$.
2. $c^2Kc^2 = 0$.
3. $(c^2xc^2)^* = c^2x^*c^2 = c^2xc^2$ *for all* $x \in R$.
4. $c^2Rc^2 = \mathcal{C}c^2$.
5. *The involution $*$ is of the first kind.*
6. *R has nonzero socle with division ring isomorphic to $\mathcal{C}$ and $*$ is of orthogonal type.*

Proof. (1) Since c is a Jordan element of K, for every $k \in K$ we have $0 = ad_c^3 k = c^3k - 3c^2kc + 3ckc^2 - kc^3 = -3(c^2kc - ckc^2)$. Since $\mathrm{char}(R) \neq 3$ this implies that $ckc^2 = c^2kc$.

(2) By (1), $c^2kc^2 = c(ckc^2) = c(c^2kc) = c^3kc = 0$.

(3) Since $x - x^* \in K$ we have $c^2(x - x^*)c^2 = 0$ and hence $c^2xc^2 = c^2x^*c^2 = (c^2xc^2)^*$.

(4) Let $x, y \in R$. Since c^2 is symmetric it follows from (3) that

$$c^2xc^2yc^2 = c^2(xc^2y)^*c^2 = (c^2y^*c^2)x^*c^2 = c^2y(c^2x^*c^2) = c^2yc^2xc^2.$$

Thus, fixed x, for every $y \in R$ we get $(c^2xc^2)y(c^2) - (c^2)y(c^2xc^2) = 0$, with $c^2 \neq 0$. Then, by Martindale's Lemma (2.3), for each $x \in R$ there is a $\lambda_x \in \mathcal{C}$ such that $c^2xc^2 = \lambda_x c^2$. Since $c^2 \neq 0$ and R is prime, $c^2Rc^2 \neq 0$ and hence $c^2Rc^2 = \mathcal{C}c^2$, since $\mathcal{C}$ is a field.

(5) By (4), given $\alpha \in \mathcal{C}$ there exists $a \in R$ such that $\alpha c^2 = c^2ac^2$. Then, by (3), $\alpha^* c^2 = c^2a^*c^2 = c^2ac^2 = \alpha c^2$, so $\alpha^* = \alpha$, proving that $*$ is of the first kind.

(6) By (4), $c^2 = c^2ac^2$ for some $a \in R$ and hence $c^2R = eR$, where $e = c^2a$ is an idempotent of R. Then $eRe = c^2Rc^2a = \mathcal{C}c^2a = \mathcal{C}e$, which proves ([1,

Proposition 4.3.3]) that eR is a minimal right ideal of R, so R has nonzero socle with associated division ring isomorphic to the field $\mathcal{C}$ ([1, Theorem 4.3.7]). Now it follows from Kaplansky's Theorem ([1, Theorem 4.6.8]) that the involution $*$ of R is either of transpose type or of symplectic type; but the latter cannot occur because c^2 is a symmetric rank-one element, so $*$ is of transpose type. ■

Let $c \in K$ be a Clifford element of R. Since c^2 is a symmetric zero-square element which is also von Neumann regular (see 3.2(4)), we have by 2.2 that there exists $d \in R$ such that

$$d^* = d,\ d^2 = 0,\ c^2dc^2 = c^2 \text{ and } d = dc^2d.$$

Such an element d will be called a *twin* of c^2. Then $e := dc^2$ is a $*$*-orthogonal idempotent*, i.e., $e^2 = e$ and $ee^* = e^*e = 0$.

Proposition 3.3. *Let $c \in K$ be a Clifford element of R, let d be a twin of c^2 and put $e := dc^2$. Then:*

1. $dKd = 0$.
2. $dRd = \mathcal{C}d$.
3. $eRe = \mathcal{C}e$, $e^*Re = \mathcal{C}c^2$, $eRe^* = \mathcal{C}d$ *and* $eKe^* = e^*Ke = 0$.
4. $ec = ce^* = 0$, $e^*c^2 = c^2e = c^2$ *and* $de^* = ed = d$.
5. $[K, K] \neq 0$.
6. $e + e^* \neq 1$ *in the unital hull* $\hat{R} = \mathcal{C}1 + R$ *of* R.

