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ORIGINAL PAPER

Stratified reduction of singularities of generalized analytic functions

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Abstract

Generalized analytic functions are naturally defined in manifolds with boundary and are built from sums of convergent real power series with non-negative real exponents. In this paper we deal with the problem of reduction of singularities of these functions. Namely, we prove that a germ of generalized analytic function can be transformed by a finite sequence of blowing-ups into a function which is locally of monomial type with respect to the coordinates defining the boundary of the manifold where it is defined.

Keywords Blowing-up morphism · Reduction of singularities · Generalized power series · Principialization of ideals

Mathematics Subject Classification 14E15 · 14P15 · 16W60 · 32C05 · 32S45

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1 Introduction

In this paper, a generalized power series (in n variables and with coefficients in some ring A) is a power series with n-tuples of non-negative real numbers as exponents and whose support is contained in a cartesian product of n well-ordered subsets of $\mathbb{R}_+ = \{r \geq 0\}$. It is worth to mention that this condition on the support is more restrictive (except for $n = 1$) than the one used to define the Hahn ring $A((\Gamma))$, where Γ is the group $\mathbb{R}^n$ with the lexicographic order (and whose elements are also called generalized power series). Introduced and studied by van den Dries and Speissegger in [6], generalized power series appear in several contexts. To mention a few: as solutions of differential/functional equations; as expressions of the Riemann zeta-function (or, more generally, the Dirichlet series) in a logarithmic chart; as asymptotic expansions of Dulac transition maps of vector fields (see for instance [12, 13]); in model theory and o-minimal geometry (the paper [6] itself, or also [17]); as parametrizations of algebraic curves in positive characteristic (see for instance [18, p. 19]).

Considering real coefficients, we have a natural notion of convergence for generalized power series, whose sums provide continuous functions on open subsets of the orthant $\mathbb{R}^n_+$, called *generalized analytic functions*. They are the local pieces to build abstract *(real) generalized analytic manifolds*, introduced and developed by Martín, Rolin and Sanz in [14]. More precisely, a generalized analytic manifold is a locally ringed space $\mathcal{M} = (M, \mathcal{G}_M)$, where M is a topological manifold with boundary and $\mathcal{G}_M$ is a sheaf of continuous functions locally isomorphic to the sheaf of generalized analytic functions on open subsets of $\mathbb{R}^n_+$. Sections of the sheaf $\mathcal{G}_M$ are called themselves generalized analytic functions on M.

The main result in [14] establishes the local reduction of singularities of generalized analytic functions, in the spirit of Zariski's local uniformization theorem of algebraic varieties [19] or Hironaka's version for analytic varieties [10]. The statement, formulated in analogous terms to those used in Bierstone–Milman's paper [4] for real analytic functions, is the following:

Local Monomialization Theorem [14]. Let f be a generalized analytic function on M and let $p \in M$. Then there exists a neighbourhood U_0 of p in M, finitely many sequences of *local blowing-ups* $\{\pi_i : \mathcal{M}_i \to U_0\}_{i=1}^r$ and compact sets $L_i \subset M_i$ satisfying that $\bigcup_i \pi_i(L_i)$ is a neighbourhood of p and such that, for every i, the total transform $f_i = f \circ \pi_i$ is of *monomial type* at every $q \in L_i$ (i.e., for some coordinates $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ centered at q, we have $f_i = \mathbf{x}^\alpha U(\mathbf{x})$ where $U(0) \neq 0$).

The centers of blowing-ups in each sequence π_i have normal crossings with the boundary, but they are defined only in some open sets of the corresponding manifold. In the standard real analytic case, we have stronger global monomialization results (typically called *Reduction of Singularities*, see [1, 5, 7]). They consist, essentially, in that in the above statement, we can take just a single sequence ($r = 1$) and the centers of blowing-ups are globally defined closed analytic submanifolds, having normal crossings with the boundary.

Such a global result is not known so far for generalized analytic functions. There are two main difficulties related to the very notion of a blowing-up morphism in the category of generalized analytic manifolds. On the one hand, a blowing-up depends on the local coordinates that we use to define it. More intrinsically, a blowing-up is not uniquely defined and depends on the choice of a *standardization* of the manifold (or at least of an open neighbourhood of the center of blowing-up). Roughly, a standardization is a subsheaf $\mathcal{O}_M$ of $\mathcal{G}_M$ such that $(M, \mathcal{O}_M)$ is a real analytic standard manifold and from which the sheaf $\mathcal{G}_M$ can be recovered by a natural completion adding generalized series (see [14], we recall this notion below). Secondly, although every generalized analytic manifold is locally standardizable, there may exist closed submanifolds which do not admit standardizable neighbourhoods; i.e., such submanifolds cannot be "geometric" centers for a blowing-up (cf. [14, Example 3.20]).

Morally, a procedure for reduction of singularities of generalized analytic functions would need to guarantee that, in the process, all closed centers susceptible to be blown-up have standardizable neighbourhoods. If this is already proved and Y is such a center, one needs to show furthermore that, among the different standardizations around Y, there exists for which the corresponding blowing-up $\pi : \widetilde{\mathcal{M}} \to \mathcal{M}$ reduces the "complexity" of the function.

In this paper, we overcome these difficulties to obtain an intermediate step towards a global result, the so-called *stratified reduction of singularities*. Let us explain it. First, we recall that, by its very definition, the boundary ∂M of a generalized analytic manifold is a normal crossing divisor; i.e., ∂M is locally given by a finite union of coordinate hyperplanes. Moreover, the number of such hyperplanes at each point provides a natural stratification of M by (standard) analytic manifolds. A generalized analytic function $f : M \to \mathbb{R}$ is said to be of *stratified monomial type* if for any given $p \in M$, if S is the stratum where p belongs, there exists a local chart $(\mathbf{x} = (x_1, x_2, \ldots, x_e), \mathbf{y})$ centered at p satisfying $S = \{x_1 = x_2 = \cdots = x_e = 0\}$ and for which

$$f(\mathbf{x}, \mathbf{y}) = \mathbf{x}^\alpha U(\mathbf{x}, \mathbf{y}), \quad \text{where } \alpha \in \mathbb{R}^e_+ \text{ and } U(0, \mathbf{y}) \not\equiv 0.$$

Thus, requiring a function to be of stratified monomial type means to require that it is of monomial type only with respect to the generalized coordinates determining equations of the components of the boundary. In particular, the condition is empty if $p \notin \partial M$. Also, it is automatic if S has codimension $e = 1$, taking in the above definition α to be the minimum of the support of the series defining f with respect to the single variable $\mathbf{x} = x_1$.

Our main result may be stated now as follows.

Theorem 1.1 (Stratified Reduction of Singularities) *Let $\mathcal{M} = (M, \mathcal{G}_M)$ be a generalized analytic manifold and let $f : M \to \mathbb{R}$ be a generalized analytic function. Let $p \in M$ and assume that the germ of f at p is not identically zero. Then, there exist a neighbourhood V_p of p in M and a sequence of blowing-ups*

$$(M_r, \mathcal{G}_{M_r}) \stackrel{\pi_{r-1}}{\to} (M_{r-1}, \mathcal{G}_{M_{r-1}}) \stackrel{\pi_{r-2}}{\to} \cdots \stackrel{\pi_1}{\to} (M_1, \mathcal{G}_{M_1}) \stackrel{\pi_0}{\to} (V_p, \mathcal{G}_M|_{V_p})$$

such that the pull-back $f' := f \circ \pi_0 \circ \cdots \circ \pi_{r-1} \in \mathcal{G}_{M_r}(M_r)$ is of stratified monomial type. Moreover, the center of each blowing-up π_j, with $j = 0, 1, \ldots, r-1$, can be chosen to be the closure of a codimension two stratum in M_j, where $M_0 := V_p$.

Our proof of Theorem 1.1 is constructive in the sense that each center, as well as the standardization used to define the respective blowing-up at each step, can be given explicitly in terms of the expression of f in some initial coordinates of $\mathcal{M}$ at p. Moreover, each blowing-up morphism is locally expressed as a purely monomial map between two domains of $\mathbb{R}^n_+$ in suitable charts. Consequently, all the process of stratified reduction of singularities can be described using only combinatorics from the starting data given simply by the minimal support (see Sect. 2 below) of a generalized power series representing f at p. The datum of minimal support is closely related to that of the Newton polyhedron of a function in the standard analytic case and therefore, our result should be compared with the combinatorial reduction of singularities stated in Molina's paper [15]. Although it has been a source of inspiration for us, we cannot apply directly the results in [15], mostly because there is no good notion of "multiplicity" in the generalized non-standard situation (any power function with positive real exponent in a generalized variable is a genuine change of variables).

We want to observe that Theorem 1.1 is already proved for dim $M = 3$ in Palma's paper [16], but with a different strategy for the choice of the sequence of blowing-ups (for instance, the centers of blowing-ups may be either corner points or closures of one-dimensional strata).

The paper is structured as follows.

In Sect. 2 we summarize the basic notions and properties of generalized power series and of the category of generalized analytic manifolds, using the mentioned references [6] and [14]. We emphasize the notion of standardization, which is crucial to define blowing-ups.

In Sect. 3 we introduce the category of *monomial (generalized or standard) analytic manifolds*. The objects of this subcategory are manifolds having at least one corner and equipped with an atlas of local charts centered at each corner point for which the change of coordinates is expressed as a monomial map between domains of the local model $\mathbb{R}^n_+$. We represent these changes of coordinates by means of a family of matrices of exponents (for a similar treatment see for instance [2, 3, 16]), a combinatorial data which codifies uniquely the structural sheaf of the manifold. We define also the class of monomial morphisms and the class of monomial standardizations of monomial manifolds. After a blowing-up using such a standardization with a center which is the closure of a stratum (a so-called *combinatorial center*), we obtain again a monomial manifold and the blowing-up morphism is a monomial morphism. The main result in this section is the abundance of monomial standardizations (Proposition 3.16 below). Furthermore, we can choose such a monomial standardization with a prescribed local expression at a given corner point. Morally, local strategies of reduction of singularities are susceptible to be "globalized". We end this section by introducing a special class of monomial manifolds, those obtained from a given one by a sequence of blowing-ups with combinatorial centers, and using only monomial standardizations. Such a sequence is called a *monomial star* and the family of such stars is called the *monomial "voûte étoilée"*, a terminology that evokes the one introduced by Hironaka in [10, 11] for sequences of local blowing-ups in complex analytic geometry.

In Sect. 4 we provide a proof of the main Theorem 1.1. Firstly, we prove a result about *principalization* of finitely generated monomial ideal sheaves in a given monomial manifold. This result (see Theorem 4.5 below) can be seen as a version for our category of a well known result on principalization of ideals in the algebraic or standard analytic situation (see for instance Goward's paper [9] for a simple proof, or see also Fernández-Duque's paper [8] for a similar statement concerning the resonances elimination for singularities of codimension-one analytic foliations). Taking into account that it suffices to obtain the principalization only at the corner points, such a result can also be regarded as a globalization of the algorithm described in van den Dries and Speissegger's paper (see [6, Lemma 4.10]) that reduces the number of elements in the minimal support of a generalized power series by monomial transformations of the variables.

Although we use certain elements and arguments of that result, and despite of what we have said above concerning the possibility to globalize a "local strategy", our proof here requires a different control invariant.

Once we have the principalization of monomial ideal sheaves, the main theorem is concluded easily in the case we start with a corner point $p \in M$. In this case, the sequence $\pi_0 \circ \pi_1 \circ \cdots \pi_{r-1}$ for Theorem 1.1 is actually a star in the voûte étoilée over the germ of M at p. Finally, the general case $p \in \partial M$ is reduced to the case of a corner point, using that around p there is a product structure of a neighbourhood of a corner point times a standard analytic manifold without boundary.

2 Preliminaries

We summarize here the basic notions about the category of generalized analytic manifolds and blowing-up morphisms in it, introduced by Martín, Rolin and Sanz in [14]. These manifolds are built from convergent generalized power series, extensively studied in a paper by van den Dries and Speissegger [6].

2.1 Formal and convergent generalized power series

Denote by $\mathbb{R}_+ = [0, \infty)$. Tuples of variables are denoted by X, Y, Z, etc., and we implicitly assume that tuples with different name have no common variables. If X has n components, we say that X is an n-tuple and so on.

Let $X = (X_1, X_2, \dots, X_n)$ be an n-tuple of variables and let A be an integral domain. A *formal generalized power series with coefficients in* A in the variables X is a map $s : \mathbb{R}_+^n \to A$, written as

$$s = \sum_{\lambda \in \mathbb{R}_+^n} s_\lambda X^\lambda, \quad \text{where } X^\lambda = X_1^{\lambda_1} X_2^{\lambda_2} \cdots X_n^{\lambda_n} \ \text{ for } \lambda = (\lambda_1, \lambda_2, \dots, \lambda_n)$$

and $s_\lambda := s(\lambda) \in A$, such that its *support* $\mathrm{Supp(s)} := \{\lambda \in \mathbb{R}_+^n : \ s_\lambda \neq 0\}$ is contained in a cartesian product of n well-ordered subsets of $\mathbb{R}$. The set of all such formal generalized power series, denoted by $A[[X^*]]$, with the usual addition and product operations of power series has an structure of an A-algebra which is also an integral domain. Moreover, if A is a field, then $A[[X^*]]$ is a local algebra (see [6, Corollary 5.6]), with maximal ideal given by $\mathfrak{m} = \{s \in A[[X^*]] : \ s_0 = 0\}$. Note that $A[[X^*]]$ is not noetherian, in fact, the ideal $\mathfrak{m}$ is never finitely generated.

The *minimal support* of a power series $s \in A[[X^*]]$ is the subset $\mathrm{Supp_{min}(s)} \subset \mathrm{Supp(s)}$ composed of the minimal tuples of $\mathbb{R}_+^n$ with respect to the (partial) *division order* $\leq_d$, that is $(\lambda_1, \lambda_2, \dots, \lambda_n) \leq_d (\mu_1, \mu_2, \dots, \mu_n)$ if and only if $\lambda_i \leq \mu_i$, for all $i \in \{1, 2, \dots, n\}$.

The condition imposed on the support of a power series s allows to show that the minimal support $\mathrm{Supp_{min}(s)}$ is finite (see [6, Lemma 4.2]). As a consequence, s admits a finite *monomial presentation*:

$$s = \sum_{\lambda \in \mathrm{Supp_{min}(s)}} X^\lambda U_\lambda(X),$$

where $U_\lambda \in A[[X^*]]$ satisfies $U_\lambda(\mathbf{0}) \neq 0$, for any $\lambda \in \mathrm{Supp_{min}(s)}$. Denote by $m(s) = \#\mathrm{Supp_{min}(s)}$. When $m(s) = 1$ or, equivalently, the monomial representation of s has a single term, we say that s is of *monomial type*.

In this paper, we are interested in real generalized power series, that is $A = \mathbb{R}$, but we use different rings when we want to distinguish some variables and put the others into the coefficients. To be precise, if Y and Z are tuples of k and $n-k$ variables, respectively, we consider $\mathbb{R}[[(Y, Z)^*]]$ as a proper $\mathbb{R}$-subalgebra of $\mathbb{R}[[Y^*]][[Z^*]]$ by the natural monomorphism

$$s = \sum_{(\lambda,\mu) \in \mathbb{R}_+^n} a_{\lambda\mu} Y^\lambda Z^\mu \ \mapsto \ s^Z = \sum_{\mu \in \mathbb{R}_+^{n-k}} A_\mu Z^\mu, \quad \text{where } A_\mu = \sum_{\lambda \in \mathbb{R}_+^k} a_{\lambda\mu} Y^\lambda. \tag{1}$$

If $\mathrm{pr} : \mathbb{R}^n \to \mathbb{R}^{n-k}$ denotes the natural projection onto the last $n-k$ coordinates, for any power series $s \in \mathbb{R}[[(Y, Z)^*]]$ we have the inclusion $\mathrm{Supp_{min}}(s^Z) \subset \mathrm{pr}(\mathrm{Supp_{min}}(s))$, and as a consequence we get the inequality

$$m(s^Z) \leq m(s). \tag{2}$$

Let us write $\mathbb{R}[[Y, Z^*]]$ to denote the subalgebra of $\mathbb{R}[[(Y, Z)^*]]$ composed by the so-called *real mixed power series*: those formal real generalized power series s in the variables (Y, Z), such that the inclusion $\mathrm{Supp}(s) \subset \mathbb{N}^k \times \mathbb{R}_+^{n-k}$ holds, or equivalently, such that $s^Z \in \mathbb{R}[[Y]][[Z^*]]$.

