


Article

Error-Correcting Codes on Projective Bundles over Deligne–Lusztig Varieties

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Abstract: The aim of this article is to give lower bounds on the parameters of algebraic geometric error-correcting codes constructed from projective bundles over Deligne–Lusztig surfaces. The methods based on an intensive use of the intersection theory allow us to extend the codes previously constructed from higher-dimensional varieties, as well as those coming from curves. General bounds are obtained for the case of projective bundles of rank 2 over standard Deligne–Lusztig surfaces, and some explicit examples coming from surfaces of type A_2 and 2A_4 are given.

Keywords: error-correcting codes; Deligne–Lusztig varieties; projective bundles; intersection theory

MSC: 94B27; 14G50; 14C17; 14L35



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1. Introduction

The theory of algebraic geometry codes arose in 1970, when V. Goppa discovered in [1] the relation between the theory of error-correcting codes and the evaluation on algebraic curves. They exhibited important properties that made researchers deepen their study. On the one hand, their good encoding–decoding algorithms led B. McEliece [2] to consider them for a public-key cryptosystem, which is considered by NIST as an alternative for the post-quantum era. On the other hand, M.A. Tsfasman et al. [3] were able to show that Goppa codes can be used to give examples of codes that go beyond the Gilbert–Varshamov bound. Since then, the interest in algebraic geometry has significantly increased. We can cite [4–9] as a few of the many existing examples of study over Hermitian, Castle, Suzuki or G-H curves.

But the definition of algebraic geometry codes can go beyond. Tsfasman and Vlăduț, in [10], suggested that higher-dimensional varieties can be used to construct these type of codes, although the number of works in this sense does not equal that of the curves, probably due to the difficulty of finding higher-dimensional varieties X , spaces of functions L and sets of rational points \mathcal{P} , that can yield codes in the sense of Definition 2, which could be interesting due to their compelling properties concerning their weight distributions, minimum distance or fast encoding–decoding algorithms. The two-dimensional case has been relatively studied, like in the case of rational, hermitian or cubic surfaces, (cf. [11–13]).

However, this is not the case in general for higher-dimensional varieties. The survey by J. Little [14] offers a rather complete vision on the study of algebraic geometry codes defined over varieties in general. Among the referred papers, we can find the work by S. Hansen [15], wherein the author makes an extensive use of the intersection theory to develop their study. In this paper, some of the provided examples concern Deligne–Lusztig varieties, whose importance in algebraic geometry comes from the fact that they are directly linked to finite groups of Lie type. Moreover, these varieties are characterized by their large number of rational points, which allows for the definition of algebraic geometry codes, as we will see. In his paper, S. Hansen applies general methods to obtain lower bounds

on the parameters of algebraic geometric error-correcting codes defined from varieties of greater dimension, as is the case of Deligne–Lusztig surfaces. Our aim in this paper is to go one step further [15] and study this kind of codes defined on projective bundles over Deligne–Lusztig surfaces. In order to obtain lower bounds on the associated parameters, we make an intensive use of the intersection theory and take advantage of the fact that, for some standard Deligne–Lusztig surfaces, all their rational points are distributed equally on the disjoint rational curves, constituting the irreducible components of a divisor D_i . This allows us to give general bounds for the case of algebraic geometric error-correcting codes on projective bundles of rank 2 as well as some explicit examples coming from surfaces of type A_2 and 2A_4 .

2. Deligne–Lusztig Varieties

In this section, we shall define Deligne–Lusztig varieties and their compactification as well as study some of their main properties.

Let (G, F) be a connected reductive algebraic group over an algebraically closed field k of positive characteristic p , equipped with an \mathbb{F}_q -structure coming from a Frobenius morphism $F : G \rightarrow G$. Let $L : G \rightarrow G$ be the corresponding Lang map taking an element $g \in G$ to $g^{-1}F(g)$. By the Lang–Steinberg Theorem (see Theorem 4.4.17 in [16]), this morphism of varieties is surjective with finite fibers. From this result, it follows that, by conjugacy of Borel subgroups, there exists an F -stable Borel subgroup B . Let $\pi : G \rightarrow G/B := X$ denote the quotient. There are then (with a slight abuse of notation) natural endomorphisms $F : W \rightarrow W$ and $F : X \rightarrow X$ of the Weyl group of G and the variety X of Borel subgroups of G . Let W be generated by the simple reflections s_1, \dots, s_n , and let $l(\cdot)$ be the length function with respect to these generators.

Let us now recall from Definition 1 in [17] the following definition of Deligne–Lusztig variety.

Definition 1. Fix an element w in the Weyl group W , and let $w = s_{i_1} \cdot \dots \cdot s_{i_r}$ be a reduced expression of w . Call w a Coxeter element if there in this expression occurs exactly one s_i from each of the orbits of F on $\{s_1, \dots, s_n\}$. Denote by δ the order of F on this set. Then, the Deligne–Lusztig variety $X(w)$ is defined as the image of $L^{-1}(B\dot{w}B)$ in G/B . That is,

$$X(w) = \pi(L^{-1}(B\dot{w}B)). \tag{1}$$

Next, we will follow the notation and definitions given in [17]. Define the closed subvariety of X^{r+1}

$$\overline{X}(s_{i_1}, \dots, s_{i_r}) = \left\{ (g_0B, \dots, g_rB) \in X^{r+1} : g_k^{-1}g_{k+1} \in B \cup Bs_{i_{k+1}}B \text{ for } 0 \leq k < r, g_r^{-1}F(g_0) \in B \right\} \tag{2}$$

In those cases wherein there is a unique product $s_{i_1} \cdot \dots \cdot s_{i_r}$ such that $s_{i_1} \cdot \dots \cdot s_{i_r} = w$, we shall write $\overline{X}(w)$ for the variety $\overline{X}(s_{i_1}, \dots, s_{i_r})$. For any subset $\{s_{j_1}, \dots, s_{j_m}\} \subset \{s_{i_1}, \dots, s_{i_r}\}$, $\overline{X}(s_{j_1}, \dots, s_{j_m})$ defines in a natural way a closed subvariety of $\overline{X}(s_{i_1}, \dots, s_{i_r})$. In particular, there are divisors

$$D_j = \overline{X}(s_{i_1}, \dots, \hat{s}_{i_j}, \dots, s_{i_r}); \quad j = 1, \dots, r. \tag{3}$$

Example 1. Let us consider the Deligne–Lusztig surface $\overline{X}(w)$ of type A_2 . In this particular case, the connected reductive algebraic group is $G = GL(3, k)$ and the Frobenius map is given by

$$F : g \longrightarrow g^q. \tag{4}$$

Moreover, the Weyl group W of G is isomorphic to the symmetric group of the three vectors in the base of k^3 , and $\overline{X}(w) = \overline{X}(s_1, s_2)$, where s_1 corresponds to the permutation of the first and the second vectors of the base and s_2 corresponds to the permutation of the second and the third vector.