Proof. Note that, by the proof of 3.2(4), $c^2Mc^2 = \mathcal{C}c^2$ for any $\mathcal{C}$-subspace M of R such that $c^2Mc^2 \neq 0$, a fact that will be used in what follows without further mention.

(1) $dKd = dc^2(dKd)c^2d = 0$, where we have used 3.2(2) and the fact that dkd is skew-symmetric for every $k \in K$.

(2) $dRd = (dc^2d)R(dc^2d) = dc^2(dRd)c^2d = d\mathcal{C}c^2d = \mathcal{C}dc^2d = \mathcal{C}d$, since $dc^2d = d$ and $c^2dc^2 = c^2$ imply that $c^2(dRd)c^2 \neq 0$.

(3) $eRe = dc^2(Rd)c^2 = d\mathcal{C}c^2 = \mathcal{C}e$, since $c^2 = c^2(dc^2d)c^2 \in c^2(Rd)c^2$ and therefore the latter is not zero. In a similar way it is proved that $e^*Re = \mathcal{C}c^2$ and $eRe^* = \mathcal{C}d$. Now $eKe^* = d(c^2Kc^2)d = 0$ by 3.2(2), and $e^*Ke = 0$ is obtained in a similar way.

(4) The identities of this item follow straightforwardly from the very definition of e.

(5) By (4), $[c, e - e^*] = ce + e^*c = cdc^2 + c^2dc \neq 0$. Otherwise $cdc^2 = -c^2dc$ would lead to the contradiction $c^2 = c^2dc^2 = -c^3dc = 0$. Since $[c, e - e^*] \in [K, K]$, $[K, K] \neq 0$.

(6) It follows from (3) and (4) that $(e + e^*)c(e + e^*) = 0$, so $e + e^* \neq 1$. ■

Remark 3.4. Twin d of c^2 are not unique. In fact, for any twin d of c and any $\lambda \in \mathcal{C}$, $\exp(\lambda \mathrm{ad}_c)d$ is a twin of c^2.

As we have seen in the proposition above, any Clifford element c of R gives rise to two orthogonal elements e and e^*, associated to a twin d of c^2. Moreover, the idempotent $e+e^*$ is not complete (see 3.3(6)), i.e., the symmetric idempotent $g := 1-e-e^*$ of the unital hull $\hat{R} = \mathcal{C}1 + R$ of R is not zero. We prove next that the complete system $\{e, e^*, g\}$ induces a 3-grading in the Lie algebra K.

Proposition 3.5. *Let $c \in K$ be a Clifford element of R, $e := dc^2$ and $g := 1-e-e^*$, where d is a twin of c^2. Then $K = K_{-1} \oplus K_0 \oplus K_1$ is a 3-grading of K, with $K_{-1} = \kappa((1-e)Ke) = \kappa((1-e)Re) = \kappa(gRe)$, $K_0 = \kappa(eRe) \oplus gKg$ and $K_1 = \kappa(eK(1-e)) = \kappa(eR(1-e)) = \kappa(eRg)$.*

Proof. Consider the complete system $\{e_0 := e^*, e_1 := g, e_2 := e\}$ of orthogonal idempotents of $\hat{R}$ and put $R_i := \bigoplus_{m-n=i} e_m R e_n$, $-2 \le i \le 2$. Then (see [15, p.174] for instance), $R = \bigoplus_{-2\le i\le 2} R_i$ is an (associative) 5-grading of R. Explicitly,

$$R = e^*Re \oplus (e^*Rg \oplus gRe) \oplus (e^*Re^* \oplus gRg \oplus eRe) \oplus (gRe^* \oplus eRg) \oplus eRe^*.$$

