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Limit analysis of planar steel frames with variable section type and arbitrary loads

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ABSTRACT

This work calculates the collapse load factor and the collapse mechanism of 2D frames with slender structural members of variable section type and arbitrary distributed loads. A Kinematic Direct Method (KDM) is used for calculations. The equilibrium equations necessary to carry out the analysis are obtained systematically. The search for the collapse mechanism and the collapse load factor is carried out using an optimization method where the load factor is maximized. The following types of load are used: uniformly distributed loads, trapezoidal and sinusoidal distributed loads.

1. Introduction

The framed structures are always the test bench, many software for this family of structure were early developed in various research centres around the world. Steel frames show a high non-linear behaviour due to the plasticity of the material and the slenderness of the members. In general, the plastic-hinge approach is adopted to capture the inelasticity of material [1].

Behaviour of steel frames has been a great topic in the research field of construction engineering. In 1914 Kazinczy [2] was the first to investigate the reserve of plastic resistance in a hyper-static beam structure, introducing the concept of the plastic hinge and the collapse mechanism. Until now, the terminology plastic hinge is used to indicate a section (zero-length) on which all points are in the plastic range. The terminology collapse mechanism is originally utilized to describe the ultimate state of a frame where it is considered as a deformable geometric system.

Plastic behaviour of structures in general and of framed structures in particular has been dealt with in many text books. The first reference to limit analysis came from Van den Broek [3], followed by the contributions of Baker and Heyman [4], Horne [5], Neal [6], and Hodge [7], all between 1955 and 1960. Considering only the bending effect, the other effects are neglected. By its simplicity, this approach is popularly applied to 2D steel frames.

In general, either the plastic-zone or the plastic-hinge approach is adopted to capture the both material inelasticity and geometric non-linearity of a framed structure. In the plastic-hinge approach, only one beam–column element per physical member can model the non-linear properties of the structures. It leads to significant reduction of computation time. Furthermore, the computer program using the plastic-hinge model is familiar the habit of engineers. However, the plastic-hinge analysis is not without inconveniences that needs then to be improved [8]. The great impulse acquired by limit analysis was possible thanks to the rigorous establishment of the basic theorems, which was carried out by Gvozdev [9], in 1938. Basic theorems: static, kinematic and uniqueness, which give rise to the kinematic or direct method based on the method of combining mechanisms [10–12].

Generally, there are two fundamental theorems: static and kinematic. It leads to two corresponding approaches: static approach and kinematic approach that are called direct methods. The terminology Direct means that the load multiplier is directly found without any intermediate state of structures. Both the static method and the kinematic method are continually exploited and improved since more than 50 years until now: classic methods and mathematical programming methods [13].

The kinematic direct method has important drawbacks from the point of view of its practical application: first, it is not systematic or general; and secondly, it requires possible collapse mechanisms to be tested, which even with few plastic hinges implies many possible collapse mechanisms that will have to be tested and verified. On the other hand, the step-by-step methods based on the matrix formulation are systematic and efficient for concentrated load cases at the nodes of the structure, and they are very inefficient and imprecise for analysing structures with uniform distributed loads [14–16].

In this work, the approach of the equilibrium equations necessary to solve the kinematic direct method has been systematized, and in addition, the search for the collapse mechanism is carried out by means

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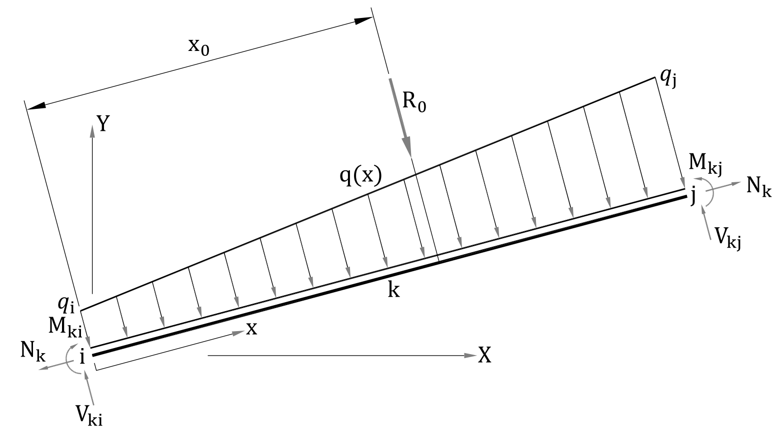

Fig. 1. Methodology. Beam/column element.

of non-linear programming, which facilitates the calculation of the collapse load factor and the structure collapse mechanism.