When G is semi-simple with connected Dynkin diagram D (with numbering of nodes and their associated simple reflections), there is a (unique) natural choice of Coxeter element: let $w = s_1 \cdot s_2 \cdot \dots \cdot s_r$ with r maximal (under the condition that s_r is not in the F -orbit of any of the previous $s_i, i < r$). When choosing this particular Coxeter element, we shall refer to $X(w)$ (or $\overline{X}(w)$) as being of standard type.

We claim that $\overline{X}(w)$ is of classical type if w is a Coxeter element for one of the following classical groups: $A_n, {}^2A_{2n}, {}^2A_{2n+1}, B_n, C_n, D_n$ or 2D_n .

For $w_1, w_2 \in W$, we shall say that w_1 and w_2 are F -conjugate if there exists $w' \in W$ such that $w_2 = w'w_1F(w')^{-1}$. It is worth noting that w and $F(w)$ are F -conjugate for any $w \in W$.

Since the morphism L is flat, it is open; hence, $\overline{L^{-1}(B\dot{w}B)} = L^{-1}(\overline{B\dot{w}B})$. Therefore, $X(w)$ is a non-singular variety of dimension n and the closure of $X(w)$ in $X, \overline{X}(w)$, is given by the disjoint union

$$\overline{X(w)} = \bigcup_{w' \leq w} X(w'), \tag{5}$$

where, as usual, \leq is the Bruhat order in W . This closure is usually singular whenever the Schubert variety $X_w = \overline{B\dot{w}B}/B$ is. But since the open subset

$$\left\{ g_0B, \dots, g_rB \in X^{r+1} : g_k^{-1}g_{k+1} \in Bs_{i_{k+1}}B, 0 \leq k < r, g_r^{-1}F(g_0) \in B \right\} \tag{6}$$

of the smooth projective variety $\overline{X}(w)$ maps isomorphically onto $X(w)$ under projection to the first factor, we have a good compactification of $X(w)$. In fact, the complement of $X(w)$ in $\overline{X}(w)$, which is easily seen to be the union of the divisors D_j defined above, is a divisor with normal crossings. If w is a Coxeter element, then $X(w)$ and $\overline{X}(w)$ are irreducible and, in fact, $X(w)$ is isomorphic to $\overline{X}(w)$, and hence non-singular.

It is worth noting that it follows from Definition 1 that, if $\overline{X}(w)$ is of classical type, then the irreducible components of any Deligne–Lusztig subvariety of $\overline{X}(w)$ are of classical type too (see Remark 1 in [17]).

The following result allows us to consider the image of standard Deligne–Lusztig varieties under a certain proper morphism as a normal strict complete intersection in a certain projective space \mathbb{P}^{N-1} as well as to compute its Picard group.

Theorem 1 (Theorem 3 in [17]). *Let $\overline{X}(w)$ be a standard Deligne–Lusztig variety of type ${}^2A_n, B_n, C_n, D_n$ or 2D_n . Assume that $\text{char}(k) \neq 2$ in the orthogonal cases. Let P be the parabolic subgroup generated by B together with the double cosets $Bs_2B, Bs_3B, \dots, Bs_nB$, and let*

$$\pi : (G/B)^{l(w)+1} \rightarrow G/P \subseteq \mathbb{P}(V) \cong \mathbb{P}^{N-1} \tag{7}$$

be the projection (projection to the first factor, followed by the quotient map). It is worth noting that the inclusion $G/P \subseteq \mathbb{P}(V)$ is an equality in the non-orthogonal cases. Denote by L^e the e -dimensional linear subspace of \mathbb{P}^{N-1} obtained by setting the $N - 1 - e$ last coordinates as equal to zero.

1. *The image $Z = \pi(\overline{X}(w))$ is a normal, strict complete intersection. In fact, Lusztig shows (see pp. 444–445 in [18]) that Z equals the support of the scheme’s theoretic complete intersection $Z' = \cap_{i=0}^{a_0(V)-1} H_i$, with $X(w)$ mapping isomorphically onto the open subset $\langle x, F_V^{a_0(V)}(x) \rangle \neq 0$ of Z (see Table 2 in [17]). In the unitary and orthogonal cases, the singular locus of Z, Z_{sing} consists of the finitely many G^F -translates of the closed subscheme $Z \cap L^{a_0(V)-1}$. Hence,*

$$\text{codim}(Z_{\text{sing}}, Z) = N + 1 - 2a_0(V) + a_0(L^{a_0(V)-1}). \tag{8}$$

In the symplectic case, Z_{sing} consists of the G^F -translates of the closed subscheme $Z \cap L^{a_0(V)-2}$, and the previous formula becomes

$$\text{codim}(Z_{\text{sing}}, Z) = 2 + a_0(L^{a_0(V)-2}). \tag{9}$$

2. For $\text{codim}(Z_{\text{sing}}, Z) \geq 4$, $\text{Pic}(Z) = \mathbb{Z}$ and consequently

$$\begin{aligned} \text{Pic}(\overline{X}(w)) = \mathbb{Z}[\pi^* H] \oplus \mathbb{Z}\{[V] : V \text{ component of } D_1\} \\ \oplus j_* A_{l(w)-1}(\bigcup_{i \in \mathcal{I}-\{1\}} D_i) \end{aligned} \tag{10}$$

where H is the hyperplane section of Z , j is the obvious inclusion and \mathcal{I} is the set of indices $\{i\}$, satisfying that some connected component of the Dynkin diagram corresponding to D_i occurs as a subgraph of the Dynkin diagram corresponding to D_1 .