Since all the components R_i are $*$-invariant subspaces, $K = \bigoplus_{-2\le i\le 2} K_i$, where $K_i := R_i \cap K = \mathrm{Skew}(R_i, *)$ for each index i and $[K_i, K_j] \subseteq [R_i, R_j] \cap [K, K] \subseteq R_{i+j} \cap K = K_{i+j}$. Thus $K = \bigoplus_{-2\le i\le 2} K_i$ is (a priori) a 5-grading of the Lie algebra K. But $K_{-2} = \kappa(e^*Re) = e^*\kappa(R)e = e^*Ke = 0$ by 3.3(3) and similarly $K_2 = e^*Ke = 0$. Moreover, the i-th homogeneous component k_i of any $k \in K$ coincides with $\bigoplus_{m-n=i} \kappa(e_m k e_n)$, so $k \in K_{-1}$ if and only if

$$\begin{aligned} gke + e^*kg &= (1-e-e^*)ke + e^*k(1-e-e^*) = (1-e)ke + e^*k(1-e^*) = \\ &(1-e)ke - ((1-e)ke)^* = \kappa((1-e)ke) \end{aligned}$$

since $e^*Ke = 0$ by 3.3(3), which proves that $K_{-1} = \kappa(gRe) = \kappa((1-e)Ke)$. Similarly, $K_1 = \kappa(eRg) = \kappa(eK(1-e))$. Therefore

$$K = \kappa((1-e)Ke) \oplus (\kappa(eRe) \oplus gKg) \oplus \kappa(eK(1-e))$$

is a 3-grading of K. Now, for any $x \in R$,

$$\kappa(gxe) = \kappa((1-e)xe) - \kappa(e^*xe) = \kappa((1-e)xe) - e^*\kappa(x)e = \kappa((1-e)xe)$$

since $e^*\kappa(x)e \in e^*Ke = 0$, which proves that $K_{-1} = \kappa((1-e)Re)$. Similarly we obtain that $K_1 = \kappa(eR(1-e))$. ■

Although the 3-grading of K has been defined by choosing a twin d of c^2, it will be seen now that the component K_{-1} only depends on the Clifford element c.

Proposition 3.6. *Let $c \in K$ be a Clifford element of R, let $e := dc^2$ where d is a a twin of c^2 and let $B := \kappa((1-e)Ke)$. Then B is an abelian inner ideal of K and we have:*

1. *If $b \in B$ then $eb = 0$ and $b = e^*b + be = \kappa((1-e)be)$.*
2. *$B = c^2 \circ K$.*
3. *$c = e^*c + ce = c^2dc + cdc^2$.*
4. *$c \in B$.*
5. *$cKc = \mathcal{C}c$.*

Proof. By Proposition 3.5, B is an extreme of a finite $\mathbb{Z}$-grading of K and hence an abelian inner ideal of K.

(1) Let $b = (1-e)ke + e^*k(1-e^*) \in B$. Then $eb = e((1-e)ke + e^*k(1-e^*)) = 0$ and $e^*b = e^*k(1-e^*)$, since $e^*e = 0$ and $e^*Ke = 0$. We also have that $be^* = 0$ and $be = (1-e)ke$. Hence $b = e^*b + be = e^*b(1-e^*) + (1-e)be = \kappa((1-e)be)$.

(2) $c^2 \circ k = \kappa(kc^2) = \kappa(ke^*c^2 - (eke^*)c^2) = \kappa((1-e)k(e^*c^2)) = \kappa((1-e)k(c^2e)) \in \kappa((1-e)Re) = \kappa((1-e)Ke) = B$ by 3.5. Conversely, let $b \in B$. Then

$$b = e^*b + be = (c^2d)b + b(dc^2) = c^2(d \circ b) + (d \circ b)c^2 = c^2 \circ (d \circ b) \in c^2 \circ K,$$

since $e^* = c^2d$, $c^2 = c^2e$ and $c^2b = (c^2e)b = c^2(eb) = 0$.

(3) Set $z := c - c^2dc - cdc^2$. We must prove that $z = 0$. For any $k \in K$ we have

$$c^2kz = c^2kc - (c^2kc^2)dc - (c^2kc)dc^2 = ckc^2 - ck(c^2dc^2) = ckc^2 - ckc^2 = 0$$