Given an n-tuple of variables X and a polyradius $\rho = (\rho_1, \rho_2, \ldots, \rho_n) \in \mathbb{R}^n_{>0}$, denote by $\mathbb{R}\{X^*\}_\rho$ the subalgebra of $\mathbb{R}[[X^*]]$ consisting on those power series s for which

$$\|s\|_\rho := \sum_{\lambda \in \mathrm{Supp}(s)} |s_\lambda| \rho^\lambda < \infty.$$

The union of the $\mathbb{R}\{X^*\}_\rho$ along all the possible polyradius $\rho \in \mathbb{R}^n_{>0}$ is again a subalgebra $\mathbb{R}\{X^*\} \subset \mathbb{R}[[X^*]]$, and its elements are called *(real) convergent generalized power series*. We have that $\mathbb{R}\{X^*\}$ is also a local algebra, whose maximal ideal is given by $\mathfrak{m} \cap \mathbb{R}\{X^*\}$. If Y, Z are tuples of k and $n-k$ variables, respectively, and $\rho \in \mathbb{R}^n_{>0}$ is a polyradius, an element $s \in \mathbb{R}[[Y, Z^*]] \cap \mathbb{R}\{(Y, Z)^*\}_\rho$ gives rise to a continuous function

$$\begin{array}{rccc} f_s : & P^\rho_{k,n-k} & \to & \mathbb{R} \\ & x = (x_1, x_2, \ldots, x_n) & \mapsto & \sum_\lambda s_\lambda x^\lambda, \end{array} \tag{3}$$

where $P^\rho_{k,n-k} = (-\rho_1, \rho_1) \times (-\rho_2, \rho_2) \times \cdots \times (-\rho_k, \rho_k) \times [0, \rho_{k+1}) \times \cdots \times [0, \rho_n) \subset \mathbb{R}^k \times \mathbb{R}^{n-k}_+$, called the *sum of the power series* s. Moreover, f_s is real analytic at any point in the interior of $P^\rho_{k,n-k}$ and its germ at $\mathbf{0} \in \mathbb{R}^n$ is uniquely determined by the series s. We define the *convergent mixed power series* to be the elements of $\mathbb{R}\{Y, Z^*\} := \mathbb{R}[[Y, Z^*]] \cap \mathbb{R}\{(Y, Z)^*\}$.

2.2 Standard and generalized analytic manifolds

Let V be an open subset of $\mathbb{R}^n_+$ and let $g : V \to \mathbb{R}$ be a continuous function. Given a point $p = (p_1, p_2, \ldots, p_n) \in V$, consider $I_p := \{i : p_i = 0\} \subset \{1, 2, \ldots, n\}$, and put $\ell = \#I_p$ and $k = n - \ell$. We say that g is *generalized analytic* (or just $\mathcal{G}$*-analytic*) at p if there exists $s \in \mathbb{R}\{Y, Z^*\}$, where Y is a k-tuple and Z is an ℓ-tuple, such that for any $x = (x_1, x_2, \ldots, x_n)$ in a sufficiently small neighbourhood of $\mathbf{0}$ in $\mathbb{R}^k \times \mathbb{R}^\ell_+$, we have

$$g(p_1 + x_{\sigma(1)}, p_2 + x_{\sigma(2)}, \ldots, p_n + x_{\sigma(n)}) = f_s(x_1, x_2, \ldots, x_n),$$

where σ is a permutation of the set $\{1, 2, \ldots, n\}$ satisfying the relation $j \in I_p$ if and only if $\sigma(j) \in \{k+1, k+2, \ldots, n\}$. We say that g is *generalized analytic in* V if it so at every point p in V. In the definition above, the series s is uniquely determined by the germ of g at p, up to permutation of the variables Y and Z, separately. Thus, the set of germs of generalized analytic functions at p defines an $\mathbb{R}$-algebra isomorphic to $\mathbb{R}\{Y, Z^*\}$. On the other hand, if g is a generalized analytic function at some point $p \in \mathbb{R}^n_+$, then it is so in a neighbourhood of p in $\mathbb{R}^n_+$. Summarizing, the assignment $\mathcal{G}_n : V \mapsto \mathcal{G}_n(V)$, where V is an open subset of $\mathbb{R}^n_+$ and $\mathcal{G}_n(V)$ is the set of generalized analytic functions in V, is a sheaf of $\mathbb{R}$-algebras of continuous functions over $\mathbb{R}^n_+$, where the stalks $\mathcal{G}_{n,p}$ are local algebras. Moreover $\mathcal{G}_n$ contains the sheaf $\mathcal{O}_n$ of analytic functions, where $\mathcal{O}_n(V)$ is the $\mathbb{R}$-algebra of real functions in V which extend to real analytic functions on some open neighbourhood of V in $\mathbb{R}^n$.

With this formalism, and taking as local models the locally ringed spaces $\mathbb{O}_n := (\mathbb{R}^n_+, \mathcal{O}_n)$ and $\mathbb{G}_n := (\mathbb{R}^n_+, \mathcal{G}_n)$, we define both the categories of *standard and generalized (real)* *analytic manifolds (with boundary and corners)*. The objects in these categories are called $\mathcal{O}$-*manifolds* and $\mathcal{G}$-*manifolds*, respectively. In order to treat both together we write $\mathcal{A}$ to make reference either to $\mathcal{O}$ or to $\mathcal{G}$, and $\mathbb{A}$ to refer either to $\mathbb{O}$ or $\mathbb{G}$. An $\mathcal{A}$-manifold of dimension n is a locally ringed space $\mathcal{M} = (M, \mathcal{A}_M)$, where M is a second countable Hausdorff topological space (the *underlying space*) and $\mathcal{A}_M$ is a subsheaf of the sheaf $\mathcal{C}^0_M$ of germs of continuous real functions on M (the *structural sheaf*), which is locally isomorphic to the local model $\mathbb{A}_n$. That is, given $p \in M$ there is an open neighbourhood V of p in M, an open subset U of $\mathbb{R}^n_+$ and a homeomorphism $\varphi : V \to U$ inducing an isomorphism of the locally ringed spaces

$$(\varphi, \varphi^{\#}) : (V, \mathcal{A}_M|_V) \xrightarrow{\sim} (U, \mathcal{A}_n|_U),$$

where $\varphi^{\#}_p : \mathcal{A}_{n,\varphi(p)} \to \mathcal{A}_{M,p}$ is given by the composition $g \mapsto g \circ \varphi$ (as germs). A morphism between two $\mathcal{A}$-manifolds is just a morphism as locally ringed spaces, induced by composition with continuous maps on the underlying spaces (with an abuse of language, we frequently identify morphisms with the corresponding continuous maps). A couple (V, φ) in the above conditions is called a *local chart* of $\mathcal{M}$ at p, the components $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ of the isomorphism $\varphi : V \to U$ are *local coordinates at* p, and a family of local charts $\{(V_j, \varphi_j)\}_{j \in J}$ such that $M = \cup_{j \in J} V_j$ is an *atlas* of $\mathcal{M}$.

Let $\mathcal{M} = (M, \mathcal{A}_M)$ be an $\mathcal{A}$-manifold. Note that the underlying space M is a topological manifold with boundary, denoted by ∂M, and that the restriction $(M \backslash \partial M, \mathcal{A}_M|_{M \backslash \partial M})$ is a standard analytic manifold without boundary (consequently, generalized analytic manifolds without boundary are also standard). Also there is a natural stratification $\mathcal{S}_{\mathcal{M}}$ of $\mathcal{M}$ described as follows. If $p \in M$, and (V, φ) is a local chart at p, the number e_p of vanishing coordinates in $\varphi(p)$ (equal to $\# I_{\varphi(p)}$) does not depend on the local chart (V, φ) chosen (see [14]). In that way, there is a well-defined map

$$e : M \to \{0, 1, \ldots, n\}, \quad p \mapsto e_p,$$

which is upper semi-continuous. The elements of $\mathcal{S}_{\mathcal{M}}$ are the connected components of the fibers of e. Given $S \in \mathcal{S}_{\mathcal{M}}$, let us write $e_S := e_p$, where p is any point in S. Observe that $(S, \mathcal{A}_{\mathcal{M}}|_S)$ is a standard analytic manifold of dimension $n - e_S$. In particular, the boundary ∂M corresponds exactly with the points $p \in M$ with $e_p > 0$, that is, ∂M is equal to the union of strata of dimension strictly smaller than n. We have also that, ∂M is a *normal crossings divisor* with respect to the structural sheaf. That is, for each $p \in \partial M$, there exists a local chart (V, φ) of $\mathcal{M}$ at p such that

$$\partial M \cap V = \{q \in V : \ x_1(q) \cdot x_2(q) \cdot \ \cdots \ \cdot x_{e_p}(q) = 0\},$$

where $(x_1, x_2, \ldots, x_n)$ are the coordinates associated to φ.

Example 2.1 Let $\bar{\mathcal{O}}_k$ be the sheaf of real (standard) analytic functions in $\mathbb{R}^k$. The locally ringed space $(\mathbb{R}^k, \bar{\mathcal{O}}_k)$ is a generalized and standard analytic manifold, with a single chart $\psi_k : \mathbb{R}^k \to (0, \infty)^k$ defined by $(a_1, a_2, \ldots, a_k) \mapsto (e^{a_1}, e^{a_2}, \cdots, e^{a_k})$.

We observe at this point that the product is defined in the category of $\mathcal{A}$-manifolds. That is, given two generalized or standard analytic manifolds $\mathcal{M}_1 = (M_1, \mathcal{A}_{M_1})$ and $\mathcal{M}_2 = (M_2, \mathcal{A}_{M_2})$ of dimensions n and m, respectively, there is a natural $\mathcal{A}$-manifold of dimension $n + m$, that we denote by $\mathcal{M}_1 \times \mathcal{M}_2 = (M_1 \times M_2, \mathcal{A}_{M_1 \times M_2})$, unique up to isomorphism, solving the "product universal property". Without too much detail, the sheaf $\mathcal{A}_{M_1 \times M_2}$ is constructed as follows. Given a point $(p, q) \in M_1 \times M_2$ and two coordinate charts φ_1 :

$V_1 \to U_1$ and $\varphi_2 : V_2 \to U_2$ at p and q respectively, we have that

$$\mathcal{A}_{M_1\times M_2,(p,q)} = \{f \circ (\varphi_1 \times \varphi_2)_{(p',q')} :\ f \in \mathcal{A}_{n+m,(p',q')}\},$$

where $(p', q') = (\varphi_1(p), \varphi_2(q))$.

Example 2.2 The product $(\mathbb{R}^k, \bar{\mathcal{O}}_k) \times (\mathbb{R}^{n-k}_+, \mathcal{A}_{n-k})$, where $\mathcal{A} \in \{\mathcal{O}, \mathcal{G}\}$, has a natural structure of $\mathcal{A}$-manifold by means of the homeomorphism $\psi_k \times \mathrm{id}$, where ψ_k has been introduced in Example 2.1. We refer to this product by writing $(\mathbb{R}^k \times \mathbb{R}^{n-k}_+, \mathcal{A}_{k,n-k})$.

Remark 2.3 Let us consider a point $p \in M$ with $e_p = k$ and let (V, φ) be a local chart of $\mathcal{M}$ at p. Up to permutation, we can assume that $\varphi(p) = (a_1, a_2, \ldots, a_k, 0, \ldots, 0)$ with $a_i \neq 0$ for all $i \in \{1, 2, \ldots, k\}$. We can split the local coordinates $\mathbf{x}$ defined by φ in two groups $\mathbf{x} = (\mathbf{y}, \mathbf{z})$, where $\mathbf{y} = (y_1, y_2, \ldots, y_k)$ are standard analytic functions at p and $\mathbf{z} = (z_{k+1}, z_{k+2}, \ldots, z_n)$ are generalized functions. By means of translations $y_i' = y_i - a_i$ in the analytic coordinates we obtain a new isomorphism

$$\varphi' : V \mapsto (\psi_k \times \mathrm{id})^{-1}(\varphi(V)) \subset \mathbb{R}^k \times \mathbb{R}^{n-k}_+.$$

We consider also φ' as a *coordinate chart centered at* p in the sense that $\varphi'(p) = \mathbf{0} \in \mathbb{R}^k \times \mathbb{R}^{n-k}_+$, and we usually assume that our charts are centered charts.

Let us recall now the expression in coordinates of the continuous maps inducing morphisms of generalized functions (details in [14, Proposition 3.16]). Consider two generalized analytic manifolds $\mathcal{M}_1 = (M_1, \mathcal{G}_{M_1})$ and $\mathcal{M}_2 = (M_2, \mathcal{G}_{M_2})$ and a continuous function $\phi : M_1 \to M_2$ inducing a morphism between $\mathcal{M}_1$ and $\mathcal{M}_2$. Given $p \in M_1$ and $q = \phi(p) \in M_2$, take $(V_p, \varphi_p), (W_q, \psi_q)$ charts centered at p and q, respectively. Following notation in Remark 2.3, denote by $\mathbf{y}$ and $\mathbf{z}$ the k standard and $n-k$ generalized coordinates defining φ_p, respectively. Up to permutation, we can assume also that the first k' coordinates defining ψ_q are standard and the other $n'-k'$ are generalized. Then, the j-th component $\tilde{\phi}_j$ of $\tilde{\phi} = \psi_q \circ \phi \circ \varphi_p^{-1}$ is a generalized analytic function and for $j = k'+1, k'+2, \ldots, n'$, we have that

$$\tilde{\phi}_j = \mathbf{z}^{\lambda_j} U_j(\mathbf{y}, \mathbf{z}), \quad U_j(\mathbf{0}, \mathbf{0}) \neq 0, \quad \lambda_j \in \mathbb{R}^{n-k}_+ \setminus \{0\}. \tag{4}$$

Moreover, if ϕ induces an isomorphism, we have that ϕ is a homeomorphism, $n = n'$, $k = k'$, the map $\mathbf{t} \in \mathbb{R}^k \mapsto (\tilde{\phi}_1(\mathbf{t}, \mathbf{0}), \tilde{\phi}_2(\mathbf{t}, \mathbf{0}), \ldots, \tilde{\phi}_k(\mathbf{t}, \mathbf{0}))$ is an analytic isomorphism, and, if we write $\lambda_j = (\lambda_{j,1}, \lambda_{j,2}, \ldots, \lambda_{j,n-k})$ in Eq. (4), up to a permutation of coordinates $\mathbf{z}$ we have

$$\lambda_{j,j-k} > 0, \quad \lambda_{j,\ell} = 0,\ \ell \in \{1, 2, \ldots, n-k\}\setminus\{j-k\}, \tag{5}$$

for all $j = k+1, k+2, \ldots, n$.

We end this section introducing some notation and definitions concerning the strata of the natural stratification $\mathcal{S}_{\mathcal{M}}$. Given a stratum S in $\mathcal{S}_{\mathcal{M}}$, denote by $\overline{S}$ the closure of S in M, and define $\dim(\bar{S}) := \dim(S)$. We write $\mathcal{Z}_{\mathcal{M}} := \{\overline{S} \subset M :\ S \in \mathcal{S}_{\mathcal{M}}\}$. For $j = 0, 1, \ldots, n$, denote by $\mathcal{Z}^j_{\mathcal{M}}$ the set of elements in $\mathcal{Z}_{\mathcal{M}}$ with dimension j, that is

$$\mathcal{Z}^j_{\mathcal{M}} = \{\bar{S} \in \mathcal{Z}_{\mathcal{M}} :\ e_S = n - j\}.$$

The elements of $\mathcal{Z}^0_{\mathcal{M}}$ are the strata of dimension 0, and are called *corner points*, the elements of $\mathcal{Z}^1_{\mathcal{M}}$ are called *edges* and the elements of $\mathcal{Z}^{n-1}_{\mathcal{M}}$ are called *components* of ∂M. Note that ∂M is the union of its components.

For each $Z \in \mathcal{Z}_{\mathcal{M}}$, we denote by $\mathcal{Z}_{\mathcal{M}}(Z)$ the subset of $\mathcal{Z}_{\mathcal{M}}$ whose elements are contained in Z, and for each $j = 0, 1, \ldots, n$, we write $\mathcal{Z}^j_M(Z) = \mathcal{Z}_{\mathcal{M}}(Z) \cap \mathcal{Z}^j_{\mathcal{M}}$. We write for short

$p \in \mathcal{Z}^0_{\mathcal{M}}$ instead of $\{p\} \in \mathcal{Z}^0_{\mathcal{M}}$, and when no confusion arises, we will put $\mathcal{Z}$ instead of $\mathcal{Z}_{\mathcal{M}}$, $\mathcal{Z}^j$ instead of $\mathcal{Z}^j_{\mathcal{M}}$, etc.

2.3 Monomial complexity along strata

We introduce in this section the concept of monomial complexity along a stratum and the definition of stratified monomial type function.

Let us consider a generalized analytic manifold $\mathcal{M} = (M, \mathcal{G}_M)$ and a stratum S of its natural stratification $\mathcal{S}$. Take a local chart (V, φ) of $\mathcal{M}$ centered at some $p \in S$, write $e = e_S$ and $k = \dim S = n - e$. We can split the coordinates defining φ, up to reorder them, as $(\mathbf{y}, \mathbf{z})$, where $\mathbf{y} = (y_1, y_2, \dots, y_k)$ are standard analytic coordinates in $S \cap V$ and $\mathbf{z} = (z_1, z_2, \dots, z_e)$ are generalized functions such that $S \cap V = \{q \in V : \ z_1(q) = z_2(q) = \cdots = z_e(q) = 0\}$. Shrinking V if necessary, the chart φ provides an isomorphism

$$\Psi^p_\varphi : \mathbb{R}\{Y, Z^*\} \to \mathcal{G}_{M,p}, \quad s \mapsto f_s \circ \varphi,$$

where Y and Z are k and e tuples, respectively, and f_s is the sum of the power series s introduced in Eq. (3). Given $f \in \mathcal{G}_{M,p}$ and $s \in \mathbb{R}\{Y, Z^*\}$ the mixed power series such that $\Psi^p_\varphi(s) = f$, we denote

$$\mathrm{Supp}_S(f; \varphi) = \mathrm{Supp}(s^Z) \subset \mathbb{R}^e_+, \quad \mathrm{Supp}_{\min,S}(f; \varphi) = \mathrm{Supp}_{\min}(s^Z) \subset \mathbb{R}^e_+, \tag{6}$$

where $s^Z \in \mathbb{R}\{Y\}\{Z^*\}$ has been introduced in Eq. (1).

