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LOCAL MONOMIALIZATION OF GENERALIZED ANALYTIC FUNCTIONS

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LOCAL MONOMIALIZATION OF GENERALIZED ANALYTIC FUNCTIONS

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Abstract. Generalized power series extend the notion of formal power series by considering exponents of each variable ranging in a well ordered set of positive real numbers. Generalized analytic functions are defined locally by the sum of convergent generalized power series with real coefficients. We prove a local monomialization result for these functions: they can be transformed into a monomial via a locally finite collection of finite sequences of local blowings-up. For a convenient framework where this result can be established, we introduce the notion of generalized analytic manifold and the correct definition of blowing-up in this category.

1. Introduction

It is well known that an analytic function defined on an analytic manifold can be transformed into normal crossings via a locally finite collection of finite sequences of local blowings-up [BM88, Theorem 4.4]. Our goal is to establish a similar result for a wider class of functions, namely the class of ***generalized analytic functions*** defined on ***generalized analytic manifolds***.

The notion of ***convergent generalized power series*** is introduced in [DS98] in order to show the existence of o-minimal expansions of the ordered real field in which the convergent Dirichlet series are definable (as our work is not really concerned with these model-theoretic notions, we do not give details here). Roughly speaking, the generalized series are essentially defined as convergent series of monomials, where the variables are raised to ***positive real powers*** (see the precise definitions and a few examples in Section 2). Hence these functions are a priori defined for positive values of the variables.

A large part of the present paper is devoted to the description of a convenient framework in vue of our monomialization result, including in particular the notion of ***generalized analytic manifold***. Indeed, although many useful results on convergent generalized series are given in [DS98], a complete treatment of these objects in the language of real analytic geometry, involving a precise definition of generalized analytic manifolds and their morphisms, admissible centers of blowings-up, and blowings-up morphisms, was still missing. Above all, although a notion of *blow-up substitution* is given in [DS98], as well as process of simplification of generalized series involving these substitutions, no local monomialization result is proved there (because such a result is not needed in their proof of o-minimality).

We consider the present work as a very first step of a more general program. Our next goal is to define generalized semianalytic and subanalytic subsets of generalized manifolds, to prove a uniformization theorem for generalized subanalytic sets, and, if possible, a rectilinearization theorem for generalized subanalytic sets in the spirit of Hironaka's rectilinearization [Hir73, Theorem 7.1]. Consequently, we do not give here the detailed proof of every proposition. We just present the framework, announce the main result and sketch the proofs. A complete and self contained text awaits a next work.

We recall in Section 2 the main definitions and properties of convergent generalized series, following the terminology and notations of [DS98]. In Section 3 we establish the formal framework needed to state our uniformization result. We give in particular the definitions of generalized manifolds (which are actually ***manifolds with boundary and corners***, since the generalized series are defined for positive values of the variables), generalized analytic maps, and blowings-up. Notice that working with manifolds with boundaries and corners leads to consider ***oriented real blowings-up***, in contrast with the ***projective real blowings-up*** used in [BM88] or [Hir73], for example. Moreover, in order to preserve the structure of the boundary as a *divisor with normal crossings*, the center of the blowing-up must not only be a smooth closed *submanifold*, but have *normal crossings* with the boundary. Such centers will be called *admissible*.

On the other hand, there is a specific problem for generalized manifolds when we want to define blowings-up with admissible center. The classical process which consists in defining blowings-up in the local models of the manifolds, and then in "lifting" these morphims on manifolds via local charts, does not work as such here. In fact, there are isomorphisms that do not lift to the blown-up space. We actually need to define blowings-up with respect to a given coordinate system. A more subtle and convenient way to overcome this difficulty is to consider only blowings-up whose centers are contained in a so called *standarizable generalized manifold*. These manifolds, defined and studied in 3.4, are essentially analytic manifolds (with boundary and corners) whose structural sheaf has been "enriched" by the introduction of ***generalized analytic functions***. We then take advantage of the fact that the notion of blowing-up morphism is well defined in the category of analytic manifolds, to extend it to the "enriched" generalized manifolds. This procedure is developed in 4.1.

It is worth to notice that the concept of standardizable manifold is pertinent: we show in 3.4 an example of generalized manifold which is not standarizable, *i.e.* that does not come from a (standard) analytic manifold by enriching its structural sheaf. It also points out that there may be closed admissible centers of a generalized manifold which can not be centers of blowing-up in any raisonable way. Examples of this kind (and obstruction to blow-up) only occur in the global setting, every generalized analytic manifold being locally standarizable.

Section 4 is devoted to our main result, which asserts the monomialization of generalized analytic functions via local blowings-up:

Theorem B. *Let M be a generalized analytic manifold, p an point of M, and f a generalized analytic function defined over M. Then there exists a finite family*

$$\Sigma = \{\pi_j \colon W_j \to M, L_j\}_{j\in J}$$

where:

(1) *each π_j is the composition of finitely many local blowings-up (with admissible centers)*

$$\pi_j \colon W_j = W_{j,n_j} \overset{\pi_{j,n_j}}{\to} W_{j,n_j-1} \overset{\pi_{j,n_j-1}}{\to} W_{j,n_j-2} \to \cdots \overset{\pi_{j,1}}{\to} W_{j,0} = M$$

(2) *each* L_j *is a compact subset of* W_j *such that* $\cup_{j\in J}\pi(L_j)$ *is a compact neighborhood of* p *in* M

such that for all $j \in J$, $f \circ \pi_j : W_j \to \mathbb{R}$ *is of monomial type at any point of* L_j.

The first step of its proof, which consists in transforming some "generalized" variables into "analytic variables" via convenient local blowings-up, follows a scheme already described in [DS98]. This procedure enables us to produce sufficiently many analytic variables so that Weierstrass Preparation Theorem with respect to one of those variables holds. We are then led to the case where f is a distinguished polynomial whose coefficients are generalized convergent series. Its second step, which consists mainly in the reduction of the degree of the distinguished polynomial, follows the great lines of the proof of Theorem 4.4 in [BM88] or Proposition 3.8 in [RSW03].

Let us add a final remark. The term "local blowing-up" means, as usual, a blowing-up with a locally closed admissible center (that is, closed in some open subset of its ambient space). One may wonder about a possible ***global resolution of singularities of generalized functions***. Namely, is it possible to improve the Local Monomialization Theorem so that the family Σ consists in a single sequence of blowing-ups ($|J| = 1$)

$$\pi\colon M_n \overset{\pi_n}{\to} M_{n-1} \overset{\pi_{n-1}}{\to} M_{n-2} \overset{\pi_{n-2}}{\to} \cdots \overset{\pi_1}{\to} M_0 = M$$

such that is π_j is "global", that is, a blowing-up with respect to a closed admissible center of the whole manifold M_{j-1}? This question could be an interesting continuation of the present work. One of the first, though not minor, tasks for this latter result, is to prove that we do not encounter examples, as the one that was mentioned above, of admissible centers which can not be blown-up (*i.e.* which admit no standarizable open neighborhood).

2. Formal and convergent generalized power series

2.1. **Formal generalized power series.**

2.1.1. *Definitions.* We recall the definitions given in [DS98]. Let m range over $\mathbb{N}$ and let $X = (X_1, \ldots, X_m)$ be a tuple of m distinct indeterminates. Consider a commutative ring A with $1 \neq 0$. A ***formal generalized power series*** in the variables X with coefficients in A is a map $s\colon [0,\infty)^m \to A$, that we write as the fomal series

$$s = s(X) = \sum_{\alpha\in[0,\infty)^m} s_\alpha X^\alpha$$

where $s_\alpha = s(\alpha) \in A$, X^α denotes the formal monomial $X_1^{\alpha_1}\cdots X_m^{\alpha_m}$, and the ***support***

$$\mathrm{supp}(s) = \{\alpha \in [0,\infty) : s_\alpha \neq 0\}$$

is a ***good set***, *i.e.* is contained in the cartesian product $S_1 \times \cdots \times S_m$ of well ordered subsets of $[0,\infty)$. In particular, every good set is finite or countable. These series are added and multiplied in the usual way, and form an A-algebra denoted by $A[[X^*]]$. The coefficient $s_0 = s(0)$ is the ***constant term*** of s. A series s is a unit in $A[[X^*]]$ if and only if s_0 is a unit in A. Examples of such series are given in 2.2.1.

We equip the set $[0,\infty)^m$ with the partial order $\alpha = (\alpha_1,\ldots,\alpha_m) \leq \beta = (\beta_1,\ldots,\beta_m) \Leftrightarrow \alpha_i \leq \beta_i$ for $i = 1,\ldots,m$ (which corresponds to the division order of the monomials, $X^\alpha | X^\beta$). For $s \in A[[X^*]]$, the ***minimal support*** $\mathrm{Supp}_{\min}(s)$ of s is the set of minimal elements of $\mathrm{Supp}(s)$ (this set is finite since $\mathrm{Supp}(s)$ is a good set). We have

$$s = \sum_{\alpha \in \mathrm{Supp}_{\min}(s)} X^\alpha u_\alpha$$

where $u_\alpha \in A[[X^*]]$ and $u_\alpha(0) \neq 0$ for every $\alpha \in \mathrm{Supp}_{\min}(s)$. This expression is called the ***monomial presentation*** of s. In particular, a series is called of ***monomial type*** if its support has exactly one minimal element.

2.1.2. *Mixed series.* The following subrings of generalized power series play an important role in the following. Let

$$(X, Y) = (X_1, \ldots, X_m, Y_1, \ldots, Y_n)$$

be a tuple of $(m+n)$ distinct variables. We denote by $A[[X^*, Y]]$ the subring of $\mathbb{A}\left[\left[(X, Y)^*\right]\right]$ whose elements are the series

$$s = \sum_{(\alpha,\beta) \in [0,\infty)^m \times \mathbb{N}^n} s_{\alpha,\beta} X^\alpha Y^\beta,$$

that is, generalized series with integers exponents in the variables Y. These series are called ***mixed series*** in the variables (X, Y). Note that $A[[X^*, Y]]$ is a *proper* subring of $A[[X^*]][[Y]]$. It is convenient to call the variables X ***generalized*** and the variables Y ***analytic***. As a matter of notation, we will write $A[[X^*, Y]]_{m,n}$ when we want to make explicit the number of generalized and analytic variables

2.1.3. *Composition morphims.* In the classical framework of (usual) formal power series, the composition makes sense: if $s \in A[[Y]]$ and $t = (t_1, \ldots, t_n) \in A[[W]]^n$ (where $W = (W_1, \ldots, W_l)$), with $t_1(0) = \cdots = t_n(0) = 0$, then we may substitute t for Y and obtain the series $s(t(W)) \in A[[W]]$.

We can proceed similarly with mixed series. If $s \in A[[X^*, Y]]$ and $t = (t_1, \ldots, t_n) \in A[[W^*]]^n$, we can substitute t for Y in s, and obtain the series $s(X, t(W)) \in A\left[\left[(X, W)^*\right]\right]$. But substitution inside generalized variables is much more delicate. Consider for example $s(X) = X^{1/2}$ and $t(W_1, W_2) = W_1 + W_2$. There is no reasonable candidate in $\mathbb{R}\left[\left[(W_1, W_2)^*\right]\right]$ to be $s(t(W_1, W_2))$, that is to be a square root of $W_1 + W_2$.