since $c^2kc = ckc^2$ and $c^2kc^2 = 0$ by 3.2, and d is a twin of c^2 (see 3). We also have $zkc^2 = (c^2kz)^* = 0$, and hence $c^2xz = c^2x^*z$ and $zxc^2 = zx^*c^2$ for every $x \in R$. Let $x, y \in R$. Then $c^2\kappa(xzy)c^2 = 0$ since $c^2Kc^2 = 0$. Thus $0 = c^2(xzy + y^*zx^*)c^2 = c^2xzyc^2 + c^2y^*zx^*c^2 = c^2xzyc^2 + c^2yzxc^2 = (c^2xz)y(c^2) + (c^2)y(zxc^2) = 0$, with $c^2 \neq 0$. By Martindale's Lemma (2.3), for every $x \in R$ there is $\lambda_x \in \mathcal{C}$ such that $c^2xz = \lambda_x c^2$. But

$$z(1-e) = (c - e^*c - ce)(1-e) = c - ce - e^*c + e^*ce - ce + ce = c - ce - e^*c = z$$

since $e^*ce \in e^*Ke = 0$. Hence $c^2xz = c^2xz(1-e) = \lambda_x c^2(1-e) = 0$, so $c^2Rz = 0$. Since R is prime and $c^2 \neq 0$ this implies that $z = 0$. Thus $c = e^*c + ce = c^2dc + cdc^2$ as required.

(4) By (3), $c = c^2dc + cdc^2 = c^2(dc + cd) + (dc + cd)c^2 \in c^2 \circ K = B$ by (2).

(5) Note that $cd + dc = \kappa(cd) \in K$ and $c(cd + dc)c = c^2dc + cdc^2 = c$ by (3). Hence $\mathcal{C}c \subseteq cKc$. Conversely, for any $k \in K$ we have $ckc = (e^*c + ce)k(e^*c + ce) = e^*cke^*c + cekce$, since $eKe^* = 0$ by 3.3(3) and $ckc \in K$. Now, again by 3.3(3), $e(kc)e = \lambda e$ for some $\lambda \in \mathcal{C}$, and hence $e^*(ck)e^* = (ekce)^* = (\lambda e)^* = \lambda e^*$, since the involution $*$ is of the first kind by 3.2(5). Then $ckc = \lambda e^*c + \lambda ce = \lambda c$, which completes the proof. ∎

The square root of d

Given a Clifford element c of R and a twin d of c^2, we put $\sqrt{d} := cd + dc$. As will be seen now, the square-root notation is absolutely justified.

Proposition 3.7. *Let $c \in K$ be a Clifford element of R and let d be a twin of c^2. Then:*

1. *$\sqrt{d} \in K_1$ in the 3-grading of 3.5. In particular $\sqrt{d}$ is a Jordan element.*
2. *$(\sqrt{d})^2 = d$.*
3. *$(\sqrt{d})^3 = 0$.*
4. *$\sqrt{d}K\sqrt{d} = \mathcal{C}\sqrt{d}$.*
5. *$\sqrt{d}c\sqrt{d} = \sqrt{d}$.*
6. *$c\sqrt{d}c = c$.*
7. *$c^2 \circ \sqrt{d} = c$.*
8. *$d \circ c = \sqrt{d}$.*
9. *$[[c, \sqrt{d}], c] = c$.*
10. *$[[\sqrt{d}, c], \sqrt{d}] = \sqrt{d}$.*
11. *$[[c, \sqrt{d}], b] = b$ for every $b \in B$.*

Proof. (1) Since $c \in K$ and $d \in H$, $\sqrt{d} = cd + dc \in K$. We have

$$\begin{aligned}\kappa(e\sqrt{d}(1-e)) &= e(cd+dc)(1-e) + (1-e^*)(dc+cd)e^* \\ &= edc(1-e) + (1-e^*)cde^* = edc - edce + cde^* - e^*cde^* \\ &= (dc^2d)c - e(dcd)c^2 + c(dc^2d) - c^2(dcd)e^* = dc + cd = \sqrt{d}\end{aligned}$$

since $ec = 0$, $e = dc^2$, $dc^2d = d$ and $dcd \in dKd = 0$. We have thus proved (see 3.5) that $\sqrt{d} \in \kappa(eK(1-e)) = K_1$. Now since K_1 is an abelian inner ideal (because it is the extreme of a finite grading), $\sqrt{d}$ is a Jordan element of K.

(2) $(\sqrt{d})^2 = (cd+dc)(cd+dc) = c(dcd) + cd^2c + dc^2d + (dcd)c = dc^2d = d$.