Lemma 2.4 *Let S be a stratum in $\mathcal{S}$ with $e = e_S$. Take an open subset U of M such that $U \cap S \neq \emptyset$, and a function $f \in \mathcal{G}_M(U)$. Consider two local charts (V_1, φ_1) and (V_2, φ_2), centered at p and q respectively, with $p, q \in S \cap U$. There exists a tuple $(\gamma_1, \gamma_2, \dots, \gamma_e) \in \mathbb{R}^e_{>0}$ such that $(\lambda_1, \lambda_2, \dots, \lambda_e) \in \mathrm{Supp}_{\min,S}(f_q; \varphi_2)$ if and only if $(\gamma_1\lambda_1, \gamma_2\lambda_2, \dots, \gamma_e\lambda_e) \in \mathrm{Supp}_{\min,S}(f_p; \varphi_1)$.*

Proof Using that S is path connected and by compactness of a given path from p to q, we can reduce the problem to the case where both points p and q belong to the same connected component W of $U \cap V_1 \cap V_2$. Write $\mathbf{y} = (y_1, y_2, \dots, y_{n-e})$, $\mathbf{z} = (z_1, z_2, \dots, z_e)$, and also $\bar{\mathbf{y}} = (\bar{y}_1, \bar{y}_2, \dots, \bar{y}_{n-e})$, $\bar{\mathbf{z}} = (\bar{z}_1, \bar{z}_2, \dots, \bar{z}_e)$, where, up to reordering, $(\mathbf{y}, \mathbf{z})$ are the coordinate functions associated to φ_1 and $(\bar{\mathbf{y}}, \bar{\mathbf{z}})$ are the ones associated to φ_2, in such a way that $\mathbf{y}|_{S\cap V_1}$, $\bar{\mathbf{y}}|_{S\cap V_2}$ are analytic coordinates in $W \cap S$. That is, we have

$$W \cap S = \{z_1 = z_2 = \cdots = z_e = 0\} = \{\bar{z}_1 = \bar{z}_2 = \cdots = \bar{z}_e = 0\}.$$

In view of Eqs. (4) and (5), up to reordering the variables $\mathbf{z}$, the change of coordinates $\varphi_2 \circ \varphi_1^{-1}$ satisfies, for any $j = 1, 2, \dots, n - e$ and $\ell = 1, 2, \dots, e$, that $\bar{y}_j = g_j(\mathbf{y}, \mathbf{z})$, and $\bar{z}_\ell = z_\ell^{\gamma_\ell} h_\ell(\mathbf{y}, \mathbf{z})$, where g_j, h_ℓ are generalized analytic functions such that $\mathbf{y} \mapsto g_j(\mathbf{y}, \mathbf{0})$ is a standard analytic non-constant function, $\gamma_\ell > 0$ and $h_\ell(\mathbf{0}, \mathbf{0}) \neq 0$. We summarize these expressions by writing $\bar{\mathbf{y}} = g$ and $\bar{\mathbf{z}} = \mathbf{z}^\gamma h$. If $\Delta_2 := \mathrm{Supp}_{\min,S}(f_q; \varphi_2) = \{\mu_1, \mu_2, \dots, \mu_t\}$, the expression of f in coordinates $(\bar{\mathbf{y}}, \bar{\mathbf{z}})$ is

$$f|_W = \bar{\mathbf{z}}^{\mu_1}\Phi_1(\bar{\mathbf{y}}, \bar{\mathbf{z}}) + \bar{\mathbf{z}}^{\mu_2}\Phi_2(\bar{\mathbf{y}}, \bar{\mathbf{z}}) + \cdots + \bar{\mathbf{z}}^{\mu_t}\Phi_t(\bar{\mathbf{y}}, \bar{\mathbf{z}}),$$

where $\Phi_j(\bar{\mathbf{y}}, \mathbf{0}) \not\equiv 0$, for any $j = 1, 2, \dots, t$. Applying the change of coordinates in order to get the expression of f in $(\mathbf{y}, \mathbf{z})$, we obtain

$$f|_W = \mathbf{z}^{\gamma\mu_1}\Psi_1(\mathbf{y}, \mathbf{z}) + \mathbf{z}^{\gamma\mu_2}\Psi_2(\mathbf{y}, \mathbf{z}) + \cdots + \mathbf{z}^{\gamma\mu_t}\Psi_t(\mathbf{y}, \mathbf{z}), \quad \Psi_k(\mathbf{y}, \mathbf{z}) = h^{\mu_k}\Phi_k(g, \mathbf{z}^\gamma h),$$

where $\gamma\mu_k := (\gamma_1\mu_{k,1}, \gamma_2\mu_{k,2}, \ldots, \gamma_e\mu_{k,e})$. Note that $\Psi_k(\mathbf{y}, \mathbf{0}) \neq 0$, so $\gamma\mu_k \in \mathrm{Supp}_S(f_p; \varphi_1)$, for any $k = 1, 2, \ldots, t$. Moreover, each $\lambda \in \mathrm{Supp}_S(f_p; \varphi_1)$ is such that $\gamma\mu_k \leq_d \lambda$, for some $k \in \{1, 2, \ldots, t\}$. Hence

$$\mathrm{Supp}_{\min,S}(f_p; \varphi_1) \subset \bar{\Delta}_2 := \{\gamma\mu_1, \gamma\mu_2, \ldots, \gamma\mu_t\}.$$

Now, recall that any pair of elements $\mu_r, \mu_s \in \Delta_2$ are incomparable for the order $\leq_d$. Then, the elements of $\bar{\Delta}_2$ are also mutually incomparable and the equality $\mathrm{Supp}_{\min,S}(f_p; \varphi_1) = \bar{\Delta}_2$ holds. □

The next definition makes sense as a result of Lemma 2.4.

Definition 2.5 Let $f : M \to \mathbb{R}$ be a generalized analytic function. The *monomial complexity$m_S(f)$ of f along$S$* is the number $m_S(f) = \#\mathrm{Supp}_{\min,S}(f_p; \varphi)$, where p is some point at S and (V, φ) is a local chart centered at p.

When $S = \{p\}$, we just write $m_p(f)$ instead of $m_{\{p\}}(f)$.

Lemma 2.6 (Horizontal stability) *Let $f : M \to \mathbb{R}$ be a generalized analytic function. Given two strata S and T such that $T \subset \bar{S}$, we have the inequality $m_S(f) \leq m_T(f)$.*

Proof It is a direct consequence of Eq. (2). □

Definition 2.7 Let $\mathcal{M} = (M, \mathcal{G}_M)$ be a generalized analytic manifold. A generalized analytic function $f : M \to \mathbb{R}$ is of *stratified monomial type* if $m_S(f) = 1$ for any stratum $S \in \mathcal{S}$.

2.4 Standardizations and blowing-ups

Let $\mathcal{M} = (M, \mathcal{A}_M)$ be a standard or generalized analytic manifold and let $Y \subset M$ be a connected closed subset of M. We say that Y is a *geometric center for $\mathcal{M}$* if at each $p \in Y$ there is a local chart (V, φ) centered at p and some $r \in \{1, 2, \ldots, n\}$, such that

$$Y \cap V = \{q \in V : \; x_1(q) = x_2(q) = \cdots = x_r(q) = 0\},$$

where $(x_1, x_2, \ldots, x_n)$ are the coordinates defined by φ. For instance, if $Z \in \mathcal{Z}^j$ for some $j \leq n-1$, then Z is a geometric center (the number r in the definition above is $n-j$, independently of the point $q \in Z$); such $Z \in \mathcal{Z}$ are called *combinatorial geometric centers*.

When $\mathcal{M} = (M, \mathcal{O}_M)$ is a standard analytic manifold (with boundary and corners), the construction of the *(real) blowing-up with center Y* is quite well-known (see details in [14]). It consists of a proper morphism of standard analytic manifolds

$$(\pi_Y, \pi_Y^{\#}) : (\widetilde{M}, \mathcal{O}_{\widetilde{M}}) \to (M, \mathcal{O}_M)$$

inducing an isomorphism between $\widetilde{M}\backslash E$ and $M\backslash Y$, where the *exceptional divisor* $E = \pi_Y^{-1}(Y)$ is a new component of $\partial\widetilde{M}$. On the contrary, when $\mathcal{M} = (M, \mathcal{G}_M)$ is a generalized analytic manifold, the blowing-up of $\mathcal{M}$ with geometric center Y may even not exist and, if it does, it depends on the so called *standardization* of $\mathcal{M}$. We devote this section to recalling this concept and what do we mean by blowing-up in the category of generalized analytic manifolds. It is worth noting that our definition is slightly different (but equivalent) to the original one in [14].

Let $\mathcal{N} = (N, \mathcal{O}_N)$ be a standard analytic manifold. Given $p \in N$ and $\varphi : V \to U$ a local chart at p, we define $\mathcal{O}_N^{\epsilon}(V)$ to be the $\mathbb{R}$-algebra of continuous functions $f : V \to \mathbb{R}$ such that $f \circ \varphi^{-1}$ belongs to $\mathcal{G}_n(U)$. Taking the sheaf associated to the presheaf $V \mapsto \mathcal{O}_N^{\epsilon}(V)$, we obtain a generalized analytic manifold $\mathcal{N}^\epsilon = (N, \mathcal{O}_N^\epsilon)$, called the *enrichment of* $\mathcal{N}$ (see [14, Proposition 3.17]). Note that the stratifications $\mathcal{S}_{\mathcal{N}^\epsilon}$ and $\mathcal{S}_\mathcal{N}$ coincide. Moreover, if Y is a geometric center for $\mathcal{N}$, then it is also a geometric center for $\mathcal{N}^\epsilon$.

The assignment $\mathcal{N} \to \mathcal{N}^\epsilon$ is not a functor from the category of $\mathcal{O}$-manifolds to the one of $\mathcal{G}$-manifolds, since the morphisms do not lift to the enrichments unless they can be expressed locally as tuples of monomial-type functions (see [14, Prop. 3.19]).

Definition 2.8 A *standardization of a generalized analytic manifold* $\mathcal{M} = (M, \mathcal{G}_M)$ is a subsheaf $\mathcal{O}$ of $\mathcal{G}_M$ such that $\mathcal{N} = (M, \mathcal{O})$ is a standard analytic manifold with $\mathcal{N}^\epsilon = \mathcal{M}$. A generalized analytic manifold $\mathcal{M}$ is said to be *standardizable* if there exists a standardization $\mathcal{O} \subset \mathcal{G}_M$ of it.

Note that a standardization is the same thing as providing an atlas $\mathfrak{A}_M = \{(V_j, \varphi_j)\}_{j \in J}$ of $\mathcal{M}$ such that, for any $i, j \in J$, the change of coordinates $\varphi_j \circ \varphi_i^{-1}$ is standard analytic in its domain of definition $\varphi_i(V_i \cap V_j) \subset \mathbb{R}^n_+$.

Remark 2.9 Let $\mathcal{M} = (M, \mathcal{A})$ be an analytic manifold without boundary (hence $\mathcal{M}$ is at the same time standard and generalized). Take a point $p \in M$, an open neigbourhood V of p, and a coordinate system $(x_1, x_2, \ldots, x_n)$ defined in V and centered at p. For each $i = 1, 2, \ldots, n$, take odd positive integers $m_i \in \mathbb{Z}_{>0}$ and consider the functions $y_i = x_i^{m_i}$. The map $\varphi : V \to \mathbb{R}^n$ defined by $\varphi(q) = (y_1(q), y_2(q), \ldots, y_n(q))$ is a homeomorphism onto $U = \varphi(V)$ and the sheaf $\tilde{\mathcal{O}}$ defined locally at $p \in V$ by

$$\tilde{\mathcal{O}}_p = \{f \circ \varphi : \ f \in \bar{\mathcal{O}}_{n,\varphi(p)}\} \subset \mathcal{A}_p$$

is a subsheaf of $\mathcal{A}|_V$ so that $(V, \tilde{\mathcal{O}})$ is a standard analytic manifold. However $\tilde{\mathcal{O}}^\epsilon = \mathcal{A}|_V$ if and only if $m_i = 1$ for all $i = 1, 2, \ldots, n$, or equivalently, if φ is a local chart if $\mathcal{M}$. In this case, there is a unique standardization for $\mathcal{M}$: the total sheaf $\mathcal{A}$ itself.

On the contrary, when $\mathcal{M} = (M, \mathcal{G})$ is a generalized analytic manifold with $\partial M \neq \emptyset$, we may have a lot of variation. For example, if we take a point $p \in \partial M$ and a small enough neighbourhood V of p, there are infinitely many standardizations of the local generalized analytic manifold $(V, \mathcal{G}|_V)$; on the other hand, there are also examples of non-standardizable generalized manifolds like the one in [14, Example 3.20].

Definition 2.10 A *center*ξ *owing-up for a generalized analytic manifold*$\mathcal{M}$ is a pair $\xi = (Y, \mathcal{O})$, where $\mathcal{O}$ is a standardization of $\mathcal{M}$ and Y is a geometric center for $(M, \mathcal{O})$.

Remark 2.11 We can have a geometric center Y for a generalized manifold $\mathcal{M}$ and a standardization $\mathcal{O}$ of $\mathcal{M}$ such that Y is not a geometric center for $\mathcal{N} = (M, \mathcal{O})$. For example, let us take the generalized analytic manifold $\mathcal{M} = (V, \mathcal{G}_{1,1}|_V)$, where $V \subset \mathbb{R} \times \mathbb{R}_+$ is a small neighbourhood of the origin $(0, 0) \in \mathbb{R}^2$. Let (y, z) be the natural coordinates in $\mathbb{R}^2$ and let Y be the closed topological subspace of V given by the zeros of $y - z^\lambda$, where $\lambda \notin \mathbb{Z}_{>0}$. Note that (y', z) with $y' = y - z^\lambda$ are also coordinates of $\mathcal{M}$ at the origin, and hence Y is a geometric center for $\mathcal{M}$. But, if we take the standardization $\mathcal{O} \subset \mathcal{G}_{1,1}|_V$ given by the local chart (y, z), then Y is not a geometric center for $\mathcal{N} = (V, \mathcal{O})$.

Now we have the ingredients to introduce the blowing-up morphisms in the category of generalized manifolds.

Definition 2.12 Let $\mathcal{M}$ be a generalized analytic manifold and let $\xi = (Y, \mathcal{O})$ be a center of blowing-up for $\mathcal{M}$. The *blowing-up*$\pi_\xi : \mathcal{M}_\xi \to \mathcal{M}$ *with center*ξ is the morphism of $\mathcal{G}$-manifolds induced by the blowing-up $\pi_Y : \widetilde{\mathcal{N}} \to \mathcal{N}$ of the standard manifold $\mathcal{N} = (M, \mathcal{O})$ with center Y (note that we are writing $\mathcal{M}_\xi = \tilde{\mathcal{N}}^\epsilon$).

If $Z \in \mathcal{Z}_{\mathcal{M}}$ is a combinatorial geometric center, and $\mathcal{O} \subset \mathcal{G}_M$ is a standardization, we can see that $\xi = (Z, \mathcal{O})$ is a center of blowing-up for $\mathcal{M}$. Such a ξ is called a *combinatorial center of blowing-up*, and we say that $\pi_\xi : \mathcal{M}_\xi \to \mathcal{M}$ is a *combinatorial blowing-up*.

3 The category of monomial analytic manifolds

We devote this section to introducing a subcategory of generalized manifolds, called *monomial generalized analytic manifolds*, which has many combinatorial properties. The objects of this subcategory are those $\mathcal{G}$-manifolds equipped with an atlas for which the change of coordinates are expressed as monomial morphisms. We codify these changes of coordinates by means of matrices of exponents, and we do the same for the morphisms. The formulation in this combinatorial language allows us to conclude that monomial generalized analytic manifolds are always standardizable. This result will be one of the keys to prove the stratified reduction of singularities.

3.1 Monomial manifolds

We consider an $\mathcal{A}$-manifold $\mathcal{M} = (M, \mathcal{A}_M)$, where $\mathcal{A} \in \{\mathcal{O}, \mathcal{G}\}$. Let us write $\partial M = \bigcup_{i \in I} E_i$, where $\{E_i\}_{i \in I}$ is the family of components of ∂M, and assume that I is a finite set. As in [15], we say that ∂M has *strong normal crossings* if for each $J \subset I$ the intersection $E_J = \bigcap_{j \in J} E_i$ is a connected set (in particular $E_\emptyset = M$ is connected). In this case, the stratification $\mathcal{S}$ and the family $\mathcal{Z}$ of closures of strata can be codified combinatorially by means of the bijection $\mathcal{H} \to \mathcal{Z}$ given by $J \mapsto E_J$, where

$$\mathcal{H} = \mathcal{H}_{\mathcal{M}} := \{J \subset I;\ E_J \neq \emptyset\}.$$

Given $Z \in \mathcal{Z}$, the element $I_Z \in \mathcal{H}$ such that $E_{I_Z} = Z$ is called the *index set of* Z. We use the notation $I_p := I_{\{p\}}$ when $p \in \mathcal{Z}^0$. Observe that $\# I_Z = \dim M - \dim Z$. In particular, if $p \in \mathcal{Z}^0$ we have that I_p is a set with $\dim M$ elements. Note also that $\tilde{Z} \in \mathcal{Z}(Z)$ if and only if $I_Z \subset I_{\tilde{Z}}$.