However, the following kinds of substitution make sense:

Proposition 2.1. *Let $X = (X_1, \ldots, X_m)$, $Y = (Y_1, \ldots, Y_n)$, $W = (W_1, \ldots, W_p)$ and $Z = (Z_1, \ldots, Z_q)$ denote multivariables.*

1. *Let $s \in A[[X^*, Y]]$ and $t = (t_1, \ldots, t_n) \in A[[W^*, Z]]^n$ with $t_1(0) = \cdots = t_n(0) = 0$. Then $s(X^*, t(W, Z)) \in A\left[\left[(X, W)^*, Z\right]\right]$. Moreover, the map $s \mapsto s(X^*, t(W, Z))$ is an A-algebra homomorphism.*
2. *Let $s \in \mathbb{R}[[X^*]]$ and $t = (t_1, \ldots, t_m) \in \mathbb{R}[[W^*]]$. If $t_i = W^{\beta^i} u_i$ with $\beta^i \neq (0, \ldots 0)$, $u_i \in \mathbb{R}[[W^*]]$ and $u_i(0) > 0$ for $i = 1, \ldots, m$ (that is, t_i is of monomial type), then $s(t(Z)) \in \mathbb{R}[[W^*]]$. Moreover, the map $s \mapsto s(t(W))$ is an $\mathbb{R}$-algebra homomorphism.*

Sketch of the proof. We use the following notion from [DS98]: a family $\{s_i\}_{i \in I}$ of generalized series (in m variables) is called a *summable family* if

- (a) for each $\alpha \in [0, \infty)^m$, there are only finitely many $i \in I$ such that $\alpha \in \mathrm{Supp}(s_i)$ and

- (b) the union $\cup_{i\in I}\mathrm{Supp}(s_i)$ is a good set of $[0,\infty)^m$. In this case, the *sum* $\sum_{i\in I} s_i$ is well defined as a generalized series.

Now, for part 1, if $s = \sum_{(\alpha,\beta)\in[0,\infty)^m\times\mathbb{N}^n} s_{(\alpha,\beta)}$, we only have to check that the family

$$\{s_{(\alpha,\beta)} t_1^{\beta_1}\cdots t_n^{\beta_n}\}_{(\alpha,\beta)\in[0,\infty)^m\times\mathbb{N}^n}$$

is a summable family. Similar arguments are needed for part 2 of the Proposition. But we have to show first that for any $a\in[0,\infty)$, the series $u_i^a\in\mathbb{R}[[W^*]]$ is well defined by

$$u_i^a = \sum_{k\in\mathbb{N}.} \begin{pmatrix} a \\ k \end{pmatrix} u_i(0)^{a-k}(u_i - u_i(0))^k.$$

□

We can complete the second point of the former proposition by a result, not mentionned in [DS98], which states that the only series which admit a Nth root for every $N\in\mathbb{N}$ are the series of monomial type in the generalized variables. It is used in section 3.3.2, in order to give the local expression of morphims between generalized analytic manifolds.

Proposition 2.2. *Let $s\in\mathbb{R}[[X^*,Y]]$ be a formal generalized power series where $X=(X_1,\dots,X_m)$, $Y=(Y_1,\dots,Y_n)$ are respectively the generalized and the analytic variables. Suppose that $s\not\equiv 0$ and that for every integer $N\in\mathbb{N}$ s admits a Nth root $s_N\in\mathbb{R}[[X^*,Y]]$. Then $s=X^\alpha u$, where $\alpha\in[0,\infty)^m$ and $u\in\mathbb{R}[[X^*,Y]]$ is a unit with $u(0,0)>0$.*

Sketch of proof. We consider s as a series $s=\sum_{\alpha\in[0,\infty)^m} s_\alpha X^\alpha\in\mathbb{R}[[Y]][[X^*]]$ in m variables. The result is clear if $m=0$: a formal series with integer powers which has N^{th}-roots for any $N$is a unit.

Assume that $m>0$. The *Newton polyhedron* $\Delta(s)$ *of* s is the convex hull in $\mathbb{R}^m$ of $\mathrm{Supp}(s)+\mathbb{R}_+^m$. In fact, it is equal to the convex hull of $\mathrm{Supp}_{\min}(s)+\mathbb{R}_+^m$ and thus $\Delta(s)$ is actually a finite convex polyhedron. It suffices to prove that $\Delta(s)$ has a unique vertex: in that case, we would have $s=X^\alpha t$ where $t\in\mathbb{R}[[Y]][[X^*]]$ with initial coefficient $t_0\in\mathbb{R}[[Y]]\neq 0$ and the hypothesis would imply that t_0 has N^{th}-roots (in $\mathbb{R}[[Y]]$) for any N. Thus t_0 would be a unit.

Assume that $\Delta(s)$ has at least two vertices. We use the following notations for any given $\rho=(\rho_1,...,\rho_m)\in(0,\infty)^m$ and any $t=\sum_\alpha t_\alpha X^\alpha\in\mathbb{R}[[Y]][[X^*]]$:

(1) The *ρ-order of* t is $\nu_\rho(t)=\min\{\rho\cdot\alpha/\alpha\in\mathrm{Supp}(t)\}$, where $\cdot$ is the usual scalar product.
(2) The *ρ-initial part of* t is $In_\rho(t)=\sum_{\alpha/\rho\cdot\alpha=\nu_\rho(t)} t_\alpha X^\alpha$.
(3) t is called *ρ-quasihomogeneous* if $t=In_\rho(t)$.

Geometrically, $\nu_\rho(t)$ is the minimum of real numbers $c\geq 0$ such that the hyperplane $H_{\rho,c}=\{\rho_1X_1+\cdots\rho_mX_m=c\}$ intersects $\Delta(t)$ and the support of $In_\rho(t)$ is equal to the set of vertices of $\Delta(t)$ contained in $H_{\rho,\nu_\rho(t)}$. Thus $\mathrm{Supp}(In_\rho(t))$ is finite for any ρ and any t. On the other hand, if t has N^{th}-roots for any N then $In_\rho(t)$ too and any N^{th}-root of $In_\rho(t)$ is ρ-quasihomogeneous with ρ-order equal to $\nu_\rho(t)/N$.

Choosing a convenient vector ρ, we can assume that our series s with all N^{th}-roots has a finite support contained in a segment $[\alpha,\beta]$ in $\mathbb{R}_+^m$, not parallel to a coordinate axis and so that α,β are both in $\mathrm{Supp}(s)$ and $\alpha\neq\beta$. Let us look for a contradiction. We write

$$s=\sum_{\lambda\in[0,1]} s_\lambda X^{(1-\lambda)\alpha+\lambda\beta},\ \ s_\lambda\in\mathbb{R}[[Y]],$$

where $\{\lambda\,/\,s_\lambda\neq 0\}$ (with some abuse, also denoted by $\mathrm{Supp}(s)$) is finite and contains $\{0,1\}$. A fixed N^{th}-root, s_N, of s writes accordingly as

$$s_N=\sum_{\lambda\in[0,1]} s_{N,\lambda} X^{(1-\lambda)\frac{\alpha}{N}+\lambda\frac{\beta}{N}},\ \ s_{N,\lambda}\in\mathbb{R}[[Y]],$$

where $\{0,1\}\subset\mathrm{Supp}(s_N)$, too. Since $(s_N)^N=s$, we have, for any $\lambda\in[0,1]$, the formula

$$s_\lambda=\sum_{\lambda_1+\cdots+\lambda_N=N\lambda} s_{N,\lambda_1}\cdots s_{N,\lambda_N}. \tag{2.1}$$

Let N big enough so that for any $\lambda\in\mathrm{Supp}(s)$, either $\lambda=0$ or $\lambda>1/N$. Considering equation 2.1 for $\lambda=1/N$, we notice that the N-tuples of the form $(\lambda_1,\ldots,\lambda_N)=(0,...,1^{(j^{th})},...,0)$ in this sum give the term $Ns_{N,1}(s_{N,0})^{N-1})\neq 0$. Since $s_{1/N}=0$, we must have a different N-tuple $(\lambda_1,...,\lambda_N)$ so that $\lambda_j\in\mathrm{Supp}(s_N)$ and $\lambda_1+\cdots+\lambda_N=1$. Thus, there exists some $0<\gamma^1<1$ in the support of s_N. Repeating the argument for $\lambda=\gamma^1/N$ we find an element $0<\gamma^2<\gamma^1$ in $\mathrm{Supp}(s_N)$. We construct inductively a strictly decreasing sequence in $\mathrm{Supp}(s_N)$, contradicting the well order of this support. □

2.1.4. *The b invariant, a numerical data towards a monomialization result.* This numerical invariant is introduced in [DS98] to measure how far a generalized series is from being of monomial type.

Let $\alpha,\beta\in[0,\infty)^m$. Put $\inf(\alpha,\beta):=(\min\{\alpha_1,\beta_1\},\ldots,\min\{\alpha_m,\beta_m\})$. If $\inf(\alpha,\beta)\in\{\alpha,\beta\}$ (which means that one of the monomials X^α, X^β divides the other one), then put $d(\alpha,\beta)=0$. Otherwise, there are two possibilities:

(1) $\inf(\alpha,\beta)=0$. Then put $d(\alpha,\beta):=a+b$, where

$$a:=|\{j\in\{1,\ldots,m\}:\alpha_j\neq 0\}|\ \text{ and } b:=|\{j\in\{1,\ldots,m\}\}:\beta_j\neq 0|\,.$$

(2) $\inf(\alpha,b)\neq 0$. Then $d(\alpha,\beta):=d(\alpha-\inf(\alpha,\beta),\beta-\inf(\alpha,\beta))$.

We note $\gcd\left(X^\alpha,X^\beta\right):=X^{\inf(\alpha,\beta)}$ and $d\left(X^\alpha,Y^\beta\right):=d(\alpha,\beta)$.

Definition 2.3. Given $s\in A[[X^*]]$, we define

$$b(s)=(b_1(s),b_2(s))=(\#\mathrm{Supp}_{\min}(s)-1,b_2(s))\in\mathbb{N}^2,$$

where

$$b_2(s)=\begin{cases}0 & \text{if } b_1(s)=0\\ \min\{d(\alpha,\beta):\alpha,\beta\in\mathrm{Supp}_{min}(s),\alpha\neq\beta\} & \text{otherwise.}\end{cases}$$

The following substitution, introduced in [DS98, 4.9], is intended to express a blowing-up transformation in some local charts. It generalizes the classical notion of *quadratic* blowing-up to the more general framework of series in monomials with positive real exponents (It is an example of the permissible substitution stated in Proposition 2.1, part 2).

Definition 2.4. For $i,j \in \{1,\dots,m\}$ and $\gamma > 0$ consider the injective A-algebra endomorphism of $A[[X^*]]$ which extends the following definition:

$$\varsigma_{i,j}^{\gamma}(X_k) = X_k \text{ for } k \neq i \text{ and } \varsigma_{i,j}^{\gamma}(X_i) = X_i X_j^{\gamma}.$$

The following simple and important fact is proved in [DS98]:

Proposition 2.5. *Let $s \in A[[X^*]]$ with $m \geq 2$.*

(1) *If $b(s) = (0,0)$ then, for every different $i,j \in \{1,\dots,m\}$ and $\gamma > 0$, we have $b\left(\varsigma_{i,j}^{\gamma}(s)\right) = 0$.*
(2) *If $b(s) \neq (0,0)$ then there exist different $i,j \in \{1,\dots,m\}$ and $\gamma > 0$ such that*
$$b\left(\varsigma_{i,j}^{\gamma}(s)\right) < b(s) \ \text{ and } b\left(\varsigma_{j,i}^{1/\gamma}(s) < b(s)\right).$$

The first point of this result garantees that if a series is of monomial type, then further transformations of the same type, possibly introduced to lower the b-invariant of other series, will not alter this property. The second point states that if a series is not of monomial type, there exists such a substitution which lowers the number of elements of its minimal support.

