ON RESTRICTED ANALYTIC GRADIENTS ON ANALYTIC ISOLATED SURFACE SINGULARITIES

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ABSTRACT. Let $(X, \mathbf{0})$ be a real analytic isolated surface singularity at the origin $\mathbf{0}$ of $\mathbb{R}^n$ and let $\mathbf{g}$ be a real analytic riemannian metric at $\mathbf{0} \in \mathbb{R}^n$. Given a real analytic function $f_0 : (\mathbb{R}^n, \mathbf{0}) \to (\mathbb{R}, 0)$ singular at $\mathbf{0}$, we prove that the gradient trajectories for the metric $\mathbf{g}|_{X\setminus\mathbf{0}}$ of the restriction $(f_0|_X)$ escaping from or ending up at $\mathbf{0}$ do not oscillate. Such a trajectory is thus a sub-pfaffian set. Moreover, in each connected component of $X \setminus \mathbf{0}$ where the restricted gradient does not vanish, there is always a trajectory accumulating at $\mathbf{0}$ and admitting a formal asymptotic expansion at $\mathbf{0}$.

1. INTRODUCTION

Let $f_0 : (\mathbb{R}^n, \mathbf{0}) \to \mathbb{R}$ be a real analytic function such that $\mathbf{0}$ is a critical point of f_0. Let $\mathbf{g}$ be a real analytic Riemannian metric defined in a neighborhood of $\mathbf{0}$. Let $\gamma : [0, +\infty[\to \mathbb{R}^n$ be a maximal solution of the gradient vector field $\nabla_{\mathbf{g}} f_0$ such that $\omega(\gamma) := \lim_{t\to\infty} \gamma(t) = \mathbf{0}$, and let $|\gamma| \subset \mathbb{R}^n$ be its image. We are not interested in any particular parameterization and we will simply call γ and $|\gamma|$ *a gradient trajectory*. Gradient trajectories $\gamma :]-\infty, 0] \to \mathbb{R}^n$ escaping from $\mathbf{0} = \lim_{t\to-\infty} \gamma(t) (= \alpha(\gamma))$ will be dealt with in same way in changing the sign of f_0.

The classical problem of the gradient is to know how, from an analytic point of view, does the solution $|\gamma|$ go to its limit point $\mathbf{0}$. For a long time remained undecided Thom's question famously known as *Thom's Gradient Conjecture*: does the trajectory have a *tangent* at its limit point, namely does $\lim_{t\to\infty} \frac{\gamma(t)}{|\gamma(t)|}$ exist ? (see [20] for an historical account by then). Eventually Kurdyka, Mostowski and Parusiński showed that the length of the radial projection of the curve $|\gamma|$ onto $\mathbb{S}^{n-1}$ is finite [17], thus proving Thom's Conjecture.

A much more challenging question about the behavior of gradient trajectories at their limit point is to decide whether they oscillate or not. A trajectory γ is (*analytically*) *non-oscillating* if given any (semi-)analytic subset $H \subset \mathbb{R}^n$ the intersection $|\gamma| \cap H$ has finitely many connected components.

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The plane case is well understood. In dimension $n \geq 3$, but a few special cases in dimension 3 [24, 9, 10], the non-oscillation of gradient trajectories is not known.

It is also worth recalling that in the case of real analytic vector fields on a 3-manifold, some very interesting properties of the Hardy field of the real analytic function germs along a given non-oscillating trajectory have been studied in [5], and thus allowing a partial reduction of the singularities result.

In the special case where a real analytic isolated surface singularity is foliated by gradient trajectories, the main result of this paper guarantees, that they do not oscillate. In fact, we will solve the following slightly more complicated problem.

Let $X \subset \mathbb{R}^n$ be a real analytic isolated surface singularity at the origin $\mathbf{0}$. Each connected component S_0 of the germ at $\mathbf{0}$ of $X \setminus \{\mathbf{0}\}$ is a real analytic submanifold of $\mathbb{R}^n$. The ambient metric $\mathbf{g}$ induces on S_0 an analytic Riemannian metric $\mathbf{h} := \mathbf{g}|_{S_0}$. The gradient vector field $\nabla_{\mathbf{h}}(f_0|_{S_0})$ of the restriction $f_0|_{S_0}$ of the function f_0 to S_0 is thus well defined. The vector field $\nabla_{\mathbf{h}}(f_0|_{S_0})$ is called *the restricted gradient vector field* of f_0 on S_0 and will be shortened as $\nabla_{\mathbf{h}}(f_0)$.

The main result of this paper is the following:

Theorem 1. *Let $\gamma : \mathbb{R}_{\geq 0} \to S_0$ be a trajectory of the restricted gradient vector field $\nabla_{\mathbf{h}} f_0$ accumulating at $\mathbf{0}$. Then γ is analytically non-oscillating.*

A pleasant and cheap consequence of this result is

Corollary 2. *The curve $|\gamma|$ is a* sub-pfaffian set.

A natural question is to ask whether there exists a trajectory γ of the restricted gradient accumulating to the origin to apply the main theorem to. Elementary topological arguments and some properties of a gradient vector field show that it is always the case:

Proposition 3. *There exists a non-stationary trajectory of $\nabla_{\mathbf{h}} f_0$ accumulating to $\mathbf{0}$ either in positive or in negative time.*

It is worth recalling that in the smooth context of an analytic gradient vector field on $(\mathbb{R}^n, \mathbf{0})$, there exists furthermore a real analytic curve through $\mathbf{0}$ invariant for the gradient vector field [20], *a real analytic separatrix*). For restricted gradients over isolated surface singularities we also prove here there always exists a *formal separatrix*:

Theorem 4. *Let S_0 be a connected component of $X \setminus \{\mathbf{0}\}$. If $\nabla_{\mathbf{h}} f_0$ does not vanish S_0, there exists a trajectory $\gamma : \mathbb{R}_{\geq 0} \to S_0$ of $\nabla_{\mathbf{h}} f_0$ accumulating to $\mathbf{0}$ which admits a formal asymptotic expansion at the origin such that the associated formal curve $\widehat{\Gamma}$ is invariant for the restricted gradient vector field.*

2. Structure of the proof

We first recall the case of an analytic Euclidean gradient in $\mathbb{R}^2$.

Trajectories of a real analytic vector field in $\mathbb{R}^2$ accumulating at the origin either "spiral" around the origin or have a tangent. In the latter case, a Rolle's type argument shows that the trajectory is non-oscillating (see [6]). The non-oscillation of a planar analytic gradient trajectory is thus given by the existence of a tangent. Although Thom's Gradient conjecture holds true ([17]), we sketch the usual simpler proof of the existence of a tangent in the plane case. This will provide a flavor of some of the arguments that makes our proof of Theorem 1 works.

Let (r, φ) be the polar coordinates at the origin of $\mathbb{R}^2$ and write

$$f_0(r\cos\varphi, r\sin\varphi) = r^k[F_k(\varphi) + O(r)]$$

where $F_k(\varphi)$ is the restriction to the unit circle of the homogeneous part of f_0 of least degree. The gradient differential equation becomes a differential equation on $\mathbb{S}^1 \times \mathbb{R}_{\geq 0}$ and, after division by r^{k-1}, writes as

$$\dot{r} = r(kF_k + O(r)) \text{ and } \dot{\varphi} = F_k' + O(r). \tag{1}$$

Since F_k is not identically zero, when it is constant we divide Equation (1) by r and find that the divided vector field is transverse to $\mathcal{C} = \mathbb{S}^1 \times 0$ at each point (*dicritical case*). If F_k is not constant, then F_k' must vanish and change sign along the circle $\mathcal{C}$. This prevents any gradient trajectory from accumulating on the whole bottom circle $\mathcal{C}$ (*non-monodromic case*).
In both cases, dicritical and non-monodromic, a plane gradient trajectory does not spiral around its limit point, therefore it has a tangent and thus does not oscillate.

The plane case is enlightening enough to provide us with some of the elements we need to prove Theorem 1. The surface S_0 on which we want to understand the behavior of the restricted gradient trajectories at their limit point $\mathbf{0}$, is analytically diffeomorphic to a cylinder $\mathbb{S}^1 \times]0, \varepsilon]$. We can carry the metric $\mathbf{h}$ over this cylinder, so that we have a well defined gradient differential equation. Our concern then becomes: how does a trajectory of this differential equation behave near the bottom circle $\mathcal{C} = \mathbb{S}^1 \times 0$? There is no canonical way to extend the inverse of the diffeomorphism onto $\mathbb{S}^1 \times [0, \varepsilon]$, and so à-priori, our differential equation is not well defined on the bottom circle $\mathcal{C}$, if defined at any point of it!

Nevertheless, we manage to prove that a limit dynamics exists on the circle $\mathcal{C}$ but at finitely many points. We first show that our setting only allows a single possible type of oscillation, that we call spiraling. To keep up with the planar situation, we actually prove that the only possible dynamics of the restricted gradient vector field will either be dicritical-like or non-monodromic-like (see Section 5 for precise definitions). Consequently, trajectories cannot spiral and will therefore be non-oscillating.