3. For any Coxeter element w' , we have

$$\text{Pic}(\overline{X}(w'))_{p'} \cong \text{Pic}(\overline{X}(w))_{p'}.$$

3. Error-Correcting Codes Construction

Tsfasman and Vlăduț introduced the following construction (generalizing the Goppa–Manin construction):

Definition 2 (H-construction p. 272 in [10]). Let X be a normal projective variety over \mathbb{F}_q . Let L be a line bundle defined over \mathbb{F}_q and let P_1, P_2, \dots, P_n be distinct \mathbb{F}_q -rational points on X . Set $\mathcal{P} = \{P_1, P_2, \dots, P_n\}$. In each P_i , choose isomorphisms of the fibers L_{P_i} with \mathbb{F}_q . The linear code $C(L, \mathcal{P})$ of length n associated to (X, L, \mathcal{P}) is the image of the germ map

$$\alpha : \Gamma(X, L) \rightarrow \bigoplus_{i=1}^n L_{P_i} \cong \mathbb{F}_q^n. \tag{11}$$

From now on, we assume that all line bundles considered actually have a non-zero global section.

Suppose L arises as the line bundle associated to a divisor D and that the P_i are not in the support of D . Then, we obtain the same code (up to isomorphism) as when evaluating the rational functions

$$L(D) = \{f \in k(X)^* : \text{div}(f) + D \geq 0\}$$

in the points \mathcal{P} .

The fundamental question is: Given a line bundle L on X , how many zeros does a section $s \in \Gamma(X, L)$ have along a fixed set \mathcal{P} of rational points?

Using the correspondence between line bundles and (Weil) divisors on a normal variety, we may reformulate the question as follows: For a fixed line bundle L , and given an effective divisor D such that $L = \mathcal{O}_X(D)$, how many points from \mathcal{P} are in its support $|D|$?

Although in the particular case of $\dim(X) = 1$, where the points $P \in \mathcal{P}$ happen to be divisors, one may apply the Riemann–Roch theorem to give a lower bound on d and a formula for k in higher dimensions; however, we have to face the task of comparing objects of different dimension. This may be remedied in two ways:

1. Make the objects have the same codimension by blowing-up at the points;
2. Make the objects have complementary dimensions, that is, make the points in some way into curves.

In the next section, we shall pursue the latter idea. In this case, the following result establishes a lower bound for the minimum distance.

Proposition 1 (Proposition 3.2 in [15]). Let X be a normal projective variety defined over \mathbb{F}_q of dimension at least two. Let C_1, C_2, \dots, C_a be (irreducible) curves on X with \mathbb{F}_q -rational points

$\mathcal{P} = \{P_1, P_2, \dots, P_n\}$. Assume the number of \mathbb{F}_q -rational points on rational C_i is less than N . Let L be a line bundle on X , defined over \mathbb{F}_q , such that $L \cdot C_i \geq 0$ for all i . Let

$$l = \sup_{s \in \Gamma(X, L)} \#\{i : Z(s) \text{ contains } C_i\}.$$

Then, the code $C(L, \mathcal{P})$ has length n and minimum distance

$$d \geq n - lN - \sum_{i=1}^a L \cdot C_i$$

If $L \cdot C_i = \eta \leq N$ for all i , then $d \geq n - lN - (a - l)\eta$.

In particular, if X is a non-singular surface, we can cite the following corollary.

Corollary 1 (Corollary 3.2 in [15]). Assume furthermore that X is a non-singular surface and that H is a nef divisor on X with $H \cdot C_i > 0$ for all i . Then,

$$l \leq \frac{L \cdot H}{\min_i \{C_i \cdot H\}} \tag{12}$$

Consequently, if $L \cdot H < C_i \cdot H$ for all i , we have $l = 0$ and

$$d \geq n - m,$$

where $m = \sum_{i=1}^a L \cdot C_i$.

4. Some EC-Codes on Projective Bundles over Standard Deligne–Lusztig Surfaces

This section is devoted to the computation of lower bounds for the parameters of certain error-correcting codes on projective bundles over standard Deligne–Lusztig surfaces.

Given a standard Deligne–Lusztig surface S of type $A_2(\mathbb{F}_q), {}^2A_3(\mathbb{F}_{q^2}), {}^2A_4(\mathbb{F}_{q^2})$ or $C_2(\mathbb{F}_q)$, for a suitable parabolic subgroup P of G we have a commutative diagram:

$$\begin{array}{ccc} S & \xrightarrow{j} & G/B \\ \rho \downarrow & & \downarrow \pi \\ Z & \xrightarrow{i} & G/P \cong \mathbb{P}^{N-1} \end{array} \tag{13}$$

where

$$Z = \begin{cases} \mathbb{P}^2 & \text{if } S = A_2(\mathbb{F}_q), \\ H_0 \equiv X_0^{q+1} + X_1^{q+1} + X_2^{q+1} + X_3^{q+1} = 0 & \text{if } S = {}^2A_3(\mathbb{F}_{q^2}), \\ H_0 \equiv X_0^q X_3 - X_0 X_3^q + X_1 X_2^q - X_1^q X_2 = 0 & \text{if } S = C_2(\mathbb{F}_q), \end{cases} \tag{14}$$

and $Z = H_0 \cap H_1$ with

$$H_0 \equiv X_0^{q+1} + X_1^{q+1} + X_2^{q+1} + X_3^{q+1} + X_4^{q+1} = 0, \tag{15}$$

$$H_1 \equiv X_0^{q^3+1} + X_1^{q^3+1} + X_2^{q^3+1} + X_3^{q^3+1} + X_4^{q^3+1} = 0, \tag{16}$$

if $S = {}^2A_4(\mathbb{F}_{q^2})$, i is an embedding, j is finite, π is locally trivial (in the Zariski topology) and ρ is birational and surjective; see Sects. 5,6,7,8 in [19].

Moreover, the surface S is isomorphic to the blow-up of (see Definition p. 163 in [20]) Z at a certain set of points in the cases $S = A_2(\mathbb{F}_q), {}^2A_3(\mathbb{F}_{q^2}), C_2(\mathbb{F}_q)$, and all its rational points are distributed equally on the disjoint rational curves B_i constituting the irreducible components of the divisor $D_1 \subset S$, whereas in the remaining case, $S = {}^2A_4$, Z is obtained

from S by contracting the disjoint hermitian curves A_i constituting the irreducible components of the divisor $D_1 \subset S$ and all the rational points of S are distributed equally on the disjoint rational curves B_i constituting the irreducible components of the divisor $D_2 \subset S$.