(3) $(\sqrt{d})^3 = (\sqrt{d})^2\sqrt{d} = d(cd+dc) = dcd + d^2c = 0$.

(4) If follows from (1), (2) and (3) that $\sqrt{d}$ is a Clifford element of R. Hence, by 3.6(5), $\sqrt{d}K\sqrt{d} = \mathcal{C}\sqrt{d}$.

(5) $\sqrt{d}c\sqrt{d} = (cd+dc)c(cd+dc) = c(dc^2d) + c(dcd)c + dc^3d + (dc^2d)c = cd + dc = \sqrt{d}$.

(6) $c\sqrt{d}c = c(cd+dc)c = c^2dc + cdc^2 = c$ by 3.6(3).

(7) $c^2 \circ \sqrt{d} = c^2(cd+dc) + (cd+dc)c^2 = c^2dc + cdc^2 = c$.

(8) $d \circ c = dc + cd = \sqrt{d}$.

(9) $[[c, \sqrt{d}], c] = 2c\sqrt{d}c - c^2 \circ \sqrt{d} = 2c - c = c$ by (6) and (7).

(10) $[[\sqrt{d}, c], \sqrt{d}] = 2\sqrt{d}c\sqrt{d} - (\sqrt{d})^2 \circ c = 2\sqrt{d} - \sqrt{d} = \sqrt{d}$ by (2), (5) and (8).

(11) $[[c, \sqrt{d}], b] = [[c, cd+dc], b] = [c^2d - dc^2, b] = [e^* - e, b] = e^*b + be = b$ by 3.6(1). ■

4. Jordan algebra at a Clifford element

As in the previous section, R will denote a centrally closed prime ring of characteristic not 2 or 3 with involution $*$. We prove here that if c is a Clifford element of R, then the abelian inner ideal $c^2 \circ K = \kappa((1-e)Ke)$ (see 3.6) can be endowed with a Jordan algebra structure of Clifford type (see 2), which happens to be isomorphic to K_c. We begin by defining a linear form and a symmetric bilinear form on the $\mathcal{C}$-vector space K (recall that $*$ is of the first kind by 3.2(5)).

Remark 4.1. By 3.6(5) there exists a linear map $\mathrm{tr} : K \to \mathcal{C}$, called the *trace*, such that

$$\mathrm{tr}(k)c = ckc$$

for every $k \in K$. Note that

1. $\mathrm{tr}(\sqrt{d}) = 1$ since $c\sqrt{d}c = c$ by 3.7(6), and hence
2. $K = \mathcal{C}\sqrt{d} \oplus \mathrm{Ker}(\mathrm{tr})$.

Remark 4.2. Since $c^2Rc^2 = \mathcal{C}c^2$ (3.2(4)) with $c^2k_1k_2c^2 = c^2k_2k_1c^2$ for all $k_1, k_2 \in K$ (3.2(2)), we have a symmetric bilinear form $\langle\, ,\rangle : K \times K \to \mathcal{C}$ defined by

$$\langle k_1, k_2\rangle c^2 = c^2k_1k_2c^2$$

for all $k_1, k_2 \in K$.

Remark 4.3. The trace can be realized from the bilinear form and vice versa. Let $k, k' \in K$:

1. $\langle\sqrt{d}, k\rangle c^2 = c^2\sqrt{d}kc^2 = c^2(cd + dc)kc^2 = c^3dkc^2 + c^2dckc^2 = c^2dckc^2 = c^2d(ckc)c = \mathrm{tr}(k)c^2dc^2 = \mathrm{tr}(k)c^2$, since $c^3 = 0$ and $c^2dc^2 = c^2$. Thus $\mathrm{tr}(k) = \langle k, \sqrt{d}\rangle$.
2. $\mathrm{tr}(\kappa(ckk'))c^2 = (c\kappa(ckk')c)c = c^2kk'c^2 + ck'kc^3 = c^2kk'c^2 = \langle k, k'\rangle c^2$. Thus $\langle k, k'\rangle = \mathrm{tr}(\kappa(ckk'))$.