2.2. **Convergent generalized power series.**

2.2.1. *Definitions and examples.* Suppose now that A is a normed ring with norm $|\cdot|$. For each polyradius $r = (r_1,\dots,r_m) \in (0,\infty)^m$ and $s \in A[[X^*]]$, we put

$$\|s\|_r = \sum |s_\alpha| r^\alpha \in [0,\infty]$$

and we let $A\{X^*\}_r$ be the normed subalgebra of $A[[X^*]]$ consisting of the s's with $\|s\|_r < \infty$, with norm given by $\|\cdot\|_r$. Each $s(X) \in A\{X\}_r$ gives rise to a continuous function (still denoted by s) $x \mapsto s(x) := \sum s_\alpha x^\alpha \colon [0,r_1] \times \cdots \times [0,r_m] \to \mathbb{R}$, analytic on the interior $(0,r_1) \times \cdots \times (0,r_m)$ of its domain.

We put

$$A\{X^*\} = \bigcup_r A\{X^*\}_r,$$

which is also a subring of $A[[X^*]]$.

Let us give a few examples of such series, with $A = \mathbb{R}$, in which we see different occurences of the notion of *good support.*

Examples. 1) Consider a real analytic germ $F \in \mathbb{R}\{x_1,\dots,x_m\}$, and m positive (and possibly irrational) numbers $\lambda_1,\dots,\lambda_m$. Then the germ $f(x_1^{\lambda_1},\dots,x_m^{\lambda_m})$ admits a representative in $\mathbb{R}\{X^*\}_r$ for some $r \in (0,\infty)^m$. Its support is included in the cartesian product $\lambda_1\mathbb{N} \times \cdots \times \lambda_m\mathbb{N}$.

2) Consider an ***ordinary Dirichlet series*** $g(t) = \sum_{j\geq 1} \frac{a_j}{j^t}$ with abscissa of convergence $t_0 \in \mathbb{R}$. We may study the behavior of g in a neighborhood of $+\infty$ in considering the logarithmic chart $t = -\log x$. Hence we consider the convergent generalized series $f\colon x \mapsto g(-\log x) = \sum_{j\geq 1} a_j x^{\log j} : [0,\mathrm{e}^{-t_0}] \to \mathbb{R}$. Its support $\{\log j \colon j \in \mathbb{N}_{>0}\}$ is an increasing sequence of real numbers, hence is obviously well ordered.

3) More generally, given an increasing sequence $(\lambda_j)_{j\geq 1}$ of real numbers whose limit is $+\infty$, the function $g\colon t \mapsto \sum_{j\geq 1} a_j \mathrm{e}^{-\lambda_j t}$ is called a ***Dirichlet series of type*** (λ_j) (see [HR64] for example). If t_0 is its abscissa of convergence, then the function

$f\colon x\mapsto g\left(-\log x\right)=\sum_{j\geq1}a_jx^{\lambda_j}\,:\,[0,\mathrm{e}^{-t_0}]\to\mathbb{R}$ is a convergent generalized series, whose support is the well ordered set $\{\lambda_j\colon j\in\mathbb{N}_{>0}\}$.

4) The supports of the foregoing examples, being cartesian products of increasing sequences whose limit is $+\infty$, are good sets for obvious reasons. Let us show that "reasonable" examples may produce series with more complicated good supports. It can easily be checked that the generalized series $f\colon x\to\sum_{i\geq0}\frac{1}{2^i}x^{2-1/2^i}$, $[0,1]\to\mathbb{R}$, is a solution of the functional equation $f\left(x\right)=x+\frac{1}{2}f\left(\sqrt{x}\right)$. Its support $\left\{2-2/2^i\colon i\in\mathbb{N}\right\}$ is a well ordered set with 2 as single limit point.

Consider now the series $g\left(x\right)=\dfrac{1}{1-x}f\left(x\right)$. It satisfies the functional equation $\left(1-x\right)g\left(x\right)=x+\left(1-\sqrt{x}\right)g\left(\sqrt{x}\right)$. Its support is the set $\left\{2+\ell-2/2^i\colon \ell,i\in\mathbb{N}\right\}$, which is well ordered, and admit all the integers greater or equal than 2 as limit points.

Mixed convergent series. We can now, for mixed variables $(X_1,\ldots,X_m,Y_1,\ldots,Y_n)$, define the rings of ***convergent mixed series***, that is

$$\begin{aligned}&A\left\{X^*,Y\right\}_{m,n}:=A\left[[X^*,Y]\right]_{m,n}\cap A\left\{\left(X,Y\right)^*\right\}\ \text{ and}\\ &A\left\{X^*,Y\right\}_{m,n;r,l}=A\left\{X^*,Y\right\}_{r,l}:=A\left[[X^*,Y]\right]_{m,n}\cap A\left\{\left(X,Y\right)^*\right\}_{r,l}\end{aligned}$$

for polyradii $r=(r_1,\ldots,r_m)$, $l=(l_1,\ldots,l_n)$. An element $s\in A\left\{X^*,Y\right\}_{r,l}$ gives rise to a continuous function (still denoted by s and called the *sum* of the convergent series) defined on

$$\bar{I}_{r,l}=[0,r_1]\times\cdots\times[0,r_m]\times[-l_1,l_1]\times\cdots\times[-l_n,l_n]$$

by $s\left(x,y\right)=\sum s_{\alpha,\beta}x^{\alpha}y^{\beta}$. This function is actually analytic on $\operatorname{int}\left(\bar{I}_{r,l}\right)$. More precisely, we have the following result [DS98, Cor. 6.7]:

Theorem 2.6. *If $(a,b)\in I_{r,l}=[0,r_1)\times\cdots\times[0,r_m)\times(-l_1,l_1)\times\cdots\times(-l_n,l_n)$, put $m':=|\{i\in\{1,\ldots,m\}:a_i=0\}|$ and consider a permutation σ of $\{1,\ldots,m\}$ such that $\sigma\left(\{i\in\{1,\ldots,m\}:a_i=0\}\right)=\{1,\ldots,m'\}$. Then, there exists a unique convergent generalized series $T_{(a,b)}(s)\in\mathbb{R}\{Z^*,W\}_{m',m+n-m'}$ such that, with the obvious action of σ on the functions, and the obvious notations:*

$$T_{(a,b)}\left(s\right)\left(z,w\right)=\sigma\left(s\right)\left(\left(a,b\right)+\left(z,w\right)\right).$$

*The series $T_{(a,b)}(s)$ is called called the **Taylor expansion** of s at (a,b). It is well defined up to a permutation of the variables (Z,W) that leaves invariant the set of generalized variables Z and that of the analytic ones W (see* [DS98, Cor. 6.7]*).*

2.2.2. *Properties of convergent series.* The behavior of convergent generalized power series under composition are summarized in the following convergent version of Proposition 2.1.

Proposition 2.7. *Let $X=(X_1,\ldots,X_m)$, $Y=(Y_1,\ldots,Y_n)$, $W=(W_1,\ldots,W_p)$ and $Z=(Z_1,\ldots,Z_q)$ denote multivariables.*

1. *Let $s\in A\left\{X^*,Y\right\}$ and $t=(t_1,\cdots,t_n)\in A\left\{W^*,Z\right\}^n$, such that $t_1\left(0\right)=\cdots=t_n\left(0\right)=0$. Then $s\left(X,t\left(W,Z\right)\right)\in A\left\{\left(X,W\right)^*,Z\right\}$, and its sum is equal to $(x,w,z)\mapsto s(x,t(w,z))$, where s,t denote the sums of the corresponding convergent series.*
2. *Let $s\in\mathbb{R}\left\{X^*\right\}$ and $t=(t_1,\ldots,t_n)\in\mathbb{R}\left\{Z^*\right\}^m$ such that $t_i=Z^{\beta^i}u_i$, with $\beta^i\neq(0,\ldots0)$, $u_i\in\mathbb{R}\left\{Z^*\right\}$ and $u_i\left(0\right)>0$ for all $i\in\{1,\ldots,m\}$. Then $s\left(t_1,\ldots,t_m\right)\in\mathbb{R}\left\{Z^*\right\}$. Moreover, as in the former part, the sum of*

this latter convergent series is given by the composition of the sums of the corresponding convergent series s and t_i.

Finally, the "generalized" version of Weierstrass preparation theorem is also proved in [DS98, Th. 5.10]:

Weierstrass preparation theorem for mixed series. *Let $n > 0$ and $s \in A\{X^*, Y\}$ be regular in Y_n of order d, that is, $s(0, 0, Y_n) = Y_n^d U(Y_n)$ where $U \in A[[Y_n]]$ is a unit. Put $Y' = (Y_1, \dots, Y_{n-1})$. Then s factors uniquely as $s = UP$, where $U \in A\{X^*, Y\}$ is a unit and $P \in A\{X^*, Y'\}[Y_n]$ is monic of degree d in Y_n.*

This statement implies the following corollary:

Implicit functions theorem. *Let $s = (s_1, s_2, \dots, s_k) \in A[[X^*, Y, W]]^k$ where $X = (X_1, X_2, \dots, X_m)$, $Y = (Y_1, Y_2, \dots, Y_n)$ and $W = (W_1, W_2, \dots, W_k)$. Suppose that $s_j(\underline{0}) = 0$ for $j = 1, 2 \dots, k$ and that the matrix $\left(\frac{\partial s_j}{\partial W_i}(\underline{0})\right)_{1\leq i,j\leq k}$ is not singular. Then there exists unique $t_1, t_2, \dots, t_k \in A[[X^*, Y]]$ with $t_i(\underline{0}) = 0$ such that*

$$s_j(X, Y, t_1(X, Y), t_2(X, Y), \dots, t_k(X, Y)) \equiv 0$$

for $j = 1, 2 \dots, k$.

Remark 2.8. Actually, the reduction of generalized series to polynomials via convenient blowings-up is the major part of the proof of o-minimality in [DS98]. This is probably why there is no attempt there to prove a monomialization result. But for different algebras of functions, where Weierstrass preparation is notoriously wrong, local monomialization is a very efficient way to prove o-minimality. It is in particular the case for so-called ***quasianalytic*** classes of functions (see for example [BM04, RSW03]).

3. Generalized analytic manifolds

We introduce in this section the language of generalized analytic manifolds and morphisms, which is the most appropriate for the statement (and the proof) of our Local Monomialization theorem. Actually, we need to introduce three types of manifolds. The simplest being the *real analytic manifolds* with boundary and corner, the richest being the *generalized analytic manifolds*. An intermediate class, the *standardizable manifolds*, is introduced because of its specific role in the blowing-up process.

In the following, k ranges over the integers.

3.1. Analytic and $\mathcal{G}$-analytic functions.

We introduce a class of functions, defined on open subsets of quadrants $\mathbb{R}^k_{\geq 0}$, and which can be represented locally by real convergent generalized power series (in the same way as the classical real analytic functions are those locally described by convergent power series). Let us recall first what is meant by an analytic function on such a quadrant:

Definition 3.1. Let V be an open subset of $\mathbb{R}^k_{\geq 0}$. A function $f: V \to \mathbb{R}$ is ***analytic on*** V if there exists an open neighborhood W of V in $\mathbb{R}^k$ and an analytic function $\tilde{f}: W \to \mathbb{R}$ such that $\tilde{f}|_V = f$. The ring of analytic functions on V is denoted by $\mathcal{O}^k(V)$.