Remark 3.1 Let $Y \in \mathcal{Z}^1$ be a compact edge. If ∂M has strong normal crossings, we have that $\mathcal{Z}^0(Y)$ consists exactly of two corner points p and q and there are exactly two different elements $i_p, i_q \in I \setminus I_Y$ such that $I_p = I_Y \cup \{i_p\}$ and $I_q = I_Y \cup \{i_q\}$.

Let us fix from now on an $\mathcal{A}$-manifold $\mathcal{M} = (M, \mathcal{A}_M)$ with at least one corner point and such that ∂M has strong normal crossings. Given $p \in \mathcal{Z}^0$, define the set

$$V_p^\star := \bigcup \{S_J \in \mathcal{S} :\ J \subset I_p\} = \bigcup \{S \in \mathcal{S} :\ p \in \overline{S}\}.$$

We have that $V_p^\star$ is an open neighbourhood of p in M homeomorphic to $\mathbb{R}_+^n$. A chart $(V_p^\star, \varphi_p)$ defined on the whole $V_p^\star$, centered at p and realizing a homeomorphism $\varphi_p : V_p^\star \to \mathbb{R}_+^n$ is called an *affine chart* or *m-chart at* p. For any $i \in I_p$, we denote by $x_{p,i} : V_p^\star \to \mathbb{R}$ the coordinate component of φ_p satisfying

$$E_i \cap V_p^\star = \{q \in V_p^\star :\ x_{p,i}(q) = 0\}.$$

The family of functions $\mathbf{x}_p = (x_{p,i})_{i \in I_p}$ is equal to the family of coordinates of φ_p. Regardless of the ordering of these coordinates, we just identify φ_p with $\mathbf{x}_p$. Even more, since the sets $V_p^\star$ are completely determined by the stratification of $\mathcal{M}$, we identify $(V_p^\star, \varphi_p)$ with $\mathbf{x}_p$ and we say simply that $\mathbf{x}_p$ is an *m-chart at* p or that $\mathbf{x}_p$ are *affine coordinates at* p.

Assume that $\mathcal{M}$ has an atlas $\mathfrak{a} = \{\mathbf{x}_p\}_{p\in\mathcal{Z}^0}$, where each $\mathbf{x}_p$ is an m-chart at p. With the convention above, note that in fact $\mathfrak{a}$ is not exactly an atlas, but an equivalence class of atlases, since we have not considered a particular ordering of the coordinates $\mathbf{x}_p$. We say that $\mathfrak{a}$ is a *monomial atlas* if all the changes of coordinates have a purely monomial expression. More precisely, given two corner points $p, q \in \mathcal{Z}^0$, for each $i \in I_q$ there exist maps $r_i : I_p \to \mathbb{R}$, given by $j \mapsto r_i(j) =: r_{ij}$ such that the change of coordinates $\mathbf{x}_q \circ \mathbf{x}_p^{-1}$ has the following expression

$$x_{q,i} = \mathbf{x}_p^{r_i}, \quad \mathbf{x}_p^{r_i} = \prod_{j\in I_p} x_{p,j}^{r_{ij}}. \tag{7}$$

Definition 3.2 A *monomial$\mathcal{A}$-manifold* is a pair $(\mathcal{M}, \mathfrak{a})$, where $\mathcal{M}$ is an $\mathcal{A}$-manifold with $\mathcal{Z}^0 \neq \emptyset$ whose boundary ∂M has strong normal crossings, and $\mathfrak{a} = \{\mathbf{x}_p\}_{p\in\mathcal{Z}^0}$ is a monomial atlas over $\mathcal{M}$.

Remark 3.3 Assume that $\mathcal{M}$ is an $\mathcal{A}$-manifold admitting two monomial atlases $\mathfrak{a}$ and $\mathfrak{a}'$, that are compatible in the sense that for all $p \in \mathcal{Z}^0$ we have that $\mathbf{x}'_p \circ \mathbf{x}_p^{-1}$ has a purely monomial expression, where $\mathbf{x}_p \in \mathfrak{a}$ and $\mathbf{x}'_p \in \mathfrak{a}'$. When $\mathcal{A} = \mathcal{O}$ we have necessarily that $\mathfrak{a} = \mathfrak{a}'$. On the contrary, when $\mathcal{A} = \mathcal{G}$ we have a lot of variation. Indeed, from the monomial manifold $(\mathcal{M}, \mathfrak{a})$, we can obtain a different monomial atlas of $\mathcal{M}$ just by replacing at a single point $p \in \mathcal{Z}^0$ the affine coordinates $\mathbf{x}_p \in \mathfrak{a}$ over $V_p^\star$, with the affine coordinates $\mathbf{y}_p$ over $V_p^\star$ defined by $y_{p,i} = x_{p,i}^{s_i}$, where $s_i \in \mathbb{R}_{>0}$, for all $i \in I_p$.

Definition 3.4 Let us consider two monomial $\mathcal{A}$-manifolds $(\mathcal{M}_1, \mathfrak{a}_1)$ and $(\mathcal{M}_2, \mathfrak{a}_2)$ and let $\phi : M_1 \to M_2$ be a continuous map. We say that ϕ is a *morphism of monomial $\mathcal{A}$-manifolds* if it provides an $\mathcal{A}$-manifolds morphism between $\mathcal{M}_1$ and $\mathcal{M}_2$, and moreover:

- For any point $p \in \mathcal{Z}^0_{\mathcal{M}_1}$ we have $\phi(p) \in \mathcal{Z}^0_{\mathcal{M}_2}$.
- The expression of ϕ in the atlases $\mathfrak{a}_1$ and $\mathfrak{a}_2$ of $\mathcal{M}_1$ and $\mathcal{M}_2$, respectively, is monomial; that is, if $p \in \mathcal{Z}^0_{\mathcal{M}_1}$ and $\bar{p} = \phi(p)$, $\bar{\mathbf{x}}_{\bar{p}} \in \mathfrak{a}_2$ and $\mathbf{x}_p \in \mathfrak{a}_1$, the composition $\bar{\mathbf{x}}_{\bar{p}} \circ \phi \circ \mathbf{x}_p^{-1}$ is written as

$$\bar{x}_{\bar{p},i} = \mathbf{x}_p^{b_i}, \quad \text{where } b_i : I_p \to \mathbb{R}_+, \quad \text{for all } i \in I_{\bar{p}}. \tag{8}$$

The first example of a monomial $\mathcal{A}$-manifold is the following one: Let $\mathcal{M} = (M, \mathcal{A}_M)$ be an $\mathcal{A}$-manifold, and let (V, φ) be an affine chart centered at some corner point p. Denote by $\mathbf{x}_p$ the tuple of components of φ. The pair $((V, \mathcal{G}_M|_V), \mathbf{x}_p)$ is a monomial $\mathcal{A}$-manifold called an *m-corner*.

3.2 Combinatorial data of monomial manifolds

In this paragraph, we codify the objects and morphisms on the category of monomial $\mathcal{A}$-manifolds through their associated *combinatorial data*. Firstly, we need to introduce some notation:

Notation 3.5 Let $\mathbb{R}^I$ be the set of maps from I to $\mathbb{R}$, where I is a finite set.

- We denote by $\mathbb{1}_I : I \to \mathbb{R}$ the element of $\mathbb{R}^I$ defined by setting $\mathbb{1}_I(i) = 1$, for all $i \in I$.
- Given $\lambda \in \mathbb{R}^I$, we denote by D_λ the element of $\mathbb{R}^{I\times I}$ defined by setting $D_\lambda(i, i) = \lambda(i)$, for all $i \in I$ and $D_\lambda(i, j) = 0$, when $i \neq j$.
- Given $A : I \times J \to \mathbb{R}$ and $B : J \times K \to \mathbb{R}$, with I, J and K finite sets, we define:

– AB to be the element of $\mathbb{R}^{I\times K}$ given by $(i,k)\mapsto \sum_{j\in J} A(i,j)B(j,k)$.
– A^{-1} as the element of $\mathbb{R}^{J\times I}$ (if it exists) such that $A^{-1}A = D_{\mathbb{1}_J}$ and $AA^{-1} = D_{\mathbb{1}_I}$.

Roughly, we consider maps $A : I \times J \to \mathbb{R}$ as matrices of size $\#I \times \#J$ with real coefficients, but without a specified order in the sets I and J. We frequently use the matrix notation $A_{ij} = A(i,j)$, for $(i,j) \in I \times J$.

Take a monomial $\mathcal{A}$-manifold $(\mathcal{M}, \mathfrak{a})$. Given two corner points $p, q \in \mathcal{Z}^0$, let $\mathbf{x}_p$ and $\mathbf{x}_q$ be the affine coordinates at p and q, respectively, belonging to $\mathfrak{a}$. We codify the change of coordinates $\mathbf{x}_q \circ \mathbf{x}_p^{-1}$ expressed by the relations in Eq. (7) by means of the *matrix of exponents*

$$C^{pq} : I_q \times I_p \to \mathbb{R}, \text{ given by } (i,j) \mapsto r_{ij}.$$

Note that we have the equality $C^{qp} = (C^{pq})^{-1}$.

Definition 3.6 The *combinatorial data of a monomial* $\mathcal{A}$*-manifold* $(\mathcal{M}, \mathfrak{a})$ is the collection $\mathfrak{C}_{(\mathcal{M},\mathfrak{a})} = \{C^{pq}\}_{p,q\in\mathcal{Z}^0}$.

Let $p, q \in \mathcal{Z}^0$ be two corner points in $\mathcal{M}$. A *path (of compact edges) from* p *to* q is a list $\mathcal{P} = (Y_1, Y_2, \ldots, Y_k)$, where each Y_j is a compact edge in $\mathcal{M}$, such that $p \in Y_1$, $q \in Y_k$, and the intersection $p_j = Y_j \cap Y_{j+1}$ is a corner point, for all $j = 1, 2, \ldots, k-1$. Note that, for any pair of corner points $p, q \in \mathcal{Z}^0$, there is always a path $\mathcal{P} = (Y_1, Y_2, \ldots, Y_k)$ from p to q. Moreover, we can assume that $Y_1 \cup Y_2 \cup \cdots \cup Y_k \subset E_J$, where $J = I_p \cap I_q$, because the strong normal crossings condition assures the connectedness of E_J. We say in this case that $\mathcal{P}$ is a *path for* p *to* q *inside* E_J. Given a path $\mathcal{P}$ from p to q inside E_J, we have the equality

$$C^{pq} = C^{p_{k-1}q} \cdots C^{p_1p_2} C^{pp_1} \in \mathbb{R}^{I_q \times I_p}, \tag{9}$$

where $p_i = Y_i \cap Y_{i+1}$, for each $i = 1, 2, \ldots, k-1$. That is, the matrices of exponents between corner points connected by edges generate the whole combinatorial data $\mathfrak{C}_{(\mathcal{M},\mathfrak{a})}$ just by taking products. From now on, if p and q are the two corner points of a compact edge Y, we simply say that p and q are *connected through* Y, and we write to emphasize $C_Y^{pq} = C^{pq}$.

Let $(\mathcal{M}, \mathfrak{a})$ be a monomial generalized analytic manifold. Let us show some properties for the combinatorial data of $(\mathcal{M}, \mathfrak{a})$.

A corner point $p \in \mathcal{Z}^0$ belongs to $E_K = \overline{S}_K$ if and only if $K \subset I_p$. Hence, if we fix two corner points p and q in $\mathcal{Z}^0$, the smallest stratum containing both p and q in its closure is S_J, with $J = I_p \cap I_q$. Moreover, we have

$$V_p^\star \cap V_q^\star = \bigcup_{K \subset J} S_K.$$

Let $\mathbf{x}_p, \mathbf{x}_q \in \mathfrak{a}$ be the affine coordinates at p and q, respectively. Given a point $a \in S_J$, we consider the coordinate systems $\mathbf{x}_p^a = \{x_{p,i}^*\}_{i\in I_p}$ and $\mathbf{x}_q^a = \{x_{q,j}^*\}_{j\in I_q}$, centered at a, defined by $x_{p,i}^* = x_{p,i} - x_{p,i}(a)$ and $x_{q,j}^* = x_{q,j} - x_{q,j}(a)$, for corresponding $i \in I_p$ and $j \in I_q$, respectively. Note that the coordinate functions $x_{p,i}^*$ (resp. $x_{q,j}^*$) are standard analytic coordinates at a if and only if $i \in I_p \setminus J$ (resp. $j \in I_q \setminus J$), hence taking into account Eqs. (4) and (5) about the local expression of morphisms in arbitrary generalized analytic manifolds, for all $i, j \in J$, we obtain

$$C_{ij}^{pq} = 0, \quad \text{if } i \neq j; \quad C_{ij}^{pq} \in \mathbb{R}_{>0}, \quad \text{if } i = j, \tag{10}$$

where $C^{pq} \in \mathfrak{C}_{(\mathcal{M},\mathfrak{a})}$.

In the following statement, we determine other entries of the matrix of exponents C^{pq} in the case where p and q are corner points connected through an edge.

Lemma 3.7 *Let Y be a compact edge in M and let $\mathcal{Z}^0(Y) = \{p, q\}$. Denote by $i_p \in I_p$ and $i_q \in I_q$ the indices such that $I_Y = I_q \backslash \{i_q\} = I_p \backslash \{i_p\}$. The map $C = C_Y^{pq} \in \mathfrak{C}_{(\mathcal{M},\mathfrak{a})}$ satisfies*

$$\text{(a) } C_{ii} \in \mathbb{R}_{>0}, \quad \text{(b) } C_{ij} = 0, \ \textit{if } i \neq j, \quad \text{(c) } C_{i_q,i} = 0, \quad \textit{for all } i, j \in I_Y.$$

Proof Assertions (a) and (b) are already established by Eq. (10). Let us prove (c). Suppose that there is an index $i \in I_Y$ such that $C_{i_q,i} \neq 0$ and let us find a contradiction. Denote by S_i the stratum such that $\overline{S}_i = E_i$, that is

$$S_i = E_i \backslash \bigcup_{j \in I \backslash \{i\}} E_j.$$

Note that $S_i \subset V_p^* \cap V_q^*$. Given $a \in S_i$, we have $x_{p,i}(a) = 0$ and by Eq. (7), we get $x_{q,i}(a) = x_{q,i_q}(a) = 0$, since we are assuming $C_{i_q,i} = r_{i_q,i} \neq 0$. This means that $a \in E_{i_q}$, which is a contradiction. □

Definition 3.8 Let $(\mathcal{M}, \mathfrak{a})$ be a monomial generalized analytic manifold, and fix two corner points $p, q \in \mathcal{Z}^0$ such that $I_p \cap I_q \neq \emptyset$. The *weight connexion function from p to q* is the map $\gamma^{pq} : I_p \cap I_q \to \mathbb{R}_{>0}$, defined by $i \mapsto C_{ii}^{pq}$, where $C^{pq} \in \mathfrak{C}_{(\mathcal{M},\mathfrak{a})}$.

Note that $i \in I_p \cap I_q$ if and only if $p, q \in E_i$. We use the notation $\gamma_i^{pq} = \gamma^{pq}(i)$. In view of Eq. (9) and Lemma 3.7, if $J = I_p \cap I_q$ and $\mathcal{P} = (Y_1, Y_2, \ldots, Y_k)$ is a path of edges from p to q inside E_J, then γ^{pq} is given by the following product

$$\gamma^{pq} = \gamma_{|J}^{p_{k-1}q} \cdot \cdots \cdot \gamma_{|J}^{p_1 p_2} \cdot \gamma_{|J}^{pp_1}, \tag{11}$$

where $p_i = Y_i \cap Y_{i+1}$, for each $i = 1, 2, \ldots, k-1$.

Lemma 3.9 *Given a boundary component $E_i \in \mathcal{Z}^{n-1}$ and two corner points $p, q \in E_i$, we have $\gamma_i^{pq}\gamma_i^{qp} = 1$.*

Proof In view of Eq. (11) it is enough to prove the result for two corner points p, q connected through a compact edge $Y \subset E_i$. Now we have that (Y, Y) is a path from p to p inside E_i, and in view of Eq. (11) again, we get

$$1 = \gamma_i^{pp} = \gamma_i^{pq}\gamma_i^{qp},$$

as we wanted. □

We end this subsection by introducing the combinatorial data associated to a morphism.

Let $\phi : (\mathcal{M}_1, \mathfrak{a}_1) \to (\mathcal{M}_2, \mathfrak{a}_2)$ be a morphism of monomial $\mathcal{A}$-manifolds. Given a corner point $p \in \mathcal{Z}_{\mathcal{M}_1}^0$ and $\bar{p} = \phi(p)$, we represent ϕ locally at p by means of the *matrix of exponents* $B_p^{\phi} : I_{\bar{p}} \times I_p \to \mathbb{R}_+$, defined by $(i, j) \mapsto b_{ij} := b_i(j)$, where $b_i : I_p \to \mathbb{R}_+$ is as in Eq. (8).