Proposition 4.4. *Let $c \in K$ be a Clifford element of R and $B := c^2 \circ K$. Then:*

1. *$B = \mathcal{C}c \oplus X$, where $X := \{c^2 \circ k : \; k \in \mathrm{Ker}(\mathrm{tr})\}$.*
2. *$B = \mathrm{ad}_c^2K$.*

Proof. (1) By 4.1(2), $K = \mathrm{Ker}(\mathrm{tr}) \oplus \mathcal{C}\sqrt{d}$. Hence

$$B = c^2 \circ K = c^2 \circ \mathrm{Ker}(\mathrm{tr}) + \mathcal{C}c^2 \circ \sqrt{d} = c^2 \circ \mathrm{Ker}(\mathrm{tr}) + \mathcal{C}c$$

since $c^2 \circ \sqrt{d} = c$ by 3.7(7). But this sum is direct since $c^2 \circ k_0 = \alpha c$, with $\mathrm{tr}(k_0) = 0$ and $\alpha \in \mathcal{C}$, implies $\alpha c^2 = c(c^2k_0 + k_0c^2) = (ck_0c)c = \mathrm{tr}(k_0)c = 0$, and hence $\alpha = 0$ since $c^2 \neq 0$ by the very definition of Clifford element.

(2) For any $k \in K$ we have

$$ad_c^2k = c^2k - 2ckc + kc^2 = c^2 \circ k - 2\mathrm{tr}(k)c = c^2 \circ k - 2\mathrm{tr}(k)(c^2 \circ \sqrt{d}) = c^2 \circ (k - 2\mathrm{tr}(k)\sqrt{d}) \in B$$

since $c = c^2 \circ \sqrt{d}$ by 3.7(7). Conversely, let $c^2 \circ k_0 + \alpha c \in B$, with $k_0 \in \mathrm{Ker}(\mathrm{tr})$ and $\alpha \in \mathcal{C}$. Then $c^2 \circ k_0 + \alpha c = \mathrm{ad}_c^2 k_0 - \alpha \mathrm{ad}_c^2 \sqrt{d} = \mathrm{ad}_c^2(k_0 - \alpha\sqrt{d})$ since $ck_0c = 0$ and $\mathrm{ad}_c^2\sqrt{d} = -c$ by 3.7(9). ■

Lemma 4.5. *The symmetric $\mathcal{C}$-bilinear form defined on X by*

$$\langle c^2 \circ k, c^2 \circ k' \rangle_0 := -\langle k, k' \rangle$$

is well defined.

Proof. Suppose that $c^2 \circ k_1 = c^2 \circ k_1'$. By multiplying the two members of this equality on the right by $k_2 c^2$ we obtain $c^2 k_1 k_2 c^2 = c^2 k_1' k_2 c^2$ since $c^2 K c^2 = 0$. This proves that $\langle\, , \rangle_0$ is well defined. ■

Remark 4.6. Consider the 3-grading $K = K_{-1} \oplus K_0 \oplus K_1$ due to $e := dc^2$ (see 3.5), with $K_{-1} = B$, $K_0 = \kappa(eKe) \oplus gKg$ and $K_1 = \kappa(eKg)$.

1. Since the pair $(d, \sqrt{d})$ plays a role symmetric to that played by (c^2, c), we also have that $K_1 = d \circ K = \{d \circ k : \ k \in K, \sqrt{d}k\sqrt{d} = 0\} \oplus \mathcal{C}c = ad^2_{\sqrt{d}} K$.

2. X can be zero in 4.4 and therefore we can have $B = \mathcal{C}c$. Let $V := H \oplus \mathbb{F}z$ be the orthogonal sum of a hyperbolic plane $H = \mathbb{F}x \oplus \mathbb{F}y$ and the line $\mathbb{F}z = H^\perp$ with z being an anisotropic vector, and let R be the simple ring $\mathrm{End}(V)$ with the adjoint as involution. For any $u, v \in V$ let $u \otimes v$ be the linear map defined by $w(u \otimes v) = \langle w, u \rangle v$ for all $w \in V$. Then $(u \otimes v)^* = v \otimes u$ and hence $c := x \otimes z - z \otimes x$ lies in the Lie algebra $K = \mathrm{Skew}(R, *)$. It is easy to check that c is a Clifford element of R such that $\mathrm{ad}_c^2 K = \mathbb{F}c$.

