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Abstract

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Keywords (separated by '-'): Real analytic vector field - First integral - Formal and analytic separatrix - Reduction of singularities - Index of vector fields

Mathematics Subject Classification (separated by '-'): 32S65 - 37F75 - 34Cxx - 14P15

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ORIGINAL PAPER

Real analytic vector fields with first integral and separatrices

Rogério Mol[1] · Fernando Sanz Sánchez[2]

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Abstract
We prove that a germ of analytic vector field at $(\mathbb{R}^3, 0)$ that possesses a non-constant analytic first integral has a real formal separatrix. We provide an example which shows that such a vector field does not necessarily have a real analytic separatrix. [1] [2]

Keywords Real analytic vector field · First integral · Formal and analytic separatrix · Reduction of singularities · Index of vector fields

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1 Introduction

In this paper we prove the following result:

Theorem 1 *Let X be a germ of real analytic vector field at $(\mathbb{R}^3, 0)$ that has an analytic first integral. Then X has a real formal separatrix. The statement is optimal in the sense that such a vector field X does not necessarily have a real analytic separatrix.*

Speaking in general terms, let X be a germ of real analytic vector field at $(\mathbb{R}^n, 0)$. A *real analytic separatrix* of X is a germ of irreducible analytic curve Γ at $0 \in \mathbb{R}^n$ which is invariant by X. If $\gamma(t) = (\gamma_1(t), \ldots, \gamma_n(t)) \in (t\mathbb{R}\{t\})^n \setminus \{0\}$ is a parametrization of Γ, the

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invariance condition is equivalent to saying that there exists $h(t) \in \mathbb{R}\{t\}$ such that $X(\gamma(t)) = h(t)\frac{d\gamma}{dt}(t)$ for any t, where $h(t) \not\equiv 0$ if and only if Γ is not contained in the singular locus $\mathrm{Sing}(X) = \{p\,;\, X(p) = 0\}$ of X. Replacing $\mathbb{R}\{t\}$ by $\mathbb{R}[[t]]$, we obtain the concept of *real formal separatrix*. On the other hand, considering the canonical complexification of X to a holomorphic vector field at $(\mathbb{C}^n, 0)$ and changing $\mathbb{R}$ to $\mathbb{C}$, we have the concepts of *complex holomorphic separatrix* and *complex formal separatrix*, seen as objects in $(t\mathbb{C}\{t\})^n\backslash\{0\}$ and $(t\mathbb{C}[[t]])^n\backslash\{0\}$, respectively.

We also recall that a first integral of X is a germ of function $f : (\mathbb{R}^n, 0) \to \mathbb{R}$ such that $df(X) = 0$. The expression "analytic first integral" in Theorem 1 could be interpreted either as "holomorphic first integral" or "real analytic first integral". In fact, if $h : (\mathbb{C}^3, 0) \to \mathbb{C}$ is a non-constant holomorphic first integral of (the complexification of) X, then one can check that the real traces of $Re(h)$ and $Im(h)$ are real analytic first integrals of X with at least one of them non-constant.

Notice that in Theorem 1 we may assume without loss of generality that X has an isolated singularity at 0, otherwise there is at least a real analytic separatrix of X contained in $\mathrm{Sing}(X)$. On the contrary, we do not assume necessarily that the singular locus $\mathrm{Sing}(df) = \{p; df(p) = 0\}$ of the first integral f of X is isolated. However, taking into account that $\mathrm{Sing}(df)$ is invariant by the vector field X, we may assume that it has no one-dimensional real components (see below in Sect. 4 for details).

Analytic or formal separatrices may of course be defined for holomorphic vector fields. They are algebraically manipulable invariant objects which play a central role in the study of the local dynamics of the vector field. Let us briefly review some avatars of the problem of existence of separatrices, related to the situation of real vector fields.

Planar case, $n = 2$. First, the Separatrix Theorem of Camacho and Sad [7] asserts that a planar vector field always has a complex holomorphic separatrix, although it may not have formal real separatrices: take, for instance, the standard vector field of *center-type*, $X = -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}$. In this example, X has an analytic first integral, showing that Theorem 1 is not true for planar vector fields. On the other hand, there are examples of planar real vector fields with real formal separatrices, none of them convergent. An explicit example could be found in [29, Example 3.7(3)]. Below, in Sect. 5, we provide other examples used for the proof of the second part of Theorem 1.

It is also known that an analytic vector field X at $0 \in \mathbb{R}^2$ with Poincaré index equal to zero has a real formal separatrix. Below, in Proposition 8, we provide a generalization of this result for vector fields defined in singular analytic surfaces, which is one of the main ingredients of the proof of Theorem 1.

Three dimensional case, $n = 3$. Camacho–Sad's Theorem is no longer valid in this case: Gómez–Mont and Luengo in [13] have constructed a family of vector fields in $(\mathbb{C}^3, 0)$ without complex separatrices. They state the result for analytic separatrices, although the same proof works in order to show that any vector field in that family is actually devoid of complex formal separatrices. An explicit member of that family with real coefficients could be found in [27, p. 333]. As a consequence of Theorem 1, vector fields in Gómez–Mont and Luengo's family with real coefficients cannot have non-constant holomorphic first integrals.

As for the planar case, there are examples of analytic vector fields at $(\mathbb{R}^3, 0)$ with formal real separatrices, none of them convergent (i.e. without real analytic separatrices). An explicit example can be found in [8, p. 3]. We construct in Sect. 5 another example which has, moreover, a non-constant analytic first integral. It will prove the second part of Theorem 1, that is, that the conclusion "formal" in the statement cannot be improved to "analytic".

We should mention that, in a recent paper, D. Cerveau and A. Lins Neto proved that a germ of complex analytic vector field in $(\mathbb{C}^3, 0)$, with isolated singularity, that is tangent

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to a holomorphic foliation of codimension one always has a complex analytic separatrix [11, Proposition 3]. This result implies in particular that any vector field X as in Theorem 1 actually *has* a complex analytic separatrix, inasmuch as X is tangent to the foliation $df = 0$, where f is a first integral. Such a complex separatrix may not be a real one (once more by our example in Sect. 5 below).

Higher dimension, $n \geq 4$. Families of holomorphic vector fields at $(\mathbb{C}^n, 0)$ without complex separatrices (neither convergent nor formal) are constructed in [21] for any dimension $n \geq 4$, generalizing the three dimensional construction carried out in [13]. Each one of these families contains an explicit example with real coefficients.

Examples of real analytic vector fields without real formal separatrices having analytic first integral can be constructed in any dimension $n \geq 4$, showing that the phenomenon depicted in Theorem 1 is exclusive for dimension three. When $n = 2p$ is even, we consider a *multicenter* vector field, written in coordinates $(x_1, y_1, \ldots, x_p, y_p)$ as $Z_n = X_1 + \cdots + X_p$, where $X_j = -y_j \frac{\partial}{\partial x_j} + x_j \frac{\partial}{\partial y_j}$. When $n = 2p + 3$ is odd, $p \geq 1$, we take coordinates $(x_1, y_1, \ldots, x_p, y_p, x, y, z)$ and set $Z_n = Z_{2p} + W$, where Z_{2p} is a multicenter vector field in the variables $(x_1, y_1, \ldots, x_p, y_p)$ and W is one of the examples of three dimensional real vector fields in Gómez-Mont and Luengo's family written in the variables (x, y, z). Notice that, in both cases, Z_n has $f(x_1, y_1) = x_1^2 + y_1^2$ as a first integral.

Finally, concerning real analytic separatrices in any dimension, it is worth mentioning Moussu's paper [26], where it is proved that an analytic gradient vector field at $(\mathbb{R}^n, 0)$ always has a real analytic separatrix. Below, we describe some arguments of that result, those which are used in our proof of Theorem 1 (concretely, in Proposition 4).

Let us sketch the proof of Theorem 1 and the plan of the article. Let X be as in the hypothesis of the statement, having isolated singularity, and assume that the first integral f of X is such that its singular locus $\mathrm{Sing}(df)$ does not have one-dimensional real components. Using a Brunella's result [6] which guarantees that X has a non-trivial orbit accumulating to the origin, we may assume, moreover, that the special fiber $Z = f^{-1}(f(0))$ of f is not reduced to the single point $0 \in \mathbb{R}^3$. Under these assumptions, we prove, in Sect. 2, a technical result (Proposition 4) which can be framed in the context of real versions of Milnor's Fibration Theorem [24]. Roughly speaking, it asserts that, in any sufficiently small neighborhood of the origin, f has regular fibers with connected components which are simply connected and which accumulate to a given two-dimensional component of the special fiber Z. Our proof of Proposition 4 requires some avatars of known results in the theory of reduction of singularities of analytic functions. We recall them in the form needed for our purposes.

In Sect. 3, we define, for any two-dimensional component L of Z, the index $I_L(X)$ of the restriction $X|_L$, a generalization to singular surfaces of the usual notion of *Poincaré index* of a planar vector field at a singular point. It is not really a new notion, it corresponds in one or another equivalent way to a particular case of standard definitions of the index of a vector field in a singular invariant variety (see [5] for more information). Pushing the restricted vector field $X|_L$ to nearby fibers, using homotopic invariance of the index and the aforementioned result about simply connected fibers, we show that $I_L(X)$ is equal to zero for at least one component L.