The following theorem constitutes the main result of this paper, where explicit lower bounds for the parameters of a certain class of codes on projective bundles over Deligne–Lusztig surfaces are given.

Theorem 2. *Let S be a standard Deligne–Lusztig surface of type $A_2(\mathbb{F}_q), {}^2A_3(\mathbb{F}_{q^2}), {}^2A_4(\mathbb{F}_{q^2})$ or $C_2(\mathbb{F}_q)$, and let V be a vector bundle of rank 2 defined over S . Consider the projective bundle $T = P(V)$ over S , that is $p : T \rightarrow S$. Then, for some $a, b > 0$, we can construct a code on T over \mathbb{F}_{q^δ} with parameters*

1. $n = \#S(\mathbb{F}_{q^\delta}) \cdot \#\mathbb{P}^1(\mathbb{F}_{q^\delta})$;
2. $k = h^0(S, \text{Symm}^b(V) \otimes \mathcal{O}_S(aD_j))$;
3. $d \geq n - (-bc_1(W_1)\#\{B_i\} + aD_j \cdot D_i)\#\mathbb{P}^1(\mathbb{F}_{q^\delta}) - (\#S(\mathbb{F}_{q^\delta}) - (-bc_1(W_1)\#\{B_i\} + aD_j \cdot D_i))b$ if $(-bc_1(W_1)\#\{B_i\} + aD_j \cdot D_i) > 0$, and $d \geq n - (\#S(\mathbb{F}_{q^\delta}))b$ otherwise,

where

$$\delta = \begin{cases} 1 & \text{if } S \text{ is of type } A_2, C_2, \\ 2 & \text{if } S \text{ is of type } {}^2A_3, {}^2A_4, \end{cases} \quad D_i = \begin{cases} D_1 & \text{if } S \text{ is of type } A_2, {}^2A_3, C_2, \\ D_2 & \text{if } S \text{ is of type } {}^2A_4, \end{cases}$$

$$D_j = \begin{cases} D_2 & \text{if } S \text{ is of type } A_2, {}^2A_3, C_2, \\ D_1 & \text{if } S \text{ is of type } {}^2A_4, \end{cases}$$

and $c_1(W_1)$ denotes the first Chern class of the line subbundle of minimum degree of the restricted vector bundle $i_{B_i}^*V$ over B_i .

Proof. We will consider \mathcal{P} to be the \mathbb{F}_{q^δ} -rational points on T . Let C_1, C_2, \dots, C_n be the fibers over the \mathbb{F}_{q^δ} -rational points of S . These disjoint lines contain all \mathbb{F}_{q^δ} -rational points of T , that is,

$$T(\mathbb{F}_{q^\delta}) = \bigcup_{P \in S(\mathbb{F}_{q^\delta})} p^{-1}(P)(\mathbb{F}_{q^\delta}). \tag{17}$$

It follows then that the length n of the code is

$$n = \#\mathcal{P} = \#S(\mathbb{F}_{q^\delta}) \cdot \#\mathbb{P}^1(\mathbb{F}_{q^\delta}) \tag{18}$$

Let L be the line bundle $L = \mathcal{O}_T(b) \otimes \mathcal{O}_T(ap^*(D_j))$ over T . From Theorem 9.6 in [21] and Proposition II.7.11 in [20], it follows that

$$\Gamma(T, L) \cong \Gamma(S, p_*L) = \Gamma(S, \text{Symm}^b(V) \otimes \mathcal{O}_S(aD_j)), \tag{19}$$

so when in the range wherein the bound on the minimum distance ensures the injectivity of the evaluation map, the dimension of the code is

$$k = h^0(S, \text{Symm}^b(V) \otimes \mathcal{O}_S(aD_j)). \tag{20}$$

Now, we will apply Proposition 1 in order to obtain the bound for the minimum distance. It is worth noting that, in the cases we are interested in, $C_i = p^{-1}(P)$ with $P \in S(\mathbb{F}_{q^\delta})$, so the maximum number of rational points on C_i will be $N = \#\mathbb{P}^1(\mathbb{F}_{q^\delta})$. Moreover, by Lemma 9.7 in [21], we have that

$$L \cdot C_i = b. \tag{21}$$

From Equation (17), we know that $\bigcup_{i=1}^{\#S(\mathbb{F}_{q^\delta})} C_i \subset p^*D_i$. Now, B_i is an irreducible component of D_i , it is rational and $T_{B_i} = p^{-1}(B_i)$ will be isomorphic to the projective bundle $p_{T_{B_i}}$:

$P(i_{B_i}^* V) \rightarrow B_i$, where $i_{B_i} : B_i \rightarrow S$ denotes the closed embedding. As a consequence, by Theorem 9.6 in [21], the Chow ring of T_{B_i} is isomorphic to

$$A^\bullet(T_{B_i}) \cong A^\bullet(B_i)[\xi]/(\xi^2 + c_1(i_{B_i}^* V)\xi) \tag{22}$$

where ξ is the hyperplane section in T_{B_i} . It is worth noting that since B_i is rational, then by Corollary V.2.14 in [20], $i_{B_i}^* V = W_1 \oplus W_2$, and we can always suppose that $\text{deg}(W_1) \leq \text{deg}(W_2)$.

Now, if we restrict the line bundle L to T_{B_i} , then $L|_{T_{B_i}} \cong \mathcal{O}_{T_{B_i}}(b) \otimes \mathcal{O}_{T_{B_i}}(p_{B_i}^* i_{B_i}^*(aD_j))$. Let $H = \xi + c_1(W_2)F$ be a nef divisor on T_{B_i} (see Theorem V.2.17 [20]). Then, since $H \cdot C_i = 1$, it follows by Corollary 1 that

$$l_{T_{B_i}} \leq H \cdot L|_{B_i} = -bc_1(W_1) + aD_j \cdot B_i, \tag{23}$$

provided that $(-bc_1(W_1) + aD_j \cdot B_i) > 0$, and $l_{T_{B_i}} = 0$ otherwise. As $D_i = \sqcup B_i$, we can conclude the following bound for the minimum distance

$$d \geq n - (-bc_1(W_1)\#\{B_i\} + aD_j \cdot D_i)\#\mathbb{P}^1(\mathbb{F}_{q^\delta}) - (\#S(\mathbb{F}_{q^\delta}) - (-bc_1(W_1)\#\{B_i\} + aD_j \cdot D_i))b \tag{24}$$

if $(-bc_1(W_1)\#\{B_i\} + aD_j \cdot D_i) > 0$ and

$$d \geq n - (\#S(\mathbb{F}_{q^\delta}))b \tag{25}$$

otherwise. \square

Before computing the explicit parameters of some families of these codes, we need the following auxiliary result.