Theorem 4.7. *Let R be a centrally closed ring with involution of characteristic not 2 or 3 and let $c \in K$ be a Clifford element of R. Then K_c is a Clifford Jordan algebra.*

Proof. Since $c = [[c, \sqrt{d}], c]$ (3.7(9)) we have by 2.1 that $K_c \cong J(c, \sqrt{d})$, the Jordan algebra defined on the $\mathcal{C}$-vector space $\mathrm{ad}_c^2 K = c^2 \circ K = \mathcal{C}c \oplus X$ (see 4.4) by the product

$$(\alpha_1 c + c^2 \circ k_1) \bullet (\alpha_2 c + c^2 \circ k_2) = [[\alpha_1 c + c^2 \circ k_1, \sqrt{d}], \alpha_2 c + c^2 \circ k_2]$$

for all $\alpha_1, \alpha_2 \in \mathcal{C}$ and $k_1, k_2 \in K$ such that $ck_1c = ck_2c = 0$. Endow the $\mathcal{C}$-vector space X with the symmetric bilinear form $\langle\, , \rangle_0$ defined in 4.5 and consider the Clifford Jordan algebra $\mathcal{C} \oplus X$ defined by $\langle\, , \rangle_0$ (see 2). We claim that the linear isomorphism $(\alpha c + c^2 \circ k) \mapsto (\alpha, c^2 \circ k)$ of $J(c, \sqrt{d})$ onto $\mathcal{C} \oplus X$ is actually an isomorphism of Jordan algebras. Since $\frac{1}{2} \in \Phi$, it suffices to check the identity

$$[[\alpha c + c^2 \circ k, \sqrt{d}], \alpha c + c^2 \circ k] = \alpha^2 c + \langle c^2 \circ k, c^2 \circ k \rangle_0 + 2\alpha(c^2 \circ k).$$

The bilinearity of the Lie product reduces the check to three products: (i) scalar by scalar, (ii) scalar by vector, and (iii) vector by vector.

(i) $[[\alpha c, \sqrt{d}], \alpha c] = \alpha^2[[c, \sqrt{d}], c] = \alpha^2 c$ by 3.7(9).

(ii) $[[\alpha c, \sqrt{d}], c^2 \circ k] = \alpha[[c, cd + dc], c^2k + kc^2] = \alpha[c^2d - dc^2, c^2k + kc^2] = \alpha(c^2 \circ k)$, where we have used $c^2dc^2 = c^2$, $c^4 = 0$ and $c^2kc^2 = c^2(dk + kd)c^2 = 0$, the latter because $c^2Kc^2 = 0$ and $(dk + kd)^* = -(kd + dk)$, since $d^* = d$ and $k^* = -k$.

(iii) $[[c^2 \circ k, \sqrt{d}], c^2 \circ k] = 2(c^2 \circ k)\sqrt{d}(c^2 \circ k) - (c^2 \circ k)^2 \circ \sqrt{d}$,
with

$$(c^2 \circ k)\sqrt{d}(c^2 \circ k) = (c^2k + kc^2)(cd + dc)(c^2k + kc^2) = (c^2kdc + kc^2dc)(c^2k + kc^2) = 0,$$

since $c^3 = 0$ and $ckc = 0$ ($\mathrm{tr}(k) = 0$), and

$$\begin{aligned}(c^2 \circ k)^2 \circ \sqrt{d} = c^2k^2c^2(cd + dc) + (cd + dc)c^2k^2c^2 = \\ c^2k^2c^2dc + cdc^2k^2c^2 = \langle k, k\rangle(c^2dc + cdc^2) = \langle k, k\rangle c\end{aligned}$$

since $c = c^2dc + cdc^2$ by 3.6(1). Therefore $(c^2 \circ k) \bullet (c^2 \circ k) = -\langle k, k\rangle c = \langle c^2 \circ k, c^2 \circ k\rangle_0 c$, which completes the proof. ■

Remark 4.8. Since $\sqrt{d}$ is a Clifford element of R (see 3.7), the theorem above also proves that $K_{\sqrt{d}}$ is a Clifford Jordan algebra. In fact, $K_{\sqrt{d}} \cong K_c$.

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