In order to define the generalized analytic functions, we need to take in consideration the difference between the "analytic variables" and the "generalized variables" (see 2.1.2). This motivates the following notations:

Notation 3.2. Let $k,m,n\in\mathbb{N}$, $A\subset\{1,\dots,k\}$ and $\xi=(\xi_1,\dots,\xi_k)\in(0,\infty)^k$.

1) We put

$$I_{A,\xi}:=B_1\times\cdots\times B_k\subset\mathbb{R}^k \text{ where } B_i=\begin{cases}[0,\xi_i) & \text{if } i\in A\\ (-\xi_i,\xi_i) & \text{if } i\notin A\end{cases}.$$

For a positive real number ε, we also write $I_{A,\varepsilon}$ for $I_{A,(\varepsilon,\dots\varepsilon)}$.

2) Let $\mathfrak{S}_k$ the group of permutations of $\{1,\dots,k\}$ and $\mathfrak{S}_{m,n}$ the subgroup of elements of $\mathfrak{S}_k$ which leave invariant the subsets $\{1,\dots,m\}$ and $\{m+1,\dots,m+n\}$. For every $\sigma\in\mathfrak{S}_k$ we also denote by σ the map $\mathbb{R}^k\to\mathbb{R}^k$: $(w_1,\dots w_k)\mapsto\left(w_{\sigma(1)},\dots,w_{\sigma(k)}\right)$.

3) From now on, we put $m=|A|$ and $n=k-m$. We put $I_{m,n,\xi}=I_{\{1,\dots,m\},\xi}$. Let $\mathfrak{S}_A$ be the subset of permutations $\sigma\in\mathfrak{S}_k$ such that $\sigma(A)=\{1,\dots,m\}$.

4) If $p=(p_1,\dots,p_k)\in\mathbb{R}^k_{\geq 0}$, we put

$$A(p):=\{i\in\{1,\dots,k\}:p_i=0\},\ m_p:=|A(p)|,\ n_p=k-m_p.$$

5) If $p\in\mathbb{R}^k_{\geq 0}$ and $\sigma\in\mathfrak{S}_{A(p)}$ then $\theta_{p,\sigma}$ is the map defined on $\mathbb{R}^{m_p}_{\geq 0}\times\mathbb{R}^{n_p}$ by

$$(w_1,\dots,w_k)\longmapsto p+\sigma(w_1,\dots,w_k)=\left(p_1+w_{\sigma(1)},\dots,p_k+w_{\sigma(k)}\right).$$

For every $\delta>0$ small enough, the map $\theta_{p,\sigma}$ restricts to an homeomoprhism from $I_{m_p,n_p,\delta}$ to $p+I_{A(p),\delta}$ with $\theta_{p,\sigma}(0)=p$.

Definition 3.3. Let V be an open subset of $\mathbb{R}^k_{\geq 0}$ and $p\in V$. A function $f\colon V\to\mathbb{R}$ is said to be ***generalized analytic*** (or ***$\mathcal{G}$-analytic*** for short) ***at*** p if there exists $\delta>0$, a convergent series $s\in\mathbb{R}\{X^*,Y\}_{m_p,n_p,\delta}$ and a permutation $\sigma\in\mathfrak{S}_{A(p)}$ such that

i) $\left(p+I_{A(p),\delta}\right)\subset V$,
ii) $s|_{I_{m_p,n_p,\delta}}=f|_{\left(p+I_{A(p),\delta}\right)}\circ\theta_{p,\sigma}$.

We say that f is ***$\mathcal{G}$-analytic*** on V if f is $\mathcal{G}$-analytic at every point $p\in V$. The ring of $\mathcal{G}$-analytic functions on V is denoted by $\mathcal{G}^k(V)$.

Notice that, although the definition of generalized analytic function at a point p does not depend on the choice of the permutation $\sigma\in\mathfrak{S}_{A(p)}$, the series $s\in\mathbb{R}\{X^*,Y\}$ depends on σ. In fact, for two such permutations, the corresponding series differ by the action on the variables (X,Y) of an element of the subgroup $\mathfrak{S}_{m_p,n_p}$ of $\mathfrak{S}_k$. Disregarding this ambiguity, we will say that the series s introduced in Definition 3.3 is the ***Taylor expansion of** f **at*** p, denoted by $T_p(f)$. Using Theorem 2.6, it is not difficult to prove the following:

Proposition 3.4. *The map $f\mapsto T_p(f)$ induces an isomorphism between the $\mathbb{R}$-algebra of germs of generalized analytic functions at p and $\mathbb{R}\{X^*,Y\}_{m_p,n_p}$.*

3.2. **The local models $\mathbb{O}^k$ and $\mathbb{G}^k$.** We introduce two locally ringed spaces on the same topological space $\mathbb{R}^k_{\geq 0}$, intended to be the local models of the manifolds considered later. They are both equipped with a sheaf of rings of continuous real functions, with the restriction of functions as restrictions morphisms .

Notation 3.5. 1) The assigment which associates to every open subset V of $\mathbb{R}^k_{\geq 0}$ the ring $\mathcal{O}^k(V)$ (*resp.* the ring $\mathcal{G}^k(V)$), together with the usual restriction maps on open subsets, defines the sheaf $\mathcal{O}^k$ (*resp.* $\mathcal{G}^k$) over $\mathbb{R}^k_{\geq 0}$.

2) In the following definitions or statements, the letter $\mathcal{A}$ denotes the letter $\mathcal{O}$ or $\mathcal{G}$, and $\mathbb{A}$ denotes the letter $\mathbb{O}$ or $\mathbb{G}$ accordingly.

Definition 3.6. The ***local model of $\mathcal{A}$-manifolds of dimension*** k is the locally ringed space $\mathbb{A}^k = \left(\mathbb{R}^k_{\geq 0}, \mathcal{A}^k\right)$.

Remark 3.7. The stalks of the sheaves $\mathcal{A}^k$ at every point $p \in \mathbb{R}^k_{\geq 0}$ are actually local rings. In fact, for $\mathcal{A} = \mathcal{O}$, the stalk at every point is isomorphic to the ring $\mathbb{R}\{X_1, ..., X_k\}$ of convergent power series. For $\mathcal{A} = \mathcal{G}$, the stalk at a point p is isomorphic to $\mathbb{R}\{X^*, Y\}_{m(p),n(p)}$ by Proposition 3.4. Hence the local models $\mathbb{A}^k$ are indeed locally ringed spaces.

3.3. Standard analytic manifolds, generalized analytic manifolds and their morphims.

3.3.1. *$\mathcal{O}$-manifolds and $\mathcal{A}$-manifolds.* Consider the category $\mathfrak{C}$ where

(1) The objects are the locally ringed spaces $X = (|X|, \mathfrak{C}_X)$ where $|X|$ is a topological space and $\mathfrak{C}_X$ is a sheaf of $\mathbb{R}$-algebras of continuous functions over $|X|$, such that for every $p \in |X|$, the stalk $\mathfrak{C}_{X,p}$ is a local $\mathbb{R}$-algebra.

(2) The morphisms between two objects $X = (|X|, \mathfrak{C}_X)$ and $Y = (|Y|, \mathfrak{C}_Y)$ are pairs $\left(\varphi, \varphi^\sharp\right)$ where $\varphi\colon |X| \to |Y|$ is an homeomorphism and $\varphi^\sharp : \mathfrak{C}_Y \to \varphi_* \mathfrak{C}_X$ is the associated morphism of sheaves determined by composition by φ; namely, if $f \in \mathfrak{C}_Y(V)$ is a section of $\mathfrak{C}_Y$ over the open subset $V \subset Y$, then $\varphi^\sharp(f) = f \circ \varphi \in \varphi_* \mathfrak{C}_X(V) = \mathfrak{C}_X\left(\varphi^{-1}(V)\right)$ (we usually drop the second component $\varphi^\sharp$ from the notation and say simply $\varphi : X \to Y$ is a morphism).

The manifolds studied in this work are the objects A of two subcategories $\mathcal{O}$ and $\mathcal{G}$ of $\mathfrak{C}$, such that A is locally isomorphic (in $\mathfrak{C}$) to one of the foregoing local models. Classically, the morphisms between manifolds are defined as morphims between objects of these categories.

Definition 3.8.

(1) An ***$\mathcal{A}$-manifold*** (or ***manifold of type*** $\mathcal{A}$) ***of dimension*** k is an object $M = (|M|, \mathcal{A}_M)$ in the category $\mathfrak{C}$, where:
   (a) $|M|$ is an Hausdorff space with a countable open basis,
   (b) Every point of $|M|$ has an open neighborhood isomorphic in $\mathfrak{C}$ to the restriction of the local model $\mathbb{A}^k|_V = \left(V, \mathcal{A}^k|_V\right)$ for some open subset $V \subset \mathbb{R}^k_{\geq 0}$.
   The $\mathcal{O}$-manifolds (respectively $\mathcal{G}$-manifolds) will be also frequently called *standard analytic manifolds* (respectively *generalized analytic manifolds*).

(2) Consider two M and N of the same type $\mathcal{A}$. A ***$\mathcal{A}$-morphism*** (or ***morphism***, for short) $\varphi\colon M \to N$ is a morphism between M and N in the category $\mathfrak{C}$.

(3) An ***open submanifold*** of a $\mathcal{A}$-analytic manifold $M = (|M|, \mathcal{A}_M))$ is the locally ringed space $(U, \mathcal{A}_M|_U)$, where U is an open subset of $|M|$. It is also obviously a $\mathcal{A}$-manifold.

The next example is intended to consider the functions associated to ***mixed series*** on $\mathcal{G}$-manifolds (see section 2.2).

Example 3.9. Consider the map $\Phi\colon \mathbb{R}^m_{\geq 0} \times \mathbb{R}^n \to \mathbb{R}^m_{\geq 0} \times \mathbb{R}^n_{>0} \subset \mathbb{R}^{m+n}_{\geq 0}$ defined by

$$\Phi(x,y) = (x, \mathrm{e}^{y_1}, \ldots, \mathrm{e}^{y_n}),\ x \in \mathbb{R}^m_{\geq 0}, y \in \mathbb{R}^n.$$

Consider the inverse image $\mathcal{G}^{m,n}$ of the sheaf $\mathcal{G}^{m+n}|_{\mathbb{R}^m_{\geq 0}\times\mathbb{R}^n_{>0}}$ by the continuous map Φ. Then the $\mathcal{G}$-manifold $\left(\mathbb{R}^m_{\geq 0} \times \mathbb{R}^n, \mathcal{G}^{m,n}\right)$ is called the (m,n)-***mixed local model***.

Definition 3.10. Let $M = (|M|, \mathcal{A}_M)$ be a $\mathcal{A}$-manifold and $p \in |M|$. A ***local chart at*** p is a pair (U, w) where U is an open neighborhood of p in $|M|$ and

$$w\colon U \to V = w(U),\ w(q) = (w_1(q), \ldots, w_k(q))$$

is a homeomorphism which induces an isomorphism between the $\mathcal{A}$-manifolds $M|_U = (U, \mathcal{O}_M|_U)$ and $\mathbb{A}^k|_V = \left(V, \mathcal{A}^k|_V\right)$.

The components $w_1, \ldots, w_k$ are called the ***local coordinates*** of M at p.