In Section 3, Proposition 14 provides a systematic way to parameterize $\mathrm{clos}(S_0)$ as the surjective image of a continuous mapping defined on $\mathbb{S}^1 \times [0,\varepsilon]$ which induces an analytic diffeomorphism between the open cylinder $\mathbb{S}^1 \times]0,\varepsilon]$ and S_0. Such a parameterization is inherited from the resolution of singularities of the analytic surface X and thus comes with some very specific properties on the bottom circle $\mathcal{C} = \mathbb{S}^1 \times 0$.
In Section 4, we use such a parameterization to express the pull-back of the restriction of the function f_0 to S_0, as well as the corresponding gradient vector field, in polar-like coordinates $(\varphi, r) \in \mathbb{S}^1 \times [0,\epsilon]$ as in Equation (1). We obtain a continuous principal part along the bottom circle that will play a similar role to that of the principal part F_k in (1).
Section 5 deals with the oscillating dynamics of a given real analytic vector field on an isolated surface singularity (such as S_0) and vanishing at the tip, which can only be spiraling around this singular point, as we have already suggested. Although of an independent nature, we use the results of the previous sections for the proof. We also describe two local dynamical situations we call "dicritical" and "non-monodromic", generalizing the planar smooth case, and show here that such dynamics are non-oscillating.
Our notion of "dicritical-ness": *there exists an arc of the bottom circle* $\mathcal{C}$ *such that each point is the* ω*-limit point of a unique trajectory*, is weaker than the usual notion requiring transversality to the exceptional divisor (here the bottom circle). Our notion of "non-monodromic-ness" is also weaker than the notion stated above: the function playing the role of F_k in Equation (1), is continuous, not constant but can fail to be differentiable at finitely many points of $\mathcal{C}$.
The proof of Theorem 1 is done in Section 6. It uses all the main results of Sections 3, 4 to obtain a differential equation on a cylinder $\mathbb{S}^1 \times [0,\varepsilon]$ which is analytic on $\mathbb{S}^1 \times]0,\varepsilon]$. Although there is a slight cost, namely a finite subset of the bottom circle where the differential equation is likely to be not defined, we know enough about it to show that only the dicritical or non-monodromic situations happen. Section 5 then guarantees the non-oscillation of the restricted gradient trajectories.
The last section deals with two not-so-unexpected consequences of our main result, Corollary 2 and Theorem 4.

3. Parameterization of real analytic surfaces

Let X be the germ, at the origin $\mathbf{0}$ of $\mathbb{R}^n$, of a real analytic surface of pure dimension 2. We will not distinguish between the germ of X at $\mathbf{0}$ and a representative in a sufficiently small neighborhood of $\mathbf{0}$.
Assume that the surface X has an isolated singularity at the origin, that is $X \setminus \{\mathbf{0}\}$ is a smooth embedded analytic surface of $\mathbb{R}^n$.

Let S_0 be a given connected component of the germ at $\mathbf{0}$ of the regular part $X \setminus \{\mathbf{0}\}$. The *tangent cone of* S_0 *at* $\mathbf{0} \in \mathbb{R}^n$ is the subset of $\mathbb{S}^{n-1}$ made of the limits of the oriented secant direction $\frac{p_k}{|p_k|}$ taken along sequences of points

$(p_k)_k$ in S_0 converging to $\mathbf{0}$. The tangent cone $C_{\mathbf{0}}(S_0)$ is a compact connected subanalytic subset of $\mathbb{S}^{n-1}$ of dimension at most one. We distinguish two cases:
- If $C_{\mathbf{0}}(S_0)$ reduces to a single point, we will speak about the *cuspidal tangent cone* case (CTC for short).
- If $C_{\mathbf{0}}(S_0)$ is a curve we will speak of the *open tangent cone* case (OTC).

For any $\varepsilon > 0$ sufficiently small, the Local Conic Structure Theorem (see [19, 2, 27]) states that X is homeomorphic to the cone with vertex $\mathbf{0}$ over $X_\varepsilon = X \cap \mathbb{S}^{n-1}_\varepsilon$, where $\mathbb{S}^{n-1}_\varepsilon$ is the Euclidean sphere of radius ε. Moreover, the surface X is transverse to $\mathbb{S}^{n-1}_\varepsilon$ so that $S_0 \cap \mathbb{S}^{n-1}_\varepsilon$ is analytically diffeomorphic to $\mathbb{S}^1$ and $S_0 \cap \operatorname{clos}(B(\mathbf{0},\varepsilon))$ is analytically diffeomorphic to $\mathbb{S}^1 \times]0,\varepsilon]$.

Definition 5. Assume $C_{\mathbf{0}}(S_0)$ consists of the single oriented direction $\eta \in \mathbb{S}^{n-1}$. A system of analytic coordinates $(\mathbf{x}, z) = (x_1, \ldots, x_{n-1}, z)$ at $\mathbf{0}$ is called *adapted for* S_0 if the half-line $\mathbb{R}_+\eta$ is the non-negative z-axis.

Given adapted coordinates $(\mathbf{x}, z)$ in the CTC case, taking the height function z instead of the distance function, the proof of the Local Conic Structure's Theorem adapts to obtain the same conclusion: the intersection $S_0 \cap \{z = \varepsilon\}$ is transverse, thus analytically diffeomorphic to $\mathbb{S}^1$ and $S_0 \cap \{0 < z \leq \varepsilon\}$ is analytically diffeomorphic to $\mathbb{S}^1 \times]0,\varepsilon]$ for $0 < \varepsilon \leq \varepsilon_0$ once ε_0 is sufficiently small.

From now on, we fix some ε_0 so that in both cases OTC or CTC, the above properties coming from the locally conic structure are satisfied. We consider a representative of S_0 in $\{0 < z < \varepsilon_0\}$, where z stands for the distance to the origin in the OTC case and for the last component of an adapted system of coordinates in the CTC case.

In what follows we will desingularize the surface S_0. First, it will be convenient for us to *open* the surface S_0 by means of a single blowing-up-like mapping β. Roughly speaking, we mean that the inverse image of S_0 by β accumulates to a one-dimensional set in the exceptional divisor.
In the OTC case, the usual polar blowing-up $\beta : (\mathbf{y}, r) \mapsto r\mathbf{y}$, for $\mathbf{y} \in \mathbb{S}^{n-1}$ and r the distance function, "opens" the surface S_0, since $\beta^{-1}(S_0)$ accumulates onto $C_{\mathbf{0}}(S_0) \subset \mathbb{S}^{n-1}$, a subanalytic curve.
The CTC case, however, requires more work. Starting with an adapted system of coordinates $(\mathbf{x}, z)$, a first and naive candidate mapping to "open" the surface is a "ramified blowing-up" of the form $\beta_s : (\mathbf{y}, w) \mapsto (w^s\mathbf{y}, w)$, where $\mathbf{y} \in \mathbb{R}^{n-1}$, for a well chosen rational exponent $s > 1$. Such an exponent s exists when the z-axis is contained in the surface S_0. However the surface $\beta_s^{-1}(S_0)$ may accumulate to a single point on the divisor $\beta_s^{-1}(z = 0)$ (or escapes to infinity) whatever the exponent s is. In such a case the surface S_0 cannot be opened with any such ramified blowing-up. In this situation, we consider a given analytic half-branch on S_0 as new non-negative z-axis, and in these new coordinates, a ramified blowing-up as above will open the surface.

The next technical lemma will detail such considerations. First, an *analytic half-branch at the origin* $\mathbf{0}$ *of* $\mathbb{R}^n$ is the germ at $\mathbf{0}$ of a connected component Γ of $Y \setminus \{\mathbf{0}\}$, where Y is a one-dimensional analytic set through $\mathbf{0}$. When Γ is contained in $\{z > 0\}$, it is parametrized as the image of an analytic mapping $z \mapsto (\theta(z), z^N), z > 0$, where $\theta = (\theta_1, \ldots, \theta_{n-1}) :]-\varepsilon, \varepsilon[\to \mathbb{R}^{n-1}$ is analytic with $\theta(0) = \mathbf{0}$ and N is a positive integer.

Lemma 6. *Assume the tangent cone $C_{\mathbf{0}}(S_0)$ is reduced to a point. Let $(\mathbf{x}, z)$ be adapted analytic coordinates at $\mathbf{0}$.*
(i) There is a unique rational number $\nu > 1$ such that the accumulation set of the mapping $S_0 \ni (\mathbf{x}, z) \to \frac{\mathbf{x}}{z^\nu} \in \mathbb{R}^{n-1}$ is a bounded subset of $\mathbb{R}^{n-1}$ and contains a point that is not $(0, \ldots, 0)$.
(ii) There exists a unique positive rational number $e \geq \nu$ such that the accumulation set of the mapping $S_0 \times S_0 \ni ((\mathbf{x}, z), (\mathbf{y}, z)) \to \frac{|\mathbf{x}-\mathbf{y}|}{z^e} \in \mathbb{R}$ is a bounded subset of $\mathbb{R}$ containing a positive number.
(iii) Let $\Gamma : z \to (\theta(z), z^N)$ be a real analytic half-branch at $\mathbf{0}$ such that $\Gamma \subset S_0$. Then, the set of accumulation values of the mapping $\tau_{e,\Gamma} : S_0 \to \mathbb{R}^{n-1}$, $(\mathbf{x}, z) \mapsto \frac{\mathbf{x}-\theta(z^{1/N})}{z^e}$ is a connected bounded subanalytic set of dimension 1.