Lemma 1. *Let Y be a variety and $V = V_1 \oplus V_2$ a vector bundle of rank 2 over Y , that is, a direct sum of two line bundles V_1 and V_2 . Then, the symmetric algebra of V satisfies*

$$\text{Symm}(V) \cong \bigotimes_{i=1}^2 \text{Symm}(V_i) = \bigoplus_{i_1, i_2 \in \mathbb{N}^2} \text{Symm}^{i_1} V_1 \otimes \text{Symm}^{i_2} V_2 \cong \bigoplus_{i_1, i_2 \in \mathbb{N}^2} V_1^{\otimes i_1} \otimes V_2^{\otimes i_2}, \tag{26}$$

Proof. The first isomorphism is a consequence of the universal property for symmetric algebras, the second equality is just by definition and the last isomorphism follows from the fact that, for a line bundle V_i , we have

$$V_i^{\otimes n} \cong \text{Symm}^n V_i. \tag{27}$$

\square

The following corollaries give explicit bounds for the parameters of certain families of codes on projective bundles of rank 2 over Deligne-Lusztig surfaces of type A_2 and 2A_4 , obtained by restricting the previous results to certain vector bundles defined over them.

Corollary 2. *Let S_1 be the Deligne–Lusztig surface of type A_2 defined over the field \mathbb{F}_q . Consider some b , such that $0 < b < (q + 1)$, and $V_i = \mathcal{O}_{S_1}(n_i H - \sum_{j=1}^{q^2+q+1} m_{i,j} B_j)$ for $i = 1, 2$, where $H = \pi^*(\mathcal{O}_{\mathbb{P}^2}(1))$, verifying for any pair $i_1, i_2 \in \mathbb{N}$ with $i_1 + i_2 = b$:*

1. $(i_1 n_1 + i_2 n_2) \leq 3(q - 1)$;
2. $(i_1 n_1 + i_2 n_2) \geq \sum_{j=1}^3 (i_1 m_{1,j} + i_2 m_{2,j})$;
3. $(i_1 m_{1,j} + i_2 m_{2,j}) \geq (i_1 m_{1,j+1} + i_2 m_{2,j+1})$;
4. $3(i_1 n_1 + i_2 n_2) > \sum_{j=1}^{q^2+q+1} (i_1 m_{1,j+1} + i_2 m_{2,j+1})$.

Let T_1 be the projective bundle $P(V_1 \oplus V_2)$ over S_1 , $p_1 : T_1 \rightarrow S_1$. Then, we can construct a code on T_1 over \mathbb{F}_q with parameters

1. $n = (q^2 + q + 1)(q + 1)^2$,
2. $k = \sum_{\substack{i_1, i_2 \in \mathbb{N}^2 \\ i_1 + i_2 = b}} \frac{1}{2} ((i_1 n_1 + i_2 n_2)((i_1 n_1 + i_2 n_2) + 3) - \sum_{j=1}^{q^2+q+1} (i_1 m_{1,j} + i_2 m_{2,j})((i_1 m_{1,j} + i_2 m_{2,j}) + 1)) + 1$,
3. $d \geq n - (q^2 + q + 1)(q + 1)b$.

Proof. Since A_2 is isomorphic to the blow-up of \mathbb{P}^2 at the set of its rational points over \mathbb{F}_q (see Sect. 5 in [19]), we have that $\#S_1(\mathbb{F}_q) = (q^2 + q + 1)(q + 1)$. Moreover, $\#\mathbb{P}^1(\mathbb{F}_q) = (q + 1)$, so we can conclude that the length n of the code is

$$n = \#\mathcal{P} = \#S_1(\mathbb{F}_q)(q + 1) = (q^2 + q + 1)(q + 1)^2 \tag{28}$$

Let L be the line bundle $L = \mathcal{O}_{T_1}(b)$. In order to compute the dimension of the code, we have $\dim \Gamma(T_1, L) = \dim \Gamma(S_1, p_{1*}L)$, where $p_{1*}L \cong \text{Symm}^b(V_1 \oplus V_2)$. As a result of Lemma 1,

$$\text{Symm}^b(V_1 \oplus V_2) = \bigoplus_{\substack{i_1, i_2 \in \mathbb{N}^2 \\ i_1 + i_2 = b}} \mathcal{O}_{S_1}(n_1 H - \sum_{j=1}^{q^2+q+1} m_{1,j} B_j)^{\otimes i_1} \otimes \mathcal{O}_{S_1}(n_2 H - \sum_{j=1}^{q^2+q+1} m_{2,j} B_j)^{\otimes i_2}. \tag{29}$$

Now, global sections of a direct sum of line bundles satisfy

$$\begin{aligned} \Gamma(S_1, \bigoplus_{\substack{i_1, i_2 \in \mathbb{N}^2 \\ i_1 + i_2 = b}} \mathcal{O}_{S_1}(n_1 H - \sum_{j=1}^{q^2+q+1} m_{1,j} B_j)^{\otimes i_1} \otimes \mathcal{O}_{S_1}(n_2 H - \sum_{j=1}^{q^2+q+1} m_{2,j} B_j)^{\otimes i_2}) &\cong \\ \bigoplus_{\substack{i_1, i_2 \in \mathbb{N}^2 \\ i_1 + i_2 = b}} \Gamma(S_1, \mathcal{O}_{S_1}(n_1 H - \sum_{j=1}^{q^2+q+1} m_{1,j} B_j)^{\otimes i_1} \otimes \mathcal{O}_{S_1}(n_2 H - \sum_{j=1}^{q^2+q+1} m_{2,j} B_j)^{\otimes i_2}), & \end{aligned} \tag{30}$$

and finally, we can conclude

$$\begin{aligned} \bigoplus_{\substack{i_1, i_2 \in \mathbb{N}^2 \\ i_1 + i_2 = b}} \Gamma(S_1, \mathcal{O}_{S_1}(n_1 H - \sum_{j=1}^{q^2+q+1} m_{1,j} B_j)^{\otimes i_1} \otimes \mathcal{O}_{S_1}(n_2 H - \sum_{j=1}^{q^2+q+1} m_{2,j} B_j)^{\otimes i_2}) &\cong \\ \bigoplus_{\substack{i_1, i_2 \in \mathbb{N}^2 \\ i_1 + i_2 = b}} \Gamma(S_1, \mathcal{O}_{S_1}((i_1 n_1 + i_2 n_2) H - \sum_{j=1}^{q^2+q+1} (i_1 m_{1,j} + i_2 m_{2,j}) B_j)) & \end{aligned} \tag{31}$$