Notation 3.11. We can associate, to every point p of a $\mathcal{A}$-manifold and every local chart (U,w) at p, an integer which is the number of zero components of $w(p)$. It can be proved (via a simple argument involving Proposition 2.2 and the Invariance of Domain Theorem) that this number does not depend on the choice of the chart (U,w). Hence it will be denoted by m_p (and we put $n_p = k - m_p$) and called the *number of boundary components of M at p.*

For such a local chart (U,w), there exists $\sigma \in \mathfrak{S}_{A(w(p))}$ (cf. notations in 2.2) such that $\theta^{-1}_{p\sigma} \circ w = (x,y) = (x_1, ..., x_{m_p}, y_1, ..., y_{n_p})$ induces an isomorphism between the open submanifold U and an open neighbourhood of the origin of the (m_p, n_p)-mixed local model. We will say that $(U,(x,y))$ is a *local (mixed) chart centered at p.*

Thus we have:

Proposition 3.12. *Let $M = (|M|, \mathcal{A}_M)$ be a $\mathcal{A}$-manifold and $p \in |M|$. Then p has an open neighborhood isomorphic to $\mathbb{A}^{m_p} \times \mathbb{R}^{n_p}$.*

Proposition 3.13. *Let $M = (|M|, \mathcal{A}|_M)$ be a $\mathcal{A}$-manifold. Then M admits an* ***$\mathcal{A}$-atlas****,* i.e. *a family $\{(U_i, \varphi_i)\}_{i\in I}$ such that:*

1. *for every $i \in I$, U_i is an open subset of $|M|$ and $\varphi_i\colon U_i \to V_i = \varphi_i(U_i) \subset \mathbb{R}^k_{\geq 0}$ is an homeomorphism,*
2. *$M = \bigcup_{i\in I} U_i$,*
3. *for every $i, j \in I$, $\varphi_i \circ \varphi_j\colon \varphi_j(U_i \cap U_j) \to \varphi_i(U_i \cap U_j)$ is a $\mathcal{A}$-isomorphism.*

Conversely, if $|M|$ is a second countable Hausdorff topological space and $\{(U_i, \varphi_i)\}_{i\in I}$ is a family satisfying 1 to 3 above, then there exists a unique structure of $\mathcal{A}$-manifold $M = (|M|, \mathcal{A}_M)$ over $|M|$ such that this family is an $\mathcal{A}$-atlas.

We can now give the local expressions of the morphims between the standard analytic manifolds and between the $\mathcal{G}$-manifolds.

3.3.2. *Local expression of morphisms between two $\mathcal{A}$-manifolds of the same type.* Consider two $\mathcal{A}$-manifolds $M = (|M|, \mathcal{A}_M)$ and $N = (|N|, \mathcal{A}_N)$ of the same type and a continous map $f\colon |M| \to |N|$ which induces a morphism bewteen M and N. Let $p \in M$, $\varphi\colon M|_{U_p} \to \mathbb{A}^{m_p} \times \mathbb{R}^{n_p}$ a local chart of M centered at p and $\psi\colon N|_{V_{\varphi(p)}} \to \mathbb{A}^{m_{\varphi(p)}} \times \mathbb{R}^{n_{\varphi(p)}}$ a local chart of N centered at $\varphi(p)$. Consider the map $h\colon = \psi \circ f \circ \varphi^{-1}\colon \mathbb{R}^{m_p}_{\geq 0} \times \mathbb{R}^{n_p} \to \mathbb{R}^{m_{\varphi(p)}}_{\geq 0} \times \mathbb{R}^{n_{\varphi(p)}}$. Then :

(1) if $\mathcal{A}$ denotes $\mathcal{O}$, then h has an analytic extension on a neighborhood of $0 \in \mathbb{R}^{m_p+n_p}$.
(2) if $\mathcal{A}$ denotes $\mathcal{G}$, then each component of h is $\mathcal{G}$-analytic at $0 \in \mathbb{R}^{m_p+n_p}$.

These considerations lead us to the following important explicit description of morphisms and isomorphisms between $\mathcal{G}$-manifolds. This description, which is in a great part a direct corollary of Proposition 2.2, together with the Implicit Function Theorem and the Invariance of Domain Theorem, shows in particular that a morphism of $\mathcal{G}$-manifolds, when expressed in any coordinate system, must be of monomial type in the generalized variables.

Proposition 3.14. *Let $m, n, m', n' \in \mathbb{N}$, $k = m+n$, $k' = m'+n'$. Let U, V be open neighborhoods of the origin in $\mathbb{R}^m_{\geq 0} \times \mathbb{R}^n$ and $\mathbb{R}^{m'}_{\geq 0} \times \mathbb{R}^{n'}$ respectively. Let $h : U \to V$ be a continuous map with $h(0) = 0$, and $h = (h_1, \dots, h_{k'})$ be the components of h. Denote by $(x, y) = (x_1, \dots, x_m, y_1, \dots y_m)$ and $(z, w) = (z_1, \dots, z_{m'}, w_1, \dots, w_{n'})$ the coordinates in $\mathbb{R}^m_{\geq 0} \times \mathbb{R}^n$ and $\mathbb{R}^{m'}_{\geq 0} \times \mathbb{R}^{n'}$. Then (always with the convention that $\mathbb{A}^m$ denotes the local model for $\mathbb{A}$-manifolds, either $\mathbb{O}^m$ or $\mathbb{G}^m$):*

(1) *h induces a morphism $\left(h, h^\sharp\right) : \mathbb{A}^m \times \mathbb{R}^n|_{U_0} \to \mathbb{A}^{m'} \times \mathbb{R}^{n'}|_{V_0}$ (where U_0 and V_0 are neighborhoods of the origin in $\mathbb{R}^m_{\geq 0} \times \mathbb{R}^n$ and $\mathbb{R}^{m'}_{\geq 0} \times \mathbb{R}^{n'}$ respectively) if and only if each component h_j is $\mathcal{A}$-analytic at the origin in $\mathbb{R}^m_{\geq 0} \times \mathbb{R}^n$, $y \mapsto (h_{m'+1}(0, y), \dots, h_{k'}(0, y))$ is analytic from $\mathbb{R}^n$ to $\mathbb{R}^{m'}$ and, for $j = 1, \dots, m'$,*
$$h_j(x, y) = x^{\alpha^j} g_j(x, y) = x_1^{\alpha_1^j} \cdots x_m^{\alpha_m^j} g_j(x, y)$$
for a certain $\alpha^j \in [0, \infty)^m$ and $\mathcal{A}$-analytic functions g_j, with $g_j > 0$ in a neighborhood of the origin in $\mathbb{R}^m_{\geq 0} \times \mathbb{R}^n$.

(2) *Assume that $k = k'$ and that h induces a morphism $\left(h, h^\sharp\right) : \mathbb{G}^m \times \mathbb{R}^n|_U \to \mathbb{G}^{m'} \times \mathbb{R}^{n'}|_V$. Then $\left(h, h^\sharp\right)$ is an isomorphism in the category $\mathcal{G}$ (up to restriction of U, V to smaller neighborhoods) if and only if $m = m'$, $n = n'$, h is an homeomorphim, the map*
$$y \mapsto (h_{m'+1}(0, y), \dots, h_{k'}(0, y))$$
induces an analytic local isomorphism of $\mathbb{R}^n$ and, for every $j = 1, \dots, m'$,
$$z_j = h_j(x, y) = x_{i(j)}^{a_j} g_j(x, y)$$
with $a_j > 0$, g_j is an analytic function at 0 such that $g(x, y) > 0$ in a neighborhood of 0 in $\mathbb{R}^m_{\geq 0} \times \mathbb{R}^n$, and $j \mapsto i(j)$ is a permutation of $\{1, \dots, m\}$.

3.4. **Standardizable manifolds.** We introduce here a special class of $\mathcal{G}$-manifolds. The elements of this class are the $\mathcal{O}$-manifolds where the structural sheaf has been "enriched" by "adding" the generalized analytic functions to the analytic ones. Let us say at once that not every $\mathcal{G}$-manifold may be obtained that way (cf. Example 3.18 below).

Proposition 3.15. *Let $A = (|A|, \mathcal{O}_M)$ be an $\mathcal{O}$-manifold, and $\mathcal{U} = (U_i, \varphi_i)$ be an $\mathcal{O}$-atlas of A. Then the subsheaf $\mathcal{G}_A$ of the sheaf of continuous functions over $|A|$, whose sections over an open subset U of A is*
$$(f : U \to \mathbb{R}) \in \mathcal{G}_A(U) \Leftrightarrow f|_{U_i \cap U} \circ \varphi_i^{-1}|_{\varphi_i(U \cap U_i)} \in \mathcal{G}^k(\varphi_i(U \cap U_i)), \forall i \in I \text{ with } U \cap U_i \neq \emptyset,$$
does not depend on the chosen atlas $\mathcal{U}$ and endows $|A|$ with a structure of $\mathcal{G}$-analytic manifold $A^e = (|A|, \mathcal{G}_A)$. Moreover, the identity map on $|A|$ induces induces a

morphism $\mathrm{Id}\colon A^e \to A$ *in the category of locally ringed spaces. In that case, we say the* A^e *is the* ***enrichment*** *of the* $\mathcal{O}$*-manifold* A.

A typical example is of course given by the local models: for every $k \in \mathbb{N}$ and every open set $U \subset \mathbb{R}^k_{\geq 0}$, we have $(\mathbb{O}^k\mid_U)^e = \mathbb{G}^k\mid_U$.

Definition 3.16. Let $M = (|M|, \mathcal{G}_M)$ be a $\mathcal{G}$-manifold. We say that M ***is standardizable*** if there exists a $\mathcal{O}$-manifold A and a $\mathcal{G}$-isomorphism $\varphi\colon M \to A^e$. In this situation, the pair (A, φ) is called a ***standardization*** of M.

The enrichment is not a functor between the category of standard analytic manifolds to the category of generalized analytic manifolds. Consider for instance $]A = \mathbb{O}^2$, $B = \mathbb{O}^1$ and the $\mathbb{O}$-morphism

$$h\colon A \to B,\ h(x_1, x_2) = x_1 + x_2.$$

Then $A^e = \mathbb{G}^2$, $B^e = \mathbb{G}^1$ but $h : \left|\mathbb{G}^2\right| \to \mid \mathbb{G}^1 \mid$ does not induce a $\mathcal{G}$-morphism (by Proposition 3.14 on the local expression of morphisms). In fact, using again the same proposition, we can prove the following:

Proposition 3.17. *Let M, N be two standarizable $\mathbb{G}$-manifolds with standarizations (A, φ) and (B, ψ) respectively. Then, given a $\mathbb{O}$-morphism $(h, h^\sharp) : A \to B$, the lifting $\tilde{h} = \psi^{-1} \circ h \circ \varphi :\mid M \mid \to \mid N \mid$ induces a $\mathcal{G}$-morphism if and only if h is* locally of monomial type, *i.e., for any point $p \in\mid A \mid$ there is a local chart (U, x) of A centered p at and a local chart (V, y) of B centered at $h(p)$ such that each component $f_j = y_j \circ f$ is of monomial type in the x-coordinates:*

$$f_j = x^{\alpha^j} g_j, \alpha^j \in [0, \infty)^{\dim(A)}, g_j(0) \neq 0.$$

Obviously, every generalized manifold is locally standarizable: if (U, w) is a local chart on a $\mathcal{G}$-manifold M then, considering the image $w(U) \subset \mathbb{R}^k_+$ as an open $\mathcal{O}$-submanifold, $(w(U), w)$ is a standarization of U.