This paper has been organized as follows: after this brief introduction, the methodology is then applied to various types of planar frames. Finally, the main conclusions and contributions of the work are summarized.

2. Methodology

In this section, the calculation hypotheses are established, the obtaining of the equilibrium equations is explained and the resolution of the problem is proposed by means of kinematic direct method [17,18], see flowchart in Fig. 2.

2.1. Hypotheses

- The beams and columns are assumed to be free of residual or initial stresses.
- Plastic collapse implies unlimited displacement at constant load, and the level of load that causes it is called the collapse load.
- The value of the maximum bending moment that the section can transmit is called plastic moment and since the section is variable, this magnitude will depend on the x coordinate, ($M_{p,k}(x)$).
- When the bending moment reaches the value of the plastic moment, the rotation of the section where it occurs can increase indefinitely.
- The plastic moment depends on the material and the section.
- The formation of each plastic hinge is supposed to take place in a sudden and concentrated way in the section where the bending moment reaches the value of the plastic moment.
- The hypothesis of small displacements and rotations of the sections of the structure at the moment of collapse is assumed; therefore, the accumulated rotations between beams or columns in the plastic hinges must also be small.

2.2. Equilibrium equations

The matrix method of analysis of beam/column structures is considered to obtain the equations of equilibrium, but in its vector formulation. The balance of each beam/column is considered as follows (see Fig. 1). The end forces are calculated and the force vector is formed:

$$R_0 = \int_0^{L_k} q(x)dx$$
$$x_0 = \frac{1}{R_0}\cdot\left(\int_0^{L_k} x\cdot q(x)dx\right) \tag{1}$$

where R_0 is the resultant of the load distribution and x_0 is the point of application of the resultant of the distributed load.

$$\boldsymbol{f}_k = \begin{pmatrix} N_{ki} \\ V_{ki} \\ M_{ki} \\ N_{kj} \\ V_{kj} \\ M_{kj} \end{pmatrix} = \begin{pmatrix} -N_k \\ R_0 + \frac{1}{L_k}\left(M_{kj} - M_{ki} - x_0\cdot R_0\right) \\ -M_{ki} \\ N_k \\ -\frac{1}{L_k}\left(M_{kj} - M_{ki} - x_0\cdot R_0\right) \\ M_{kj} \end{pmatrix} \tag{2}$$

where N_{ki}, N_{kj} are the axial forces, V_{ki}, V_{kj} are the shear forces and M_{ki}, M_{kj} are the bending moments at the ends of the beam/column (k); the first subscript indicates the element beam/column (k) and the second subscript indicates the node (i, j), $q(x)$ is the transversal distributed load. All magnitudes are expressed as function of the axial force N_k, the values of the bending moments in both end sections M_{ki}, M_{kj}, the resultant force R_0 and section of application of the resultant force x_0.

The previous vector is expressed in the coordinates (x, y) of the beam/column, and must be expressed in a global coordinate system (X, Y) common to the structure, through the corresponding coordinate transformation ($\boldsymbol{T}(\alpha)$) [19]:

$$\boldsymbol{F}_k = \boldsymbol{T}^{\boldsymbol{T}}(\alpha)\cdot\boldsymbol{f}_k \tag{3}$$

where ($\boldsymbol{F}_k$) are the forces (and moments) at the ends of the beam/column k, expressed in a common system for all the members of the structure, and ($\boldsymbol{T}^{\boldsymbol{T}}(\alpha)$) indicates the operation of transposing rows and columns in the matrix ($\boldsymbol{T}(\alpha)$) of change of coordinates.

Finally, the vector of internal forces ($\boldsymbol{F}_{int}$) must be assembled. It balances the external loads ($\boldsymbol{F}_{ext}$) applied at the nodes of the structure:

$$\boldsymbol{F}_{int} = \boldsymbol{F}_{ext} \tag{4}$$

In the case of point loads, it is known that the sections of the structure that are candidates for forming a possible plastic hinge are: the nodes (joints between bars), the fixed supports, the section of application of the loads and section changes [20,21].

In the case of beams with distributed loads, plastic hinges can additionally be formed in the intermediate sections of the beams/columns. Logically, it is then necessary to carry out the corresponding checks from the bending moments calculated in the element [22,23].