Proof. The uniqueness of ν and e are clear.
For (i), let $h(z) := \sup\{|\mathbf{x}| \text{ for } (\mathbf{x}, z) \in S_0\}$. The function h is subanalytic and extends continuously to $z = 0$ by $h(0) = 0$. Writing it as a Puiseux's series $h(z) = az^\nu + \cdots$ with $a \neq 0$, the exponent ν satisfies the required properties: the cuspidal nature of S_0 and the definition of adapted coordinates imply that $\nu > 1$.
We show the existence of e of point (ii) similarly to point (i): We take this time the function h to be defined as $h(z) := \sup\{|\mathbf{x} - \mathbf{y}| \text{ for } (\mathbf{x}, z), (\mathbf{y}, z) \in S_0\}$.
For (iii), let Λ be the set of accumulation values of the mapping $\tau = \tau_{e,\Gamma}$. Since Γ is contained in S_0, the origin $\mathbf{0}$ of $\mathbb{R}^{n-1}$ is in Λ. By definition of the exponent e of point (ii), Λ is bounded and contains a point $p \neq \mathbf{0}$. The connectedness and subanalyticity of Λ follow from the connectedness of S_0 and the subanalyticity of τ. $\square$

Remark 7. The numbers ν, e of Lemma 6 depend on the adapted system of coordinates. Take in $\mathbb{R}^3$ the revolution surface $x^2 + y^2 - z^5 = 0$, then $e = \nu = 5/2$. Consider now the change of coordinates $(x', y', z') = (x + z^2, y, z)$, then $e' = \nu' = 2$.

The next result synthesizes the discussion about the possible *opening* of the surface S_0 by a single blowing-up-like mapping. Its proof follows from Lemma 6.

Proposition 8. *In the OTC case, let $M = \mathbb{S}^{n-1} \subset \mathbb{R}^n$ with coordinates $\mathbf{y} = (y_1, \ldots, y_n)$. In the CTC case, let $M = \mathbb{R}^{n-1}$ with coordinates $\mathbf{y} = (y_1, \ldots, y_{n-1})$. Let $(\mathbf{x}, z)$ be adapted coordinates for S_0 at $\mathbf{0}$ and let $e \in \mathbb{Q}_{>1}$*

be the exponent of point (ii) in Lemma 6 for these adapted coordinates and let $z \mapsto (\theta(z), z^N)$ *be a parametrization of an analytic half-branch* Γ *in* S_0 *such that* $eN \in \mathbb{N}$*. Consider the following analytic mapping*

$$\begin{array}{rcll} \beta: & M \times [0,\varepsilon_0] & \to & \mathbb{R}^n \\ & (\mathbf{y}, z) & \mapsto & \begin{cases} z\mathbf{y}, & \text{OTC case,} \\ (z^{eN}\mathbf{y} + \theta(z), z^N), & \text{CTC case} \end{cases} \end{array} \tag{2}$$

Then β *induces a diffeomorphism from* $M \times]0, \varepsilon_0]$ *onto its image. Let* $S := \beta^{-1}(S_0)$*,* $D := \{z = 0\} \subset M \times \mathbb{R}$ *and* $E := \operatorname{clos}(S) \cap D$*. Then* E *is a closed bounded connected subanalytic curve of* D *of dimension one.*

A mapping β as in (2) is called an *opening blow-up of* $\operatorname{clos}(S_0)$. In the CTC case, β depends on the adapted system of coordinates, on the given curve Γ on S_0 and on the number N in the parametrization of Γ. As we will see, the choice of all these parameters will not matter for our purpose, so we do not need the notation β to carry these parameters.

For the rest of this section, assume that we have picked an opening blow-up β of the surface S_0. A key element in our result relies on the construction of an explicit diffeomorphism between S and the open cylinder $\mathbb{S}^1 \times]0, \varepsilon_0]$, which extends to a global parameterization of $\operatorname{clos}(S) = S \cup E$: a surjective continuous mapping $\Phi : \mathbb{S}^1 \times [0, \varepsilon_0] \to \operatorname{clos}(S)$. For this purpose, we first resolve the singularities of the surface $\operatorname{clos}(S)$, also providing a resolution of the singularities of $\operatorname{clos}(S_0)$ (up to ramification). Several formulations are possible. The version we use is stated in the following theorem, an avatar of the general theory on reduction of singularities of real analytic space as found in Hironaka & Al. [13, 1] (see also [4]).

Theorem 9 (Reduction of singularities of S)**.** *There exists a non-singular real analytic surface* $\widetilde{S}$*, a normal crossing divisor* $\widetilde{E} \subset \widetilde{S}$ *and a proper analytic mapping* $\sigma : \widetilde{S} \to \mathcal{U}$ *where* $\mathcal{U}$ *is an open neighborhood* $\mathcal{U}$ *of* E *in* $M \times \mathbb{R}$ *such that:*

(i) $\operatorname{clos}(S) \cap \mathcal{U} \subset \sigma(\widetilde{S})$ *and* $\sigma^{-1}(E) \subset \widetilde{E}$*,*
(ii) $S' = \sigma^{-1}(S)$ *is an open submanifold of* $\widetilde{S}$ *and the restricted mapping* $\sigma|_{S'} : S' \to S \cap \mathcal{U}$ *is an isomorphism,*
(iii) *If* $E' = \operatorname{clos}(S') \cap \widetilde{E}$*, then* $E' = \operatorname{clos}(S') \setminus S'$*, it is a compact subanalytic connected curve of* $\widetilde{S}$ *and* $\sigma(E') = E$*.*
(iv) *If* $p \in E'$*, there is a fundamental system of neighborhoods* $\{\mathcal{W}_k\}$ *of* p *in* $\widetilde{S}$ *such that any connected component of* $\mathcal{W}_k \setminus \widetilde{E}$ *is either contained in* S' *or has empty intersection with* S'*.*

Proof. Let $X_1 = \operatorname{clos}(\beta^{-1}(X \setminus \{\mathbf{0}\}))$ be the strict transform of X by the opening blowing-up β and let $Z = (X_1 \cup D) \cap \mathcal{U}$ on some open neighborhood $\mathcal{U}$ of E in $M \times \mathbb{R}$. The general reduction of singularities applied to the real closed analytic set Z states there exists a proper surjective analytic mapping $\pi : \widetilde{M} \to \mathcal{U}$, composition of finitely many blowing-ups with closed analytic smooth centers, such that the total transform $\pi^{-1}(Z)$ has only normal crossings. Moreover, the smooth centers of blowing-ups are chosen either to be contained in the singular locus of the corresponding strict transform of Z or in the divisors created along the resolution process. Since $\mathrm{sing}(Z) \cap \mathrm{clos}(S) \subset D$, the mapping π induces an isomorphism from $\pi^{-1}(\mathcal{U} \setminus D)$ onto $\mathcal{U} \setminus D$. Let $\widetilde{S}$ be the irreducible component of $\pi^{-1}(Z)$ containing $\pi^{-1}(S)$. Let $\widetilde{E} = \pi^{-1}(D) \cap \widetilde{S}$ and put $\sigma = \pi|_{\widetilde{S}}$. Since $\widetilde{S}$ is closed in $\widetilde{M}$ and σ is proper, we obtain the first inclusion in point (i). The second inclusion is given by construction. Since $S \cap \mathcal{U} \subset \mathcal{U} \setminus D$ and π is an isomorphism on $\pi^{-1}(\mathcal{U} \setminus D)$ we get point (ii). To prove point (iii), we first remark that $E' = \mathrm{clos}(S') \setminus S'$ as an easy consequence of (i). The properness of σ ensures that E' is the Hausdorff limit as $\varepsilon \to 0$ of the subanalytic family of compact sets $\widetilde{\mathcal{C}}_\varepsilon = \sigma^{-1}(S \cap \{z = \varepsilon\})$, each analytically diffeomorphic to the circle, and so E' is subanalytic, compact and connected. It cannot be reduced to a single point p since, otherwise the curve selection lemma would show that $S' \cup \{p\} \subset \widetilde{S}$ is locally open at p and thus p would be isolated in $\widetilde{E}$ which cannot be. The properness of σ is used again to prove that $\sigma(E') = E$. Finally, for point (iv), let $\mathcal{W}$ be an affine chart at p, isomorphic to $\mathbb{R}^2$, such that $\mathcal{W} \cap \widetilde{E}$ is either one or the two coordinate axis. Let $\mathcal{W}_k = [-1/k, 1/k]^2$. A connected component of $\mathcal{W}_k \setminus \widetilde{E}$ is either a half-space or a quadrant. Each contains a single connected component of $\mathcal{W}_{k+1} \setminus \widetilde{E}$. If the property described in point (iv) does not hold, there will be points in $\mathcal{W}_k \setminus \widetilde{E}$ which belong to the boundary $E' = \mathrm{clos}(S') \setminus S'$ of S', thus impossible since $E' \subset \widetilde{E}$. $\square$

A triple $\mathcal{R} = (\widetilde{S}, \widetilde{E}, \sigma)$ satisfying the properties (i-iv) of Theorem 9 will be called a *(total) resolution of singularities of* S. The curve $\widetilde{E}$ will simply be called the *divisor of the resolution* $\mathcal{R}$. The surface $S' = \sigma^{-1}(S)$ and $E' = \mathrm{clos}(S') \cap \widetilde{E}$ will be respectively called the *strict transform* and the *strict divisor of the resolution*. We will also speak of $\mathcal{R}' = (S', E', \sigma' = \sigma|_{S'})$ as the *strict resolution of* S *(associated to* $\mathcal{R}$*)*.