But this is not the case in the global setting, as the following example shows.

Example 3.18. Consider two copies of $\mathbb{R}_+ \times \mathbb{R}$, denoted by U_1 and U_2, with coordinates (x, y) and (z, w) respectively. Let α be a positive real number and consider C_α the topological space obtained as the quotient of the disjoint union of U_1 and U_2 under the relation

$$(x, y) \sim (z, w) \iff \begin{cases} x = z,\ y = 1/w, & \text{if } y > 0; \\ x = z^\alpha,\ y = 1/w & \text{if } y < 0. \end{cases}$$

It is easy to see that C_α is homeomorphic to the cylinder $\mathbb{S}^1 \times \mathbb{R}$. On the other hand, denoting by $U_{i,\alpha}$ the image of U_i into C_α for $i = 1, 2$, we have that

$$\{(U_{1\alpha}, (x, y)), (U_{2,\alpha}, (z, w))\}$$

is a topological atlas on C_α so that the change of variables determines an isomorphism between the open $\mathcal{G}$-submanifold $\mathbb{G}^1 \times (\mathbb{R} \setminus \{0\})$ and itself. Thus, we can endow C_α with a unique structure of $\mathcal{G}$-manifold $\mathcal{C}_\alpha = (C_\alpha, \mathcal{G}_{\mathcal{C}_\alpha})$ in such a way that that atlas is a $\mathcal{G}$-atlas.

Claim.- $\mathcal{C}_\alpha$ is standarizable if and only if $\alpha = 1$.

Moreover, if $\alpha \neq 1$ then there is no open neighborhood of the boundary ∂C_α which is a standarizable $\mathcal{G}$-manifold. The "if" part of the claim is quite clear: $\mathcal{C}_1$ is isomorphic to the (generalized) product manifold $\mathbb{G}^1 \times \mathbb{R}$, which is the enrichment of $\mathbb{O}^1 \times \mathbb{R}$. In order to prove the "only if" part of the claim we look at the following statement about $\mathcal{C}_\alpha$:

> (*) There exists a $\mathcal{G}$-functions h_i in a neighborhood of $\partial C_\alpha \cap U_{i,\alpha}$ for $i = 1, 2$, having the boundary ∂C_α as their zero locus and such that the quotients h_1/h_2 and h_2/h_1 remain bounded in a neighborhood of any point of ∂C_α, except possibly for a discrete subset of points.

The claim is finished once we prove the following results:

(1) If $\mathcal{C}_\alpha$ is standarizable then (*) holds: let (A, ϕ) be a standardization of $\mathcal{C}_\alpha$. Consider, for $i = 1, 2$, a point $q_i \in \partial A \cap \phi(U_{i,\alpha})$ and take an analytic coordinate chart (x_i, y_i) at q_i such that $\partial A = \{x_i = 0\}$. We consider h_i so that $h_i \circ \phi^{-1}$ is the analytic continuation in $\phi(U_{i,\alpha})$. Analyticity implies that at any point $q \in \partial A$ where defined, the function $h_i \circ \phi^{-1}$ writes in analytic coordinates (x, y) at q for which $x = 0$ is the boundary as

$$h_i \circ \phi^{-1}(x, y) = xH(x, y), \text{ where } H(0, y) \not\equiv 0$$

and this proves the required properties on the quotients h_1/h_2 and h_2/h_1.

(2) If $\alpha \neq 1$ then (*) does not hold: Suppose that (*) holds. Write the functions h_1, h_2 in the atlas $(U_{1,\alpha}, (x, y))$, $(U_{2,\alpha}, (z, w))$ as

$$h_1(x, y) = x^{\beta_1} H_1, \; h_2(z, w) = z^{\beta_2} H_2,$$

where $\beta_i \in \mathbb{R}_+$ and H_i is a $\mathcal{G}$-analytic function in a neighborhood of $\partial C_\alpha \cap U_{i,\alpha}$ such that $H_i|_{\partial C_\alpha} \not\equiv 0$ (thus, since this restriction is analytic, its zero locus is a discrete subset of $\partial C_\alpha \cap U_{i,\alpha}$). Now, consider an open set Ω^ϵ, for $\epsilon = +$ or $-$, contained in $\partial C_\alpha \cap \{\epsilon y > 0\}$ where neither H_1 or H_2 vanishes. Taking into account the expression of the change of variables between the two charts, we can write

$$h_1 = x^{\beta_1} H_1 = z^{\beta_1} H_1 \text{ in } \Omega^+,$$

$$h_1 = x^{\beta_1} H_1 = z^{\alpha\beta_1} H_1 \text{ in } \Omega^-.$$

If the condition about the quotients h_1/h_2 and h_2/h_1 is true in both open sets Ω^+ and Ω^- then we must have $\beta_2 = \beta_1 = \alpha\beta_1$, which implies $\alpha = 1$.

4. Local monomialization of generalized analytic functions

4.1. **Blowings-up on $\mathcal{G}$-manifolds.** The standard approach in the definition of the blowing-up of a point in a manifold consists in defining the blowing-up of a point in the local model of the manifold and then to use a system of coordinates. This method holds for $\mathcal{O}$-manifolds, but not for $\mathcal{G}$-manifolds. The main problem is that it is not possible for $\mathcal{G}$-manifolds to define the blowings-up independently of the system of coordinates, which leads us to define it *relatively to some coordinates*. However, this goal will be achieved if we consider only blowings-up with an *admissible center* Y inside a standarizable manifold M (defined in 3.4) with respect to a standarization $\varphi : M \to A$ of M that sends Y to an admissible center in the category $\mathcal{O}$.

Our definition proceeds in introducing the following notions:

- The admissible centers to be blown-up: in particular, the *standardizable* ones play a specific role in the process.

- The blowings-up of admissible centers on $\mathcal{O}$-manifolds, based on blowings-up of local models and local charts.
- The blowings-up of standardizable admissible centers on $\mathcal{G}$-manifolds.

4.1.1. *Admissible centers.*

Definition 4.1. Let $M = (|M|, \mathcal{A}_M)$ and $N = (|N|, \mathcal{A}_N)$ two $\mathcal{A}$-manifolds of the same type.

(1) A $\mathcal{A}$-morphism $\varphi\colon N \to M$ is a ***submanifold*** if φ is injective and if for each $p \in |N|$, the induced homomorphism of germs $\varphi_p^\sharp\colon \mathcal{G}_{M,\varphi(p)} \to \mathcal{G}_{N,p}$ is surjective.
(2) The submanifold $\varphi\colon N \to M$ is called ***closed*** if $\varphi(|N|)$ is a closed subset of $|M|$ and ***regular*** if $\varphi\colon\ |N| \to \varphi(|N|)$ is an homeomorphism.

Definition 4.2. Let $M = (|M|, \mathcal{A}_M)$ be an $\mathcal{A}$-manifold and $|Y|$ be a connected subset of $|M|$. Suppose that for every $p \in |Y|$ there exist a $\mathcal{A}$-local chart $(U_p, \varphi_p = (x_1, \ldots, x_k))$ at $p \in |Y|$ and $J_p \subset \{1, \ldots, k\}$ such that

$$\varphi_p (|Y| \cap U_p) = \{q \in U_p \colon x_j (q) = x_j (p), j \in J_p\}. \tag{4.1}$$

Let $l_p = k - \sharp J_p$, $\pi_p\colon\ (x_1, \ldots, x_k) \in \mathbb{R}^k \mapsto (x_j)_{j \notin J_p} \in \mathbb{R}^{l_p}$ and $\psi_p = \pi_p \circ \varphi_p$. Then the closed regular $\mathcal{A}$-submanifold $(Y, \mathcal{A}_Y)$ of M (for the inclusion map $i : |Y| \hookrightarrow |M|$) defined by the atlas $\{(V_p = |Y| \cap U_p), \psi_p\}_{p \in |Y|}$ is called an $\mathcal{A}$-***admissible center***.

For example, consider the subset Y in $\mathbb{R}^2_{\geq 0}$ given by the graph of the function $x \mapsto x^\lambda$ with $\lambda \neq 1$. Then Y is a $\mathcal{G}$-admissible center of $\mathbb{G}^2$ but it is not a $\mathcal{O}^2$-admissible center of $\mathbb{O}^2$.

Thus, in the case of $\mathcal{G}$-manifolds, we need an improvement of the foregoing notion.

Definition 4.3. Let M be a $\mathcal{G}$-manifold and Y be an admissible center. We say that ***the pair*** (M, Y) ***is standardizable*** if there exists a standardization $\varphi : M \to A$ such that $|Z| = \varphi(|Y|) \subset |A|$ has the property 4.1. In this situation, we say that φ is a standardization of the pair (M, Y).

4.1.2. *Blowing-up of a point of the standard model $\mathbb{R}^k$ of analytic manifold (without boundary).* Consider a point $p \in \mathbb{R}^k$ and put

$$\pi_p^k\colon \mathbb{R}_{\geq 0} \times \mathbb{S}^{k-1} \to \mathbb{R}^k, \quad (r, (x_1, \ldots, x_k)) \mapsto p + (rx_1, \ldots, rx_k).$$

Let $\widetilde{\mathbb{R}^k} := \mathbb{O}^1 \times \mathbb{S}^{k-1}$ the product in the category $\mathcal{O}$ of $\mathbb{O}^1$ and $\mathbb{S}^{k-1}$. The maps π_p^k is continuous, proper and it induces a morphism from $\widetilde{\mathbb{R}^k}$ to $\mathbb{R}^k$ and an isomorphism from $\widetilde{\mathbb{R}^k} \setminus \{0\} \times \mathbb{S}^{k-1}$ to $\mathbb{R}^k \setminus \{0\}$. The pair $\left(\widetilde{\mathbb{R}^k}, \pi_p^k\right)$ is called the ***blowing-up in*** $\mathbb{R}^k$ ***with center the point*** p (in fact, this is the *polar blowing-up* in contrast with the usual *projective blowing-up* which does not create any boundary; we will only use here the polar blowing-up, so we will call it simply blowing-up). As usual, $\left(\pi_p^k\right)^{-1}(p)$ is called the ***exceptional divisor*** of this blow-up. It is the boundary of the $\mathcal{O}$-manifold $\widetilde{\mathbb{R}^k}$.

4.1.3. *Blowing-up of a point of the standard model $\mathbb{O}^m \times \mathbb{R}^n$ of $\mathcal{O}$-manifold.* Let $p \in \mathbb{R}^m_{\geq 0} \times \mathbb{R}^n$. Then $\widetilde{\mathbb{R}^{m,n}_p} = \left(\pi^{m+n}_p\right)^{-1}\left(\mathbb{R}^m_{\geq 0} \times \mathbb{R}^n\right) \subset \mathbb{R}_{\geq 0} \times \mathbb{S}^{m+n-1}$ is a regular closed submanifold of $\widetilde{\mathbb{R}^{m+n}}$, and the map

$$\pi^{m,n}_p = \pi^{m+n}_p|_{\widetilde{\mathbb{R}^{m,n}_p}} \colon \widetilde{\mathbb{R}^{m,n}_p} \to \mathbb{O}^m \times \mathbb{R}^n,$$

is a proper morphism and a local isomorphism at any point except for those in the regular $\mathcal{O}$-submanifold $\left(\pi^{m,n}_p\right)^{-1}(p)$ or $\widetilde{\mathbb{R}^{m,n}_p}$ (which is of codimension 1). The pair $\left(\widetilde{\mathbb{R}^{m,n}_p}, \pi^{m,n}_p\right)$ is called the ***blowing-up of*** $\mathbb{O}^m \times \mathbb{R}^n$ ***at the point*** p, and $\left(\pi^{m,n}_p\right)^{-1}(p)$ is called the ***exceptional divisor*** of this blowing-up.