It is important to bear in mind that if a plastic hinge is produced in an intermediate section, then its location at the beam (given by parameter x_k) can be modified during the plasticizing process, up until the formation of the collapse mechanism.

$$x_k = f_1(M_{ki}, M_{kj}, q_i, q_j)$$
$$M_k = f_2(x_k) \tag{5}$$

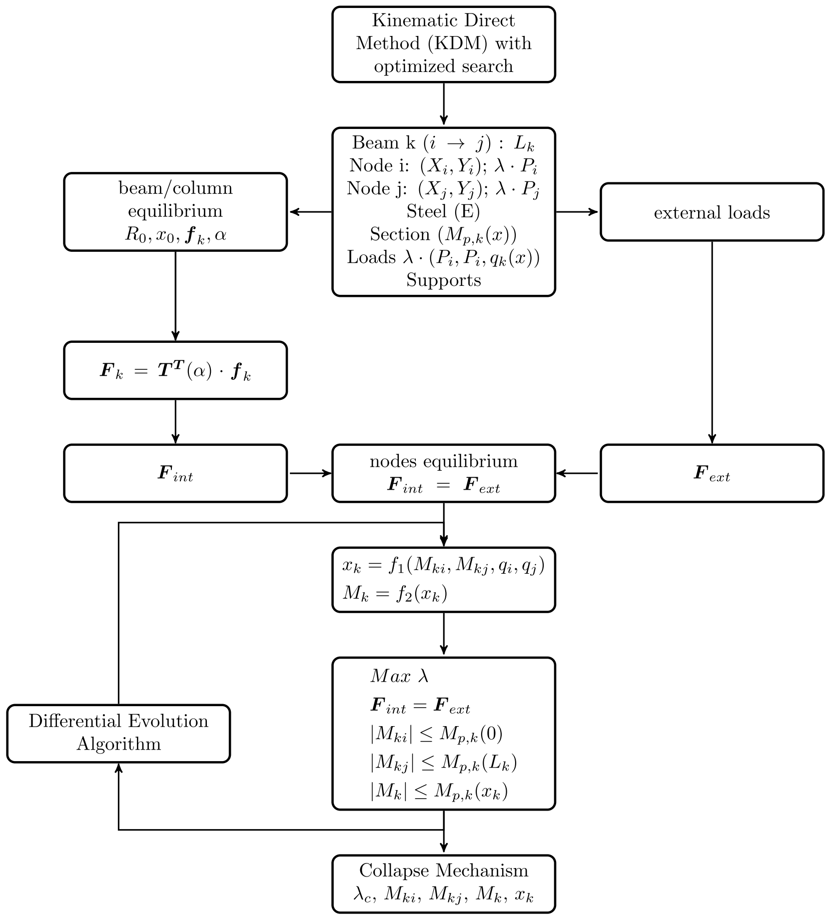

Fig. 2. Methodology flowchart.

where M_k is the maximum bending moment in the beam/column k and x_k is the section where the maximum value occurs, see Annex A.

2.3. Search for the collapse mechanism

The collapse mechanism must comply with the equilibrium equations obtained according to the previous section and the bending moment of any section must not exceed the value of the plastic moment. The search for the collapse mechanism is carried out by posing and solving an optimization problem. The optimization method used has been the Differential Evolution Algorithm [24] with the Wolfram Mathematica tool (NMaximize function) [25].

The objective function consists of maximizing the load factor subject to: the equilibrium equations of the problem and inequality constraints, the conditions of absolute value of bending moment less than or equal to the plastic moment. Posed optimization problem:

$$\begin{aligned} &Max\ \lambda \\ &\boldsymbol{F}_{int} = \boldsymbol{F}_{ext} \\ &|M_{ki}| \leq M_{p,k}(0) \\ &|M_{kj}| \leq M_{p,k}(L_k) \\ &|M_k| \leq M_{p,k}(x_k) \end{aligned} \tag{6}$$

where λ is the load factor; $\boldsymbol{F}_{int}$ are the internal forces; $\boldsymbol{F}_{ext}$ are the external loads; M_{ki}, M_{kj} are the bending moments at the ends of the beam/column k; $M_{p,k}(0)$ is the plastic moment of the beam/column k at node i (at $x = 0$); $M_{p,k}(L_k)$ is the plastic moment of the beam/column k at node j (at $x = L_k$, being L_k the length of the beam/column k); $M_{p,k}(x_k)$ is the plastic moment of the beam/column k at $x = x_k$ and M_k is the relative maximum or minimum bending moment in the beam/column k.

The final value that the objective function takes is the collapse load factor λ_c. In the sections where the value of the plastic moment is reached, a plastic hinge is formed. After forming a sufficient number of plastic hinges, the structure becomes a mechanism, the collapse mechanism.