Let $\mathcal{R} = (\widetilde{S}, \widetilde{E}, \sigma)$ be a resolution and p be a point of $\widetilde{E}$. Let $\sigma_p : \widetilde{S}^1 \to \widetilde{S}$ be the blowing-up of $\widetilde{S}$ at p. This provides a new triple $\mathcal{R}_p = (\widetilde{S}^1, \sigma_p^{-1}(\widetilde{E}), \sigma \circ \sigma_p)$ which is a new resolution of singularities of S.

Definition 10. Let $\mathcal{R}^1, \mathcal{R}^2$ be two resolutions of the surface $S = \beta^{-1}(S_0)$. If $\mathcal{R}^2$ is obtained from $\mathcal{R}^1$ by finitely many successive points blowing-ups at points in the successive corresponding divisors, we will say that $\mathcal{R}^2$ *dominates* $\mathcal{R}^1$ and will write $\mathcal{R}^2 \succeq \mathcal{R}^1$.

A resolution dominating a given one will be obtained when we want to "monomialize" one or several functions on S which are restrictions of analytic functions.

Definition 11. Let $H = (h_1, \dots, h_k)$ be a k-uple of real analytic functions in a neighborhood of E in $M \times \mathbb{R}_{\geq 0}$. A resolution $\mathcal{R} = (\widetilde{S}, \widetilde{E}, \sigma)$ of S is *adapted to* H (or briefly a (S, H)*-resolution*) if, for any j, the composition $\widetilde{h}_j = h_j \circ \sigma$ has a *monomial representation* at any point $p \in \widetilde{S}$: There are analytic coordinates (u, v) of $\widetilde{S}$ at p such that $\widetilde{h}_j = u^a v^b G_j(u, v)$, where $a, b \in \mathbb{N}$, G_j is analytic and $G_j(0, 0) \neq 0$.

Corollary 12. *Let $H = (h_1, \dots, h_k)$ be as above and suppose that the restriction $h_j|_S$ has no critical point. Then there exists a resolution $\mathcal{R}$ of S such that, for any $\mathcal{R}^1 \succeq \mathcal{R}$, $\mathcal{R}^1$ is a (S, H)-resolution.*

Proof. From classical results in local monomialization of analytic functions in a smooth analytic manifolds (see for instance [3]): just consider a resolution of S and blow-up the points of the divisor where the corresponding total transform of the h_j have not yet a monomial representation. □

The following terminology is needed to state the principal result of this section. Let N be a real analytic manifold with real analytic smooth boundary ∂N and $f : N \to \mathbb{R}$ be a continuous map. The function f is *ramified-analytic at a point p of ∂N*, if there are $l \in \mathbb{N}$ and analytic coordinates $(\mathbf{x}, z)$ at p for which $N = \{z \geq 0\}$ and $\partial N = \{z = 0\}$, such that the mapping $(\mathbf{x}, z) \mapsto f(\mathbf{x}, z^l)$ is analytic at $(\mathbf{0}, 0)$. If $h : N \to M$ is a continuous mapping into an analytic manifold M, the mapping h will be called *ramified-analytic at $p \in \partial N$* if, in some analytic coordinates of M, its components are ramified-analytic at p.

Remark 13. Let $f : N \to \mathbb{R}$ be a ramified-analytic function at some point $p \in \partial N$, with analytic coordinates $(\mathbf{x}, z)$ at p for which $N = \{z \geq 0\}$ and $\partial N = \{z = 0\}$. The function $z\partial_z f$ extends continuously, in a neighborhood $\mathcal{V}$ of p, into a function which is ramified-analytic at p and, moreover, vanishes along the boundary $\mathcal{V} \cap \partial N$.

Proposition 14. *Let $\mathcal{R} = (\widetilde{S}, \widetilde{E}, \sigma)$ be a (S, z)-resolution and $\mathcal{R}' = (S', E', \sigma')$ be the associated strict resolution. There exist $\varepsilon > 0$ and a continuous mapping $\widetilde{\Phi} : \mathbb{S}^1 \times [0, \varepsilon] \to \widetilde{S}$ with the following properties:*

(i) *It maps $\mathbb{S}^1 \times \{r\}$ onto $\sigma^{-1}(S \cap \{z = r\})$ for $0 < r \leq \varepsilon$ and induces an analytic diffeomorphism between $\mathbb{S}^1 \times]0, \varepsilon]$ and $\sigma^{-1}(S \cap \{0 < z \leq \varepsilon\})$.*

(ii) *It maps surjectively $\mathbb{S}^1 \times [0, \varepsilon]$ onto $\operatorname{clos}(S') = S' \cup E'$ and it maps $\mathcal{C} = \mathbb{S}^1 \times \{0\}$ onto E'.*

(iii) *The set $\Omega = \Omega(\widetilde{\Phi}) = (\widetilde{\Phi})^{-1}(E' \cap \operatorname{sing}(\widetilde{E})) \subset \mathcal{C}$ is finite and $\widetilde{\Phi}$ is* uniformly ramified-analytic *at any point of $\mathcal{C} \setminus \Omega$: there exists $l \in \mathbb{N}$ such that $(\varphi, r) \mapsto \widetilde{\Phi}(\varphi, r^l)$ is analytic at every point of $\mathcal{C} \setminus \Omega$.*

Using Theorem 9, points (i), (ii) and (iii) are true for $\Phi := \sigma \circ \widetilde{\Phi}$ when replacing the strict transforms S' and E' with the initial subsets $S$and E respectively. Namely, Φ maps surjectively $\mathbb{S}^1 \times [0, \varepsilon]$ onto $\operatorname{clos}(S) = S \cup E$,

$\mathcal{C}$ onto E and $\mathbb{S}^1\times]0,\varepsilon]$ diffeomorphically onto S, sending $\mathbb{S}^1\times\{r\}$ onto $S\cap\{z=r\}$. Moreover, Φ is uniformly ramified analytic at every point of $\mathcal{C}\setminus\Omega$. A mapping $\widetilde{\Phi}$ (or Φ) satisfying points (i) to (iii) of Proposition 14 is called a *parameterization associated to the resolution* $\mathcal{R}$, and the subset Ω in (iii) and is called the *exceptional set of the parameterization* $\widetilde{\Phi}$ *(or* Φ*).*

Proof. Let $\mathcal{R}=(\widetilde{S},\widetilde{E},\sigma)$ be a (S,z)-resolution. We construct a retraction of a neighborhood of $\widetilde{E}$ in $\widetilde{S}$ onto $\widetilde{E}$ by integration of a certain analytic vector field. It is just an avatar of the construction of a *Clemens structure* on an analytic manifold equipped with a normal crossings divisor (see [8, 23]).

Let $\widetilde{g}$ be an analytic Riemannian metric on $\widetilde{S}$, whose existence is guaranteed by Grauert's Theorem on the analytic embedding of analytic manifolds in Euclidean spaces [11]. Let $\tilde{z}:=z\circ\sigma:\widetilde{S}\to\mathbb{R}$. Let $\xi=\nabla_{\widetilde{g}}(-\tilde{z}^2)$ be the gradient vector field of $-\tilde{z}^2$ w.r.t the metric $\widetilde{g}$. Its singular set is exactly the divisor $\widetilde{E}=\{\widetilde{z}=0\}$.

Let ε be small enough so that σ induces a diffeomorphism from $\sigma^{-1}(S\cap\{0<z\le\varepsilon\})$ to $S\cap\{0<z\le\varepsilon\}$. We can now consider S just as being $S\cap\{0<z\le\varepsilon\}$.