Furthermore, by the hypothesis of the theorem, $\mathcal{O}_{S_1}((i_1 n_1 + i_2 n_2) H - \sum_{j=1}^{q^2+q+1} (i_1 m_{1,j} + i_2 m_{2,j}) B_j)$ is excellent with respect to the exceptional configuration of $\pi : S_1 \rightarrow \mathbb{P}^2$ (see p. 215 in [22]), so if we denote $F = \mathcal{O}_{S_1}((i_1 n_1 + i_2 n_2) H - \sum_{j=1}^{q^2+q+1} (i_1 m_{1,j} + i_2 m_{2,j}) B_j)$, then by Theorem 1.1 in [22]:

$$\begin{aligned} h^0(S_1, F) &= \frac{1}{2}(f \cdot f - f \cdot k_{S_1}) + 1 \\ &= \frac{1}{2}((i_1 n_1 + i_2 n_2)((i_1 n_1 + i_2 n_2) + 3) - \sum_{j=1}^{q^2+q+1} (i_1 m_{1,j} + i_2 m_{2,j})((i_1 m_{1,j} + i_2 m_{2,j}) + 1)) + 1. \end{aligned} \tag{32}$$

This allows us to conclude that, when in the range wherein the bound on the minimum distance ensures the injectivity of the evaluation map, the dimension of the code is

$$k = \sum_{\substack{i_1, i_2 \in \mathbb{N}^2 \\ i_1 + i_2 = b}} \frac{1}{2} ((i_1 n_1 + i_2 n_2)((i_1 n_1 + i_2 n_2) + 3) - \sum_{j=1}^{q^2+q+1} (i_1 m_{1,j} + i_2 m_{2,j})((i_1 m_{1,j} + i_2 m_{2,j}) + 1)) + 1. \tag{33}$$

Finally, by Proposition 9.4 in [21], any section of $L|_{T_{1B_i}}$ (recall that $T_{1B_i} = p_1^{-1}(B_i)$) intersects a fiber $C_i \subset T_{1B_i}$ in a hyperplane, so $l = 0$ and we can conclude the following bound for the minimum distance:

$$d \geq n - (q^2 + q + 1)(q + 1)b. \tag{34}$$

□

Corollary 3. *Let S_2 be the Deligne–Lusztig surface of type 2A_4 defined over the field \mathbb{F}_{q^2} . Consider $V_i = \pi^* i^* \mathcal{O}_{\mathbb{P}^4}(t_i)$ for $i = 1, 2$, and let T_2 be the projective bundle $P(V_1 \oplus V_2)$ over S_2 , $p_2 : T_2 \rightarrow S_2$. Then, for some $0 < b < (q^2 + 1)$ and for any $t_1, t_2 \in \mathbb{N}$ such that $(q + 1) < (i_1 t_1 + i_2 t_2) < (q^3 + 1)$ for any pair $i_1, i_2 \in \mathbb{N}$ with $i_1 + i_2 = b$, we can construct a code on T_2 over \mathbb{F}_{q^2} with parameters*

1. $n = (q^5 + 1)(q^3 + 1)(q^2 + 1)^2$;
2. $k = \sum_{\substack{i_1, i_2 \in \mathbb{N}^2 \\ i_1 + i_2 = b}} \binom{4+i_1 t_1 + i_2 t_2}{i_1 t_1 + i_2 t_2} - \binom{4+i_1 t_1 + i_2 t_2 - (q+1)}{i_1 t_1 + i_2 t_2 - (q+1)}$;
3. $d \geq n - (q^5 + 1)(q^3 + 1)(q^2 + 1)b$.

Proof. Since all the rational points of S_2 are distributed equally on the disjoint rational curves B_i constituting the irreducible components of the divisor $D_2 \subset S_2$, then $\#S_2(\mathbb{F}_{q^2}) = (q^5 + 1)(q^3 + 1)(q^2 + 1)$. Moreover, $\#\mathbb{P}^1(\mathbb{F}_{q^2}) = (q^2 + 1)$, so we can conclude that the length n of the code is

$$n = \#P = \#S_2(\mathbb{F}_{q^2})(q^2 + 1) = (q^5 + 1)(q^3 + 1)(q^2 + 1)^2. \tag{35}$$

Let L be the line bundle $L = \mathcal{O}_{T_2}(b)$. In order to compute the dimension of the code, we have that $\dim \Gamma(T_2, L) = \dim \Gamma(S_2, p_{2*}L)$, with $p_{2*}L \cong \text{Sym}^b(V_1 \oplus V_2)$. As a result of Lemma 1

$$\text{Sym}^b(V_1 \oplus V_2) = \bigoplus_{\substack{i_1, i_2 \in \mathbb{N}^2 \\ i_1 + i_2 = b}} \pi^* i^* \mathcal{O}_{\mathbb{P}^4}(t_1)^{\otimes i_1} \otimes \pi^* i^* \mathcal{O}_{\mathbb{P}^4}(t_2)^{\otimes i_2}. \tag{36}$$