Finally, by way of validation a simple example has been resolved with a step-by-step method [26] and with the methodology proposed in this work, the same results are reached as summarized in Annex B.

3. Numerical results and discussion

In this section, the methodology is applied to the study of three application problems with a total of four cases.

3.1. Numerical data

The numerical data in common for all the problems are: $E = 2.1 \cdot 10^8\,\mathrm{kN/m^2}$; $f_y = 275.0 \cdot 10^3\,\mathrm{kN/m^2}$; $q = 1.0\,\mathrm{kN/m}$; where E is Young's module, f_y the yield strength of the steel and q is the load value.

The columns are uniform section type IPE300 or variable section type IPEvar300, see Annex C. The properties of the **IPE300** section type are: $W_{pc} = 602.10 \cdot 10^{-6}\,\mathrm{m^3}$; $M_{pc} = W_{pc} \cdot f_y = 165.577\,\mathrm{kN\ m}$ where W_{pc} is the column section plastic module and M_{pc} is the column plastic moment. The properties of the **IPEvar300** section type are: $W_{pc}(0) = 192.68 \cdot 10^{-6}\,\mathrm{m^3}$; $W_{pc}(L_k) = 1126.53 \cdot 10^{-6}\,\mathrm{m^3}$; $M_{pc}(0) = 52.988\,\mathrm{kN\ m}$; $M_{pc}(L_k) = 309.797\,\mathrm{kN\ m}$ where $W_{pc}(0)$ is the column section plastic module at node i (at $x = 0$); $W_{pc}(L_k)$ is the column section plastic

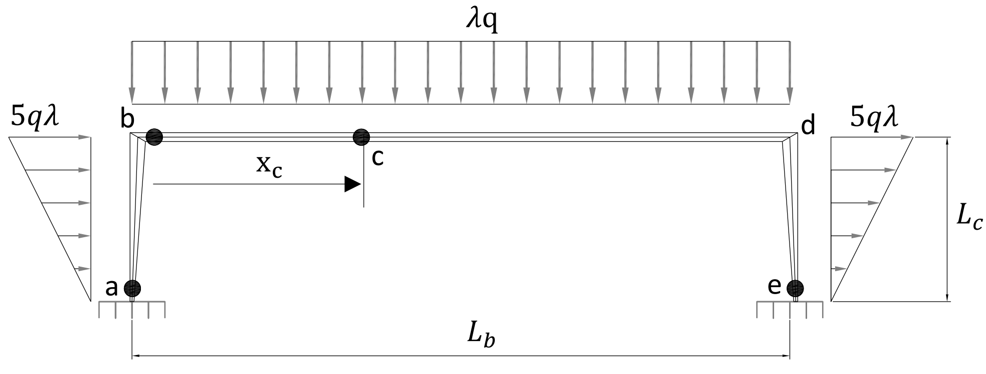

Fig. 3. Flat frame. Plastic analysis results.

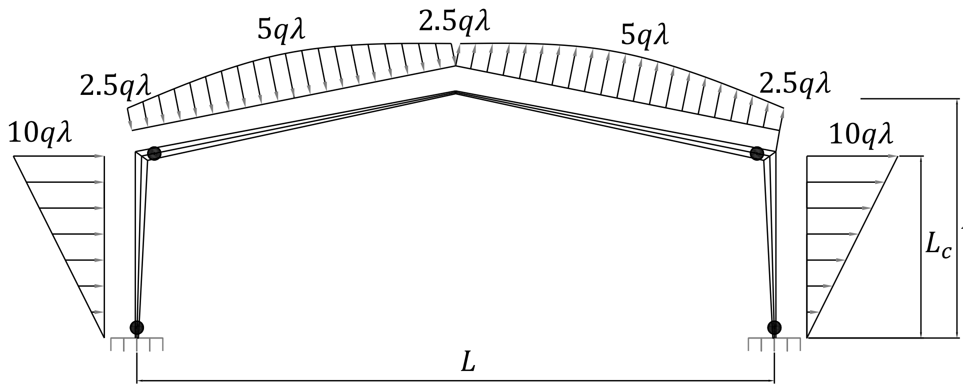

Fig. 4. Gabled frame. Plastic analysis results.

module at node j (at $x = L_k$); $M_{pc}(0)$ is the column plastic moment at node i (at $x = 0$) and $M_{pc}(L_k)$ is the column plastic moment at node j (at $x = L_k$).