For $r\in]0,\varepsilon]$, let $\widetilde{\mathcal{C}_r}=\tilde{z}^{-1}(r)=\sigma^{-1}(S\cap\{z=r\})$. It is an embedded curve in $\widetilde{S}$ isomorphic to the circle $\mathbb{S}^1$. Let $\rho:\mathbb{S}^1\to\widetilde{\mathcal{C}_\varepsilon},\quad\varphi\mapsto\rho(\varphi)$ be an analytic diffeomorphism. For $p\in S'$, let γ_p be the maximal integral curve of ξ with initial data $\gamma_p(0)=p$. The parameterized curve γ_p is defined for times $t\ge0$ and stays in S'. Since the function $t\mapsto\tilde{z}(\gamma_p(t))$ strictly decreases to 0 as t goes to infinity γ_p cuts (orthogonally) each curve $\widetilde{\mathcal{C}_r}$ for $r\in]0,\tilde{z}(p)]$ only once. Thanks to Łojasiewicz's Gradient Inequality [18], the omega-limit set $\omega(\gamma_p)$ consists of a single point $R(p)\in E'$ and the mapping $R:\widetilde{S}\to E'$ is continuous since $\widetilde{E}$ is compact. The following mapping is thus well defined:

$$\widetilde{\Phi}:\mathbb{S}^1\times[0,\varepsilon]\to\widetilde{S},\quad\widetilde{\Phi}(\varphi,r)=\begin{cases}\widetilde{\mathcal{C}_r}\cap|\gamma_{\rho(\varphi)}|, & \text{if } r\neq0;\\ R(\rho(\varphi)), & \text{if } r=0,\end{cases}\tag{3}$$

where $|\gamma_p|\subset\widetilde{S}$ is the image set of γ_p. The restriction of $\widetilde{\Phi}$ to the open cylinder $\mathbb{S}^1\times]0,\varepsilon]$ is an analytic diffeomorphism onto S', proving point (i).

In order to obtain the continuity of $\widetilde{\Phi}$ and properties (ii) and (iii), we will show that for $p\in E'$ there exists $\varphi_0\in\mathbb{S}^1$ such that $\widetilde{\Phi}(\varphi_0,0)=p$, $\widetilde{\Phi}$ is continuous at $(\varphi_0,0)$ and ramified-analytic if $p\in E'\setminus\operatorname{sing}(\widetilde{E})$.

Let $p\in E'\setminus\operatorname{sing}\widetilde{E}$. Let (u,v) be analytic coordinates at p such that $\tilde{z}(u,v)=v^m$ with $m\ge1$ and $\widetilde{E}=\{v=0\}$. From point (iv) of Theorem 9, there is a neighborhood $\mathcal{V}$ of p such that the half-space $\{v>0\}$ is contained in S'. The metric writes $\widetilde{g}=A\mathrm{d}u^2+2B\mathrm{d}u\mathrm{d}v+C\mathrm{d}v^2$, and we obtain

$$\xi=2(\det\widetilde{g})^{-1}(Bmv^{2m-1}\tfrac{\partial}{\partial u}-Amv^{2m-1}\tfrac{\partial}{\partial v}).$$

Since $A(p) \neq 0$, the divided vector field $\xi' := v^{1-2m}\xi$ is not singular, transverse to the divisor $\widetilde{E}$ at p and generates the same foliation as ξ on $\{v \neq 0\}$. Thus there exists a trajectory $|\gamma|$ of ξ with $\omega(\gamma) = p$ which extends smoothly and analytically through of p as a trajectory $|\gamma'|$ of ξ'. Going backwards in time, $|\gamma|$ cuts $\widetilde{C}_\varepsilon$ at a point $\rho(\varphi_0)$ for some $\varphi_0 \in \mathbb{S}^1$. Thus $p = R(\rho(\varphi_0)) = \widetilde{\Phi}(\varphi_0, 0)$. Let γ'_q be the trajectory of ξ' through a point $q \in \mathcal{V}$. Since ξ' is not singular in $\mathcal{V}$ and transverse to the fibers $v = cst$, up to shrinking $\mathcal{V}$, the following mapping

$$H : \mathcal{V} \times]-\delta, \delta[\to \widetilde{S},\ (q,t) \mapsto H(q,t) := v^{-1}(t) \cap |\gamma'_q|,$$

is analytic. Fix $v_0 > 0$ such that γ cuts $v^{-1}(v_0)$ inside $\mathcal{V}$ and denote $\psi : \mathbb{S}^1 \to \widetilde{C}_{v_0^{1/m}}$, $\psi(\varphi) = \widetilde{\Phi}(\varphi, v_0^{1/m})$, an analytic diffeomorphism. By construction the mapping we are looking for satisfies

$$\widetilde{\Phi}(\varphi, r) = H(\psi(\varphi), r^{1/m})$$

in some neighborhood of $(\varphi_0, 0)$ and thus is ramified analytic at that point. The number m can be chosen constant for each connected component of $E \setminus \operatorname{sing}(\widetilde{E})$, which are finitely many. Thus there is a uniform ramification index l along $\mathcal{C} \setminus \Omega$. So we get (iii).

Let $p \in E' \cap \operatorname{sing} \widetilde{E}$. Let (u,v) be analytic coordinates at p such that $\tilde{z}(u,v) = u^l v^m$ with $l, m \geq 1$ and $\widetilde{E} = \{uv = 0\}$. From point (iv) of Theorem 9 we assume that the first quadrant $Q = \{u > 0, v > 0\}$ is contained in S' for u, v small enough. The metric writes as $\widetilde{g} = A\mathrm{d}u^2 + 2B\mathrm{d}u\mathrm{d}v + C\mathrm{d}v^2$, and we obtain

$$\xi = 2(\det \widetilde{g})^{-1} u^{2l-1} v^{2m-1} [(-lCv + mBu)\tfrac{\partial}{\partial u} + (lBv - mAu)\tfrac{\partial}{\partial v}].$$

Since $\widetilde{g}$ is positive definite, the divided vector field $\xi' = u^{1-2l} v^{1-2m} \xi$ has a saddle-type singularity at p: its linear part L_p at p has two non-zero eigenvalues with opposite sign. Moreover, each eigen-direction is transverse to the u-axis and v-axis, namely the components of $\widetilde{E}$ at p. The only trajectories of ξ' with ω-limit point p are the two connected components of $W^s \setminus \{p\}$, where W^s is the local stable manifold at p. Since ξ and ξ' are positively proportional on Q, the separatrix $W^s \cap Q \subset S'$ is a trajectory $|\gamma_q|$ of ξ and thus $\omega(\gamma_q) = p$. Going backwards in time, $|\gamma_q|$ cuts $\widetilde{C}_\varepsilon$ at a point $\rho(\varphi_0)$ for some $\varphi_0 \in \mathbb{S}^1$ and thus $\widetilde{\Phi}(\varphi_0, 0) = p$. Let $H : \operatorname{clos}(Q) \times [0, \delta[\to \widetilde{S}$, where $H(q,t)$ is the intersection point of the trajectory of ξ' through the point q with the level curve $\{u^l v^m = t\}$. As in the previous case, continuity at p of the mapping $\widetilde{\Phi}$ will follow from the continuity at $p = (0,0)$ of the mapping H. This property is easily obtained by explicit computation when the vector field ξ' is linear, and we can reduce to this case using Hartman-Grobman Theorem (see for instance [22]). □

Definition 15. Let Ω be a finite subset of $\mathcal{C}$ (such as the exceptional set of a parameterization $\widetilde{\Phi}$ in the proposition above). An analytic mapping $F : \mathbb{S}^1\times]0,\varepsilon] \to N$, is called *uniformly almost ramified-analytic (with respect to Ω)* if there exists some $l \in \mathbb{N}$ such that $(\varphi, r) \mapsto F(\varphi, r^l)$ can be extended as an analytic mapping at any point of $\mathcal{C} \setminus \Omega$. To be shorter, we will either write Ω-u-a-r-a or simply u-a-r-a if the subset Ω is understood.

Part (iv) of Proposition 14 says that $\widetilde{\Phi}$ (or Φ) is an u-a-r-a mapping with respect to the exceptional set Ω. Since ramified-analyticity at any point of $\mathcal{C} \setminus \Omega$ is inherited from the construction of Φ and uniformity comes from the compactness of E, another typical situation example we will come across in the sequel is the following: if h is a continuous function in a neighborhood of $E \subset M \times \mathbb{R}_{\geq 0}$ which is ramified-analytic along E (with respect to $D = M \times \{0\}$), the composite mapping $h^\Phi = h \circ \Phi$ is Ω-u-a-r-a.

4. Asymptotic expansions of restricted functions

A $\mathbb{Q}$-*generalized (real) formal power-series* is a formal expansion $G(T) = \sum_{k\geq 0} a_k T^{\alpha_k}$, where $(\alpha_k)_{k\geq 0}$ is a strictly increasing sequence of non-negative rational numbers and each coefficient a_k is a real number. It is said *convergent* if there exists $t_0 > 0$ such that the sequence of m-partial sum functions $G_m : \mathbb{R}_{\geq 0} \to \mathbb{R}$, $G_m(t) = \sum_{k=0}^m a_k t^{\alpha_k}$, converges uniformly in $[0, t_0]$, thus given rise to a continuous function, also denoted $G : [0, t_0] \to \mathbb{R}$, analytic for $t > 0$, called *the sum of the convergent series.* If the exponents α_k are in $\frac{\mathbb{N}}{l}$ for some positive integer l, then $G(T)$ is called a *Puiseux series.* If all but finitely many coefficients a_k are non-zero then $G(T)$ is a $\mathbb{Q}$-*generalized real polynomial.*

Let $X \subset \mathbb{R}^n$ be an analytic isolated surface singularity at $\mathbf{0}$ and let S_0 be a connected component of $X \setminus \{\mathbf{0}\}$. Let $\beta : M \times \mathbb{R}_{\geq 0} \to \mathbb{R}^n$ be an opening blowing-up of S_0 and denote $S = \beta^{-1}(S_0)$, $D = \{z = 0\} = M \times \{0\}$, $E = \operatorname{clos}(S) \cap D$ as in the previous section.