4.1.4. *Blowing-up at a point of an $\mathcal{O}$-manifold.* In a natural way, we define the blowing-up at a point of an $\mathcal{O}$-manifold by carrying the blowing-up at 0 of the local model $\mathbb{O}^m \times \mathbb{R}^n$.

Consider an $\mathcal{O}$-manifold $(A, \mathcal{O}_A)$, a point $p \in |A|$ and a local chart $\varphi\colon U \to \mathbb{R}^m_{\geq 0} \times \mathbb{R}^n$ centered at p. Let $\widetilde{A}(\varphi)$ be the $\mathcal{O}$-manifold obtained by taking the quotient of the disjoint union of $A \setminus \{p\}$ and $\widetilde{\mathbb{R}^{m,n}}$ under the relation

$$a \sim q \text{ iff } a \in U, q \in \widetilde{\mathbb{R}^{m,n}} \text{ and } \varphi(a) = \pi^{m,n}_0(q).$$

The map $\pi^A_p(\varphi) : \widetilde{A}(\varphi) \to A$ defined as the inclusion on $A \setminus \{p\}$ and by $\varphi^{-1} \circ \pi^{m,n}_0$ on $\widetilde{\mathbb{R}^{m,n}}$ is a proper analytic morphism and induces an isomorphism from $\widetilde{A} \setminus (\pi^A_p)^{-1}(p)$ to $A \setminus \{p\}$. A ***blowing-up of*** A ***at*** p is any pair $\left(\widetilde{A}, \pi^A_p\right)$ where $\widetilde{A}$ is an $\mathcal{O}$-manifold and $\pi^A_p \colon \widetilde{A} \to A$ is a morphism such that there exists an isomorphism $\theta\colon \widetilde{A} \to \widetilde{A}(\varphi)$, with $\pi^A_p(\varphi) \circ \theta = \pi^A_p$.

It may be proved that two such pairs associated to different charts are isomorphic.

4.1.5. *Blowing-up an admissible center of a $\mathcal{O}$-manifold.* Consider an admissible center $Z = (|Z|, \mathcal{O}_Z)$ of an $\mathcal{O}$-manifold $A = (|A|, \mathcal{O}_A)$. Every point $p \in |Z|$ has an open neighborhood in A which is a *normalizing domain* for the submanifold Z, that is

$$A|_U \simeq Z|_U \times \mathbb{O}^{m(U)} \times \mathbb{R}^{n(U)}.$$

Since $|Z|$ is a closed subset of $|A|$, if $p \notin |Z|$ we consider an open neighborhood U of p which does not intersect $|Z|$ (so that $Z|_U = \emptyset$). Define $\pi^A_Z(U) := \left(\mathrm{id}, \pi^{m(U)\times n(U)}_0\right) : Z|_U \times \widetilde{\mathbb{O}^{m(U)} \times \mathbb{R}^{n(U)}} \to Z|_U \times \mathbb{O}^{m(U)} \times \mathbb{R}^{n(U)}$.

For two normalizations $\varphi_U : U \cap |Z| \to Z|_U \times \mathbb{O}^{m(U)} \times \mathbb{R}^{n(U)}$ and $\varphi_V : U \cap |Z| \to Z|_V \times \mathbb{O}^{m(V)} \times \mathbb{R}^{n(V)}$ the isomorphism $\theta = \varphi_V \circ \varphi_U^{-1}$ lifts to an isomorphism $\tilde{\theta}$ of the blown-up spaces. Consider now the topological space

$$\left|\widetilde{A}\right| = \coprod_{U \text{ normalizing chart}} |U \cap Z| \times \widetilde{\mathbb{O}^{m(U)} \times \mathbb{R}^{n(U)}} \Big/ \sim$$

where two elements $p = (a, x) \in |U \cap Z| \times \widetilde{\mathbb{O}^{m(U)} \times \mathbb{R}^{n(U)}}$ and $q = (b, y) \in |V \cap Z| \times \widetilde{\mathbb{O}^{m(V)} \times \mathbb{R}^{n(V)}}$ are equivalent for the relation $\sim$ iff $\tilde{\theta}(a, x) = (b, y)$. The collection of charts $\{|Z \cap U|, \mathcal{U}(U)\}$, where U describes the normalizing charts of Z, endows $\left|\widetilde{A}\right|$ with a structure of $\mathcal{O}$-manifold.

A ***blowing-up of*** A ***with center*** Z is a pair (B,π) where B is an $\mathcal{O}$-manifold and $\pi\colon B\to A$ is a morphism for which there exists an isomorphism $\theta\colon B\to\widetilde{A}$ such that $\pi_Z^A\circ\theta=\pi$. A ***local blowing-up on*** A is a pair (B,π) where B is an $\mathcal{O}$-manifold and $\pi=i\circ\tau\colon B\to U\hookrightarrow A$ where $i\colon U\hookrightarrow A$ is an open submanifold and $\tau\colon B\to U$ is a blowing-up of A with an admissible center $Z\subset U$ closed in U.

4.1.6. *Blowing-up on $\mathcal{G}$-manifolds.* Unlike the analytic framework, the definition of the blowing-up in a generalized manifold with a closed admissible center may depend on the choice of the local chart at p. For instance, a (a priori) good candidate for the blowing-up of the origin in $\mathbb{G}^2$ could be the enrichment $(\widetilde{\mathbb{R}^{2,0}})^e$ of the blowing-up of the origin of the standard local model. But a change of coordinates in $\mathbb{G}^2$ of the form $\theta: x\mapsto x^\lambda, y\mapsto y$ with $\lambda\neq 1$ can not be lifted to an isomorphism on the blown-up space $(\widetilde{\mathbb{R}^{2,0}})^e$.

This inconvenience forces us to specify a standarization of an open submanifold containing the center which is to be blown-up. And, consequently, we have the possibility that we do not have *a priori* the possibility to blow-up the center if such a standarization does not exists (see Example 3.18 above).

Consider a $\mathcal{G}$-manifold M and a closed (connected) admissible center $Y\subset M$ such that the pair (M,Y) is *standardizable* by means of the standardization $\varphi: M\to A$. The manifold $Z=\varphi(Y)$ is an admissible center in the $\mathcal{O}$-manifold A. Let $\left(\tilde{A},\pi_Z^A\right)$ be a blowing-up on A with center Z, and $\tilde{M}$ be the enrichment of $\tilde{A}$. Let $\tilde{\varphi}\colon\tilde{M}\to\tilde{A}$ be the morphism induced by the identity map. Moreover, being locally of monomial type, the morphism π_Y^A lifts to a morphism $\pi_Y^M:\tilde{M}\to M$. The triple $\left(\tilde{M},\pi_Y^M,\tilde{\varphi}\right)$ is called the ***blowing-up of*** M ***with center*** Y ***relatively to the standardization*** φ. The inverse image $D=\left(\pi_Y^M\right)^{-1}(Y)$ is a regular manifold of codimension 1, called the ***exceptional divisor*** of this blowing-up. We have the relation $\pi_Z^A\circ\tilde{\varphi}=\varphi\circ\pi_Y^M$. Finally, π_Y^M is a proper, surjective morphism which restricts to an isomorphism from $\tilde{M}\setminus D$ to $M\setminus Y$.

Consider now a locally closed subset Y of a $\mathcal{G}$-manifold M. A ***local blowing-up on*** M ***with center*** Y is any triple (N,π,φ), where N is a $\mathcal{G}$-manifold,

$$\pi=i\circ\tau\colon N\to U\hookrightarrow M,$$

where $i\colon U\hookrightarrow M$ is an open submanifold such that Y is closed in U, and $\tau\colon N\to U$ is a blowing-up morphism on U with the admissible center $Y\subset U$.

Example 4.4. A useful situation occurs when Y is of codimension 2. Considering a normalizing chart of Y in some open neighborhood of $p\in Y$, we can assume that $U=\mathbb{G}^m\times\mathbb{R}^n$, and that $Y\mid_U=\{x_i=x_j=0\}$ for two generalized coordinates x_i,x_j of $\mathbb{G}^m\times\mathbb{R}^n$. Let $\gamma>0$ and consider the standardization of the pair (U,Y) given by

$$\varphi_\gamma\colon U\to\mathbb{O}^m\times\mathbb{R}^n,\ \varphi_\gamma(x,y)=\left(\ldots,x_i^\gamma,\ldots,x_j,\ldots,y\right).$$

Let $\pi\colon\tilde{M}\to U\hookrightarrow M$ the local blowing-up with center $Y|_U$ relatively to the standardization φ_γ. If, for instance, the two variables x_i,x_j are generalized, then $\tilde{M}$ is covered by two charts (x',y') and (x'',y'') with values in $\mathbb{R}^m_{\geq 0}\times\mathbb{R}^n$ such that the expression of the blowing-up morphism is

$$\pi\left(x',y'\right)=\left(x_1',\ldots,x_i',\ldots,\left(x_i'\right)^{\gamma}x_j',\ldots,y'\right)$$
$$\pi\left(x'',y''\right)=\left(x_1'',\ldots,x_j'',\ldots,\left(x_j''\right)^{1/\gamma}x_i'',\ldots,y''\right)$$

Actually, our monomialization process is entirely based on blowings-up with centers of codimension at most 2. We may observe that these local expressions of blowings-up correspond to the transformations ς_{ij}^{γ} and $\varsigma_{ji}^{1/\gamma}$ of 2.1.4.

Example 4.5. With our definitions, the blowing-up with an admissible center of codimension one may produce some effect, as opposed to the situation in standard analytic manifolds without boundary where the used blowing-up is a projective one. Suppose for instance that Y is an admissible center of codimension 1 with normalizing chart in some open set U

$$\varphi : U \to Y\mid_U \times \mathbb{R}$$

then the local blowing up with center $Y\mid_U$ (with respect to the foregoing normalizing chart considered as a standardization) is the following morphism

$$\pi : Y\mid_U \times \mathbb{G}^1 \bigsqcup Y\mid_U \times \mathbb{G}^1 \to Y\mid_U \times \mathbb{R}$$

defined by $\pi(q,x) = (q,\pm x)$ where the sign of x is taken different in the two different copies $Y\mid_U \times \mathbb{G}^1$. Geometrically, we add two new boundary components of codimension one, $\{x=0\}$ in each copy, the exceptional divisor.

4.2. **Local monomialization of $\mathcal{G}$-functions.** The following definition is inspired by Hironaka's notion of "voûte étoilée" [Hir73, Chapter 3]:

Definition 4.6. Let M be a $\mathcal{G}$-manifold and $p \in |M|$. A ***proper *-neighborhood of*** p is a finite family $\Sigma = \{\pi_j : W_j \to M, L_j\}_{j\in J}$ where

(1) each π_j is the composition of a finite sequence of finitely many local blowings-up (with admissible centers):

$$\pi_j \colon W_j = W_{j,n_j} \overset{\pi_{j,n_j}}{\to} W_{j,n_j-1} \overset{\pi_{j,n_j-1}}{\to} W_{j,n_j-2} \to \cdots \overset{\pi_{j,1}}{\to} W_{j,0} = M$$

(2) each L_j is a compact subset of $|W_j|$ such that $\bigcup_{j\in J}\pi_j\left(L_j\right)$ is a compact neighborhood of p in $|M|$.