The beams are uniform section type IPE270 or variable section type IPEvar270, see Annex C. The properties of the **IPE270** section type are: $W_{pb} = 460.54 \cdot 10^{-6}\,\text{m}^3$; $M_{pb} = W_{pb} \cdot f_y = 126.648\,\text{kN m}$ where W_{pb} is the beams section plastic module and M_{pb} is the beam plastic moment. The properties of the **IPEvar270** section type are: $W_{pb}(0) = 147.33 \cdot 10^{-6}\,\text{m}^3$; $W_{pb}(L_k) = 860.35 \cdot 10^{-6}\,\text{m}^3$; $M_{pb}(0) = 40.516\,\text{kN m}$; $M_{pb}(L_k) = 236.597\,\text{kN m}$ where $W_{pb}(0)$ is the beam section plastic module at node i (at $x = 0$); $W_{pb}(L_k)$ is the beam section plastic module at node j (at $x = L_k$); $M_{pb}(0)$ is the beam plastic moment at node i (at $x = 0$) and $M_{pb}(L_k)$ is the beam plastic moment at node j (at $x = L_k$).

3.2. Problem 1: flat frame

In this section, a flat frame is solved, the base of columns are fixed, beam/column joints are perfectly rigid and loads are triangular and uniform distributed type. Geometric properties and loads are: $L_c = 5.0\,\text{m}$; $L_b = 20.0\,\text{m}$; $q = 1.0\,\text{kN/m}$ where L_c is the length of the columns; L_b is the length of the beams and q is the load value.

In the case studied, IPEvar300 variable section columns and IPE270 uniform section beams are considered. The loads applied to the columns are trapezoidal distributed loads.

The methodology in Section 2 is applied and systematically solves the plastic problem, only the equilibrium equations of the problem are required. In beams/columns with load distributed in the element, an intermediate plastic hinge can be produced between extreme sections.

Collapse mechanism is indicated in Fig. 3 and collapse load factor results $\lambda_c = 4.38$ and $x_b = 6.79\,\text{m}$. The iterative process is detailed in Table 1. It is verified that the total number of iterations is low and the computing time is very reduced.

Table 1
Flat frame. Optimization iteration process.

Step	λ_c	M_a
1	2.80192	9143.72
2	3.51975	−21229.2
3	3.79445	−16065.8
4	3.99569	−21229.2
5	4.37592	−52987.8
6	4.37592	−52987.8
CPU time spent: 0.5 s		

3.3. Problem 2: gabled frame

In this section, this work methodology is used to solve a gabled frame. The bases of the columns are perfectly fixed. And all the loads on the structural elements are distributed loads, simulating for example a wind type load from left to right. Geometric properties and loads are: $L_c = 7.0\,\text{m}$; $H = 9.5\,\text{m}$ $L = 25.0\,\text{m}$; $q = 1.0\,\text{kN/m}$ where L_c is the length of the columns; H is the height of the frame; L is distance between supports and q is the load value.

Variable section beams and columns, IPEvar270 section beams and IPEvar300 section columns are considered, see Fig. 4.

Fig. 4 not only shows the definition of the problem but also the plastic hinges that give rise to the collapse mechanism of the structure. The collapse load factor is $\lambda_c = 1.18$. This case represents a material saving of 88% compared to all IPE270 section beams and all IPE300 section columns.

3.4. Problem 3: double gabled frame

The last problem solved is a double gabled frame. Again the bases of the columns are perfectly fixed, the loads are of the uniformly distributed type of intensity q. Geometric properties and loads are:

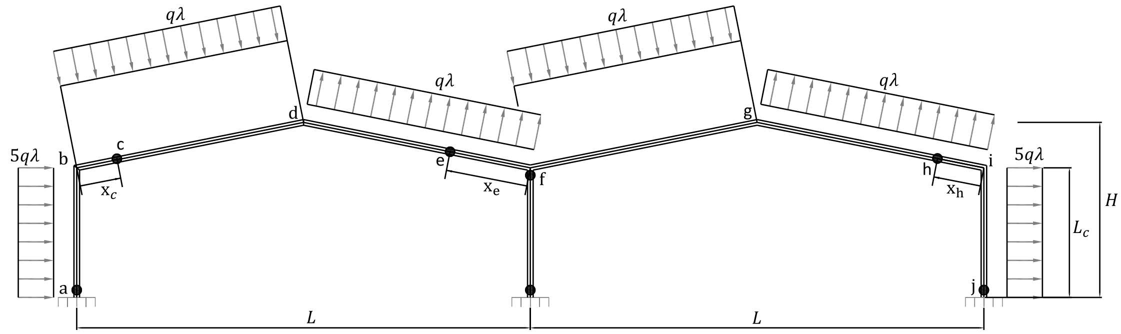

Fig. 5. Double gabled frame. Case a. Plastic analysis results.