Let $f : \mathcal{U} \to \mathbb{R}$ be a continuous function in $\mathcal{U}$, a neighborhood of E in $M \times \mathbb{R}_{\geq 0}$, which is *ramified-analytic along D*. Let $f_S : \operatorname{clos}(S) \to \mathbb{R}$ be the restriction of f to $\operatorname{clos}(S) = S \cup E$. Given a (S, z)-resolution $\mathcal{R} = (\widetilde{S}, \widetilde{E}, \sigma)$ and an associated parameterization $\widetilde{\Phi} : \mathbb{S}^1 \times [0, \varepsilon] \to \widetilde{S}$ as in Proposition 14, we denote by $f^\Phi := f_S \circ \Phi = f_S \circ \sigma \circ \widetilde{\Phi} : \mathbb{S}^1 \times [0, \varepsilon] \to \mathbb{R}$.

This Section is devoted to prove the following result, establishing an asymptotic expansion of the restricted function f_S w.r.t. the height coordinate $z : M \times \mathbb{R} \to \mathbb{R}_{\geq 0}$ (let again (φ, r) be the standard coordinates on $\mathbb{S}^1 \times [0, \varepsilon]$).

Proposition 16. *Assume that f is not identically vanishing on S. One and only one of the following two properties is satisfied:*
(a) There exists a $\mathbb{Q}$-generalized real formal power-series $G(T) = \sum_{k\geq 0} a_k T^{\alpha_k}$ which is an asymptotic expansion *of f_S in the following sense: for any positive integer m, there exists a neighborhood $\mathcal{V}_m$ of E in $\operatorname{clos}(S)$ and a bounded*

function $g_m : \mathcal{V}_m \to \mathbb{R}$ such that, for any $(\mathbf{y}, z) \in \mathcal{V}_m$ with $z \neq 0$,

$$f_S(\mathbf{y}, z) = \sum_{k=0}^{m-1} a_k z^{\alpha_k} + z^{\alpha_m} g_m(\mathbf{y}, z). \tag{4}$$

Moreover, the formal power series $G(T)$ is a convergent Puiseux series and $f_S(\mathbf{y}, z) = G(z)$ for any $(\mathbf{y}, z) \in S$ in a neighborhood of E.

(b) Given an initial resolution of S_0, there exists a dominating resolution $\mathcal{R}^0$ adapted to the function z, a $\mathbb{Q}$-generalized polynomial $P(T) = \sum_{k=0}^{m} a_k T^{\alpha_k}$ and a rational number $\alpha > \alpha_m$ such that, for any resolution $\mathcal{R} \succeq \mathcal{R}^0$ and any associated parameterization $\widetilde{\Phi}$, the mapping $f^\Phi : \mathbb{S}^1 \times [0, \varepsilon] \to \mathbb{R}$ writes as

$$f^\Phi(\varphi, r) = P(r) + r^\alpha F(\varphi, r), \tag{5}$$

where F is a continuous function on $\mathbb{S}^1 \times [0, \varepsilon]$ and its restriction to $\mathcal{C} := \mathbb{S}^1 \times \{0\}$ is not constant. Moreover, F is u-a-r-a with respect to the exceptional set Ω of Φ.

The proof will follow from the following lemma.

Lemma 17. *With the hypotheses and notations of Proposition 16, we find:*

(i) There exists a unique $\alpha = \alpha(f_S) \in \mathbb{Q}_{\geq 0}$ such that the quotient f_S/z^α is bounded on S and cannot have the value 0 as a single accumulation value as $z \to 0^+$. The number α is called the exponent of the restricted function f_S (with respect to E).

(ii) Given an initial resolution of S_0, there exists a dominating (S, z)-resolution $\mathcal{R}^0$ such that, for any other resolution $\mathcal{R} \succeq \mathcal{R}^0$ and any associated parameterization $\Phi : \mathbb{S}^1 \times [0, \varepsilon] \to S$, the quotient function $f^\Phi/r^\alpha = (f/z^\alpha) \circ \Phi$ is well defined and analytic on $\mathbb{S}^1 \times]0, \varepsilon]$ and extends to a continuous function on $\mathbb{S}^1 \times [0, \varepsilon]$. Its restriction to the bottom circle $\mathcal{C}$ will be denoted by $\mathrm{in}^\Phi(f)$ and called the initial part of the restricted function f_S (relative to Φ).

(iii) An initial part $\mathrm{in}^\Phi(f)$ like in point (ii) is constant if and only f_S/z^α has a unique accumulation value as $z \to 0$.

Proof. By definition of a ramified-analytic function along D and since E is a compact subset of D, there exists a positive integer $l \in \mathbb{N}$ such that the function $\overline{f} : (\mathbf{y}, z) \mapsto f(\mathbf{y}, z^l)$ is analytic in a neighborhood of E in $M \times \mathbb{R}$. If we prove the Lemma for the analytic function $(\overline{f})_S := \overline{f}|_S$, we obtain the exponent $\overline{\alpha}$. Then $\alpha := \bar{\alpha}/l$ is the exponent of f_S with respect to E and it satisfies (i)-(iii). For the rest of the proof, we suppose that f is analytic in a neighborhood of E in $M \times \mathbb{R}$.

Proof of (i). The uniqueness of the exponent α is immediate from its definition. Consider the following function

$$\mu(t) = \max\{|f(\mathbf{y}, t)| \text{ for } (\mathbf{y}, t) \in S\}.$$

It is well defined since $S \cap \{z = t\}$ is compact for $t > 0$. The function μ is subanalytic, continuous and identically zero only if f_S is. So assuming that f_S does not vanish identically on S, there exists a positive real number a and a non-negative rational number α such that $t^{-\alpha}\mu(t) \to a$ as $t \to 0$. This proves the claim.

Proof of (ii). Assume we are given a first resolution. Let $\mathcal{R}^0$ be a (S, f, z)-resolution dominating it. Any other resolution $\mathcal{R} = (\widetilde{S}, \widetilde{E}, \sigma)$ dominating $\mathcal{R}^0$ is still a resolution adapted to f and z. Let $\widetilde{\Phi} : \mathbb{S}^1 \times [0, \varepsilon] \to \widetilde{S}$ be a parameterization associated with $\mathcal{R}$. The function $h = z^{-\alpha} f_S$ is analytic, continuous and bounded on $S \cap \{0 < z < \varepsilon\}$ for some $\varepsilon > 0$. Let $S' = \sigma^{-1}(S)$, $E' = \mathrm{clos}(S') \setminus S'$ be respectively the strict transform of S and the strict divisor of the resolution $\mathcal{R}$ (see the notations of Theorem 9). Let $h' = h \circ \sigma : S' \to \mathbb{R}$. Thus $r^{-\alpha} f^{\Phi} = h' \circ \widetilde{\Phi}$. Since $\widetilde{\Phi}$ is continuous and maps $\mathcal{C}$ onto E', there is just to prove that h' extends to a continuous function up to E'. We also write $h' = \widetilde{z}^{-\alpha} \widetilde{f_S}$ where $\widetilde{f_S} = f_S \circ \sigma$ and $\widetilde{z} = z \circ \sigma$.

First, let $p' \in E' \setminus \mathrm{sing}(\widetilde{E})$. There are analytic coordinates (u, v) of $\widetilde{S}$ at p' such that $\widetilde{E} = \{v = 0\}$ and $\{v > 0\} \subset S'$ using (iv) of Theorem 9. Since $\mathcal{R}$ is a resolution adapted to f and z, we write

$$\widetilde{f_S}(u, v) = u^{l_1} v^{m_1} U_1(u, v), \quad \widetilde{z} = v^{m_2} U_2(u, v)$$

for some integers $l_1, m_1, m_2 \in \mathbb{N}$ and invertible analytic functions U_1, U_2 with $U_2(0, 0) > 0$. For (u, v) close to $p' = (0, 0)$ with $v > 0$, we find

$$h'(u, v) = v^{m_1 - \alpha m_2} \frac{u^{l_1} U_1(u, v)}{U_2(u, v)^{\alpha}}. \tag{6}$$

Since h' is bounded on $\{v > 0\}$ necessarily $m_1 \geq \alpha m_2$ and the right hand term in Equation (6) defines a continuous function on $\{v \geq 0\}$. If $\mathrm{clos}(S') \subset \{v \geq 0\}$ nearby p', we get the desired conclusion. If instead $\{v < 0\} \subset S'$, necessarily m_2 is even since $\widetilde{z}$ is positive on S'. In this case, the monomial $v^{m_1 - \alpha m_2}$ in expression (6) must be read as $v^{m_1}/(v^{m_2})^{\alpha}$. The function h' turns out to be continuous in a neighborhood of $p' = (0, 0)$.