Now, global sections of a direct sum of line bundles satisfy

$$\Gamma(S_2, \bigoplus_{\substack{i_1, i_2 \in \mathbb{N}^2 \\ i_1 + i_2 = b}} \pi^* i^* \mathcal{O}_{\mathbb{P}^4}(t_1)^{\otimes i_1} \otimes \pi^* i^* \mathcal{O}_{\mathbb{P}^4}(t_2)^{\otimes i_2}) \cong \bigoplus_{\substack{i_1, i_2 \in \mathbb{N}^2 \\ i_1 + i_2 = b}} \Gamma(S_2, \pi^* i^* \mathcal{O}_{\mathbb{P}^4}(t_1)^{\otimes i_1} \otimes \pi^* i^* \mathcal{O}_{\mathbb{P}^4}(t_2)^{\otimes i_2}), \tag{37}$$

and since pull-back commutes with tensor product, it follows then that

$$\bigoplus_{\substack{i_1, i_2 \in \mathbb{N}^2 \\ i_1 + i_2 = b}} \Gamma(S_2, \pi^* i^* \mathcal{O}_{\mathbb{P}^4}(t_1)^{\otimes i_1} \otimes \pi^* i^* \mathcal{O}_{\mathbb{P}^4}(t_2)^{\otimes i_2}) \cong \bigoplus_{\substack{i_1, i_2 \in \mathbb{N}^2 \\ i_1 + i_2 = b}} \Gamma(S_2, \pi^* i^* \mathcal{O}_{\mathbb{P}^4}(i_1 t_1 + i_2 t_2)) \tag{38}$$

Furthermore, as π is birational onto Z , we obtain (see Theorem 2.31 in [23])

$$\Gamma(S_2, \pi^* i^* \mathcal{O}_{\mathbb{P}^4}(i_1 t_1 + i_2 t_2)) \cong \Gamma(Z, i^* \mathcal{O}_{\mathbb{P}^4}(i_1 t_1 + i_2 t_2)) \tag{39}$$

Since Z is the complete intersection of two hyper-surfaces $H_0, H_1 \subset \mathbb{P}^4$ of degrees $q + 1$ and $q^3 + 1$, respectively, we obtain short exact sequences of line bundles

$$0 \rightarrow \mathcal{O}_{\mathbb{P}^4}(t - (q + 1)) \rightarrow \mathcal{O}_{\mathbb{P}^4}(t) \rightarrow \mathcal{O}_{H_0}(t) \rightarrow 0, \tag{40}$$

and

$$0 \rightarrow \mathcal{O}_{H_0}(t - (q^3 + 1)) \rightarrow \mathcal{O}_{H_0}(t) \rightarrow \mathcal{O}_{H_0 \cap H_1}(t) \rightarrow 0, \tag{41}$$

(see Section 7.3 in [23]). These sequences give long exact sequences of cohomology groups,

$$\begin{aligned} 0 \rightarrow H^0(\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(t - (q + 1))) \rightarrow H^0(\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(t)) \rightarrow H^0(H_0, \mathcal{O}_{H_0}(t)) \rightarrow \\ H^1(\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(t - (q + 1))) \rightarrow H^1(\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(t)) \rightarrow \\ H^1(H_0, \mathcal{O}_{H_0}(t)) \rightarrow H^2(\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(t - (q + 1))) \rightarrow \dots, \end{aligned} \tag{42}$$

and

$$\begin{aligned} 0 \rightarrow H^0(H_0, \mathcal{O}_{H_0}(t - (q^3 + 1))) \rightarrow H^0(H_0, \mathcal{O}_{H_0}(t)) \rightarrow \\ H^0(H_0 \cap H_1, \mathcal{O}_{H_0 \cap H_1}(t)) \rightarrow H^1(H_0, \mathcal{O}_{H_0}(t - (q^3 + 1))) \rightarrow \dots \end{aligned} \tag{43}$$

By the formulas of Theorem III.5.1 in [20] for the cohomology of projective space, the first long exact sequence reduces (for any t) to

$$\begin{aligned} 0 \rightarrow H^0(\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(t - (q + 1))) \rightarrow H^0(\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(t)) \rightarrow \\ H^0(H_0, \mathcal{O}_{H_0}(t)) \rightarrow 0 \rightarrow 0 \rightarrow H^1(H_0, \mathcal{O}_{H_0}(t)) \rightarrow 0 \rightarrow \dots \end{aligned} \tag{44}$$

Hence, for any t , $H^1(H_0, \mathcal{O}_{H_0}(t)) = 0$ and

$$\dim H^0(H_0, \mathcal{O}_{H_0}(t)) = \dim H^0(\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(t)) - \dim H^0(\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(t - (q + 1))) \tag{45}$$

$$= \binom{4 + t}{t} - \binom{4 + t - (q + 1)}{t - (q + 1)}. \tag{46}$$

Consequently, for any $q + 1 < t < q^3 + 1$, the last sequence then gives

$$\Gamma(Z, \mathcal{O}_Z(t)) \cong H^0(H_0, \mathcal{O}_{H_0}(t)), \tag{47}$$

so, when in the range wherein the bound on the minimum distance ensures the injectivity of the evaluation map, the dimension of the code is

$$k = \sum_{\substack{i_1, i_2 \in \mathbb{N}^2 \\ i_1 + i_2 = b}} \binom{4 + i_1 t_1 + i_2 t_2}{i_1 t_1 + i_2 t_2} - \binom{4 + i_1 t_1 + i_2 t_2 - (q + 1)}{i_1 t_1 + i_2 t_2 - (q + 1)}. \tag{48}$$

Finally, by Proposition 9.4 in [21], any section of $L|_{T_{2B_i}}$ (recall that $T_{2B_i} = p_2^{-1}(B_i)$) intersects $C_i \subset T_{4B_i}$ in an hyperplane, so $l = 0$ and we can conclude the following bound for the minimum distance:

$$d \geq n - (q^5 + 1)(q^3 + 1)(q^2 + 1)b. \tag{49}$$

□

Finally, we compute the parameters of some of the codes in Corollary 2 and Corollary 3 for the binary case, $q = 2$, as this is the most common finite field within applications.

Example 2. Let us consider the particular case $q = 2$, with $b = 1$, $n_1 = n_2 = 3$, $m_{1,j} = m_{2,j} = 1$ for $j = 1, 2, 3$, and $m_{1,j} = m_{2,j} = 0$ otherwise, for the family of codes presented in Corollary 2. Then, we obtain codes with the following parameters:

1. $n = 63$;
2. $k = 14$;
3. $d \geq 42$.

Example 3. For $q = 2$, let us consider the particular case $b = 2$, $t_1 = t_2 = 4$ for the family of codes presented in Corollary 3. Then we obtain a code with the following parameters:

1. $n = 7425$,
2. $k = 1107$,
3. and $d \geq 4455$.