In Sect. 4, we conclude the proof of the first part of Theorem 1 proving that, given a two-dimensional component L of Z, either there exists a formal separatrix of X inside L or $I_L(X) \neq 0$ (Proposition 8 below). Incidentally, we use again the reduction of singularities as presented in Sect. 2 for the proof of this result. As mentioned before, it generalizes a known result of planar vector fields to the situation of vector fields in singular surfaces. It is related to

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Bendixson's formula for the computation of the Poincaré index using hyperbolic and elliptic sectors of the vector field at the singularity.

Finally, in Sect. 5, we provide an explicit example of a vector field X with isolated singularity at $0 \in \mathbb{R}^3$ which has an analytic first integral but which does not have any real analytic separatrix. The difficult part to check is that the formal real separatrix of such an example does really diverge. For that, we use the Martinet–Ramis moduli of planar holomorphic foliations of saddle-node type [18,23] and the computation of the tangent of the moduli map in Elizarov's work [12]. We thank Loïc Teyssier for his comments and decisive remarks concerning these arguments and techniques.

2 About the fibers of a real analytic function

The main result in this section is Proposition 4 below, a result on the geometry of the fibers of a real analytic function in $\mathbb{R}^3$. We provide a proof adapted to our situation which employs the reduction of singularities of analytic functions. Some of the arguments are inspired on those of the paper [15] and also on a part of Roche's work [30] concerning *Real Clemens Structures*.

Our starting point is the following result (see Aroca et al. [2], Hironaka [16] or Bierstone and Milman [3,4]).

Theorem 2 *Let $f : (\mathbb{R}^n, 0) \to (\mathbb{R}, 0)$ be a non-zero real analytic function. There exists a neighborhood U of $0 \in \mathbb{R}^n$ and a sequence of blow-ups with closed analytic non-singular centers*

$$\pi : M_m \xrightarrow{\pi_m} M_{m-1} \xrightarrow{\pi_{m-1}} \cdots \xrightarrow{\pi_2} M_1 \xrightarrow{\pi_1} U \tag{1}$$

such that $f \circ \pi : M_m \to \mathbb{R}$ is everywhere locally of monomial type, i.e. it can be written locally as a monomial times a unit in analytic coordinates. Moreover, if Y_{j-1} is the center of π_j for $j = 1, \ldots, m$, and we define recursively the total divisor E_j at stage j by $E_j = \pi_j^{-1}(E_{j-1} \cup Y_{j-1})$ with $E_0 = \emptyset$, then Y_j has normal crossing with E_j and it is contained in the singular locus $\mathrm{Sing}(df_j)$ of $f_j = f \circ \pi_1 \circ \cdots \circ \pi_j$, for $j \geq 0$, where $f_0 = f$.

In particular, if $Z = f^{-1}(0)$ and $\tilde{Z} = Z \setminus \mathrm{Sing}(df)$ is assumed to be non-empty (thus $\tilde{Z}$ is a smooth analytic hypersurface), then π restricts to an analytic isomorphism from $\pi^{-1}(\tilde{Z})$ to $\tilde{Z}$.

For our purposes, we will use *real blow-ups* instead of the usual (projective) blow-ups π_j in Theorem 2. In order to define properly a real blow-up, we must consider the category of real analytic manifolds with boundary and corners; i.e. manifolds locally defined in coordinate charts $(x_1, \ldots, x_n)$ as quadrants $\{x_{i_1} \geq 0, x_{i_2} \geq 0, \ldots, x_{i_r} \geq 0\}$ and so that the changes of coordinates are analytic isomorphisms preserving the quadrants. The point is that a real blow-up (also called a "polar blow-up") produces a boundary in the blown-up manifold, namely the inverse image of the center by the blow-up (called *exceptional divisor*), which corresponds to the set of *half-directions* (instead of directions) in the normal bundle of the center as a submanifold of the ambient space. Subsequent real blow-ups produce new boundaries which intersect old boundaries along corners.

Let us recall the main definitions here (see for instance the recent reference [22] for details). First, we define the real blow-up, with closed non-singular center Y, on a real manifold without boundary M. Let $\pi : M_1 \to M$ be the usual blow-up of M with center Y and let $\tau : M_1^+ \to M_1$ be the orientable double covering of M_1. The composition $\pi \circ \tau$ is an analytic map which ramifies along the divisor $E = \pi^{-1}(Y)$. Then the real blow-up of M with

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center Y is the restriction $\sigma : M_1' \to M$ of $\pi \circ \tau$ to only one sheet, so that M_1' is an analytic manifold with boundary $\partial M_1' = E$. Next, more generally, if M is a real analytic manifold with boundary and corners and $Y \subset M$ is a non-singular analytic submanifold having normal crossings with ∂M, we may consider first M immersed in a real analytic manifold $\widetilde{M}$ with no boundaries or corners of the same dimension (the immersion is locally uniquely determined up to analytic isomorphisms), so that ∂M becomes a normal crossing divisor of $\widetilde{M}$ and such that Y is sent into a non-singular submanifold $\widetilde{Y} \subset \widetilde{M}$ with normal crossings with ∂M inside $\widetilde{M}$. The real blow-up $\sigma : M' \to M$ with center $Y \subset M$ is the restriction of the real blow-up $\widetilde{\sigma} : \widetilde{M}' \to \widetilde{M}$ with center $\widetilde{Y}$ to $M' = \overline{\widetilde{\sigma}^{-1}(M \setminus Y)}$.

With this construction in mind, we adapt Theorem 2 to obtain a version which uses real blow-ups and which will be more convenient for us. Although we can consider general statements, we will concentrate on three-dimensional analytic functions with some extra condition concerning its singular locus.

Fix a germ $f : (\mathbb{R}^3, 0) \to (\mathbb{R}, 0)$ of analytic function. Consider the prime decomposition $f = f_1^{n_1} f_2^{n_2} \cdots f_r^{n_r}$, where each f_j is an irreducible germ of analytic function, and let $h = Red(f) = f_1 f_2 \cdots f_r$. Notice that $Z = f^{-1}(0) = h^{-1}(0)$. Assume the following property, that we call *Reduced Isolated Singularity*:

(RIS). The germ of analytic set $Z = f^{-1}(0)$ is not reduced to $\{0\}$ and $\mathrm{Sing}(dh) \subset \{0\}$.

Note that the hypothesis (RIS) implies that, in some neighborhood of the origin, the set $Z \setminus \{0\}$ is a non-singular two-dimensional analytic submanifold and that the irreducible components $f_j^{-1}(0)$ of Z, as germs of analytic sets, only intersect at 0. (The converse of this result is not true: take $f = Red(f) = y^3 - x^6$ for which the special fiber $Z = \{y - x^2 = 0\}$ is a non-singular surface at every point and the z-axis is contained in $\mathrm{Sing}(df)$.) To be more precise, let $\varepsilon > 0$ be sufficiently small such that f is defined and analytic in a neighborhood of the closed ball $V = \overline{B}(0, \varepsilon)$, and such that $Z \cap V$ cuts transversally the boundary of V. By the Conic Structure Theorem (see Milnor [24] or vdDries [32] for a more general statement), the set $(Z \setminus \{0\}) \cap V$ has finitely many connected components, denoted by $L_1, L_2, \ldots, L_r$, where each L_i is a non-singular analytic surface immersed in V whose closure in V is homeomorphic to the cone at 0 over the link $C_i = \partial V \cap L_i$ (a curve homeomorphic to $\mathbb{S}^1$). The germs of the components L_i at 0 are well defined and do not depend on ε. We will use the same notation L_i for both the components of $(Z \setminus \{0\}) \cap V$ (for any given sufficiently small ε) and their germs. They will be called *local components* of the special fiber $Z = f^{-1}(0)$.

Proposition 3 *Let $f : (\mathbb{R}^3, 0) \to (\mathbb{R}, 0)$ be a germ of analytic function that satisfies the hypothesis (RIS). Then, if $\varepsilon > 0$ is sufficiently small and $V = \overline{B}(0, \varepsilon)$, there is a sequence of real blow-ups (independent of ε)*

$$\sigma : M_m' \xrightarrow{\sigma_m} M_{m-1}' \xrightarrow{\sigma_{m-1}} \cdots \xrightarrow{\sigma_2} M_1' \xrightarrow{\sigma_1} V, \tag{2}$$

such that the composition $f \circ \sigma$ is everywhere locally of monomial type and such that, if L is a local component of $Z = f^{-1}(0)$, we have:

(i) *$\sigma^{-1}(L)$ is diffeomorphic to the half-open cylinder $[0, 1) \times \mathbb{S}^1$, where the boundary $\{0\} \times \mathbb{S}^1$ corresponds to the link $C = L \cap \partial V$.*

(ii) *The strict transform $L' = \overline{\sigma^{-1}(L)}$ of $\overline{L} = L \cup \{0\}$ is a real analytic submanifold of M_m' with boundary and corners, homeomorphic to the closed cylinder $[0, 1] \times \mathbb{S}^1$.*

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(iii) *Denoting* $\partial L' = C_\infty \cup \sigma^{-1}(C)$ *the two connected components of the boundary of* L', *we have that* L' *cuts transversally the total divisor* E'_m *along* C_∞, *which is a piecewise smooth analytic curve homeomorphic to* $\mathbb{S}^1$.