Suppose now that $p' \in E' \cap \mathrm{sing}\widetilde{E}$. There are analytic coordinates (u, v) of $\widetilde{S}$ at p' with $\widetilde{E} = \{uv = 0\}$ and $\{u > 0, v > 0\} \subset S'$ and such that we can write

$$\widetilde{f_S}(u, v) = u^{l_1} v^{m_1} U_1(u, v), \quad \widetilde{z} = u^{l_2} v^{m_2} U_2(u, v)$$

for some $l_1, m_1, l_2, m_2 \in \mathbb{N}$ and analytic functions U_1, U_2 with $U_1(0, 0) \neq 0$, $U_2(0, 0) > 0$. This time, for small and positive u, v, we have

$$h'(u, v) = u^{l_1 - \alpha l_2} v^{m_1 - \alpha m_2} \frac{U_1(u, v)}{U_2(u, v)^{\alpha}}. \tag{7}$$

Since the function h' is bounded in a neighborhood of p' in S', $l_1 - \alpha l_2$ and $m_1 - \alpha m_2$ are both non-negative. The continuity of h' follows by the same arguments as in the previous case.

Proof of (iii). It follows by continuity of f^Φ/r^α, proved in (ii), the properness of Φ and that Φ maps $\mathcal{C}$ onto $E = \mathrm{clos}(S) \cap \{z=0\}$. $\square$

Proof of Proposition 16. Let $\alpha_0 \in \mathbb{Q}_{\geq 0}$ be the exponent of f with respect to E. Let $\mathcal{R}^0$ be a (S,z)-resolution and Φ^0 be an associated parameterization satisfying the properties of (ii) in Lemma 17.
If the initial part $\mathrm{in}^{\Phi^0}(f)$ is not constant then we are in case (b) of the proposition with $P=0$ and $\alpha = \alpha_0$.
Assume now $\mathrm{in}^{\Phi^0}(f) \equiv a_0 \in \mathbb{R}^*$. The function $f_1 := f - a_0 z^{\alpha_0}$ is ramified-analytic along D. If $f_1|_S \equiv 0$ then we are in case (a). Otherwise, using Lemma 17, let $\alpha_1 \in \mathbb{Q}_{\geq 0}$ be the exponent of f_1 w.r.t E. By definition of the exponent, we find $\alpha_1 > \alpha_0$. Let $\mathcal{R}^1$ be a (S,z)-resolution with $\mathcal{R}^1 \succeq \mathcal{R}^0$ and Φ^1 an associated parameterization for which the initial part $\mathrm{in}^{\Phi^1}(f_1)$ of f_1 exists as in part (ii). If $\mathrm{in}^{\Phi^1}(f_1)$ is not constant we are in case (b) as above and we are done, otherwise we continue this process.
Suppose there exists a sequence of (S,z)-resolutions $\{\mathcal{R}^k\}_{k\geq 0}$ with $\mathcal{R}^{k+1} \succeq \mathcal{R}^k$, associated parameterizations Φ^k and a $\mathbb{Q}$-generalized power series $G(T) = \sum_{k\geq 0} a_k T^{\alpha_k}$ such that, for any $m \geq 0$, α_m is the exponent of the function $f_m = f - \sum_{k=0}^{m-1} a_k z^{\alpha_k}$ and the principal part $in^{\Phi^m}(f_m)$ is a constant function equal to $a_m \neq 0$. The definition of the exponent α gives directly the asymptotic expansion of f_S as in equation (4). Let $\Gamma \subset S$ be an analytic half-branch accumulating to a single point in E, parameterized by the variable z. Let $L :]0,\varepsilon] \to \mathbb{R}$ defined as $L(z) = f_S(\Gamma(z))$. By (4), we have for any $m \geq 0$ and z sufficiently small,

$$L(z) - \textstyle\sum_k^{m-1} a_k z^{\alpha_k} = O(z^{\alpha_m}),$$

that is, that $G(T)$ is the asymptotic expansion of L as $z \to 0^+$. Since L is a semi-analytic function, $G(T)$ is a convergent Puiseux series. Thus $L(z) = G(z)$, where G is considered here as the sum of the expansion $G(T)$. We define $G_S : S \to \mathbb{R}$ by $G_S(\mathbf{y}, z) = G(z)$, an analytic function on S which depends only on z. We have shown that the restrictions of f_S and G_S on Γ coincide. Since Γ can be chosen arbitrarily, $f_S = G_S$ on the whole surface S. This proves statement (a) of the Proposition.
Finally, $F = \frac{f^\Phi - P}{r^\alpha}$ is u-a-r-a since both f^Φ and P are so. $\square$

Remark 18. Although F depends on the resolution $\mathcal{R}$ and on the associated parameterization Φ, we insist it is of the special following form:

$$F = g \circ \Phi \quad \text{with} \quad g := \left(\frac{f^\Phi - P}{r^\alpha}\right).$$

The function g is continuous in a neighborhood of E in $M \times \mathbb{R}_{\geq 0}$, ramified-analytic along E, and depends on f and the opening blowing-up β only.

5. Oscillation vs Spiraling in singular surfaces

Let $\gamma : [0, +\infty[\to \mathbb{R}^n$ be an analytically parameterized curve such that $\lim_{t\to+\infty} \gamma(t) = \mathbf{0} \in \mathbb{R}^n$ and $\mathbf{0}$ does not belong to $|\gamma|$, the image of γ.

Definition 19. A parameterized curve γ is said *(analytically) non-oscillating* if for any semi-analytic subset H of $\mathbb{R}^n$, either $|\gamma|$ is contained in the subset H or the intersection $|\gamma| \cap H$ consists at most of finitely many points. If, on the contrary, there exists a semi-analytic set H such that $|\gamma|$ is not contained in H and the intersection $|\gamma| \cap H$ has infinitely many points then we will say that γ is *oscillating relatively to H*.

The notion of oscillation clearly depends only on the germ at $\mathbf{0}$ of the image $|\gamma|$ of the parameterized curve γ, not on any given parameterization.

In dimension 2, the notion of *spiraling* around a given point is a special case of oscillation for a curve. A convenient definition is found in [6]. We generalize this notion for a curve $|\gamma|$ contained in an analytic isolated surface singularity $X \subset \mathbb{R}^n$ at the origin $\mathbf{0}$ and accumulating at $\mathbf{0}$.

Let X be an analytic surface with an isolated singularity at $\mathbf{0} \in \mathbb{R}^n$. Let S_0 be a connected component of $X \setminus \{\mathbf{0}\}$. Let Γ be an analytic half-branch at $\mathbf{0}$ contained in S_0. For a small enough simply connected neighborhood $\mathcal{V}$ of (the germ at $\mathbf{0}$ of) Γ in S_0, the curve $\Gamma \cap \mathcal{V}$ separates $\mathcal{V} \setminus \Gamma$ into two connected components which we call the *two local sides of* Γ *in* S_0.

Definition 20. The curve $\gamma : [0, +\infty[\to S_0 \subset X \setminus \{\mathbf{0}\}$ *spirals in* X if, for any analytic half-branch Γ at $\mathbf{0}$ in S_0, there exists an increasing sequence $(t_k)_{k\in\mathbb{N}} \subset \mathbb{R}_{>0}$ with $t_k \to +\infty$ such that for each k:

$$\gamma([t_k, t_{k+1}[) \cap \Gamma = \{\gamma(t_k)\}, \quad \gamma(t_k - \varepsilon_k) \in \mathcal{V}^- \ \text{ and } \ \gamma(t_k + \varepsilon_k) \in \mathcal{V}^+,$$

for $\varepsilon_k > 0$ small and where $\mathcal{V}^-$, $\mathcal{V}^+$ are the local sides of Γ in S_0.

When γ is a trajectory of a real analytic vector field in a neighborhood of $\mathbf{0} \in \mathbb{R}^2$, a Rolle-Khovanskii's argument proves that the only oscillating dynamics at $\mathbf{0}$ is spiraling (see [6]). Proposition 21 below extends this result to analytic isolated surfaces singularities.

Let ξ_0 be an analytic vector field on S_0 which extends continuously and subanalytically to the origin by $\xi_0(\mathbf{0}) = 0$, as a mapping from $\mathrm{clos}(S_0)$ to $T\mathbb{R}^n|_{\mathrm{clos}(S_0)}$.

Proposition 21. *Assume that ξ_0 does not vanish in S_0. Let $\gamma : [0, +\infty[\to S_0$ be a non-trivial trajectory of ξ_0 accumulating at $\mathbf{0}$. Then γ is oscillating if and only if it spirals in X.*

Proof. If γ spirals then it is oscillating. Suppose that γ does not spiral. There exists an analytic half-branch Γ in S_0 such that either
(a) the germ at $\mathbf{0}$ of the intersection $|\gamma| \cap \Gamma$ is empty, or
(b) $|\gamma| \cap \Gamma$ is infinite but γ does not cross Γ from one fixed local side of Γ to the other side at those intersection points.
If (b) happens, a Rolle's argument implies that Γ is tangent to ξ_0 at infinitely many points accumulating to $\mathbf{0}$. The subanalyticity of ξ_0 implies that the half-branch Γ is a trajectory of ξ_0, contradicting the oscillation of γ relatively to Γ. So (b) is impossible.