We can now state our main result:

Theorem 4.7. (**Local monomialization of $\mathcal{G}$-functions**) *Let M be a generalized analytic manifold, $f \in \mathcal{G}(M)$ and $p \in M$. Then there exists a proper *-neighborhood $\Sigma = \{\pi_j \colon W_j \to M, L_j\}_{j\in J}$ of p such that for all $j \in J$, $f\circ\pi_j \colon W_j \to \mathbb{R}$ is locally monomial at every point of L_j. Moreover, we may supposed that the admissible center of every local blowing-up involved in Σ is of codimension ≤ 2.*

The proof is achieved in two main steps:

(1) Reducing the function f into a Weierstrass polynomial.
(2) Proving the result for Weierstrass polynomials.

4.2.1. *The reduction to the case of a Weierstrass polynomial.* Consider a $\mathcal{G}$-manifold M, $p \in |M|$ and a $\mathcal{G}$-analytic function f at p. Let $(U, \varphi = (x, y))$ be a chart of M centered at p. We can define $b\,(f, p, (U, \varphi)) :\ = b\,(s)$, where $s \in \mathbb{R}\{X^*, Y\}$ is the Taylor expansion of f at p with respect to the coordinates (x, y) and s is considered as an element in $\mathbb{R}\{Y\}[[X^*]]$. It is not difficult to prove that $b\,(f, p, (U, \varphi))$ does not depend on the local chart (U, φ). We let $b\,(f, p) = (b_1\,(f, p), b_2\,(f, p)) \in \mathbb{N}^2$ be the invariant of the $\mathcal{G}$-function f at p. Hence the data $I\,(f, p) = (m_p, n_p, b\,(f, p)) \in \mathbb{N}^4$ is a well defined numerical invariant depending only on f and p (actually, m_p and n_p depends only on p and M). Recall also that $I(f, p)$ is upper semicontinuous (for the lexicographic order in $\mathbb{N}^4$) as a function of the point p.

The next statement explains how the invariant $I\,(f, p)$ can be lowered when $b_2\,(f, p) > 0$:

Proposition 4.8. *Let $f \in \mathcal{G}\,(M)$ and $p \in |M|$, and assume that $b_2\,(f, p) > 0$. Then there exists a local blowing-up $\pi \colon \tilde{M} \to M$ with admissible center Y through p, of codimension 2, such that, for every point $q \in \pi^{-1}\,(p)$, the function $\tilde{f} = f \circ \pi \in \mathcal{G}(\tilde{M})$ satisfies*

$$I(\tilde{f}, q) < I\,(f, p)\,.$$

Proof. Consider a local chart $(U, (x, y))$ centered at p with $x = (x_1, \ldots, x_m)$, $y = (y_1, \ldots, y_m)$, $m = m_p$, $n = n_p$, and let $s \in \mathbb{R}\{X^*, Y\}_{m,n}$ be the Taylor expansion of f in these coordinates. Then $b\,(f, p) = b\,(s)$. By Proposition 2.5 there exists $\gamma > 0$ and two different indices $i, j \in \{1, \ldots, m\}$ such that the transformations ζ_{ij}^{γ} and $\zeta_{ji}^{1/\gamma}$ applied to s gives series with smaller b-invariant.

Hence if we consider the closed admissible center $Y = \{x_i = x_j = 0\}$ inside U, the standardization of the pair (U, Y) given by

$$\varphi \colon U \to \mathbb{O}^m \times \mathbb{R}^n, \varphi\,(x, y) = (\ldots, x_i^{\gamma}, \ldots, x_j, \ldots, y)$$

and the local blowing-up morphism $\pi \colon \tilde{M} \to U \hookrightarrow M$ with center Y associated with the standardization φ, straightforward computations show the following:

(1) If $q \in \pi^{-1}\,(p)$ is the origin of one of the two canonical charts of $\tilde{M}$ which describes π (cf. Example 4.4), then $b\,(f \circ \pi, q) < b\,(f, p)$.
(2) If q is any other point of $\pi^{-1}\,(p)$, then $m_q\,(f \circ \pi) = m_p\,(f) - 1$ and hence $I\,(f \circ \pi, q) < I\,(f, p)$.

□

Using the fact that $I(f, p)$ is upper semicontinuous and properness of the blowing-up morphism, this result implies that it is enough to prove our main theorem when $b_2\,(f, p) = 0$. But in this situation, we have two possibilities:

(1) $n_p = 0$. In that case, f is of monomial type at p.
(2) $n_p > 0$. There exists a local chart (U, φ) centered at p, $\alpha \in [0, \infty)^{m_p}$ and $g \in \mathcal{G}_M\,(U)$ such that

$$f\,(x, y) = x^{\alpha} g\,(x, y)$$

with $g\,(0, y) \not\equiv 0$ in $\varphi\,(U)$. There exists a change of coordinates involving only the y variables which transforms g in a function regular in the last variable y_{n_p}.

Hence the reduction of Theorem 4.7 to the case of a distinguished polynomial is a consequence of Weierstrass Preparation Theorem, together with the following easy fact:

Assertion. *Let $h = x^\alpha$ be a monomial in the generalized variables and let $\pi : \widetilde{M} \to M$ be a local blowing-up with admissible center. Then, at any point $q \in \pi^{-1}(p)$, the Taylor expansion of the total transform $h \circ \pi$ with respect to any choice of local coordinates (z, w) centered at q (z being the generalized variables) is of monomial type of the form $Z^\beta U(Z, W)$ with $U(0, 0) \neq 0$.*

4.2.2. *The case of a Weierstrass polynomial.* Consider a $\mathcal{G}$-manifold M of dimension k, $\dim M = k \geq 2$ and $f \in \mathcal{G}(M)$. Assume that the main theorem is true in dimension less than k (the result for $k = 1$ being trivial). Moreover, assume that the number m_p of boundary components of M at p is less than k and that there exists a local chart $(U, \varphi = (\underline{x}, y))$ where y is an analytic variable such that

$$f(\underline{x}, y) = y^d + a_1(\underline{x})\, y^{d-1} + \cdots + a_d(\underline{x})$$

with $a_i \in \mathcal{G}(U)$ and $a_i(p) = 0$ for all i.

The rest of the proof follows the main lines of the proof of Theorem 4.4 in [BM88, p.24] and [RSW03, Prop. 3.8]. It consists mainly in lowering the order d by convenient blowings-up.

If $d = 1$, the change of coordinates $x_1 = x$, $y_1 = y - a_1(\underline{x})$ gives a new local chart for which f is of monomial type at p.

If $d > 1$, the classical Tschirnhausen change of coordinates $y \rightsquigarrow y - \frac{a_1(d)}{d}$ leads to the following expression for f:

$$f(\underline{x}, y) = y^d + b_2(\underline{x})\, y^{d-2} + \cdots + b_d(\underline{x}).$$

The hypothesis induction made on the dimension k applied to the function $\prod_{i=2, b_i \neq 0}^{d} b_i$ implies that there exists a *-neighborhood of p,

$$\Sigma = \left\{\pi_j \colon W_j = W_j' \times (-\delta, \delta) \to M, L_j = L_j' \times [-\delta/2, \delta/2]\right\}$$

such that for every j, the local expression of f at any point $(q, 0) \in L_j' \times \{0\}$ is

$$(f \circ \pi_j)(x', t) = t^d + (x')^{\alpha_2}\, u_2'(x')\, t^{d-2} + \cdots + (x')^{\alpha_d}\, u'(x')$$

where $u_i'(q, 0) \neq 0$ and $\alpha_i \in [0, \infty)^{k-1}$.

In that case, after possible blowings-up with (codimension one) centers at the coordinate hyperplanes $x_i' = 0$ (as in Example 4.5), we can suppose that the number of boundary components of q in W_j' is maximal, equal to $m_q = k - 1$. In this case, $\left(b_l \circ \pi_j'\right)^{1/l} = (x')^{\alpha_l/l}\, (u')_l^{1/l}$ is a $\mathcal{G}$-analytic function. The foregoing argument applied to the product of all nonzero functions and of all the nonzero differences among the family $\left\{\left(b_l \circ \pi_j'\right)^{1/l}\right\}_l$ leads to the case where the local expression of f is

$$f(x, y) = y^d + x^{\alpha_2} u_2(x)\, y^{d-2} + \cdots + x^{\alpha_d} u_d(x)$$

and the set of vectors $\{\alpha_l/l\}_{l=2,\ldots,d}$ is totally ordered for the division order (the proof of this statement is just the same as that of Lemma 4.7. in [BM88]). We follow the classical argument which consists in picking the index r such that $\alpha_r/r \leq \alpha_j/j$ for all j, $2 \leq j \leq d$, and an index l such that $\alpha_{r,l} \neq 0$. Consider the admissible center $Y = \{y = x_l = 0\} \subset U$, closed in U and of codimension 2, together with the standardization of the pair (U, Y) given by

$$\varphi \colon U \to \mathbb{R}_{\geq 0}^{k-1} \times \mathbb{R}, \varphi = \left(x_1, \ldots, x_{l-1}, x_l^{\alpha_{r,l}/r}, x_{l+1}, \ldots, x_{k-1}, y\right).$$

The corresponding (local) blowing-up $\pi_Y^U\colon \tilde{U} \to M$ with center Y and with respect to this standardization is such that $\tilde{U}$ is covered by two charts (x',y') and (x'',y'') with values in $\mathbb{R}_{\geq 0}^{k-1}\times\mathbb{R}$, so that the exceptional divisor $\left(\pi_Y^U\right)^{-1}(Y)$ has equations $\{x'_l=0\}$ and $\{y''=0\}$ and such that the morphism π_Y^U has the following expressions:

$$\pi_Y^U(x',y')=\left(x',(x'_l)^{\alpha_{r,l}/r}\,y'\right),\quad \pi_Y^U(x'',y'')=\left(x''_1,\cdots,(y'')^{r/\alpha_{r,l}}\,x''_l,\cdots,y''\right).$$

Let $q\in\left(\pi_Y^U\right)^{-1}(Y)$. There are three cases. In each of them, straightforward computations lead to the corresponding conclusions:

(1) q is the origin of the chart (x'',y''). Then $f\circ\pi_Y^U$ is equal to $(y'')^d$ times a unit, thus of monomial type.
(2) q is in the domain of the chart (x',y') but not the origin of the chart. Then $f\circ\pi_Y^U$ is the product of a monomial and a function which is regular in y of order less that d. This is the case where Tschirnhausen transformation is helpful.
(3) q is the origin of the chart (x',y'). Then $f\circ\pi_Y^U$ is equal to the product of a power of the variable x'_l and a Weierstrass polynomial in y'. In this polynomial, the coefficient $(x')^{\alpha'_r}u'_r(x')$ of $(y')^{d-r}$ is such that $\alpha'_{r,l}=0$ and $\alpha'_{r,i}=\alpha_{r,i}$ if $i\neq l$. Thus, if $\alpha_{r,i}=0$ for $i\neq l$, we are led to a function regular in y' of order $d-r<d$. Otherwise, if $\alpha_{r,i}\neq 0$ for some $i\neq r$, making a local blowing-up with center $\{y'=x'_i=0\}$, the same arguments lead to one of the favorable situations (either of monomial type or regular in an analytic variable of order less that d) except, possibly, in the case where we are placed at the origin of the first chart of this new blowing-up. But in that case, the exponent $\alpha_{r,i}$ vanishes. A simple induction argument allows to conclude.

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