(iv) *The strict transforms of two different local components do not intersect.*

Moreover, σ_1 *is the real blow-up with center* $Y'_0 = \{0\}$ *and, if* Y'_{j-1} *is the center of* σ_j *for* $j = 2, \ldots, m$, *and we define recursively the total divisor* E'_j *at stage* j *by* $E'_j = \sigma_j^{-1}(E'_{j-1} \cup Y'_{j-1})$ *with* $E'_0 = \emptyset$, *then, for any* $j \geq 1$, $Y'_j \subset E'_j$ *and* E'_j *is homeomorphic to the sphere* $\mathbb{S}^2$.

Proof Let $f = f_1^{n_1} f_2^{n_2} \cdots f_r^{n_r}$ be the prime decomposition of f as a germ and put $h = Red(f) = f_1 f_2 \cdots f_r$. Let $\varepsilon > 0$ be sufficiently small such that f is defined in a neighborhood of a closed ball $V = \overline{B}(0, \varepsilon)$, and such that Z cuts transversally the boundary of V. Assume moreover that V is contained in a neighborhood where Theorem 2 applies to h, so that we obtain a sequence of blow-ups π as in (1) with centers $Y_0, Y_1, \ldots, Y_{m-1}$, such that $h \circ \pi$ is everywhere locally of monomial type. Therefore, the composition $f \circ \pi$ is also everywhere locally of monomial type. Define the sequence (2) recursively as follows: $\sigma_1 : M'_1 \to V$ is the real blow-up of V with center $Y'_0 = Y_0$, $\sigma_2 : M'_2 \to M'_1$ the real blow-up with center $Y'_1 = Y_1 \cap M'_1$, and so on. Since $\mathrm{Sing}(dh) \subset \{0\}$, by the hypothesis (RIS), and since the center Y_{j-1} of π_j is contained in the singular locus of $h_{j-1} = h \circ \sigma_1 \circ \cdots \circ \sigma_{j-1}$, we have that $Y'_0 = \{0\}$ and that $Y'_j \subset E'_j$ for $j \geq 1$. We deduce then that $E'_j \cong \mathbb{S}^2$ by recurrence on j, using the definition of real blow-up.

Property (i) is a consequence of the mentioned Conic Structure Theorem, together with the fact that $\sigma : M'_m \backslash E'_m \to V \backslash \{0\}$ is a diffeomorphism since each center Y'_j is contained in E'_j for $j \geq 0$. To prove properties (ii) and (iii) we use the conclusion that $\pi^{-1}(Z \cap V)$ is a normal crossing divisor, so that L' is contained in one of its components, a non-singular analytic surface which cuts transversally the components of the total divisor E'_m. Finally, for property *(iv)*, notice that if L'_1, L'_2 are the strict transforms of two different local components L_1, L_2 of Z and $L'_1 \cap L'_2 \neq \emptyset$, then necessarily $L'_1 \cap L'_2 \not\subset E'_m$ (since L'_1, L'_2 and any component of E'_m are components of the normal crossing divisor $\sigma^{-1}(Z \cap V)$). Hence $\sigma^{-1}(L_1) \cap \sigma^{-1}(L_2) \neq \emptyset$ and also $L_1 \cap L_2 \neq \emptyset$, which is impossible by the hypothesis (RIS). □

Proposition 4 *Let* $f : (\mathbb{R}^3, 0) \to (\mathbb{R}, 0)$ *be a germ of analytic function satisfying the hypothesis (RIS). Then there is a local component* L *of the special fiber* Z *and a neighborhood base* $\mathcal{B}$ *of* $0 \in \mathbb{R}^3$ *such that each* $U \in \mathcal{B}$ *is compact and satisfies the following property: there exists a family* $\{F_\lambda^U\}_{\lambda \in (0,\delta)}$, *where* F_λ^U *is a connected component of a non-singular fiber of* $f|_U$, *such that* F_λ^U *is homeomorphic to a closed disc and such that* $F_\lambda^U \xrightarrow{\lambda \to 0} (L \cup \{0\}) \cap U$ *in the Hausdorff topology.*

Proof We prove that any closed ball $V = \overline{B}(0, \varepsilon)$, with $\varepsilon > 0$ sufficiently small for which Proposition 3 holds, contains a neighborhood U with the required properties of the statement. We use notation of Proposition 3 so that, if $L_1, \ldots, L_r$ are the local components of the singular fiber $Z = f^{-1}(0)$ and L'_j is the strict transform of $\overline{L_j}$, then L'_j is homeomorphic to the cylinder $[0, 1] \times \mathbb{S}^1$ and $L'_j \cap E'_m$ is a curve homeomorphic to $\mathbb{S}^1$. Moreover, $L'_i \cap L'_j \cap E'_m = \emptyset$ if $i \neq j$. Let j_0 be such that one of the connected components of $E'_m \backslash L'_{j_0} \cap E'_m$, say D, contains no curve $L'_j \cap E'_m$ for $j \neq j_0$. Then $\Omega = L'_{j_0} \cup D$ is homeomorphic to a closed disc.

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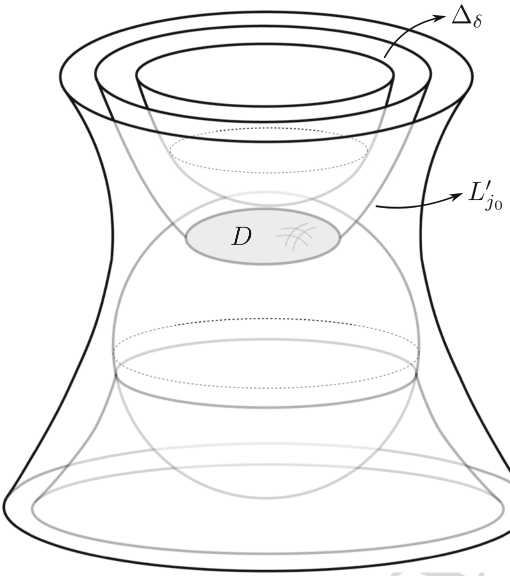

Let us prove the statement for the component $L = L_{j_0}$. Consider $\widetilde{f} = f \circ \sigma : M'_m \to \mathbb{R}$, whose singular fiber is given by

$$\widetilde{Z} = \widetilde{f}^{-1}(0) = \sigma^{-1}(Z \cap V) = L'_1 \cup L'_2 \cup \cdots \cup L'_r \cup E'_m,$$

and thus $\Omega \subset \widetilde{Z}$. By construction, there is a unique connected component of $M'_m \backslash \widetilde{Z} = \sigma^{-1}(V \backslash f^{-1}(0))$, denoted by K, whose topological frontier in M'_m is exactly Ω. We assume, without lost of generality, that $\widetilde{f}$ is positive on K. Denote also by $\dot{M}'_m = \sigma^{-1}(V \backslash \partial V)$ (a manifold with boundary where $\partial \dot{M}'_m = E'_m$).

Let g be an analytic riemannian metric on M'_m (the existence of such a metric is guaranteed by Grauert's Analytic Immersion Theorem [14]) and let $\xi = -\nabla_g(\widetilde{f}^2)$ be the gradient vector field of $\widetilde{f}^2$ with respect to g. The square and the sign "−" are taken in order to guarantee both that $\widetilde{f}$ decreases along any trajectory of ξ and that $\widetilde{Z}$ is exactly the singular locus of ξ. By a Łojasiewicz's result (see [20]), there exists an open neighborhood H of $\widetilde{Z} \cap \dot{M}'_m$ in $\dot{M}'_m$ such that for any $p \in H$, the integral curve γ_p of ξ with $\gamma_p(0) = p$ is defined on $[0, \infty)$ and the limit

$$R_\xi(p) = \lim_{t \to \infty} \gamma_p(t)$$

exists and belongs to $\widetilde{Z} \cap \dot{M}'_m$. Moreover, the map $R_\xi : H \to \widetilde{Z} \cap \dot{M}'_m$ is a continuous retraction.

In our particular case where $\widetilde{f}$ is locally of monomial type, one can show, moreover, the following:

Claim *For any $q \in \Omega \cap \dot{M}'_m$, there exists a neighborhood B_q of q and a unique orbit of ξ in $B_q \cap K$ that accumulates to q.*

Assume that the Claim is true. Put $\Omega' = \Omega \cap \sigma^{-1}(\overline{B}(0, \varepsilon/2))$. Using the existential part of the Claim, the fact that $\widetilde{f}$ decreases along integral curves of ξ and the compactness of Ω', there exists a fiber $\Delta_\delta = \widetilde{f}^{-1}(\delta) \cap K$ of $\widetilde{f}|_K$, for some $\delta > 0$, such that $\Omega' \subset R_\xi(\Delta_\delta)$.

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(Notice that fibers of $\widetilde{f}|_K$ and of $\widetilde{f}^2|_K$ coincide since we have assumed that $\widetilde{f}$ is positive on K). On the other hand, by the uniqueness property stated in the Claim and since an orbit of ξ can intersect at most once any fiber of $\widetilde{f}$, if $\widetilde{F}_\delta = R_\xi^{-1}(\Omega') \cap \Delta_\delta$ then

$$R_\xi|_{\widetilde{F}_\delta} : \widetilde{F}_\delta \to \Omega'$$

is bijective, and hence a homeomorphism. Observe that all the conclusions above also hold for any λ with $\lambda \in (0, \delta]$, using the flow of ξ, which provides, by restriction, a diffeomorphism from $\widetilde{F}_\delta$ to $\widetilde{F}_\lambda$ for every such λ. In particular, $\widetilde{F}_\lambda = R_\xi^{-1}(\Omega') \cap \Delta_\lambda$ is homeomorphic to a closed disc for any $\lambda \in (0, \delta]$.