Definition 3.10 The *combinatorial data* $\mathfrak{B}_\phi$ *of a morphism* $\phi : (\mathcal{M}_1, \mathfrak{a}_1) \to (\mathcal{M}_2, \mathfrak{a}_2)$ is the family of matrices of exponents $\mathfrak{B}_\phi = \{B_p^{\phi}\}_{p \in \mathcal{Z}_{\mathcal{M}_1}^0}$.

Remark 3.11 Once we fix an m-chart $(V_p^{\star}, \mathbf{x}_p)$ at some corner point $p \in \mathcal{Z}^0$, we can recover the whole monomial atlas $\mathfrak{a}$ of $\mathcal{M}$ from $\mathfrak{C}_{(\mathcal{M},\mathfrak{a})}$ using Eq. (7). Moreover, a morphism ϕ between monomial $\mathcal{A}$-manifolds is completely determined by its combinatorial data $\mathfrak{B}_\phi$.

3.3 Abundance of standardizations of monomial manifolds

In this section, we define m-standardizations, we give a characterization for their combinatorial data and we prove a result of abundance of m-standardizations of a fixed monomial $\mathcal{G}$-manifold.

Let us fix a monomial generalized analytic manifold $(\mathcal{M}, \mathfrak{a})$. A *local m-standardization* of $(\mathcal{M}, \mathfrak{a})$ at a corner point p is just an m-chart $\mathbf{u}_p$ defined in the whole open set $V_p^\star$, such that if $\mathbf{x}_p \in \mathfrak{a}$, then $\mathbf{u}_p \circ \mathbf{x}_p^{-1}$ is given by monomial relations of the form

$$u_{p,i} = x_{p,i}^{\alpha_{p,i}}, \quad \text{where } \alpha_{p,i} \in \mathbb{R}_{>0}, \quad \text{for all } i \in I_p. \tag{12}$$

We represent this change of coordinates by means of the map $\alpha_p : I_p \to \mathbb{R}_{>0}$ defined by $i \mapsto \alpha_{p,i}$. In that way, the change of coordinates $\mathbf{u}_p \circ \mathbf{x}_p^{-1}$ is codified by the *matrix of exponents* $D_{\alpha_p} : I_p \times I_p \to \mathbb{R}_{>0}$, where we recall that (once an order in I_p is fixed) D_{α_p} is a diagonal matrix with the elements $\alpha_{p,i}$ in the diagonal.

Definition 3.12 An *m-standardization of* $(\mathcal{M}, \mathfrak{a})$ is a pair $(\mathcal{O}, \mathfrak{b})$, where $\mathcal{O}$ is a standardization of $\mathcal{M}$ and $\mathfrak{b} = \{\mathbf{u}_p\}_{p \in \mathcal{Z}^0}$ is a monomial atlas of $\mathcal{N} = (M, \mathcal{O})$ such that $\mathbf{u}_p$ is a local m-standardization of $(\mathcal{M}, \mathfrak{a})$ for every corner point $p \in \mathcal{Z}^0$. The *combinatorial data of an m-standardization* $(\mathcal{O}, \mathfrak{b})$ is the collection of maps $\Lambda_{(\mathcal{O},\mathfrak{b})} = \{\alpha_p\}_{p \in \mathcal{Z}^0}$.

Remark 3.13 If $(\mathcal{O}, \mathfrak{b})$ and $(\mathcal{O}, \mathfrak{b}')$ are m-standardizations of $(\mathcal{M}, \mathfrak{a})$, then $\mathfrak{b}$ necessarily that $\mathfrak{b} = \mathfrak{b}'$ as we have already noted in Remark 3.3. Note also that the m-standardization $(\mathcal{O}, \mathfrak{b})$ is completely determined by the combinatorial data $\Lambda_{(\mathcal{O},\mathfrak{b})}$.

Lemma 3.14 *A collection of maps $\Lambda = \{\alpha_p : I_p \to \mathbb{R}_{>0}\}_{p \in \mathcal{Z}^0}$ is the combinatorial data of an m-standardization of $(\mathcal{M}, \mathfrak{a})$ if and only if for any pair of corner points $p, q \in \mathcal{Z}^0$ the following relations hold:*

$$\alpha_{p,\ell} = \gamma_\ell^{pq} \alpha_{q,\ell}, \quad \textit{for all } \ell \in I_p \cap I_q, \tag{13}$$

where γ^{pq} is the weight connexion function from p to q.

Proof Let us assume first that $\Lambda = \Lambda_{(\mathcal{O},\mathfrak{b})}$, where $(\mathcal{O}, \mathfrak{b})$ is an m-standardization of $(\mathcal{M}, \mathfrak{a})$. Let us denote $\mathcal{N} = (M, \mathcal{O})$ and let $\mathfrak{C}_{(\mathcal{N},\mathfrak{b})}$ be the combinatorial data of the monomial standard analytic manifold $(\mathcal{N}, \mathfrak{b})$. In view of Eq. (11), it is enough to prove Eq. (13) for two corner points p and q connected through a compact edge Y. Let us consider the m-charts $\mathbf{u}_p, \mathbf{u}_q \in \mathfrak{b}$ at p and q, respectively. The change of coordinates $\mathbf{u}_p \circ \mathbf{u}_q^{-1}$ is codified by a matrix of exponents $A = A_Y^{pq} \in \mathfrak{C}_{(\mathcal{N},\mathfrak{b})}$. This change must be standard analytic in its domain of definition $\mathbf{u}_q(V_p^\star \cap V_q^\star) = \mathbb{R} \times \mathbb{R}_+^{n-1}$, and this implies

$$A_{i\ell} \in \mathbb{Z}_+, \quad (A^{-1})_{j\ell} \in \mathbb{Z}_+, \quad \text{for all } i \in I_q,\ j \in I_p,\ \ell \in I_Y. \tag{14}$$

Let $C = C_Y^{pq} \in \mathfrak{C}_{(\mathcal{M},\mathfrak{a})}$ and $\alpha_p, \alpha_q \in \Lambda_{(\mathcal{O},\mathfrak{b})}$. Note that A is obtained as the product

$$A = D_{\alpha_q} C D_{\alpha_p}^{-1} : I_q \times I_p \to \mathbb{R}.$$

When $\ell \in I_p \cap I_q = I_Y$, in view of Lemmas 3.7 and 3.9, we have

$$A_{\ell\ell} = \frac{\gamma_\ell^{pq} \alpha_{q,\ell}}{\alpha_{p,\ell}} \in \mathbb{Z}_+, \quad (A^{-1})_{\ell\ell} = (A_Y^{qp})_{\ell\ell} = \frac{\gamma_\ell^{qp} \alpha_{p,\ell}}{\alpha_{q,\ell}} = \frac{\alpha_{p,\ell}}{\gamma_\ell^{pq} \alpha_{q,\ell}} = 1/A_{\ell\ell} \in \mathbb{Z}_+,$$

which shows $A_{\ell\ell} = (A^{-1})_{\ell\ell} = 1$. From here we get $\alpha_{p,\ell} = \gamma_\ell^{pq}\alpha_{q,\ell}$, and hence Λ satisfies Eq. (13) as we wanted.

Assume now that Λ satisfies Eq. (13) for any pair of corner points $p, q \in \mathcal{Z}^0$. At each corner point $p \in \mathcal{Z}^0$, consider the m-chart $\mathbf{u}_p$ defined on $V_p^\star$ such that the change of coordinates $\mathbf{u}_p \circ \mathbf{x}_p^{-1}$ satisfies $u_{p,j} = x_{p,j}^{\alpha_{p,j}}$, for all $j \in I_p$, where $\alpha_p \in \Lambda$ and $\mathbf{x}_p \in \mathfrak{a}$. In that way, we get a new monomial atlas $\mathfrak{b} = \{\mathbf{u}_p\}_{p\in\mathcal{Z}^0}$ of $\mathcal{M}$. Let us see that the changes of coordinates $\mathbf{u}_q \circ \mathbf{u}_p^{-1}$ are standard analytic for any pair of corner points p and q. In view of Eq. (9) it is enough to suppose that p and q are connected through an edge Y. Defining the matrix $A = D_{\alpha_q} C D_{\alpha_p}^{-1}$, the change of coordinates $\mathbf{u}_q \circ \mathbf{u}_p^{-1}$ is given by

$$u_{q,i} = \prod_{j\in I_p} u_{p,j}^{A_{ij}}, \quad \text{for any } i \in I_q.$$

It suffices to show that A satisfies the conditions in Eq. (14). Indeed, if $A_{i\ell} \in \mathbb{Z}_+$ for $i \in I_q$ and for all $\ell \in I_Y$, then $u_{q,i}$ in the above equation is standard analytic in terms of the variables $\mathbf{u}_p$ in the domain $V_p^\star \cap V_q^\star = \{u_{p,i_p} \neq 0\} \cap \{u_{q,i_q} \neq 0\}$ (the same interchanging p and q if $(A^{-1})_{j\ell} \in \mathbb{Z}_+$ for $j \in I_q$ and any $\ell \in I_Y$). Applying Lemma 3.7 we get that $A_{r\ell} = 0$ and $(A^{-1})_{r\ell} = 0$, for all $r, \ell \in I_Y$ with $r \neq \ell$. Moreover, the same lemma assures that $C_{i_q\ell} = 0$ and that $(C_{i_p\ell})^{-1} = C_{i_p\ell}^{qp} = 0$, for all $\ell \in I_Y$; hence, for any such index $\ell \in I_Y$ we obtain

$$A_{i_q\ell} = C_{i_q\ell}\alpha_{q,i_q}/\alpha_{p,\ell} = 0, \quad (A^{-1})_{i_p\ell} = (C_{i_p\ell})^{-1}\alpha_{p,i_p}/\alpha_{q,\ell} = 0.$$

Again by Lemma 3.7 we get

$$A_{\ell\ell} = \frac{C_{\ell\ell}\alpha_{q,\ell}}{\alpha_{p,\ell}} = \frac{\gamma_\ell^{pq}\alpha_{q,\ell}}{\alpha_{p,\ell}}, \quad (A^{-1})_{\ell\ell} = \frac{(C^{-1})_{\ell\ell}\alpha_{p,\ell}}{\alpha_{q,\ell}} = \frac{\gamma_\ell^{qp}\alpha_{p,\ell}}{\alpha_{q,\ell}},$$

for all $\ell \in I_Y$. Using Lemma 3.9 and Eq. (13) we conclude $A_{\ell\ell} = (A^{-1})_{\ell\ell} = 1$. As a conclusion, the atlas $\mathfrak{b}$ defines a standard analytic structure $\mathcal{N} = (M, \mathcal{O})$ over M, where $\mathcal{M} = (M, \mathcal{G}_M)$; thus $\mathcal{O} \subset \mathcal{G}_M$ is a standardization of $\mathcal{M}$. Moreover, by definition of $\mathfrak{b}$, we have that $(\mathcal{O}, \mathfrak{b})$ is an m-standardization of $(\mathcal{M}, \mathfrak{a})$ with $\Lambda_{(\mathcal{O},\mathfrak{b})} = \Lambda$. ◻

In the sequel, a collection of maps $\Lambda = \{\alpha_p : I_p \to \mathbb{R}_{>0}\}_{p\in\mathcal{Z}^0}$ is called *realizable for* $(\mathcal{M}, \mathfrak{a})$ if Eq. (13) holds for any pair of corner points $p, q \in \mathcal{Z}^0$.

Definition 3.15 Let $\mathbf{u}_p$ be a local m-standardization of $(\mathcal{M}, \mathfrak{a})$ at a given corner point $p \in \mathcal{Z}^0$. An *extension of* $\mathbf{u}_p$ is a (global) m-standardization $(\mathcal{O}, \mathfrak{b})$ of $(\mathcal{M}, \mathfrak{a})$ such that $\mathbf{u}_p \in \mathfrak{b}$; we say also that $(\mathcal{O}, \mathfrak{b})$ *extends* $\mathbf{u}_p$. We denote by $\mathcal{E}(\mathbf{u}_p)$ the set of extensions of $\mathbf{u}_p$.

Proposition 3.16 *Let* $(\mathcal{M}, \mathfrak{a})$ *be a monomial generalized analytic manifold. Then:*

(a) *There is a bijection between the set of m-standardizations of* $(\mathcal{M}, \mathfrak{a})$ *and* $\mathbb{R}^N_{>0}$*, where* N *is the number of boundary components of* ∂M.
(b) *Given a corner point* $p \in \mathcal{Z}^0$ *and a local m-standardization* $\boldsymbol{u}_p$ *at* p*, there is a bijective map* $\mathbb{R}^{N-n}_{>0} \to \mathcal{E}(\boldsymbol{u}_p)$*, where* n *is the dimension of* $\mathcal{M}$.

Proof We start with the proof of the first assertion (a). Let I be the set of indices labelling the components of ∂M, that is $\partial M = \bigcup_{i\in I} E_i$, where $N = \#I$, and let us fix a collection of corner points $\mathfrak{q} = \{q_i\}_{i\in I}$ in such a way that $q_i \in E_i$ for each $i \in I$. Given a map $\beta \in \mathbb{R}^I_{>0}$, we take $\Lambda_\beta = \{\alpha_p : I_p \to \mathbb{R}_{>0}\}_{p\in\mathcal{Z}^0}$ to be the family of maps defined by

$$\alpha_{p,\ell} = \gamma_\ell^{pq_\ell}\beta_\ell, \quad \text{for all } p \in \mathcal{Z}^0 \text{ and } \ell \in I_p.$$

Let us see that Λ_β is a realizable family of maps. Fix two corner points p and q, and let $\ell \in I_p \cap I_q$. By Eq. (11) we have that $\gamma_\ell^{pq_\ell}\gamma_\ell^{q_\ell q} = \gamma_\ell^{pq}$. Moreover, by the definition of $\alpha_{q,\ell}$ and as a consequence of Lemma 3.9, we have that $\beta_\ell = \gamma_\ell^{q_\ell q}\alpha_{q,\ell}$. Then we obtain

$$\alpha_{p,\ell} = \gamma_\ell^{pq_\ell}\beta_\ell = \gamma_\ell^{pq_\ell}\gamma_\ell^{q_\ell q}\alpha_{q,\ell} = \gamma_\ell^{pq}\alpha_{q,\ell}$$

which is the required condition for Λ_β to be realizable. Now, in view of Lemma 3.14, there exists a unique m-standardization $(\mathcal{O}^\beta, \mathfrak{b}^\beta)$ with $\Lambda_{(\mathcal{O}^\beta,\mathfrak{b}^\beta)} = \Lambda_\beta$. Finally, we show that the map

$$\Psi_{\mathfrak{q}} : \mathbb{R}^I_{>0} \to \left\{ \begin{matrix} \text{m-standardizations} \\ \text{of } (\mathcal{M}, \mathfrak{a}) \end{matrix} \right\}, \quad \beta \mapsto (\mathcal{O}^\beta, \mathfrak{b}^\beta)$$

is a bijection. Indeed, if $\beta \neq \beta'$, we have that $\Lambda_\beta \neq \Lambda_{\beta'}$ and hence $(\mathcal{O}^\beta, \mathfrak{b}^\beta) \neq (\mathcal{O}^{\beta'}, \mathfrak{b}^{\beta'})$ taking into account Remark 3.13. On the other hand, given an m-standardization $(\mathcal{O}, \mathfrak{b})$ with combinatorial data $\Lambda = \{\alpha_p\}_{p\in\mathcal{Z}^0}$, we have that $(\mathcal{O}, \mathfrak{b}) = \Psi_{\mathfrak{q}}(\beta)$, where β is defined by $\beta_i = \alpha_{q_i,i}$, for all $i \in I$. The proof of (a) is finished.

Let us prove now the second assertion (b). Denote by $\alpha_p : I_p \to \mathbb{R}_{>0}$ the map of exponents defining $\mathbf{u}_p$, that is

$$u_{p,i} = x_{p,i}^{\alpha_{p,i}}, \quad \text{for all } i \in I_p, \quad \text{where } \mathbf{x}_p \in \mathfrak{a}.$$

Consider the injective map $i_{\alpha_p} : \mathbb{R}^{I\setminus I_p}_{>0} \hookrightarrow \mathbb{R}^I_{>0}$, defined by

$$\delta \mapsto i_{\alpha_p}(\delta) := \beta^\delta, \quad \text{where } \beta_i^\delta = \begin{cases} \delta_i & \text{if } i \in I\setminus I_p, \\ \alpha_{p,i} & \text{if } i \in I_p. \end{cases}$$

Take a collection of corner points $\mathfrak{q}_p = \{q_i\}_{i\in I}$ such that $q_i = p$, for each $i \in I_p$, and $q_i \in E_i$, for each $i \in I\setminus I_p$. Using the notations in item a) above, we have that $\Psi_{\mathfrak{q}_p}(\beta) \in \mathcal{E}(\mathbf{u}_p)$ if and only if $\beta|_{I_p} = \alpha_p$, or equivalently $\beta = i_{\alpha_p}(\beta|_{I\setminus I_p})$. In other words, we have the equality

$$\mathcal{E}(\mathbf{u}_p) = \mathrm{Im}(\Psi_{\mathfrak{q}_p} \circ i_{\alpha_p}),$$

and hence we have the bijection $\mathbb{R}^{I\setminus I_p}_{>0} \to \mathcal{E}(\mathbf{u}_p)$ mapping δ into $\Psi_{\mathfrak{q}_p}(i_{\alpha_p}(\delta))$. We finish just by noting that $\#I_p = n$. □

3.4 The monomial Voûte Etoilée

In this section we give the definition of m-combinatorial blowing-up and we introduce the concept of "monomial voûte étoilée" over an m-manifold, whose elements, called m-stars, are sequences of monomial blowing-ups starting from that m-manifold. The terminology is inspired by Hironaka [10, 11].