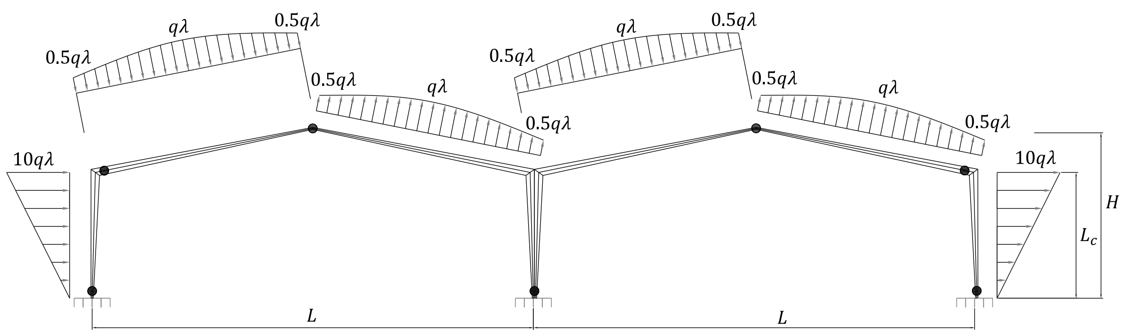

Fig. 6. Double gabled frame. Case b. Plastic analysis results.

$L_c = 7.0\,\text{m}$; $H = 9.5\,\text{m}$ $L = 25.0\,\text{m}$; $q = 1.0\,\text{kN/m}$ where L_c is the length of the columns; H is the height of the frame; L is distance between supports and q is the load value.

Two cases will be considered: case a, beams and columns of uniform section, beams of section IPE270 and columns of section IPE300, see Fig. 5; and case b, variable section beams and columns, IPEvar270 section beams and IPEvar300 section columns, see Fig. 6.

Figs. 5 and 6 shows the plastic analysis results, the collapse mechanism and the load factor associated. For case a, the collapse load factor is $\lambda_c = 2.94$ and $x_c = 2.21\,\text{m}$; $x_e = 4.42\,\text{m}$ and $x_h = 2.56\,\text{m}$. For case b, the load factor is $\lambda_c = 2.84$. One of the advantages of the methodology is that distributed loads case are solved using the same discretization of nodes and elements that for concentrated loads case.

4. Conclusions

The classic formulation for plastic methods of planar frames is very unsystematic. It is based on the Virtual Works Principle (VWP) and use equilibrium equations to find the structure's collapse mechanism. To obtain these equilibrium equations, the VWP is formulated using virtual problems in displacements (virtual mechanisms). This analysis technique is based on testing possible mechanisms until the collapse mechanism is found and this procedure is inefficient.

However, the present work uses the Kinematic Direct Method to carry out the first order plastic analysis of planar frames. One of the objectives achieved is that the entire methodology has been simplified, since the necessary equilibrium equations are obtained systematically. It leads directly to the collapse mechanism corresponding to the structure with given loads, geometry and boundary conditions.

The second advantage is that the search for the final state of the structure (collapse mechanism) is an optimization method quickly and efficiently, since the objective function is simple, the equality constraints are the equilibrium equations of the problem, which are linear algebraic equations that depend on the proportional load factor parameter λ (which is the value to be maximized), which allows solving practical problems with a large number of unknown parameters.

The third advantage of this approach is that the same methodology is used for the plastic calculation regardless of the type of load applied in the beams/columns. This allows solving with the same discretization of elements that we would use to solve the same problem but only with point loads, that is, one element per beam/column and without intermediate nodes in the beams/columns with distributed load applied.

Also indicate that the results obtained have been compared with those resulting from applying other methods, such as the step-by-step methods for limit analysis, and very similar results have been obtained, although the method presented here is much simpler.

CRediT authorship contribution statement

M. Cacho-Pérez: Conceptualization, Investigation, Methodology, Resources, Software, Supervision, Validation, Visualization, Writing – original draft, Writing – review & editing.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

No data was used for the research described in the article.