We finally consider the set

$$\widetilde{U} = (M'_m \backslash K) \cup (R_\xi^{-1}(\Omega') \cap \widetilde{f}^{-1}([0, \delta]) \cap \overline{K}),$$

which is a compact neighborhood of the total divisor E'_m in M'_m, and we put $U = \sigma(\widetilde{U})$. Then U is a neighborhood of the origin contained in V with the required properties for the family $\{F_\lambda^U = \sigma(\widetilde{F}_\lambda)\}_{\lambda \in (0,\delta]}$.

Proof of the Claim. Fix $q \in \Omega \cap \dot{M}'_m$ and denote by $e = e(q)$ the number of components of $\widetilde{Z}$, considered as a normal crossing divisor, which meet at q. We analyze separately the three possible values of $e \in \{1, 2, 3\}$. First, it is worth recalling that the expression of the vector field $\xi = -\nabla_g \widetilde{f}^2 = A\partial/\partial x + B\partial/\partial y + C\partial/\partial z$ in analytic coordinates $w = (x, y, z)$ at q is computed by the formula

$$(A\ B\ C) = -\frac{\partial \widetilde{f}^2}{\partial w} \left(h_{ij}\right), \tag{3}$$

where (h_{ij}) is the inverse of the matrix of the metric g in the coordinates w and $\partial \widetilde{f}^2/\partial w$ is the row vector of partial derivatives of $\widetilde{f}^2$. □

Case $e = 1$. We choose an analytic chart $(B_q, (x, y, z))$ centered at q so that $\widetilde{f} = x^m$, with $m > 0$, and $B_q \cap K = \{x > 0\}$. Inside the domain B_q of the chart, using (3), we may write $\xi = x^{2m-1}\bar{\xi}$, where $\bar{\xi}$ is a vector field, which is non-singular at q. Moreover, $\bar{\xi}$ is transversal to $\widetilde{Z} = \{x = 0\}$ in a neighborhood of q. Thus, the orbit of $\bar{\xi}$ through q is the unique orbit that may accumulate to q and cuts $\{x > 0\}$. Since orbits of ξ in $\{x > 0\}$ are contained in orbits of $\bar{\xi}$, we conclude the claim.

Case $e = 2$. In this case, we choose an analytic chart $(B_q, (x, y, z))$ such that $\widetilde{f} = x^m y^n$, with $m, n > 0$, and $B_q \cap K = \{x > 0,\ y > 0\}$. Then, using (3), we may write $\xi = 2x^{2m-1}y^{2n-1}\bar{\xi}$, where

$$\bar{\xi} = -(myh_{11} + nxh_{21})\frac{\partial}{\partial x} - (myh_{12} + nxh_{22})\frac{\partial}{\partial y} - (myh_{13} + nxh_{23})\frac{\partial}{\partial z}. \tag{4}$$

Since the orbits of ξ and $\bar{\xi}$ coincide in $B_q \cap K$, it suffices to prove the Claim for $\bar{\xi}$. Using the fact that (h_{ij}) is positive definite, we have that $\mathrm{Sing}(\bar{\xi}) = \{x = y = 0\}$ and that the linear part $D\bar{\xi}(q)$ of $\bar{\xi}$ at q has (real) eigenvalues $\{0, \lambda, \mu\}$, where $\lambda < 0 < \mu$. Let W^s, W^u be the stable and unstable manifolds of $\bar{\xi}$ at q. They are invariant smooth curves (in fact real analytic separatrices of $\bar{\xi}$, see [10]) tangent to the eigendirections E_λ, E_μ corresponding to λ and μ, respectively. Also, $W^c = \mathrm{Sing}(\bar{\xi})$ is a center manifold of $\bar{\xi}$ at q. The *Theorem of Reduction to the Center Manifold* (see [9,17]) implies that $\bar{\xi}$ is topologically equivalent, in a neighborhood of q, to the linear vector field $D\bar{\xi}(q) = \lambda u\partial/\partial u + \mu v\partial/\partial v$ in $\mathbb{R}^3$, where u and v are linear coordinates on E_λ and E_μ, respectively. As a consequence, the four connected components of $(W^s \cup W^u)\backslash\{q\}$ are the unique non-trivial orbits of $\bar{\xi}$ which accumulate to q.

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It suffices to show that $(W^u \cup W^s) \cap K = W^s \cap K \neq \emptyset$ (notice that in that case only one of the components of $W^s \backslash \{q\}$ may be contained in K, since W^s, W^u are transversal at q to the components $\{x = 0\}$ and $\{y = 0\}$ of $\widetilde{Z}$, both contained in $Fr(K)$).

Let us show that $W^u \cap K = \emptyset$. Notice that the integral curve of $\bar{\xi}$ at any point of $W^u \backslash \{q\}$ is defined in an interval of the form $(-\infty, a)$ and converges to q for $t \to -\infty$. This would also be the case for an integral curve of ξ if $W^u \cap K \neq \emptyset$, since the sense of parametrization of integral curves of ξ and $\bar{\xi}$ coincide in $B_q \cap K$. However, this is impossible because $-\widetilde{f}^2$ grows along integral curves of ξ in K and $\widetilde{f}(q) = 0$.

Let us show that $W^s \cap H \neq \emptyset$. Denote by $\Delta = E_\lambda \oplus E_\mu$, a linear plane invariant by the linear vector field $D\bar{\xi}(q)$. Let Q be the cone inside Δ bounded by the half-lines ℓ_x, ℓ_y of Δ which correspond to the tangent directions of $\{y = 0, x \geq 0\} \cap \Delta$ and $\{x = 0, y \geq 0\} \cap \Delta$, respectively. If $W^s \cap K = \emptyset$ then we would have $Q \cap (E_\lambda \cup E_\mu) = \{0\}$. In this case, we could see that the vector field $D\bar{\xi}(q)$, that is everywhere transversal to the boundary of Q, *enters* Q through one of the half-lines ℓ_x, ℓ_y while it *escapes* from Q through the other one. This is impossible by comparing $D\bar{\xi}(q)$ with $\bar{\xi}$, since this last vector field escapes from K through any point of $Fr(K) \backslash \{x = y = 0\} = \{y = 0, x > 0\} \cup \{x = 0, y > 0\}$.

Case e $= 3$. We use the result by Kurdyka et al. [19] that solves Thom's Conjecture : Let $h : (\mathbb{R}^n, 0) \to \mathbb{R}$ be an analytic function and let g be a real analytic riemannian metric at 0. Then any non-trivial orbit Γ of the analytic gradient vector field $\nabla_g h$ that converges to $0 \in \mathbb{R}^n$ has a well defined limit tangent

$$\nu_\Gamma = \lim_{x \in \Gamma, x \to 0} \frac{x}{\|x\|} \in \mathbb{S}^{n-1}.$$

Also, we use the following results from Moussu's paper [26, Theorems 1 and 3]: if $g(0)$ is the Euclidean metric in $\mathbb{R}^n = T_0\mathbb{R}^n$ (or also a scalar positive multiple of it) and $H_k = h_k|_{\mathbb{S}^{n-1}}$, where h_k is the first non-zero homogeneous polynomial (of degree k) in the Taylor expansion of h at $0 \in \mathbb{R}^n$, then we have:

(a) If $\nu_\Gamma \in \mathbb{S}^{n-1}$ is the limit tangent of an orbit Γ of $\nabla_g h$ that converges to the origin, then $\nu_\Gamma \in \mathrm{Sing}(dH_k)$.[1]

(b) Assume that $h \geq 0$ in a neighborhood of the origin and denote by $S_0 \subset \mathbb{S}^{n-1}$ the set of points $\nu \in \mathbb{S}^{n-1}$ that satisfy

$$\nu \in \mathrm{Sing}(dH_k), \quad H_k(\nu) < 0, \quad kH_k(\nu) < \inf\{\lambda_1(\nu), \ldots, \lambda_{n-1}(\nu)\},$$

where the $\lambda_j(\nu)$ are the eigenvalues of the hessian matrix of H_k at ν (with respect to the standard metric in $\mathbb{S}^{n-1}$). Then, for any $\nu \in S_0$ there exists a unique orbit Γ of $\nabla_g h$ converging to the origin such that $\nu = \nu_\Gamma$ (in fact, Γ is an analytic separatrix of $\nabla_g h$).