Let $(\mathcal{M}, \mathfrak{a})$ be a monomial generalized analytic manifold. An *m-combinatorial center of blowing-up for* $(\mathcal{M}, \mathfrak{a})$ is a tripet $(Z, \mathcal{O}, \mathfrak{b})$, where Z is a combinatorial geometric center for $\mathcal{M}$ and $(\mathcal{O}, \mathfrak{b})$ is an m-standardization of $(\mathcal{M}, \mathfrak{a})$. Given such an m-combinatorial center $(Z, \mathcal{O}, \mathfrak{b})$, we consider the blowing-up $\pi_\xi : \mathcal{M}_\xi \to \mathcal{M}$ with center $\xi = (Z, \mathcal{O})$. Let I be an index set labelling the components of ∂M. We write $\infty \notin I$ to label the exceptional divisor $E_\infty := \pi_\xi^{-1}(Z)$, and we put $I_\xi = I \cup \{\infty\}$ as an index set for the components of ∂M_ξ. More precisely, given $i \in I$, it represents both the boundary component E_i of ∂M and its strict transform

$$E_i' = \overline{\pi_\xi^{-1}(E_i \setminus Z)} \subset \partial M_\xi,$$

belonging to $\mathcal{Z}^{n-1}_{\mathcal{M}}$ and $\mathcal{Z}^{n-1}_{\mathcal{M}_\xi}$, respectively. The index $\infty \in I_\xi$ represents $E_\infty \in \mathcal{Z}^{n-1}_{\mathcal{M}_\xi}$.

Proposition 3.17 *There is a monomial atlas $\mathfrak{a}_\xi$ of $\mathcal{M}_\xi$ in such a way that π_ξ defines a morphism of monomial $\mathcal{G}$-manifolds from $(\mathcal{M}_\xi, \mathfrak{a}_\xi)$ to $(\mathcal{M}, \mathfrak{a})$.*

Proof Take a corner point p' in $\mathcal{M}_\xi$ and let $p = \pi_\xi(p')$. Note that p is a corner point in $\mathcal{M}$. Let $\mathbf{x}_p \in \mathfrak{a}$ be the m-chart of the atlas $\mathfrak{a}$ at p. We distinguish two situations:

Case $p' \notin E_\infty$. We have that $I_{p'} = I_p$ and the blowing-up π_ξ induces an isomorphism between V^*_p and $V^*_{p'}$. We take affine coordinates $\mathbf{x}'_{p'}$ over $V^\star_{p'}$ defined by

$$\tilde{x}'_{p',i} = x_{p,i} \circ \pi_\xi|_{V^*_{p'}}, \text{ for all } i \in I_p.$$

Thus, the expression of π_ξ in coordinates $\mathbf{x}'_{p'}$ and $\mathbf{x}_p$ is purely monomial. This expression can be codified with the matrix of exponents $B_{p'} : I_p \times I_p \to \mathbb{R}_{\geq 0}$ given by

$$B_{p'}(i, j) = \delta_{ij}, \quad i, j \in I_p, \tag{15}$$

where δ_{ij} is the Kronecker delta symbol. In other words, $B_{p'} = D_{\mathbb{1}_{I_p}}$.

Case $p' \in E_\infty$. We have $I_{p'} = I_p \backslash \{j\} \cup \{\infty\}$, for some $j \in I_Z$ (see for instance [15] for details in the combinatorial treatment of blowing-ups). By hypothesis, the pair $(\mathcal{O}, \mathfrak{b})$ is an m-standardization of $(\mathcal{M}, \mathfrak{a})$; in particular, $\mathfrak{b}$ is a monomial atlas of the standard analytic manifold $\mathcal{N} = (M, \mathcal{O})$. Using this information, together with the definition of blowing-up centered at ξ, we get that there exists an m-chart $\mathbf{x}'_{p'}$ defined in $V^\star_{p'}$ such that the map $\mathbf{x}'_{p'} \circ \pi_\xi \circ \mathbf{x}_p^{-1}$ is purely monomial with associated matrix of exponents $B_{p'} : I_p \times I_{p'} \to \mathbb{R}_+$ given by

$$(r, s) \mapsto \begin{cases} 1 & \text{if } \ r = s \quad & \text{and } \ r \in I_p \backslash \{j\}, \\ \alpha_{p,j}/\alpha_{p,r} & \text{if } \ s = \infty & \text{and } \ r \in I_Z, \\ 0 & \quad \text{otherwise.} & \end{cases} \tag{16}$$

where $\alpha_p \in \Lambda_{(\mathcal{O},\mathfrak{b})}$. With an appropriate order of rows and columns, $B_{p'}$ can be seen as the upper triangular matrix

$$\left(\begin{array}{c|c|c} \mathrm{Id}_{n-s} & 0 & 0 \\ \hline 0 & \mathrm{Id}_{s-1} & \mathbf{a} \\ \hline 0 & 0 & 1 \end{array}\right) \in \mathbb{R}^{n \times n}_+,$$

where $s = \# I_Z$ and $\mathbf{a} \in \mathbb{R}^{s-1}_{>0}$ is a column vector whose entries are defined by the quotients $\alpha_{p,j}/\alpha_{p,r}$, with $r \in I_Z \backslash \{j\}$.

The collection $\mathfrak{a}_\xi = \{\mathbf{x}'_{p'}\}_{p' \in \mathcal{Z}^0_\xi}$ with $\mathcal{Z}^0_\xi = \mathcal{Z}^0_{\mathcal{M}_\xi}$ is thus a monomial atlas in $\mathcal{M}_\xi$. Moreover, the blowing-up π_ξ induces a morphism from $(\mathcal{M}_\xi, \mathfrak{a}_\xi)$ to $(\mathcal{M}, \mathfrak{a})$ and the associated combinatorial data is $\mathfrak{B}_{\pi_\xi} = \{B_{p'}\}_{p' \in \mathcal{Z}^0_\xi}$. □

From now on, given an m-combinatorial center of blowing-up $(Z, \mathcal{O}, \mathfrak{b})$ for a monomial generalized analytic manifold $(\mathcal{M}, \mathfrak{a})$, and the blowing-up morphism $\pi_\xi : \mathcal{M}_\xi \to \mathcal{M}$, with center at $\xi = (Z, \mathcal{O})$, we always consider $\mathcal{M}_\xi$ endowed with the monomial atlas $\mathfrak{a}_\xi$ constructed in Proposition 3.17. Moreover, we also write

$$\pi_\xi : (\mathcal{M}_\xi, \mathfrak{a}_\xi) \to (\mathcal{M}, \mathfrak{a}),$$

to emphasize that the morphism π_ξ is considered also as a morphism of monomial generalized analytic manifolds, and we call it an *m-combinatorial blowing-up of* $(\mathcal{M}, \mathfrak{a})$. The associated combinatorial data $\mathcal{B}_{\pi_\xi} = \{B_{p'}\}_{p' \in \mathcal{Z}^0_\xi}$ of this morphism has been made explicit in Eq. (15), for points $p' \in \mathcal{Z}^0_\xi$ with $p' \notin E_\infty$ and in Eq. (16), for points $p' \in \mathcal{Z}^0_\xi(E_\infty)$.

Definition 3.18 Let $(\mathcal{M}, \mathfrak{a})$ be a monomial generalized analytic manifold. An *m-star over* $(\mathcal{M}, \mathfrak{a})$ is the composition $\sigma = \pi_0 \circ \pi_1 \circ \cdots \circ \pi_{r-1}$ of a finite sequence of m-combinatorial blowing-ups. That is

$$\sigma : (\mathcal{M}_r, \mathfrak{a}_r) \xrightarrow{\pi_{r-1}} (\mathcal{M}_{r-1}, \mathfrak{a}_{r-1}) \xrightarrow{\pi_{r-2}} \cdots \xrightarrow{\pi_0} (\mathcal{M}_0, \mathfrak{a}_0) = (\mathcal{M}, \mathfrak{a}),$$

where for each $k = 0, 1, 2, \ldots, r-1$, the morphism π_k is an m-combinatorial blowing-up of $(\mathcal{M}_k, \mathfrak{a}_k)$. The integer r and the monomial generalized analytic manifold $(\mathcal{M}_r, \mathfrak{a}_r)$ are called, respectively, the *age* and the *end* of the m-star σ. The collection $\mathcal{V}^{\mathrm{m}}_{(\mathcal{M},\mathfrak{a})}$ of all the m-stars over $(\mathcal{M}, \mathfrak{a})$ is called the *monomial voûte étoilée of* $(\mathcal{M}, \mathfrak{a})$.

4 Stratified reduction of singularities via principalization of m-ideals

We devote this section to introducing the concept of m-ideal, in order to prove a theorem of principalization. With this result we prove the stratified reduction of singularities for a global function defined in generalized analytic manifolds admitting a monomial structure. Finally, we apply this result to prove the main result of this paper stated in Theorem 1.1.

4.1 Principalization of m-ideals

Let us fix a monomial generalized analytic manifold $(\mathcal{M}, \mathfrak{a})$, where $\mathcal{M} = (M, \mathcal{G}_M)$.

Take a global generalized analytic function $f \in \mathcal{G}_M(M)$ and two corner points $p, q \in \mathcal{Z}^0$. Let $\mathbf{x}_q, \mathbf{x}_q \in \mathfrak{a}$ be the m-charts at p and q, respectively, and let $C^{pq} \in \mathfrak{C}_{(\mathcal{M},\mathfrak{a})}$ be the matrix of exponents codifying the change of coordinates $\mathbf{x}_q \circ \mathbf{x}_p^{-1}$. The relation between the supports of f at p and q with respect to these coordinates is given by:

$$\mathrm{Supp}_p(f; \mathbf{x}_p) = \{\lambda_q C^{pq} : \lambda_q \in \mathrm{Supp}_q(f; \mathbf{x}_q)\} \subset \mathbb{R}^{I_p}_+. \tag{17}$$

Definition 4.1 A generalized analytic global function $\mathfrak{m} \in \mathcal{G}_M(M)$ is said to be an *m-function in* $(\mathcal{M}, \mathfrak{a})$ if for each $p \in \mathcal{Z}^0$, there is a map $\lambda_p : I_p \to \mathbb{R}_+$, such that

$$\mathfrak{m}|_{V_p^\star} = \mathbf{x}_p^{\lambda_p}, \quad \text{where } \mathbf{x}_p \in \mathfrak{a}.$$

The *combinatorial data of* $\mathfrak{m}$ is the list $\mathcal{L}_\mathfrak{m} = \{\lambda_p\}_{p\in\mathcal{Z}^0}$.

Let us consider an m-function $\mathfrak{m}$ in $(\mathcal{M}, \mathfrak{a})$ with combinatorial data $\mathcal{L}_\mathfrak{m} = \{\lambda_p\}_{p\in\mathcal{Z}^0}$. By Eq. (17), for any pair of corner points $p, q \in \mathcal{Z}^0$ we have the relation $\lambda_p = \lambda_q C^{pq}$, where $C^{pq} \in \mathfrak{C}_{(\mathcal{M},\mathfrak{a})}$. In particular, we get that

$$\lambda_{p,\ell} = \lambda_{q,\ell}\gamma^{pq}_\ell, \quad \text{for any } \ell \in I_p \cap I_q, \tag{18}$$

where γ^{pq} is the weighted connexion function from p to q. Indeed, in view of Eq. (11), it is enough to check Eq. (18) for the case where p and q are connected through an edge Y. For this case, it holds as a consequence of Lemma 3.7.

Remark 4.2 Given a list of maps $\mathcal{L} = \{\lambda_p : I_p \to \mathbb{R}_+\}_{p\in\mathcal{Z}^0}$ satisfying $\lambda_p = \lambda_q C^{pq}$, for any pair of corner points $p, q \in \mathcal{Z}^0$, there exists an m-function $\mathfrak{m}$ such that $\mathcal{L}_\mathfrak{m} = \mathcal{L}$.

Definition 4.3 A *finitely generated m-ideal in* $(\mathcal{M}, \mathfrak{a})$ is a sheaf of ideals $\mathcal{J} \subset \mathcal{G}_M$ generated by finitely many m-functions. That is,

$$\mathcal{J} = \mathfrak{m}_1\mathcal{G}_M + \mathfrak{m}_2\mathcal{G}_M + \cdots + \mathfrak{m}_k\mathcal{G}_M =: (\mathfrak{m}_1, \mathfrak{m}_2, \ldots, \mathfrak{m}_k),$$

where $\mathfrak{m}_1, \mathfrak{m}_2, \ldots, \mathfrak{m}_k$ are m-functions called *m-generators of* $\mathcal{J}$.

Notation 4.4 Let I be a finite index set and let A be a finite subset of $\mathbb{R}^I$. We denote by $A^{\min}$ the set of elements in A that are minimal with respect to the division order $\leq_d$ in $\mathbb{R}^I$.

Let $\mathcal{J}$ be an m-ideal in $(\mathcal{M}, \mathfrak{a})$ with set of m-generators $G = \{\mathfrak{m}_1, \mathfrak{m}_2, \ldots, \mathfrak{m}_k\}$. For each $i = 1, 2, \ldots, k$, let us write $\mathcal{L}_{\mathfrak{m}_i} = \{\lambda^i_p\}_{p\in\mathcal{Z}^0}$. Given a corner point $p \in \mathcal{Z}^0$ and $\mathbf{x}_p \in \mathfrak{a}$ the m-chart at p, the restriction $\mathcal{J}|_{V^\star_p}$ is an m-ideal in the m-corner $(\mathcal{M}|_{V^\star_p}, \mathbf{x}_p)$ with set of m-generators equal to $G|_{V^\star_p} := \{\mathfrak{m}_1|_{V^\star_p}, \mathfrak{m}_2|_{V^\star_p}, \ldots, \mathfrak{m}_k|_{V^\star_p}\}$. Consider the set

$$\Gamma_{G,p} := \{\lambda^1_p, \lambda^2_p, \ldots, \lambda^k_p\} \subset \mathbb{R}^{I_p}_+.$$

Note that if $(\Gamma_{G,p})^{\min} = \{\mu^1_p, \mu^2_p, \ldots, \mu^{k_p}_p\}$, then

$$\mathcal{J}|_{V^\star_p} = (\mathbf{x}_p^{\mu^1_p}, \mathbf{x}_p^{\mu^2_p}, \cdots, \mathbf{x}_p^{\mu^{k_p}_p}). \tag{19}$$

The sheaf of ideals $\mathcal{J}$ is called *locally principal* if at each point $a \in M$, the stalk $\mathcal{J}_a \subset \mathcal{G}_{M,a}$ is a principal ideal. Using the definition of m-ideal, it is enough to ask this property for the corner points. In terms of the set introduced above, we have that $\mathcal{J}$ is locally principal if and only if $(\Gamma_{G,p})^{\min}$ is a singleton for any $p \in \mathcal{Z}^0$.

Let $\mathfrak{m}$ be an m-function in $(\mathcal{M}, \mathfrak{a})$ and take an m-star $\sigma : (\mathcal{M}', \mathfrak{a}') \to (\mathcal{M}, \mathfrak{a})$. The *total transform* $\sigma^*\mathfrak{m} = \mathfrak{m} \circ \sigma$ is a again an m-function in $(\mathcal{M}', \mathfrak{a}')$. More precisely, if $p' \in \mathcal{Z}^0_{\mathcal{M}'}$ and $p = \sigma(p')$, then $\lambda'_{p'} \in \mathcal{L}_{\sigma^*\mathfrak{m}}$ is given by

$$\lambda'_{p'} = \lambda_p B^\sigma_{p'}, \tag{20}$$

where $\lambda_p \in \mathcal{L}_{\mathfrak{m}}$ and $B^\sigma_{p'} \in \mathfrak{B}_\sigma$ is the matrix of exponents codifying σ at p'. If $\mathcal{J}$ is an m-ideal generated by $G = \{\mathfrak{m}_1, \mathfrak{m}_2, \ldots, \mathfrak{m}_k\}$, then the total transform $\sigma^*\mathcal{J}$ is also an m-ideal in $(\mathcal{M}', \mathfrak{a}')$ generated by $\sigma^*G := \{\sigma^*\mathfrak{m}_1, \sigma^*\mathfrak{m}_2, \ldots, \sigma^*\mathfrak{m}_k\}$.

The main result in this section is the following one about principalization of m-ideals.

Theorem 4.5 *Let $\mathcal{J}$ be a finitely generated m-ideal in a monomial generalized analytic manifold $(\mathcal{M}, \mathfrak{a})$. There exists an m-star $\sigma \in \mathcal{V}^m_{(\mathcal{M},\mathfrak{a})}$ such that $\sigma^*\mathcal{J}$ is locally principal.*

To prove this theorem, we can reduce ourselves to the case where $\mathcal{J}$ is generated by two m-functions by considering a clear finite recurrence and the following lemma.