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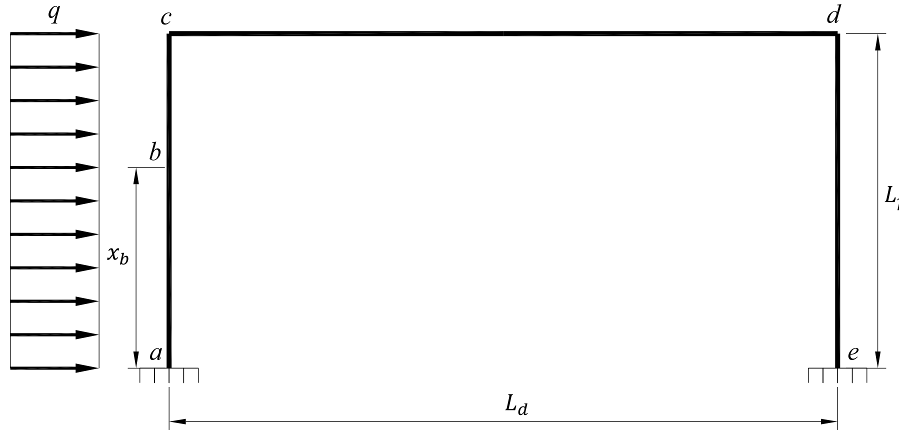

Fig. 7. Validation problem. Fixed-fixed frame. Uniform distributed load.

Annex A. Loads types

In this section the parameters used for each type of load will be indicated.

Uniformly distributed load

The parameters used are:

$$
\begin{aligned}
R_0 &= q \cdot L_k \\
x_0 &= \frac{L_k}{2} \\
x_k &= \frac{2M_{kj} - 2M_{ki} + q \cdot L_k^2}{2L_k \cdot q} \\
M_k &= \frac{4M_{ki}^2 + \left(2M_{kj} + q \cdot L_k^2\right) + M_{ki}\left(4q \cdot L_k^2 - 8M_{kj}\right)}{8q \cdot L_k^2}
\end{aligned}
\tag{7}
$$

where R_0 is the resultant of the applied load; x_0 is the point of application of the load resultant; x_k is the point on the beam/column where the bending moment is a relative maximum or minimum and M_k is the value of the relative maximum or minimum bending moment.

Trapezoidal distributed load

The parameters used for this load case are:

$$
\begin{aligned}
R_0 &= \frac{1}{2} L_k \left(q_i + q_j\right) \\
x_0 &= \frac{L_k \left(q_i + 2q_j\right)}{3\left(q_i + q_j\right)} \\
x_k &= \frac{q_i + \frac{\sqrt{L_k^2 q_i q_j + L_k^2 q_j^2 - L_k^2 q_i^2 + 6M_{ki} q_i - 6M_{ki} q_j - 6M_{kj} q_i + 6M_{kj} q_j}}{L_k}}{3\left(\frac{q_i}{L_k} - \frac{q_j}{L_k}\right)} \\
M_k &= \frac{L_k^3 q_i \left(4q_i^2 - 3q_i q_j - 3q_j^2\right)}{54L_k(q_i - q_j)^2} \\
&+ \frac{2L_k^2\left(q_i^2 - q_i q_j - q_j^2\right)\sqrt{L_k^2\left(-q_i^2 + q_i q_j + q_j^2\right) + 6M_{ki}(q_i - q_j) - 6M_{kj}(q_i - q_j)}}{54L_k(q_i - q_j)^2} \\
&- \frac{12(M_{ki} - M_{kj})(q_i - q_j)\sqrt{L_k^2\left(-q_i^2 + q_i q_j + q_j^2\right) + 6M_{ki}(q_i - q_j) - 6M_{kj}(q_i - q_j)}}{54L_k(q_i - q_j)^2} \\
&+ \frac{18L_k(q_i - q_j)(2M_{ki} q_i - 3M_{ki} q_j + M_{kj} q_i)}{54L_k(q_i - q_j)^2}
\end{aligned}
\tag{8}
$$

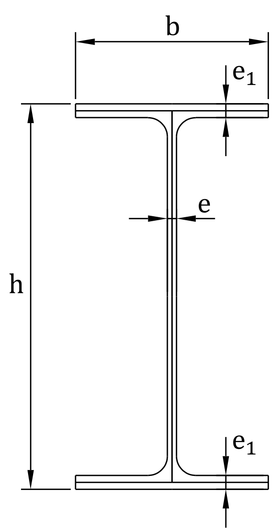

Fig. 8. IPE section type.

Table 2
Validation problem. Iteration process.