In order to apply these results to our gradient vector field $\xi = \nabla_g(-\widetilde{f}^2)$, we consider an analytic chart $(B_q, w = (x, y, z))$, centered at q, such that the matrix of the metric g in the coordinates w at $w = 0$ is equal to the identity and, moreover, $\widetilde{f}$ is written in the form $\widetilde{f} = \ell_1^{m_1} \ell_2^{m_2} \ell_2^{m_3}$, where ℓ_1, ℓ_2, ℓ_3 are linearly independent homogeneous polynomials of degree one in the variables (x, y, z). To show that we can choose such an analytic chart, first take coordinates $\bar{w} = (\bar{x}, \bar{y}, \bar{z})$ so that $\widetilde{f} = \bar{x}^{m_1} \bar{y}^{m_2} \bar{z}^{m_3}$ and then take a linear change of variables $w = P\bar{w}$ so that $g(0)$ has the identity matrix in the coordinates w. We may assume also that $K \cap B_q$ is described by the set $\{\ell_1, \ell_2, \ell_3 > 0\}$ and, possibly allowing $g(0)$ to be a positive multiple of the identity, that the range $w(B_q)$ of the chart contains a neighborhood of $[-1, 1]^3$, so that the sphere $\mathbb{S}^2 = \{x^2 + y^2 + z^2 = 1\}$ in the coordinates w is well defined as a "sphere" inside B_q. Consider

[1] This result, essentially, was already established by Martinet and Thom.

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$$F = (-\ell_1^{2m_1}\ell_2^{2m_2}\ell_2^{2m_3})|_{\mathbb{S}^2},$$

an analytic non-constant function on $\mathbb{S}^2$. According to Moussu's results (a) and (b) above, it suffices to prove that:

(i) F has a unique singular point ν_0 in $\mathbb{S}^2 \cap K$, which is a local minimum for F (thus the hessian of F at ν_0 is positive semidefinite and hence ν_0 belongs to the set S_0 defined in (b) above).
(ii) No point of the frontier of $\mathbb{S}^2 \cap K$ in $\mathbb{S}^2$ can be the limit tangent of an orbit of ξ contained in K.

Property (i) is an exercise in convex geometry: For any $c > 0$, the function $\tilde{f}^2 - c = \ell_1^{2m_1}\ell_2^{2m_2}\ell_3^{2m_3} - c$ restricted to $K = \{\ell_1, \ell_2, \ell_3 > 0\}$ is such that its epigraph $\{\tilde{f}^2 \geq c\} \cap K$ is strictly convex. Thus, if $\nu_0 \in \mathbb{S}^2$ is a singular point of $\tilde{f}^2|_{\mathbb{S}^2 \cap K}$ then the tangent plane of $\mathbb{S}^2$ at ν_0 equals the tangent plane of the fiber $(\tilde{f}^2)^{-1}(\tilde{f}(\nu_0)^2)$ at ν_0 and separates $\mathbb{S}^2$ from the epigraph $\{\tilde{f}^2 \geq \tilde{f}(\nu_0)^2\}$. Thus ν_0 is a global maximum of $\tilde{f}^2$ in restriction to $\mathbb{S}^2 \cap K$, which shows (i).

Let us show (ii). The set $T = \mathbb{S}^2 \cap \overline{K}$ is a spherical triangle T determined by the lines $\ell_j \cap \mathbb{S}^2$, $j = 1, 2, 3$. We consider the real blow-up $\sigma_q : \widetilde{M} \to M'_m$ at q so that the divisor $E = \sigma_q^{-1}(q)$ is identified with the sphere $\mathbb{S}^2$. The transformed vector field $\sigma_q^* \xi$ is singular along E but it can be divided by an equation of E so that we obtain a new vector field $\widetilde{\xi}'$ on $\widetilde{M}$, which leaves invariant the divisor E and so that $E \not\subset \mathrm{Sing}(\widetilde{\xi}')$. A calculation (which, this time, is easier assuming that the coordinates are chosen so that $\tilde{f} = x^{m_1}y^{m_2}z^{m_3}$) shows that $\mathrm{Sing}(\widetilde{\xi}') \cap T$ is the set of vertices of T. This proves that if an orbit of $\widetilde{\xi}'$, not contained in the divisor E, accumulates to a single point of T, then this point must be a vertex. On the other hand, if v is a vertex of T, one can see that $\mathrm{Sing}(\widetilde{\xi}')$ is a non-singular curve at v transversal to the divisor E and that the restriction of $\widetilde{\xi}'$ to E is a linear vector field (in standard charts for the blow-up) with real eigenvalues of different sign. Thus the stable and unstable manifolds of $\widetilde{\xi}'$ at v are contained in E, whereas $\mathrm{Sing}(\widetilde{\xi}')$ is a center manifold. Using the Theorem of Reduction to the Center Manifold in a way analogous to the case $e = 2$, we conclude that no orbit of $\widetilde{\xi}'$ outside E can accumulate to v. This proves (ii), as wanted. □

3 The Poincaré–Hopf index

Let $f : (\mathbb{R}^3, 0) \to \mathbb{R}$ be a germ of analytic function which satisfies the hypothesis (RIS) and let $\varepsilon > 0$ be sufficiently small so that Proposition 3 holds for f. Put $Z = f^{-1}(0) \cap \overline{B}(0, \varepsilon)$ and let L be one of the local components of Z. Consider in L the orientation induced by the normal vector field $\nabla f|_L$. Let $C = L \cap \partial \overline{B}(0, \varepsilon) \cong \mathbb{S}^1$ be the corresponding link with the usual orientation as a boundary of L. By the conic structure of L, there exists a homeomorphism $\Psi : \overline{L} \to \mathbb{D}$, where $\mathbb{D}$ is the unit closed disc in $\mathbb{R}^2$ centered at the origin, such that $\Psi(0) = 0$ and which restricts to a diffeomorphism from L into $\mathbb{D}\backslash\{0\}$. We can suppose that Ψ is orientation preserving. Moreover, we can suppose that the tangent map $T\Psi : TL \subset T\mathbb{R}^3 \to T\mathbb{R}^2$ of Ψ over L is uniformly bounded for the usual norm of tangent vectors of $\mathbb{R}^n$ (for instance, change Ψ by $g(\|\Psi\|)\Psi$ where $g : [0, 1] \to [0, 1]$ is a convenable monotonic C^1-function). Thus $\Psi_*(X|_L)$ extends to a continuous vector field $\widetilde{X}_L$ on the disc $\mathbb{D}$ with isolated singularity at 0. We define the *index of X along L*, denoted by $I_L(X)$, to be the Poincaré–Hopf index of $\widetilde{X}_L$ at the origin of $\mathbb{R}^2$. It can be computed as the degree of the map $\widetilde{X}_L/||\widetilde{X}_L|| : \mathbb{S}^1 \to \mathbb{S}^1$. It is well known that $I_L(X)$ does not depend on ε or on the homeomorphism Ψ (as long as it satisfies the mentioned properties).

Proposition 5 *Let X be a real analytic vector field at $(\mathbb{R}^3, 0)$ having a real-analytic first integral f. Assume that f satisfies the hypothesis (RIS) and that X has an isolated singularity at $0 \in \mathbb{R}^3$. Then there exists a local component L of $Z = f^{-1}(0)$ such that $I_L(X) = 0$.*

Proof Take a local component L of Z satisfying the properties stated in Proposition 4, i.e., in every neighborhood of L there are fibers of f that have connected components which are simply connected. Assume, without loss of generality, that there are such fibers with positive values of f. Consider a diffeomorphism $\Psi : L \to \mathbb{D}\backslash\{0\}$ and the vector field $\widetilde{X}_L = \Psi_*(X|_L)$ as in the paragraph above, so that $I_L(X)$ is the Poincaré-Hopf index of $\widetilde{X}_L$ at $0 \in \mathbb{R}^2$.

We shift the link $C = L \cap \partial \overline{B}(0, \varepsilon)$ of L to nearby fibers of f in the following way. Notice first that, by the condition (RIS), if $f = f_1^{n_1} \cdots f_r^{n_r}$ is the decomposition of f in irreducible factors, there is a unique j so that f_j vanishes along C and, if $k \neq j$, then $f_k(x) \neq 0$ for any $x \in C$. Let $m = n_j$. In a sufficiently small neighborhood W of C in $\mathbb{R}^3$, the function $\beta : W \to \mathbb{R}$ defined by

$$\beta = f_j \prod_{k \neq j} |f_k|^{\frac{n_k}{m}}$$

satisfies $\mathrm{Sing}(d\beta) = \emptyset$, $W \cap L = \{\beta = 0\}$ and $f|_W = \epsilon \beta^m$, where $\epsilon = \pm 1$. Up to changing the sign of β we can assume that $\epsilon = +1$ (notice that $\epsilon = -1$ and m even cannot occur since we suppose that f takes positive values near L). Notice that the fibers of β are contained in the fibers of $f|_W$. Put $Y = \nabla\beta / \|\nabla\beta\|^2$, a vector field which is transversal to the fibers of f in W, in particular to $L \cap W$. Moreover, if $\phi_t(x)$ is the flow of Y, we have $\beta(\phi_t(x)) = t$ for every $x \in L \cap W$ and every $t \in \mathbb{R}$ sufficiently small. Let us denote by L_t the fiber $f = t^m$ in $\overline{B}(0, \varepsilon)$ and by C_t the curve $\phi_t(C)$. There exists a small $\rho > 0$ such that, for each fixed $|t| < \rho$, the flow $\phi_t(x)$ defines a diffeomorphism Φ_t between an open neighborhood A of C in L and an open neighborhood A_t of C_t in L_t, which restricts to a diffeomorphism from C to C_t. Moreover, if $t > 0$, Φ_t preserves the orientation induced by the gradient ∇f on the fibers of f.