Lemma 4.6 *Let $\mathcal{J} = (\mathfrak{m}_1, \mathfrak{m}_2, \ldots, \mathfrak{m}_k)$ be an m-ideal in $(\mathcal{M}, \mathfrak{a})$. Assume that $\mathcal{J}^{rs} := (\mathfrak{m}_r, \mathfrak{m}_s)$ is locally principal for any pair of indices $r, s \in \{1, 2, \ldots, k\}$. Then $\mathcal{J}$ is locally principal.*

Proof Assume that there is a point $p \in \mathcal{Z}^0$ such that $\mathcal{J}_p$ is not principal. There exist indices $r, s \in \{1, 2, \ldots, k\}$ such that $\lambda^r_p \in \mathcal{L}_{\mathfrak{m}_r}$ and $\lambda^s_p \in \mathcal{L}_{\mathfrak{m}_s}$ are not comparable for the division order $\leq_d$ in $\mathbb{R}^{I_p}$. Note that $\Gamma_{G^{rs},p} = (\Gamma_{G^{rs},p})^{min} = \{\lambda^r_p, \lambda^s_p\}$, where $G^{rs} = \{m_r, m_s\}$, and hence $\mathcal{J}^{rs}$ is not locally principal, which is a contradiction. □

The case of two generators. Let us assume that $\mathcal{J} = (\mathfrak{m}, \mathfrak{n})$ is an m-ideal in $(\mathcal{M}, \mathfrak{a})$ generated by two m-functions and write $\mathcal{L}_\mathfrak{m} = \{\lambda_p\}_{p\in\mathcal{Z}^0}$ and $\mathcal{L}_\mathfrak{n} = \{\mu_p\}_{p\in\mathcal{Z}^0}$.

We introduce first several definitions, mainly inspired by the "b-invariant" introduced in [6] by van den Dries and Speissegger.

Let $Z \in \mathcal{Z}^{n-2}$ be a codimension two combinatorial geometric center for $\mathcal{M}$. We know that the index set I_Z has just two elements, say $I_Z = \{i, j\}$. Let $p \in \mathcal{Z}^0(Z)$ be a corner point in Z. We say that Z is *uncoupled fo $\mathcal{J}$ at p* if

$$(\lambda_{p,i} - \mu_{p,i})(\lambda_{p,j} - \mu_{p,j}) < 0.$$

We say that Z is *uncoupled for $\mathcal{J}$* if it is so at each $p \in \mathcal{Z}^0(Z)$.

Lemma 4.7 *A combinatorial geometric center $Z \in \mathcal{Z}^{n-2}$ is uncoupled for $\mathcal{J}$ if and only if there is a corner point $q \in \mathcal{Z}^0(Z)$ such that Z is uncoupled for $\mathcal{J}$ at q.*

Proof Assume that Z is uncoupled for $\mathcal{J}$ at a corner point $q \in \mathcal{Z}^0(Z)$ and take any other point $p \in \mathcal{Z}^0(Z)$. By Eq. (18), we have $\lambda_{p,\ell} = \lambda_{q,\ell}\gamma_\ell^{pq}$, for all $\ell \in I_p \cap I_q$, where γ^{pq} is the weighted connexion function from p to q. Since $p, q \in Z$, we know that $I_Z = \{i, j\} \subset I_p \cap I_q$. Then

$$(\lambda_{p,i} - \mu_{p,i})(\lambda_{p,j} - \mu_{p,j}) = \gamma_i^{pq}\gamma_j^{pq}(\lambda_{q,i} - \mu_{q,i})(\lambda_{q,j} - \mu_{q,j}) < 0,$$

since $\gamma_i^{pq} > 0$ and $\gamma_j^{pq} > 0$. Hence Z is uncoupled for $\mathcal{J}$ also at p, and we conclude that Z is uncoupled for $\mathcal{J}$. □

Observe that if $p \in \mathcal{Z}^0$ and there are no uncoupled centers for $\mathcal{J}$ passing through p, then we necessarily have that $\lambda_p \leq_d \mu_p$ or $\mu_p \leq_d \lambda_p$, that is $\mathcal{J}_p$ is a principal ideal.

Definition 4.8 Let $\Omega_\mathcal{J}$ be the family of codimension two combinatorial geometric centers in $\mathcal{M}$ that are uncoupled for $\mathcal{J}$, and define *the invariant of $\mathcal{J}$* to be $\mathrm{Inv}_\mathcal{J} := \#\Omega_\mathcal{J}$.

We have that $\mathrm{Inv}_\mathcal{J} = 0$ if and only if $\mathcal{J}$ is locally principal. Thus, the objective now is to find an m-star $\sigma \in \mathcal{V}^m_{(\mathcal{M},\mathfrak{a})}$ such that $\mathrm{Inv}_{\sigma^*\mathcal{J}} = 0$.

Suppose that $\mathrm{Inv}_\mathcal{J} > 0$ and fix $Z \in \Omega_\mathcal{J}$. Take a corner point $p \in Z$ and pick a local m-standardization $\mathbf{u}_p$ of $(\mathcal{M}, \mathfrak{a})$ at p defined by the map $\alpha_p : I_p \to \mathbb{R}_{>0}$. We say that $\mathbf{u}_p$ is *adapted to $\mathcal{J}$ with respect to Z* if

$$\alpha_{p,j}(\lambda_{p,i} - \mu_{p,i}) + \alpha_{p,i}(\lambda_{p,j} - \mu_{p,j}) = 0.$$

A global m-standardization $(\mathcal{O}, \mathfrak{b} = \{u_p\}_{p\in Z^0})$ of $(\mathcal{M}, \mathfrak{a})$ is said to be *adapted to $\mathcal{J}$ with respect to Z* if u_p is adpated to $\mathcal{J}$ with respect to Z for every $p \in \mathcal{Z}^0(Z)$.

Lemma 4.9 *An m-standardization $(\mathcal{O}, \mathfrak{b})$ is adapted to $\mathcal{J}$ with respect to Z if and only if there is a corner point $q \in Z$ such that $\mathbf{u}_q \in \mathfrak{b}$ is adapted to $\mathcal{J}$ with respect to Z.*

Proof Denote $\Lambda = \Lambda_{(\mathcal{O},\mathfrak{b})}$. Assume that there is a corner point $q \in Z$ such that $\mathbf{u}_q \in \mathfrak{b}$ is adapted to $\mathcal{J}$ with respect to Z. Take any other corner point $p \in Z$. In view of the realizability of Λ established in Lemma 3.14 we know that $\alpha_{q,\ell} = \gamma_\ell^{qp}\alpha_{p,\ell}$, for all $\ell \in I_p \cap I_q$. Since $p, q \in Z$ we have that $I_Z = \{i, j\} \subset I_p \cap I_q$. Then, by Eq. (18), we get

$$\begin{aligned}
\alpha_{p,j}(\lambda_{p,i} - \mu_{p,i}) + \alpha_{p,i}(\lambda_{p,j} - \mu_{p,j}) &= \gamma_j^{pq}\alpha_{q,j}\gamma_i^{pq}(\lambda_{q,i} - \mu_{q,i}) \\
&\quad + \gamma_i^{pq}\alpha_{q,i}\gamma_j^{pq}(\lambda_{q,j} - \mu_{q,j}) \\
&= \gamma_i^{pq}\gamma_j^{pq}[\alpha_{q,j}(\lambda_{q,i} - \mu_{q,i}) \\
&\quad + \alpha_{q,i}(\lambda_{q,j} - \mu_{q,j})] = 0.
\end{aligned}$$

As a consequence, the local m-standardization $\mathbf{u}_p \in \mathfrak{b}$ is adapted to $\mathcal{J}$ with respect to Z at p. We conclude that $(\mathcal{O}, \mathfrak{b})$ is adapted to $\mathcal{J}$ with respect to Z. □

A codimension two combinatorial center of blowing-up $\xi = (Z, \mathcal{O}, \mathfrak{b})$ is *adapted to* $\mathcal{J}$ if $Z \in \Omega_{\mathcal{J}}$ and $(\mathcal{O}, \mathfrak{b})$ is an m-standardization adapted to $\mathcal{J}$ with respect to Z. The next result assures the existence of such a center.

Lemma 4.10 *Assume that* $\mathrm{Inv}_{\mathcal{J}} > 0$. *Then, there exist codimension two combinatorial centers of blowing-up adapted to* $\mathcal{J}$.

Proof By definition $\mathrm{Inv}_{\mathcal{J}} > 0$ if and only if $\Omega_{\mathcal{J}} \neq \emptyset$. Fix an element $Z \in \Omega_{\mathcal{J}}$ and let us see that there are m-standardizations adapted to $\mathcal{J}$ with respect to Z. In view of Lemma 4.9, it is enough to prove the existence of an m-standardization $(\mathcal{O}, \mathfrak{b})$ adapted to $\mathcal{J}$ at a corner point $p \in Z$. Fix any corner point $p \in Z$. Since Z is uncoupled for $\mathcal{J}$, we can assume, up to exchanging the indices i and j, that

$$\ell_i = \lambda_{p,i} - \mu_{p,i} > 0, \quad \text{and} \quad \ell_j = \mu_{p,j} - \lambda_{p,j} > 0.$$

Take $\alpha_p : I_p \to \mathbb{R}_{>0}$ to be a map such that $\alpha_{p,i} = \ell_i$ and $\alpha_{p,j} = \ell_j$, and take the m-chart $\mathbf{u}_p = \mathbf{x}_p^{\alpha_p}$ defined in $V_p^\star$. Any m-standardization extending $\mathbf{u}_p$ is adapted to $\mathcal{J}$ with respect to Z at the point p because of the definition of α_p. Moreover, such an extension exists as a consequence of Proposition 3.16. □

We conclude by applying finitely many times the following result.

Proposition 4.11 *Let* $\mathcal{J} = (\mathfrak{m}, \mathfrak{n})$ *be an m-ideal with* $\mathrm{Inv}_{\mathcal{J}} > 0$. *Given an m-combinatorial center of blowing-up* $\xi = (Z, \mathcal{O}, \mathfrak{b})$ *adapted to* $\mathcal{J}$, *the blowing-up* $\pi_\xi : \mathcal{M}_\xi \to \mathcal{M}$ *centered at* ξ *satisfies* $\mathrm{Inv}_{\pi_\xi^* \mathcal{J}} = \mathrm{Inv}_{\mathcal{J}} - 1$.

Proof Let us write $\mathcal{Z}_\xi = \mathcal{Z}_{\mathcal{M}_\xi}$, $\pi = \pi_\xi$, $E_\infty = \pi^{-1}(Z)$ and $I_Z = \{i, j\}$. Denote also

$$\mathcal{L}_{\mathfrak{m}} = \{\lambda_p\}_{p \in \mathcal{Z}^0}, \ \mathcal{L}_{\mathfrak{n}} = \{\mu_p\}_{p \in \mathcal{Z}^0}, \quad \mathcal{L}_{\pi^*\mathfrak{m}} = \{\lambda'_{p'}\}_{p' \in \mathcal{Z}_\xi^0}, \ \mathcal{L}_{\pi^*\mathfrak{n}} = \{\mu'_{p'}\}_{p' \in \mathcal{Z}_\xi^0}.$$

Let T be a codimension two combinatorial geometric center in $\mathcal{M}$ different from Z. Denote by S_T the stratum in $\mathcal{S}_{\mathcal{M}}$ such that $\overline{S_T} = T$. The closure T' of $\pi^{-1}(S_T)$ is a codimension two geometric center in $\mathcal{M}_\xi$ having index set $I_{T'} = I_T = \{r, s\}$. Given a corner point $p' \in T'$, let $p = \pi_\xi(p')$. We have

$$\lambda_{p',r} = \lambda_{p,r}, \quad \mu_{p',r} = \mu_{p,r}, \quad \lambda_{p',s} = \lambda_{p,s}, \quad \mu_{p',s} = \mu_{p,s},$$

in view of the relation between $\lambda_{p'}$, λ_p and $\mu_{p'}$, μ_p established in Eq. (20), and the expression of $B^\pi_{p'} \in \mathfrak{B}_\pi$ given in Eqs. (15) and (16). Then, we have that $T' \in \Omega_{\pi^*\mathcal{J}}$ if and only if $T \in \Omega_{\mathcal{J}}$, that is T' is uncoupled for $\pi^*\mathcal{J}$ if and only if T is uncoupled for $\mathcal{J}$. Let us see now that any element in $\Omega_{\pi^*\mathcal{J}}$ is among the ones considered before. That is, let us show that there is no codimension two combinatorial geometric center Z' uncoupled for $\pi^*\mathcal{J}$ contained in E_∞.

Take a codimension two combinatorial geometric center $Z' \subset E_\infty$ and a point $p' \in \mathcal{Z}_\xi^0(Z')$. In view of Lemma 4.7, it is enough to prove that Z' is not uncoupled for $\pi^*\mathcal{J}$ at p'. More precisely, if we write $I_{Z'} = \{k, \infty\}$, we want to show that

$$(\lambda_{p',k} - \mu_{p',k})(\lambda_{p',\infty} - \mu_{p',\infty}) \geq 0.$$

Let us consider $p = \pi(p')$ and the local data $\alpha_p \in \Lambda_{(\mathcal{O},\mathfrak{b})}$. The corner point p belongs to Z, and the affine coordinates $\mathbf{u}_p \in \mathfrak{b}$ define a local m-standardization adapted to $\mathcal{J}$ with respect to Z at p, that is, we have the relation $\alpha_{p,j}(\lambda_{p,i}-\mu_{p,i})+\alpha_{p,i}(\lambda_{p,j}-\mu_{p,j})=0$. We know that $I_{p'}=I_p\backslash\{j\}\cup\{\infty\}$, up to exchanging the indices i and j. Hence, the matrix $B':=B_{p'}:I_p\times I_{p'}\to\mathbb{R}_+$ satisfies, using Eq. (16):

$$B'_{\ell\ell}=1,\ \text{for}\ \ell\in I_p\backslash\{j\},\quad B'_{j\infty}=1,\quad B'_{i\infty}=\frac{\alpha_{p,j}}{\alpha_{p,i}}=\frac{\mu_{p,j}-\lambda_{p,j}}{\lambda_{p,i}-\mu_{p,i}},\quad B'_{rs}=0\ \text{otherwise}.$$

By Eq. (20) we get the relations $\lambda_{p',k}=\lambda_{p,k}$, $\mu_{p',k}=\mu_{p,k}$, and

$$\lambda_{p',\infty}=\lambda_{p,j}+B'_{i,\infty}\lambda_{p,i}=\frac{\lambda_{p,i}\mu_{p,j}-\lambda_{p,j}\mu_{p,i}}{\lambda_{p,i}-\mu_{p,i}}=\mu_{p,j}+B'_{i,\infty}\mu_{p,i}=\mu_{p',\infty}.$$

Thus $(\lambda_{p',k}-\mu_{p',k})(\lambda_{p',\infty}-\mu_{p',\infty})=0$, and we are done. □

4.2 Stratified reduction of singularities in monomial manifolds

We use the result of principalization of m-ideals in order to prove the following statement:

Proposition 4.12 *Let $(\mathcal{M},\mathfrak{a})$ be a monomial generalized analytic manifold with $\mathcal{M}=(M,\mathcal{G}_M)$. Given a generalized analytic function $f\in\mathcal{G}_M(M)$, there is an m-star $\sigma\in\mathcal{V}^m_{(\mathcal{M},\mathfrak{a})}$ such that the pull-back $f'=f\circ\sigma$ is of stratified monomial type.*

For the proof of Proposition 4.12 we associate to f a finitely generated m-ideal $\mathcal{J}_f$, and we prove that the principalization of $\mathcal{J}_f$ gives rise to the stratified reduction of singularities of f.

Given $q\in\mathcal{Z}^0$ and $\lambda_q\in\mathrm{Supp}_q(f;\mathbf{x}_q)$, it makes sense to define the m-function $\mathfrak{m}_{\lambda_q}$ as the one having the collection of maps $\mathcal{L}_{\mathfrak{m}_{\lambda_q}}=\{\lambda_q C^{pq}\}_{p\in\mathcal{Z}^0}$ as a combinatorial data, by Remark 4.2 and Eq. (17). The *m-ideal $\mathcal{J}_f$ associated to f* is the ideal sheaf generated by the finite set of m-functions

$$G_f=\bigcup_{q\in\mathcal{Z}^0}\left\{\mathfrak{m}_{\lambda_q}\ :\ \lambda_q\in\mathrm{Supp}_{\min,q}(f;\mathbf{x}_q)\right\}.$$

By definition, notice that for any corner point $q\in\mathcal{Z}^0$, we have

$$(\Gamma_{G_f,q})^{\min}=\mathrm{Supp}_{\min,q}(f;\mathbf{x}_q). \tag{21}$$

where the the notation $\Gamma_{G_f,q}$ was introduced in Sect. 4.1 above.

Lemma 4.13 *Given an m-star $\tau:(\mathcal{M}',\mathfrak{a}')\to(\mathcal{M},\mathfrak{a})$, we have $\tau^*\mathcal{J}_f=\mathcal{J}_{f'}$, where $f'=f\circ\tau$.*

Proof In view of Eq. (19), it is enough to prove that for any corner point $p'\in\mathcal{Z}^0_{\mathcal{M}'}$ the equality $(\Gamma_{\tau^*G_f,p'})^{\min}=(\Gamma_{G_{f'},p'})^{\min}$ holds. Denote for short $\Gamma_1=\Gamma_{\tau^*G_f,p'}$ and $\Gamma_2=\Gamma_{G_{f'},p'}$.