5. Conclusions

By means of an intensive use of the intersection theory, we extend some of the previous results for codes over Deligne–Lusztig surfaces to the case of codes on projective bundles of rank 2 over standard Deligne–Lusztig surfaces. In particular, we compute the length, dimension and give a lower bound for the minimum distance in Theorem 2 for the cases of codes on projective bundles over Deligne–Lusztig surfaces of type $A_2(\mathbb{F}_q)$, ${}^2A_3(\mathbb{F}_{q^2})$, ${}^2A_4(\mathbb{F}_{q^2})$ or $C_2(\mathbb{F}_q)$. Moreover, in Corollary 2 and Corollary 3, we focus on some special families of these codes, by restricting our results to certain vector bundles over Deligne–Lusztig surfaces of type A_2 and 2A_4 , in order to give a more explicit computation of their dimension. Finally, we give two examples of binary codes, as seen in Example 2 and Example 3, motivated by the fact that $q = 2$ is the most common framework within applications. In Example 2, we obtain an information rate $k/n = 14/63$ and an error-correcting rate $\delta = d/n \geq 42/63$, and in Example 3 we obtain an information rate $k/n = 1107/7425$ and an error-correcting rate $\delta = d/n \geq 4455/7425$, concluding that both codes exhibit high error-correcting rates.

Some interesting problems would be to extend these results to the case of projective bundles of higher rank over standard Deligne–Lusztig surfaces following the techniques used in [24], and to establish a more explicit understanding on how the morphism $\pi : \bar{X}(w) \rightarrow Z$ of Theorem 1 behaves for Deligne–Lusztig varieties of dimension ≥ 3 in order to construct new algebraic geometric error-correcting codes over them.

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References

1. Goppa, V.D. A new class of linear error-correcting codes. *Probl. Inf. Transm.* **1970**, *6*, 300–304.
2. McEliece, R.J. A public-key cryptosystem based on algebraic coding theory. *DSN Prog. Rep. Jet Propuls. Lab. Pasadena* **1978**, *42*, 114–116.
3. Tsfasman, M.A.; Vladut, S.G.; Zink, T. Modular curves, Shimura curves, and Goppa codes, better than Varshamov–Gilbert bound. *Math. Nachr.* **1982**, *109*, 21–28. [\[CrossRef\]](#)
4. Barelli, E.; Beelen, P.; Datta, M.; Neiger, V.; Rosenkilde, J. Two-point codes for the generalized GK curve. *IEEE Trans. Inform. Theory* **2008**, *64*, 6268–6276. [\[CrossRef\]](#)
5. Christensen, R.B.; Munuera, C.; Pereira, F.; Ruano, D. An algorithmic approach to entanglement-assisted quantum error-correcting codes from the Hermitian curve. *Adv. Math. Commun.* **2023**, *17*, 78–97. [\[CrossRef\]](#)
6. Hansen, J.P.; Stichtenoth, H. Group codes on certain algebraic curves with many rational points. *Appl. Algebra Engrg. Comm. Comput.* **1990**, *1*, 67–77. [\[CrossRef\]](#)
7. Xing C.; Chen, H. Improvements on parameters of one-point AG codes from Hermitian curves. *IEEE Trans. Inform. Theory* **2002**, *48*, 535–537. [\[CrossRef\]](#)
8. Munuera, C.; Sepúlveda, A.; Torres, F. Algebraic Geometry codes from Castle curves. In *Coding Theory and Applications. Lecture Notes in Computer Science*; Barbero, A., Ed.; Springer: Berlin/Heidelberg, Germany, 2008; pp. 117–127.
9. Stichtenoth, H. A note on Hermitian codes over $\text{GF}(q^2)$. *IEEE Trans. Inf. Theory* **1988**, *34*, 1345–1348. [\[CrossRef\]](#)
10. Tsfasman, M.A.; Vladut, S.G. *Algebraic-Geometric Codes*, 3rd ed.; Springer Kluwer: Dordrecht, The Netherlands, 1991.
11. Couvreur, A. Construction of rational surfaces yielding good codes. *Finite Fields Appl.* **2011**, *17*, 424–441. [\[CrossRef\]](#)
12. Edokou, F. Codes defined by forms of degree 2 on hermitian surfaces and Sensen’s conjecture. *Finite Fields Appl.* **2007**, *13*, 616–627. [\[CrossRef\]](#)

13. Voloch, J.F. Algebraic geometric codes on surfaces. In *Arithmetics, Geometry, and Coding Theory (AGCT 2005)*; Volume 21; Séminaires et Congrès-Société Mathématique de France: Paris, France, 2010; pp. 211–216.
14. Little, J.B. Algebraic geometry codes from higher dimensional varieties. In *Advances in Algebraic Geometry Codes*; Serie Coding Theory Cryptology; Martinez-Moro, E., Munuera, C., Ruano, D., Eds.; World Scientific Publishing: Hackensack, NJ, USA, 2008; Volume 5, pp. 257–293.
15. Hansen, S.H. Error-correcting codes from higher-dimensional varieties. *Finite Fields Appl.* **2001**, *7*, 531–552. [[CrossRef](#)]
16. Springer, T.A. *Linear Algebraic Groups*; Birkhäuser Boston: Boston, MA, USA, 1998.
17. Hansen, S.H. Picard groups of Deligne-Lusztig varieties—With a view toward higher codimensions. *Beiträge Algebra Geom.* **2002**, *43*, 9–26.
18. Lusztig, G. On the Green polynomials of classical groups. *Proc. Lond. Math. Soc.* **1976**, *33*, 443–475. [[CrossRef](#)]
19. Rodier, F. Nombre de points des surfaces de Deligne et Lusztig. *J. Algebra* **2000**, *227*, 706–766. [[CrossRef](#)]
20. Hartshorne, R. *Algebraic Geometry*; Springer: New York, NY, USA; Heidelberg, Germany, 1977.
21. Eisenbud, D.; Harris, J. *3264 and All That—A Second Course in Algebraic Geometry*; Cambridge University Press: Cambridge, UK, 2016.
22. Harbourne, B. Complete linear systems on rational surfaces. *Trans. Amer. Math. Soc.* **1985**, *289*, 213–226. [[CrossRef](#)]
23. Iitaka, S. *Algebraic Geometry. An Introduction to Birational Geometry of Algebraic Varieties*; Springer: New York, NY, USA; Berlin, Germany, 1982.
24. Nakashima, T. Error-correcting codes on projective bundles. *Finite Fields Appl.* **2006**, *12*, 222–231. [[CrossRef](#)]

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