If $|t| < \rho$, the map $\Psi_t = \Psi \circ \Phi_t^{-1}$ takes A_t diffeomorphically into a neighborhood of $\mathbb{S}^1$ in $\mathbb{R}^2$, sending C_t to $\mathbb{S}^1$. We define $\widetilde{X}_t = \Psi_{t*}(X|_{A_t})$. The map $s \mapsto \widetilde{X}_{st}$, for $s \in [0, 1]$, defines a homotopy between $\widetilde{X}_L = \widetilde{X}_0$ and $\widetilde{X}_t$. Moreover, if t is sufficiently small, we may assume that $\widetilde{X}_{st}$ never vanishes over $\mathbb{S}^1$. Thus we have

$$I_L(X) = \text{degree } (\widetilde{X}_L / ||\widetilde{X}_L|| : \mathbb{S}^1 \to \mathbb{S}^1) = \text{degree } (\widetilde{X}_t / ||\widetilde{X}_t|| : \mathbb{S}^1 \to \mathbb{S}^1).$$

By our choice of the local component L and Proposition 4, if $t > 0$ is sufficiently small, C_t is contained in a connected component of L_t that is simply connected. Hence, the curve C_t is the boundary of a submanifold D_t in L_t diffeomorphic to the unit disc $\mathbb{D}$ via a diffeomorphism $h : D_t \to \mathbb{D}$, which can be extended to a neighborhood of C_t in L_t and satisfies $h(C_t) = \mathbb{S}^1$. On the one hand, the vector field $\xi = h_*(X|_{D_t})$ in $\mathbb{D}$ has Poincaré index equal to 0, since it never vanishes. On the other hand, such an index can be calculated as the degree of the map $\xi/||\xi|| : \mathbb{S}^1 \to \mathbb{S}^1$ and this is equal to the degree of $\widetilde{X}_t / ||\widetilde{X}_t|| : \mathbb{S}^1 \to \mathbb{S}^1$, since ξ and $\widetilde{X}_t$ are related by the diffeomorphism $h \circ \Psi_t^{-1}$ defined in a neighborhood of $\mathbb{S}^1$ in $\mathbb{R}^2$. This concludes the proof of the Proposition. □

4 Proof the main theorem

In this section we complete the proof of the first part of Theorem 1.

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Let X be a germ of analytic vector field with an isolated singularity at $0 \in \mathbb{R}^3$ having a non-constant analytic first integral $f : \mathbb{R}^3 \to \mathbb{R}$ with $f(0) = 0$. As mentioned in the introduction, we may assume that the singular locus $\text{Sing}(df) = \{p \in \mathbb{R}^3 : df(p) = 0\}$ of f has no components of real dimension equal to one at 0. This is a consequence of the following result (whose proof given in [25] for the complex case generalizes, without changes, to the real case) and the fact that X is tangent to the foliation given by $df = 0$.

Proposition 6 *Let Y be a germ of real analytic vector field having an isolated singularity at $0 \in \mathbb{R}^3$. Let ω be a germ of real analytic integrable 1-form at $0 \in \mathbb{R}^3$ such that $\omega(Y) = 0$. Then the one-dimensional components of* $\text{Sing}(\omega) = \{p : \omega(p) = 0\}$ *are invariant by Y.*

Under the hypothesis that f is a first integral of X, we obtain that the special fiber $Z = f^{-1}(0)$ of f is not reduced to a single point, i.e., that $Z\backslash\{0\} \neq \emptyset$. This is a consequence of Brunella's result [6] which asserts that X has a non trivial orbit τ accumulating to the origin. In our case, we have necessarily that $\tau \subset Z$ and hence $Z \neq \emptyset$, since the orbits of X are contained in the fibers of f.

Moreover, the following lemma implies that we may assume that the function f satisfies the (RIS) hypothesis.

Lemma 7 *If f does not satisfy the (RIS) hypothesis then X has a real analytic separatrix.*

Proof Consider the prime decomposition $f = f_1^{n_1} f_2^{n_2} \cdots f_r^{n_r}$ where the f_j are two by two different irreducible germs of real analytic functions and let $h = Red(f) = f_1 f_2 \cdots f_r$. Notice that $Z = f^{-1}(0) = h^{-1}(0)$. We have already shown that $Z \not\subset \{0\}$, so if f does not satisfy the (RIS) hypothesis then $\text{Sing}(dh) \not\subset \{0\}$. In this case, there exists a component H of the analytic set $\text{Sing}(dh)$ of positive (real) dimension accumulating to the origin. Necessarily $H \subset \text{Sing}(dh) \cap Z$ and hence H is one-dimensional, since $\text{Sing}(dh) \cap Z$ has no component of codimension one. Indeed, in a neighborhood of the origin, a point p belongs to $\text{Sing}(dh) \cap Z$ if and only it satisfies at least one of the following conditions:

(i) there is a pair of indices i, j, with $i \neq j$, such that $p \in \{f_i = f_j = 0\}$;
(ii) there is an index j such that $p \in \{f_j = 0\} \cap \text{Sing}(df_j)$.

It suffices to prove that H is invariant by X. For this, we consider the real analytic 1-form obtained by canceling the poles of the logarithmic derivative of f:

$$\omega_f = Red(f)\frac{df}{f} = \sum_{j=1}^{r} n_j f_1 \cdots \widehat{f_j} \cdots f_r df_j.$$

We have that $\text{Sing}(\omega_f) \cap Z = \text{Sing}(dh) \cap Z$ since both sets are described by the same properties (i) and (ii) above, and thus H is a one-dimensional component of $\text{Sing}(\omega_f) \cap Z$. Finally, since $\omega_f(X) = 0$, the set $\text{Sing}(\omega_f) \cap Z$ (and hence H) is invariant by X by Proposition 6, as wanted. □

Assume now that the first integral f satisfies the hypothesis (RIS), so that we can apply to f the constructions and the results described in the preceding sections. In particular, let $L_1, \ldots, L_r \subset \overline{B}(0, \varepsilon)$ be the local components of $Z = f^{-1}(0)$, where ε is sufficiently small, as defined in Sect. 2 and let $I_{L_j}(X)$ be the index of X along L_j as defined in Sect. 3.

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Theorem 1 is a consequence of Proposition 5 and of the following result:

Proposition 8 *For any $j \in \{1, 2, \ldots, r\}$, either there is a formal real separatrix of X contained in L_j or $I_{L_j}(X) \neq 0$.*

Proof Fix $j \in \{1, 2, \ldots, r\}$ and put for simplicity $L = L_j$, $C = C_j$ etc. Assume that ε is sufficiently small so that Proposition 3 holds for $V = \overline{B}(0, \varepsilon)$. That is, there exists a sequence of real blow-ups $\sigma : M' \to V$ such that $L' = \overline{\sigma^{-1}(L)}$ is a real analytic surface with boundary and corners, homeomorphic to a closed cylinder $[0, 1] \times \mathbb{S}^1$, such that σ induces a diffeomorphism between $\sigma^{-1}(L)$ and L. The boundary of L' consists of the two components $C' = \sigma^{-1}(C)$ (the transform of the link of L by σ) and $D' = L' \cap E'$, where $E' = \sigma^{-1}(0)$ is the exceptional divisor of σ. While C' is a smooth analytic curve, D' is only piecewise smooth analytic. Denote by $J \subset D'$ the set of corners of D', i.e., the set of points where D' is not smooth. Consider in L' the orientation induced from that of L by σ. Up to considering another surface diffeomorphic to L', we may assume that L' is a submanifold with boundary and corners inside the euclidean plane $\mathbb{R}^2$, with the standard orientation.

The transformed vector field $X' = \sigma^*(X|_L)$ in $L' \setminus D'$ defines a one-dimensional singular analytic foliation $\mathcal{F}'$ which can be extended analytically to D' as an oriented foliation (i.e., at any point $p \in D'$, there is an analytic vector field X'_p in a neighborhood V_p of p in L', with isolated singularities, generating $\mathcal{F}'$ and such that X'_p and X' are equally oriented in $V_p \setminus D'$) whose set of singular points $\mathrm{Sing}(\mathcal{F}')$ is finite and contained in D'. Moreover, using Seidenberg's Theorem on reduction of singularities [31], and up to considering new blow-ups on L' at points of D', we can assume that any point of $\mathrm{Sing}(\mathcal{F}')$ is a simple singularity (that is, the eigenvalues λ, μ of the corresponding linear part are real and satisfy $\mu \neq 0$ and $\lambda/\mu \notin \mathbb{Q}_{>0}$) and that any connected component of $D' \setminus J$ is either invariant for $\mathcal{F}'$ or everywhere transversal to $\mathcal{F}'$.