Fix a point $p'\in\mathcal{Z}^0_{\mathcal{M}'}$ and consider $p=\tau(p')$. Write $\Delta:=\mathrm{Supp}_{\min,p}(f;\mathbf{x}_p)$, where $\mathbf{x}_p\in\mathfrak{a}$, and let $B^\tau_{p'}\in\mathfrak{B}_\tau$ be the matrix of exponents codifying τ at p'. Let $\Theta:=\{\lambda_p B^\tau_{p'};\ \lambda_p\in\Delta\}$. We prove that both $\Gamma_1^{\min}$ and $\Gamma_2^{\min}$ are equal to $\Theta^{\min}$.

Step 1: $\Gamma_2^{\min}=\Theta^{\min}$. Recall by Eq. (21) that $\Gamma_2^{\min}=\Delta':=\mathrm{Supp}_{\min,p'}(f';\mathbf{x}'_{p'})$, where $\mathbf{x}'_{p'}\in\mathfrak{a}'$. Let $\Delta=\{\lambda^1,\lambda^2,\dots,\lambda^k\}\subset\mathbb{R}^{I_p}$, that is, the function f around the corner point p has the finite presentation $f|_{V_p^\star}=\mathbf{x}_p^{\lambda^1}U_1+\mathbf{x}_p^{\lambda^2}U_2+\cdots\mathbf{x}_p^{\lambda^k}U_k$, where $U_i(p)\neq 0$, for all

$i = 1, 2, \ldots, k$. Taking into account that $\tau(V^\star_{p'}) \subset V^\star_p$, the function $f' = f \circ \tau$ is written in the chart $\mathbf{x}_{p'} \in \mathfrak{a}'$ as

$$f'|_{V^\star_{p'}} = \mathbf{x}_{p'}^{\tilde{\lambda}^1}(U_1 \circ \tau|_{V^\star_{p'}}) + \mathbf{x}_{p'}^{\tilde{\lambda}^2}(U_2 \circ \tau|_{V^\star_{p'}}) + \cdots \mathbf{x}_{p'}^{\tilde{\lambda}^k}(U_k \circ \tau|_{V^\star_{p'}}),$$

where $\tilde{\lambda}^i = \lambda^i B^\tau_{p'}$. Since $B^\tau_{p'}$ is an invertible matrix, we can assure that $\tilde{\lambda}^r \neq \tilde{\lambda}^s$ for any pair of different indices $r, s \in \{1, 2, \ldots, k\}$. This implies that $\Delta' = \{\tilde{\lambda}^1, \tilde{\lambda}^2, \ldots, \tilde{\lambda}^k\}^{\min}$ by definition of minimal support of f' at p'. We are done, since $\Theta = \{\tilde{\lambda}^1, \tilde{\lambda}^2, \ldots, \tilde{\lambda}^k\}$.

Step 2: $\Gamma_1^{\min} = \Theta^{\min}$. Recall that $\Delta \subset \Gamma_{G_f,p}$ and denote $\tilde{\Delta} = \Gamma_{G_f,p} \backslash \Delta$. By Eq. (21) we know that for any $\mu \in \tilde{\Delta}$ there exists $\lambda \in \Delta$ such that $\lambda \leq_d \mu$. Note that $\Gamma_1 = \Theta \cup \tilde{\Theta}$, where

$$\tilde{\Theta} := \{\mu B^\tau_{p'};\ \mu \in \tilde{\Delta}\}.$$

Therefore, we need only to prove the following claim: *If* $\lambda, \mu : I_p \to \mathbb{R}_+$ *satisfy* $\lambda \leq_d \mu$, *then* $\lambda B^\tau_{p'} \leq_d \mu B^\tau_{p'}$. For that, it is enough to consider the case where $\tau = \pi_\xi$ is a single m-combinatorial blowing-up with center $\xi = (Z, \mathcal{O}, \mathfrak{b})$. Denote, as usual, $E_\infty = \pi_\xi^{-1}(Z)$. If $p' \notin E_\infty$ or equivalently $p \notin Z$, we have that $I_{p'} = I_p$ and $B^\tau_{p'} = D_{\mathbb{1}_{I_p}}$, and we are done. Assume that $p' \in E_\infty$, and let j be the index in I_Z such that $I_{p'}\backslash\{\infty\} = I_p\backslash\{j\}$. Denote $\lambda' = \lambda B^\tau_{p'}$ and $\mu' = \mu B^\tau_{p'}$. By Eq. (16), we have that $\lambda'_\ell = \lambda_\ell$, $\mu'_\ell = \mu_\ell$, and thus $\lambda'_\ell \leq \mu'_\ell$, for all $\ell \in I_{p'}\backslash\{\infty\}$; whereas

$$\lambda'_\infty = \sum_{\ell \in I_Z} \frac{\alpha_{p,j}}{\alpha_{p,\ell}} \lambda_\ell \leq \sum_{\ell \in I_Z} \frac{\alpha_{p,j}}{\alpha_{p,\ell}} \mu_\ell = \mu'_\infty,$$

where $\alpha_p \in \Lambda_{(\mathcal{O},\mathfrak{b})}$, as we wanted. □

Proof of Propostion 4.12 In view of Theorem 4.5, we can take an m-star $\sigma : (\mathcal{M}', \mathfrak{a}') \to (\mathcal{M}, \mathfrak{a})$ such that $\sigma^*\mathcal{J}_f$ is locally principal. By Lemma 4.13 we know also that $\sigma^*\mathcal{J}_f = \mathcal{J}_{f\circ\sigma}$. Hence, since $G_{f\circ\sigma}$ is a set of generators of $\mathcal{J}_{f\circ\sigma}$, we have that $(\Gamma_{G_{f\circ\sigma},p'})^{\min}$ is a singleton for all $p' \in \mathcal{Z}^0_{\mathcal{M}'}$. Finally, by Eq. (21) we obtain

$$m_{p'}(f) = \#\mathrm{Supp}_{\min,p'}(f'; \mathbf{x}'_{p'}) = \#(\Gamma_{G_{f\circ\sigma},p'})^{\min} = 1$$

for all $p' \in \mathcal{Z}^0_{\mathcal{M}'}$. Since $\mathfrak{a}$ is a monomial atlas, we know that $M' = \bigcup_{p' \in \mathcal{Z}^0_{\mathcal{M}'}} V^\star_{p'}$. Thus, given a stratum $S \in \mathcal{S}_\mathcal{M}$, there is a corner point $p' \in \mathcal{Z}^0_{\mathcal{M}'}$ such that $p' \in \bar{S}$. In view of the horizontal stability property for the monomial complexity established in Lemma 2.6, we get $m_S(f) \leq m_{p'}(f) = 1$. We conclude that f' is of stratified monomial type. □

4.3 Proof of the main statement

We end end here the proof of the stratified reduction of singularities for generalized analytic functions, as stated in Theorem 1.1.

Recall that we have a generalized analytic manifold $\mathcal{M} = (M, \mathcal{G}_M)$, and a generalized analytic function $f \in \mathcal{G}_M(M)$ in $\mathcal{M}$. Given a point $p \in M$, we want to prove that there exist an open neighbourhood $V \subset M$ of p and a finite sequence of blowing-ups

$$\sigma : \mathcal{M}_r \xrightarrow{\pi_{r-1}} \mathcal{M}_{r-1} \xrightarrow{\pi_{r-2}} \cdots \xrightarrow{\pi_0} \mathcal{M}_0 = (V, \mathcal{G}_M|_V),$$

such that $f' = f \circ \sigma$ is of stratified monomial type in $\mathcal{M}_r$. Moreover, we are going to see that it can be done by taking blowing-ups with combinatorial centers of codimension two.

Let S be the stratum of $\mathcal{S}_{\mathcal{M}}$ containing p, put $k := \dim S$ and $e := n - k$, where n is the dimension of M. Take a local chart

$$\varphi : V \to \mathbb{R}^k \times \mathbb{R}^e_+,$$

centered at p and such that $\varphi(S) = \mathbb{R}^k \times \{\mathbf{0}\}$. If the minimal support of f along S with respect to φ is $\Delta_0 := \mathrm{Supp}_{\min,S}(f;\varphi) = \{\lambda^1, \lambda^2, \ldots, \lambda^t\} \subset \mathbb{R}^e_+$, we can write

$$f \circ \varphi^{-1} = \mathbf{z}^{\lambda^1} U_1(\mathbf{y}, \mathbf{z}) + \mathbf{z}^{\lambda^2} U_2(\mathbf{y}, \mathbf{z}) + \cdots + \mathbf{z}^{\lambda^t} U_t(\mathbf{y}, \mathbf{z}),$$

where $\mathbf{y} = (y_1, y_2, \ldots, y_k)$ and $\mathbf{z} = (z_1, z_2, \ldots, z_e)$ are the natural coordinates in $\mathbb{R}^k$ and $\mathbb{R}^e$, respectively, $U_i(\mathbf{y}, \mathbf{0}) \not\equiv 0$, for all $i = 1, 2, \ldots, t$, and the elements of Δ_0 are two-by-two incomparable with respect to the division order $\leq_d$ in $\mathbb{R}^e$.

Let us consider the m-corner $(\mathbb{G}_e = (\mathbb{R}^e_+, \mathcal{G}_e), \mathbf{z})$. Let $\mathcal{J}_0$ be the m-ideal in $(\mathbb{G}_e, \mathbf{z})$ generated by the family of m-functions $G_0 = \{\mathbf{z}^{\lambda^1}, \mathbf{z}^{\lambda^2}, \ldots, \mathbf{z}^{\lambda^t}\}$. By Theorem 4.5, there exists a sequence of m-combinatorial blowing-ups

$$\bar{\sigma} : (\bar{\mathcal{M}}_r, \mathfrak{a}_r) \xrightarrow{\bar{\pi}_{r-1}} (\bar{\mathcal{M}}_{r-1}, \mathfrak{a}_{r-1}) \xrightarrow{\bar{\pi}_{r-2}} \cdots \xrightarrow{\bar{\pi}_0} (\bar{\mathcal{M}}_0, \mathfrak{a}_0) = (\mathbb{G}_e, \mathbf{z}),$$

such that $\mathcal{J} = \bar{\sigma}^* \mathcal{J}_0$ is locally principal. Let us write $\tilde{\sigma} = \mathrm{id} \times \bar{\sigma}$, and $\mathcal{W} = (\mathbb{R}^k, \bar{\mathcal{O}}_k)$, where the sheaf $\bar{\mathcal{O}}_k$ has been introduced in Example 2.1. We are going to show that the map

$$\sigma = \varphi^{-1} \circ \tilde{\sigma} : \mathcal{W} \times \bar{\mathcal{M}}_r \to (V, \mathcal{G}_V)$$

is the composition of a finite sequence of combinatorial blowing-ups such that $f' = f \circ \sigma$ is of stratified monomial type.

Step 1: The map σ is a sequence of combinatorial blowing-ups.

Given an index $i \in \{0, 1, \ldots, r\}$, write $\bar{\mathcal{M}}_i = (\bar{M}_i, \mathcal{G}_{\bar{M}_i})$ and consider the product manifold $\mathcal{M}_i = \mathcal{W} \times \bar{\mathcal{M}}_i = (M_i, \mathcal{G}_{M_i})$, where $M_i = \mathbb{R}^k \times \bar{M}_i$. We want to prove that, for $i \neq r$, the map $\pi_i = \mathrm{id} \times \bar{\pi}_i : \mathcal{M}_{i+1} \to \mathcal{M}_i$ is a combinatorial blowing-up. Let $\bar{\xi}_i = (\bar{Z}_i, \mathcal{O}_{\bar{M}_i})$ be the center of blowing-up of $\bar{\pi}_i$, and denote $\bar{\mathcal{N}}_i = (\bar{M}_i, \mathcal{O}_{\bar{M}_i})$. By definition of standardization $\mathcal{O}^{\epsilon}_{\bar{M}_i} = \bar{\mathcal{G}}_{M_i}$. Define $Z_i = \mathbb{R}^k \times \bar{Z}_i$, and let $\mathcal{O}_{M_i}$ be the structural sheaf of the standard analytic manifold $\mathcal{W} \times \bar{\mathcal{N}}_i$. Note that $\mathcal{O}^{\epsilon}_{M_i} = \mathcal{G}_{M_i}$ and that $Z_i \in \mathcal{Z}_{\mathcal{M}_i}$, and thus $\xi_i = (Z_i, \mathcal{O}_{M_i})$ is a combinatorial center of blowing-up for $\mathcal{M}_i$. Finally, we have that π_i is the blowing-up of $\mathcal{M}_i$ centered at ξ_i. Indeed, just note that the blowing-up centered at Z_i of the standard analytic manifold $(M_i, \mathcal{O}_{M_i})$ is $\pi_{Z_i} = \mathrm{id} \times \pi_{\bar{Z}_i}$, where $\pi_{\bar{Z}_i} : \tilde{\bar{\mathcal{N}}}_i \to \bar{\mathcal{N}}_i$ is the blowing-up centered at $\bar{Z}_i$ of $\bar{\mathcal{N}}_i$.

Let us see now that the composition $\varphi^{-1} \circ \pi_0 : (M_1, \mathcal{G}_{M_1}) \to (V, \mathcal{G}_V)$ is a blowing-up of the generalized analytic manifold $(V, \mathcal{G}_V)$. The combinatorial geometric center is $\tilde{Z}_0 = \varphi^{-1}(Z_0)$ and the standardization is the sheaf $\mathcal{O}_V$ given locally at any $a \in V$ by

$$\mathcal{O}_{V,a} = \{g \circ \varphi : \ g \in \mathcal{O}_{M_0, \varphi(a)}\}.$$

Indeed, note that $M_0 = \mathbb{R}^k \times \mathbb{R}^e_+$, and that $\mathcal{O}^{\epsilon}_{M_0} = \mathcal{G}_{M_0} = \mathcal{G}_{k,e}$, where $\mathcal{G}_{k,e}$ has been introduced in Example 2.2. Since φ is an isomorphism we have $\mathcal{O}_V \subset \mathcal{G}_V$ and also $\mathcal{O}^{\epsilon}_V = \mathcal{G}_V$. Then $\xi = (\tilde{Z}_0, \mathcal{O}_V)$ is a combinatorial center of blowing-up. Since the blowing-up with center at $\tilde{Z}_0$ of the standard analytic manifold $(V, \mathcal{O}_V)$ is $\pi_{\tilde{Z}_0} = \pi_{Z_0} \circ \varphi^{-1}$, we get $\pi_\xi = \pi_0 \circ \varphi^{-1}$ as we wanted to prove.

Step 2: The function $f' : M_r \to \mathbb{R}$ is of stratified monomial type.

Denote $\mathcal{Z}_r^0 = \mathcal{Z}_{\bar{\mathcal{M}}_r}^0$. Recall that $\mathcal{M}_r = \mathcal{W} \times \bar{\mathcal{M}}_r$, then there is a bijection between $\mathcal{Z}_r^0$ and the strata of dimension k in $\mathcal{S}_{\mathcal{M}_r}$ sending a corner point q into the stratum $S_q = \mathbb{R}^k \times \{q\}$. Let us prove that

$$m_{S_q}(f') = 1, \tag{22}$$

for each $q \in \mathcal{Z}_r^0$. If we do it, we are done. Indeed, any other stratum in $S \in \mathcal{M}_r$ contains S_q in its closure for some $q \in \mathcal{Z}_r^0$, and by the horizontal stability for the monomial complexity stated in Lemma 2.6 we have $m_S(f') \leq m_{S_q}(f') = 1$.

Fix a corner point $q \in \bar{M}_r$ and $\mathbf{z}_q \in \mathfrak{a}_r$. Let us prove Eq. (22). Take $B_q \in \mathfrak{B}_{\tilde{\sigma}}$ be the matrix of exponents representing $\bar{\sigma}$ at q and denote $\Delta_q = \{\tilde{\lambda}^1, \tilde{\lambda}^2, \ldots, \tilde{\lambda}^t\}$, where $\tilde{\lambda}^i = \lambda^i B_q$ for all $i = 1, 2, \ldots, t$. The expression of f' around S_q is:

$$f'|_{\mathbb{R}^k \times V_q^\star} = \mathbf{z}_q^{\tilde{\lambda}^1}(U_1 \circ \tilde{\sigma}|_{\mathbb{R}^k \times V_q^\star}) + \mathbf{z}_q^{\tilde{\lambda}^2}(U_2 \circ \tilde{\sigma}|_{\mathbb{R}^k \times V_q^\star}) + \cdots \mathbf{z}_q^{\tilde{\lambda}^t}(U_t \circ \tilde{\sigma}|_{\mathbb{R}^k \times V_q^\star}),$$

so that $\mathrm{Supp}_{\min, S_q}(f'; (\mathbf{z}_q, \mathbf{y})) = (\Delta_q)^{\min}$. On the other hand, using Eq. (20) we have $\Gamma_{\bar{\sigma}^* G_0, q} = \Delta_q$. Since $\bar{\sigma}^* G_0|_{V_q^\star}$ is a set of generator of $\mathcal{J}|_{V_q^\star}$ and $\mathcal{J}$ is locally principal, we get that $(\Delta_q)^{\min}$ is a singleton by Eq. (19). We conclude $m_{S_q}(f') = 1$, as we wanted. □

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