Step	q_c	M_c
1	85 184	120 727.0
2	105 416	81 403.5
3	10 673	96 565.4
4	115 938	126 509.0
5	129 386	81 403.5
6	137 127	98 971.6
7	138 187	121 009.0
8	143 218	126 381.0
9	143 221	126 394.0
10	143 228	126 425.0
CPU time spent: 1.078 s		

Sinusoidal distributed load

The parameters used are:

$$
\begin{aligned}
R_0 &= \frac{L_k((\pi - 2)q_i + 2q_j)}{\pi} \\
x_0 &= \frac{L_k}{2} \\
x_k &= \frac{\pi L_k^2 q_i - 2L_k^2 q_i + 2L_k^2 q_j - 2\pi M_{ki} + 2\pi M_{kj}}{2\pi L_k q_i}
\end{aligned}
$$

Table 3
Beams/columns of uniform height. IPE series.

IPE S275	h (mm)	b (mm)	e (mm)	e_1 (mm)	W_p (cm^3)	M_p (N m)
270	270	135	6.6	10.2	460.54	126 648.00
300	300	150	7.1	10.7	602.10	165 577.00

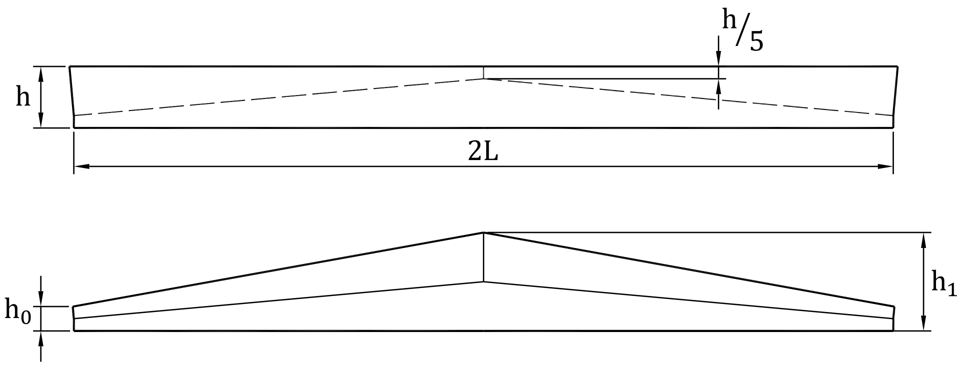

Fig. 9. Variable height beams/columns. IPEvar series.

Table 4
Variable height beams/columns. IPEvar series.

IPEvar S275	h_0 (mm)	h_1 (mm)	b (mm)	e (mm)	e_1 (mm)	$W_{p,0}$ (cm^3)	$M_{p,0}$ (N m)	$W_{p,1}$ (cm^3)	$M_{p,1}$ (N m)
270	108	432	135	6.6	10.2	147.33	40 516.40	860.35	236 597.00
300	120	480	150	7.1	10.7	192.68	52 987.80	1126.53	309 797.00

$$M_k = M_{ki} - \frac{\left(\pi L_k^2 q_i - 2L_k^2 q_i + 2L_k^2 q_j - 2\pi M_{ki} + 2\pi M_{kj}\right)^2}{8\pi^2 L_k^2 q_i} + \frac{\left(\frac{M_{kj}-M_{ki}}{L_k} + \frac{L_k((\pi-2)q_i+2q_j)}{2\pi}\right)\left(\pi L_k^2 q_i - 2L_k^2 q_i + 2L_k^2 q_j - 2\pi M_{ki} + 2\pi M_{kj}\right)}{2\pi L_k q_i} \tag{9}$$

Annex B. Validation problem

In this section, a basic frame fixed ended in the base of both columns with an uniform load is applied in the left column of the frame (see Fig. 7). Solution by a step by step method an be consulted in Ref. [26] and summarized here. The data in this case are: $L_p = 3\,\mathrm{m}$; $L_d = 5\,\mathrm{m}$, where L_p is the height of the column and L_d is the beam length.

As pointed out above, one more unknown (x_b, position of the plastic hinge) appears for each additional distributed load. The value of the bending moment in section b is expressed as a function of the bending moments at the ends of the column (M_a, M_c) and the applied load (q), and the coordinate x_b, see Annex A. The mechanism involves plastic hinges in the following sections: **a, e, b** and **d**. The collapse load results $q_c = 143.2\,\mathrm{kN/m}$. The intermediate plastic hinge (section b) in the element requested by the distributed load is $x_b = 2.196\,\mathrm{m}$.

On the other hand, the methodology of this work provides the following results: $q_c = 143228\,\mathrm{N/m}$, $x_b = 2.196\,\mathrm{m}$ and $M_c = 126425\,\mathrm{Nm}$. Table 2 shows the variables during the iterative resolution process.

Annex C. Variable height beam/column of the IPE series

See Figs. 8 and 9 and Tables 3 and 4.

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