Suppose that there is no formal real separatrix of X inside L. Then, at any point $p \in D'$, the formal separatrices of $\mathcal{F}'$ at p (of a generator X'_p of $\mathcal{F}'$) are contained in D'. In particular, any connected component of $D' \setminus J$ is invariant for $\mathcal{F}'$. Also taking into account that a simple singularity of a two-dimensional real vector field, with real eigenvalues, has exactly two transversal formal separatrices (both real, non-singular and tangent to the corresponding eigendirections), we have necessarily that $\mathrm{Sing}(\mathcal{F}') = J$ and that the only formal separatrices of $\mathcal{F}'$ at any $p \in J$ are the two components of D' through the point p (thus, they are analytic separatrices). Notice that, since D' is contained in the boundary of L', there are exactly two connected components of $D' \setminus J$ locally at $p \in J$, each of them is part of one of the separatrices of $\mathcal{F}'$ at p. Each connected component ℓ of $D' \setminus J$ is a non-singular oriented leaf of $\mathcal{F}'$, going from $\alpha(\ell)$ to $\omega(\ell)$, both points in J. A singular point $p \in J$ is either a *sink*, a *source* or a *saddle*, depending if $\omega(\ell) = \omega(\ell') = p$, $\alpha(\ell) = \alpha(\ell') = p$ or the remaining cases, respectively, where ℓ, ℓ' are the two components of $D' \setminus J$ which accumulate to p. Sinks an sources are jointly called *nodes*. A *node connection* is a union

$$\tau = \overline{\ell_1} \cup \cdots \cup \overline{\ell_r}$$

where the ℓ_j are connected components of $D' \setminus J$ satisfying $\alpha(\tau) := \alpha(\ell_1)$ is a source, $\omega(\tau) := \omega(\ell_r)$ is a sink and $\omega(\ell_j) = \alpha(\ell_{j+1})$ for $j = 1, \ldots, r-1$ (which are saddle

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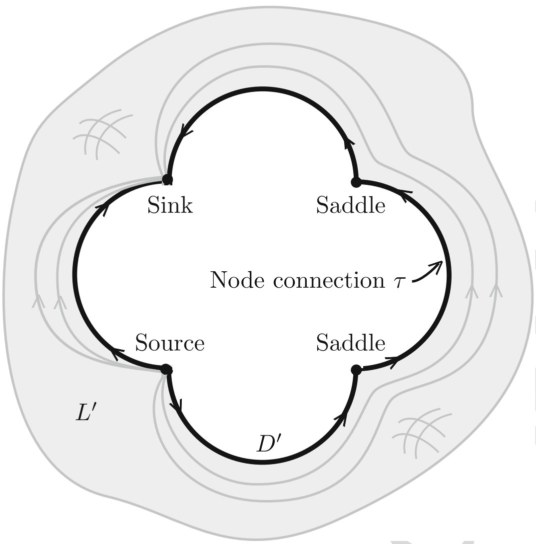
points). By construction, there is a continuum of trajectories of X' in $L' \backslash D'$ accumulating to τ and having $\alpha(\tau)$, $\omega(\tau)$ as the α and ω-limit set, respectively. See the figure below:



By the nature of real blow-ups, the cardinal of J is even and, if $J \neq \emptyset$, the number of connected components of $D' \backslash J$ and the number of node connections are also even.

To conclude the proposition, let us prove that in this situation we have $I_L(X) > 0$. The index $I_L(X)$ can be calculated as follows. Let $\mathcal{S}$ be a closed simple curve in $L' \backslash D'$ surrounding D' and homotopic to C' in $L' \backslash D'$ with the standard orientation and let $\phi : \mathbb{S}^1 \to \mathcal{S}$ be an orientation preserving homeomorphism. Then $I_L(X)$ is equal to the degree of the map

$$\theta : \mathbb{S}^1 \to \mathbb{S}^1, \; p \mapsto \frac{X'(\phi(p))}{\|X'(\phi(p))\|}.$$

Suppose, moreover, that $\mathcal{S}$ is a differentiable curve having only finitely many tangencies with X' and let i and e be, respectively, the number of *interior* and *exterior* tangencies (i.e., at such a tangency point q, the orbit of X, devoid of q, stays locally at q in the interior or in the exterior of $\mathcal{S}$, respectively). Then, from Poincaré (see also Pugh's work [28]), we can calculate the degree of the map θ above as

$$\deg(\theta) = 1 + \frac{i - e}{2}. \tag{5}$$

We will finish by constructing a differentiable curve $\mathcal{S}$ with a positive (even) number of interior tangencies and no exterior tangencies with the vector field X'.

Let τ be a node connection in D' and let γ_1, γ_2 be two trajectories of X' in $L' \backslash D'$, both having α and ω-limit equal to $\alpha(\tau)$ and $\omega(\tau)$, respectively, and such that γ_2 is inside the circle $\tau \cup \overline{\gamma_1}$. Using the flow-box theorem, we can construct a differentiable arc of curve $\mathcal{S}_\tau$ connecting two different points of γ_1, lying inside $\tau \cup \overline{\gamma_1}$ except for its extremities and everywhere transversal to X' except for a point a_τ where it touches γ_2. This is depicted in the figure:

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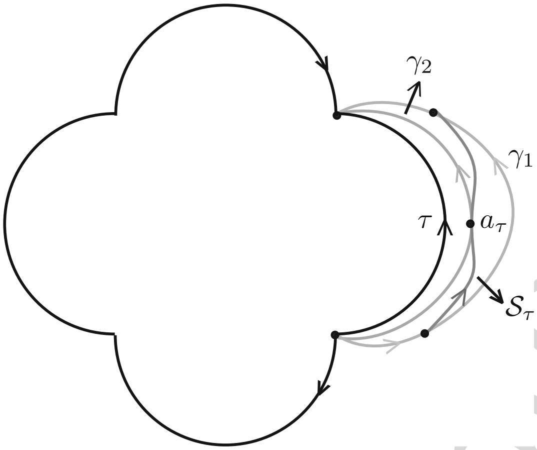

We choose the extremities of $\mathcal{S}_\tau$ sufficiently near the corresponding singular points $\alpha(\tau)$, $\omega(\tau)$, so that, in sufficiently small neighborhoods of the the node singularities of D', the arcs $\mathcal{S}_\tau$ can be jointed in a smooth way by small arcs transversal to X'. Thus, we produce a simple closed curve $\mathcal{S}$ surrounding D' with the required properties: $\mathcal{S}$ is everywhere transversal to X' except for the points a_τ, which are in fact interior tangencies of $\mathcal{S}$ with the vector field X', and there are as many of them as the number of node connections (an even number). □

Remark 9 It is worth noticing that formula (5) is closely related to Bendixson's formula for the index of a planar analytic vector field X at the origin in $\mathbb{R}^2$:

$$I(X) = 1 + \frac{e-h}{2}$$

where e is the number of "elliptic" sectors and h is the number of "hyperbolic" sectors of X' at the origin (see Andronov et al. [1]). In fact, in our situation, if we collapse L' into a neighborhood of $0 \in \mathbb{R}^2$ sending D' to the origin, the push-forward of X' gives a vector field (which can be continuously extended to the origin) having as many elliptic sectors as the number of node connections in D' and no hyperbolic sectors. This is an alternative proof of the last part of Proposition 8, after the observation that Bendixson's formula extends to continuous vector fields which have finitely many sectors of elliptic, hyperbolic or parabolic type.

5 Examples

In this section we prove the second part of Theorem 1, that is, we provide examples of analytic vector fields at $0 \in \mathbb{R}^3$ having an analytic non-constant first integral but not having analytic separatrices. Our examples are obtained as a one-parameter unfolding of a two-dimensional vector field which has a unique formal real separatrix which is not convergent. In the introduction, we have already discussed the existence of planar vector fields with such a property, for instance, Risler's example in [29]. We need to modify such example in order that its unfolding produces a three-dimensional vector field with isolated singularity.

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Proposition 10 *Let $a = a(x) \in \mathbb{R}\{x\}$ be a convergent series in one variable such that $a(0) = a'(0) = 0$ and consider the planar analytic vector field*

$$Y_a = (y^2 + x^4)\frac{\partial}{\partial x} + \left(-xy + x^3 a(x) + \frac{a(x)}{x} y^2\right)\frac{\partial}{\partial y}. \tag{6}$$

Then Y_a has a unique real formal separatrix Γ_a at $0 \in \mathbb{R}^2$ and, for a convenient (in fact generic) choice of the series $a(x)$, Γ_a is not convergent.

Using this proposition, we construct our desired examples in $\mathbb{R}^3$ as follows.

Example 11 Given $a(x) \in \mathbb{R}\{x\}$ with $a(0) = a'(0) = 0$, consider the vector field in $\mathbb{R}^3$, expressed in coordinates (x, y, z) as

$$X_a = Y_a + z^2\frac{\partial}{\partial x},$$

where Y_a is given in (6). The vector field X_a is in fact a family of planar vector fields in the parameter z. In other words, the function $f = z$ is an analytic first integral of X_a. Moreover, since the coefficient of $\partial/\partial x$ is $y^2 + x^4 + z^2$, the origin is an isolated singularity of X_a. Hence, the real formal separatrices of X_a are those contained in the fiber $z = 0$. More specifically, they are the separatrices of the restriction $X_a|_{z=0} = Y_a$. By Proposition 10, X_a has a unique real formal separatrix Γ_a, which is not convergent for a convenient choice of the series $a(x)$.

Proof of Proposition 10 If Γ is a formal real separatrix of Y_a then its tangent line corresponds to a root of the tangent cone of Y_a at the origin, which is given by the equation $y^3 + yx^2 = y(y^2 + x^2) = 0$. Thus Γ is tangent to $\ell = (y = 0)$. Let $\pi_1 : M_1 \to \mathbb{R}^2$ be the blow-up at the origin and let p_1 be the point in the exceptional divisor $E_1 = \pi_1^{-1}(0)$ corresponding to ℓ. The strict transform $\overline{\Gamma}$ of Γ by π_1 is a formal separatrix of the the strict transform $\overline{Y}_a$ of Y_a at p_1. A computation using usual coordinates $(x, y_1) = (x, y/x)$ of the blow-up π_1 shows that $\overline{Y}_a$ has a saddle-node singularity at p_1 for which the divisor E_1 is the *strong* separatrix (tangent to the non-zero eigenvalue) and thus $\overline{\Gamma}$ is the *weak* formal separatrix (tangent to the zero eigenvalue). This proves the uniqueness of $\Gamma = \Gamma_a$.

Let us prove that Γ_a is not convergent for some choice of the series $a(x)$. For that, we consider the blow-up $\pi_2 : M_2 \to M_1$ at the point p_1 and the point p_2 in the exceptional divisor $E_2 = \pi_2^{-1}(p_1)$ corresponding to the tangent of $\overline{\Gamma}_a$ at p_1. We put usual coordinates at p_2 of the form $(x, y_2) = (x, y_1/x) = (x, y/x^2)$ and compute the strict transform of $\overline{Y}_a$ as

$$\overline{\overline{Y}}_a = x^3(1 + y_2^2)\frac{\partial}{\partial x} + \left(-y_2(1 + 2x^2(1 + y_2^2)) + a(x)(1 + y_2^2)\right)\frac{\partial}{\partial y_2}.$$

Again $\overline{\overline{Y}}_a$ has a saddle-node singularity for which the divisor $E_2 = (x = 0)$ is the strong separatrix and the strict transform $\overline{\overline{\Gamma}}_a$ of $\overline{\Gamma}_a$ by π_2 is the weak separatrix. To finish, let us show that $\overline{\overline{\Gamma}}_a$ is not convergent for a convenient choice of a. Let us assume that $a(x) = \alpha(2x^2)$ for some $\alpha(z) \in z\mathbb{R}\{z\}$. After dividing $\overline{\overline{Y}}_a$ by $1 + y_2^2$, we consider the ramification $z = 2x^2$ and rename $w = y_2$, obtaining the saddle-node vector field

$$\xi_\alpha = z^2\frac{\partial}{\partial z} + \left(-w(1 + z) + \frac{w^3}{1 + w^2} + \alpha(z)\right)\frac{\partial}{\partial w}. \tag{7}$$

It suffices to prove the following:

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Assertion *There is a choice of the series $\alpha(z)$ so that, for any $\delta > 0$ sufficiently small, the weak formal separatrix of the saddle-node vector field $\xi_{\delta\alpha}$ is not convergent.*

We use the Martinet-Ramis moduli for analytic orbital classification of holomorphic foliations generated by saddle-node vector fields at the origin of $\mathbb{C}^2$ (see [23] and also [18]). In our particular case, any vector field ξ_α of the form (7) is formally orbitally equivalent to the vector field in normal form

$$N = z^2 \frac{\partial}{\partial z} + (-w(1+z)) \frac{\partial}{\partial w}.$$

If we denote by $\mathcal{N}$ the class of vector fields formally orbitally equivalent to N, the moduli map associates to any $\eta \in \mathcal{N}$ is a couple $G(\eta) = (g(\eta), \psi(\eta))$ where $g(\eta) \in \mathbb{C}$ and $\psi(\eta)$ is a germ of a tangent to the identity biholomorphism at $(\mathbb{C}, 0)$ in such a way that two vector fields η, η' are orbitally analytically equivalent if and only if $G(\eta) = G(\eta')$. On the other hand, if $\eta \in \mathcal{N}$ then the weak formal separatrix of η is convergent if and only if the constant part $g(\eta)$ of the moduli is equal to zero [23, Theorem III.4.4]. Moreover, if we have a family $\{\eta_\lambda\}$ of vector fields in $\mathcal{N}$ depending analytically on $\lambda \in \mathbb{C}^m$ then $\lambda \mapsto g(\eta_\lambda)$ is also analytic [18, Theorem 1, p. 33].

In order to prove the assertion, put $\delta = \varepsilon^{3/2}$ for $\varepsilon \in \mathbb{R}_{>0}$ and write the vector field $\xi_{\delta\alpha}$ under the change of variable $w = \sqrt{\varepsilon}\bar{w}$ as

$$\eta_{\varepsilon,\alpha} = z^2 \frac{\partial}{\partial z} + \left(-\bar{w}(1+z) + \varepsilon\left(\bar{w}^3 + \alpha(z)\right) - \varepsilon^2\bar{w}^5 + \varepsilon^3\bar{w}^7 - \cdots\right) \frac{\partial}{\partial \bar{w}}.$$

Hence $g(\xi_{\varepsilon^{3/2}\alpha}) = g(\eta_{\varepsilon,\alpha})$ and it suffices to show that there exists a series $\alpha = \alpha(z)$ so that

$$\frac{d(g(\eta_{\varepsilon,\alpha}))}{d\varepsilon} \,|_{\varepsilon=0} \neq 0. \tag{8}$$

(Notice that this gives the assertion since the weak separatrix of $\xi_0 = \eta_{0,\alpha}$ is $\bar{w} = 0$ and hence $g(\xi_0) = 0$). First, put

$$\overline{\eta}_{\varepsilon,\alpha} = N + \varepsilon(\bar{w}^3 + \alpha(z)) \frac{\partial}{\partial \bar{w}} = z^2 \frac{\partial}{\partial z} + \left(-\bar{w}(1+z) + \varepsilon(\bar{w}^3 + \alpha(z))\right) \frac{\partial}{\partial \bar{w}},$$

so that (changing the notation $w = \bar{w}$) we have $\eta_{\varepsilon,\alpha} = \overline{\eta}_{\varepsilon,\alpha} + \varepsilon Y_\varepsilon$ where

$$Y_\nu = (-\nu w^5 + \nu^2 w^7 - \cdots) \frac{\partial}{\partial w}.$$

In other words, if we put $\zeta_{\varepsilon,\nu,\alpha} = \overline{\eta}_{\varepsilon,\alpha} + \varepsilon Y_\nu$ then we have $\eta_{\varepsilon,\alpha} = \zeta_{\varepsilon,\varepsilon,\alpha}$. Notice that, for any series α, we have that $g(\zeta_{\varepsilon,\nu,\alpha})$ is analytic in (ε, ν), $g(\zeta_{0,\nu,\alpha}) = 0$ for any ν and $\zeta_{\varepsilon,0,\alpha} = \overline{\eta}_{\varepsilon,\alpha}$ for any ε. Hence we obtain

$$\frac{d(g(\eta_{\varepsilon,\alpha}))}{d\varepsilon} \,|_{\varepsilon=0} = \frac{d(g(\overline{\eta}_{\varepsilon,\alpha}))}{d\varepsilon} \,|_{\varepsilon=0} .$$

Thus, to prove (8), it suffices to show that $\frac{d(g(\overline{\eta}_{\varepsilon,\alpha}))}{d\varepsilon} \,|_{\varepsilon=0} \neq 0$ for some choice of α.

The derivative of $g(\overline{\eta}_{\varepsilon,\alpha})$ at $\varepsilon = 0$ (considered as a component of the tangent of the moduli map G) can be computed explicitly from Elizarov's paper [12] as follows. Make the change of variables $z \mapsto -z$ and multiply by -1, getting the new expression for the family

$$\overline{\eta}_{\varepsilon,\alpha} = z^2 \frac{\partial}{\partial z} + \left(w(1-z) - \varepsilon(w^3 + \alpha(-z))\right) \frac{\partial}{\partial w}.$$

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To put it in Elizarov's pattern, we have to divide it by $1-z$ so that the family $\bar{\eta}_{\varepsilon,\alpha}$ becomes the family $v_{p,\lambda}+\varepsilon P\partial_w$ considered in equation [12, Eq. 1.8], where

$$v_{p,\lambda}=\frac{z^2}{1-z}\frac{\partial}{\partial z}+w\frac{\partial}{\partial w}\quad \text{(and hence } p=1 \text{ and } \lambda=-1)$$

and

$$P(z,w)=-\frac{\alpha(-z)+w^3}{1-z}=-(\alpha(-z)+w^3)(1+z+z^2+\cdots).$$

Choose $\alpha(z)$ such that $\alpha(0)=\alpha'(0)=0$ and write

$$-\alpha(-z)(1+z+z^2+\cdots)=\sum_{k\geq 2}c_kz^k.$$

This corresponds to $f_{-1}(z)$ in the expansion in power series in [12, Eq. 1.9]. The constant part g of the moduli map corresponds in our case to the component $a_{0,-1}$ in equation [12, Eq. 1.3] (that is, $j=0$ and $l=-1$).

From all these data, and computing the sequence $m_k(l)=m_k(-1)$ in [12, Eq. 1.7] for the corresponding *Borel transform*, we conclude from Elizarov's formula in [12, Theorem 1] that

$$\frac{d(g(\overline{\eta}_{\varepsilon,\alpha}))}{d\varepsilon}\,|_{\varepsilon=0}=u\sum_{k=2}^{\infty}c_k\frac{k}{\Gamma(k+2)},$$

where Γ is the Euler's Gamma function and u is some non-zero constant which does not depend on α (if we want to be precise, we can check that in fact $u=-1$). Therefore, $\frac{d(g(\overline{\eta}_{\varepsilon,\alpha}))}{d\varepsilon}\,|_{\varepsilon=0}\neq 0$ for a generic choice of $\alpha(z)$, as we wanted. This ends the proof. □

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