Assume we are in case (a). Since the surface S_0 is analytically diffeomorphic to a cylinder, $S_0 \setminus \Gamma$ is a simply connected analytic manifold. Using Haefliger's Theorem [12, 16, 21]), we deduce that any leaf of the real analytic foliation induced by ξ_0 in $S_0 \setminus \Gamma$ is a *Rolle's leaf.* In particular, the curve $|\gamma| \subset S_0 \setminus \Gamma$ is a Rolle's leaf and cannot cut infinitely many times any analytic half-branch contained in $S_0 \setminus \Gamma$. Thus γ is non-oscillating. □

Despite of the similarities between spiraling in a smooth surface and in an analytic isolated surface singularity, there is however a very important difference. The existence, for a trajectory γ, of a *tangent at the origin*, that is the limit of secants $\lim_{t\to\infty} \frac{\gamma(t)}{|\gamma(t)|}$ exists, prevents, in the smooth surface situation, from spiraling around the origin. For an isolated surface singularity, although in the OTC case this argument is still valid, in the CTC situation, the curve γ will always have a tangent at the origin corresponding to the direction of the tangent cone, regardless if it is spiraling or not
A criterion stronger than the existence of tangent to imply non-spiraling is that the lifting of γ by a reduction of singularities of the surface accumulates to a single point on the exceptional divisor.
We will use this criterion through its lifting on $\mathbb{S}^1 \times [0, \varepsilon]$ via a parameterization as in section 3.

Criterion for non-spiraling. Let $\mathcal{R}$ be a resolution of $S = \beta^{-1}(S_0)$ where β is an opening blowing-up of S_0. Let $\Phi : \mathbb{S}^1 \times [0, \varepsilon] \to S$ be a parameterization associated to $\mathcal{R}$. Assume that $|\gamma| \subset S_0$ and suppose the ω-limit set $\omega(\overline{\gamma})$ of the lifted curve $\overline{\gamma} = (\beta \circ \Phi)^{-1} \circ \gamma$ is such that $\mathcal{C} \setminus \omega(\overline{\gamma})$ contains an open non-empty arc. Then γ does not spiral in X.

The proof is easy: the stated property will imply that $\overline{\gamma}$ does not intersect a given analytic half-branch $\overline{\Gamma}$ on $\mathbb{S}^1 \times [0, \varepsilon]$ through a point $p \in \mathcal{C} \setminus (\omega(\overline{\gamma}) \cup \Omega)$ where Ω is the exceptional set of Φ. Therefore, γ does not intersect the curve $\Gamma = (\beta \circ \Phi)(\overline{\Gamma}) \subset S_0$, which is an analytic half-branch by properness of the resolution and the property that Φ is ramified-analytic at p. Thus γ does not spiral in X.

The next result describes, for a vector field ξ_0 on S_0, two types of dynamics ensuring that none of its trajectories accumulating at the origin is spiraling. These types correspond to either "dicritical" or "non-monodromic" dynamics similar to those in the plane gradient case met in Section 2.

Proposition 22. *Assume that ξ_0 does not vanish in S_0. Suppose that the transformed vector field $\overline{\xi} = (\beta \circ \Phi)^* \xi_0$ on the open cylinder $\mathbb{S}^1 \times]0, \varepsilon]$ satisfies one of the following non-exclusive situations:*

(a) Dicritical case: *There exist a point $p \in \mathcal{C} \setminus \Omega$ and a neighborhood $\mathcal{U}$ of p in $\mathbb{S}^1 \times [0, \varepsilon]$ disjoint from Ω in which $\overline{\xi}$ writes as*

$$\left\{ \begin{array}{lcl} \dot{r} & = & r^{\mu} H(r, \varphi) \\ \dot{\varphi} & = & r^{\mu - 1 + \eta} G(r, \varphi) \end{array} \right. \tag{8}$$

where $\mu, \eta \in \mathbb{Q}_{>0}$ *and* H, G *are continuous on* $\mathcal{U}$ *and ramified-analytic at any point of* $\mathcal{U} \cap \mathcal{C}$ *and such that* H *is negative on* $\mathcal{U}$.

(b) Non-monodromic case: *There exist* $\mu \in \mathbb{Q}_{\geq 0}$, *u-a-r-a functions* $G_1, G_2 : \mathbb{S}^1 \times]0, \varepsilon] \to \mathcal{R}$ *so that* G_2 *vanishes on* $\mathcal{C} \setminus \Omega$ *and an u-a-r-a function* H *continuous on the whole cylinder* $\mathbb{S}^1 \times [0, \varepsilon]$ *such that the restricted function* $H|_{\mathcal{C}}$ *is not constant, in such a way that* $\overline{\xi}$ *writes in the open cylinder* $\mathbb{S}^1 \times]0, \varepsilon]$ *as*

$$
\left\{ \begin{array}{rcl} \dot{r} & = & r^{\mu+1} G_1 \\ \dot{\varphi} & = & r^{\mu}[\frac{\partial H}{\partial \varphi} + G_2]. \end{array} \right. \tag{9}
$$

Then any trajectory γ *of* ξ_0 *accumulating to the origin is non-spiraling and therefore is non-oscillating.*

Proof. It suffices to show that any trajectory γ of ξ_0 accumulating to the origin satisfies the non-spiraling criterion above.

In the dicritical situation (a) we prove a slightly stronger result: there exists a non-empty arc $I \subset \mathcal{U} \cap \mathcal{C}$ such that each point in I is the unique ω-limit point of a trajectory of the transformed vector field $\overline{\xi}$.
When $\mu - \eta + 1 \geq \mu$ in Equation (8), dividing $\overline{\xi}$ by r^μ, gives a vector field which extends to $\mathcal{U} \cap \mathbb{S}^1 \times [0, \varepsilon]$ as a ramified-analytic vector field transverse to $\mathcal{C} \cap \mathcal{U}$. Thus any point of $\mathcal{C} \cap \mathcal{U}$ is the unique accumulation point of a trajectory of $\overline{\xi}$ living in the open cylinder $\mathbb{S}^1 \times]0, \varepsilon]$.
Assume now that $\mu - \eta + 1 < \mu$ in (8). We suppose that $\mathcal{U}$ is of the form $\mathcal{U} =]\varphi_1, \varphi_2[\times [0, \delta] \in \mathbb{S}^1 \times [0, \varepsilon]$ for some $\delta > 0$ small enough. Dividing $\overline{\xi}$ by $r^{\mu-1}|H|$, our vector field provides the following equations in $\mathcal{U}$:

$$
\left\{ \begin{array}{rcl} \dot{r} & = & -r \\ \dot{\varphi} & = & r^{\eta} \frac{G}{|H|} \end{array} \right. \tag{10}
$$

Up to shrinking $\mathcal{U}$, and since G is ramified-analytic, we furthermore assume that G does not vanish on $\mathcal{U}$, up to increasing the exponent η. If $G(p) = 0$ but $G|_{\mathcal{U} \cap \mathcal{C}} \not\equiv 0$, then there are points of $\mathcal{C}$ close to p at which G does not vanish. Thus we can also suppose $G(p) \neq 0$, for instance that G is positive on $\mathcal{U}$. Up to shrinking $\mathcal{U}$ again, we know that $K_1 \leq \frac{G}{|H|} \leq K_2$ on $[\varphi_1, \varphi_2] \times [0, \delta]$ for some positive constants K_1, K_2. The solution of (10) through a point $(\varphi_0, r_0) \in [\varphi_1, \varphi_2] \times]0, \delta]$, as long as it is in that domain, lies between the solutions through (φ_0, r_0) of the systems of equations $\dot{r} = -r$ and $\dot{\varphi} = K_i r^\mu$ for $i = 1, 2$. These last curves are parameterized by

$$
\varphi \mapsto r(\varphi) = [r_0^\eta - \tfrac{\eta}{K_i}(\varphi - \varphi_0)]^{1/\eta}, \ i = 1, 2.
$$

We deduce that any point of $]\varphi_1, \varphi_2[\times 0 \subset \mathcal{C}$ is the unique accumulation point of a trajectory of the system (10), lying in $\{r > 0\}$.

Consider now the non-monodromic situation (b). The hypothesis about H implies its partial derivative $\partial_\varphi H$ is u-a-r-a and continuous along $\mathcal{C}\setminus\Omega$. Let $\mathrm{crit}^*(H|_{\mathcal{C}})$ be the critical locus of $H|_{\mathcal{C}}$ in $\mathcal{C}\setminus\Omega$, and let

$$\Omega' = \Omega \cup (H|_{\mathcal{C}})^{-1}(H(\mathrm{crit}^*(H|_{\mathcal{C}})).$$

Since $H|_{\mathcal{C}}$ is not constant, $\mathcal{C}\setminus\Omega'$ has non empty interior. To show the criterion for non-spiraling for any trajectory γ of ξ_0, it is enough to check that the limit set $\omega(\overline{\gamma})$ of any trajectory $\overline{\gamma}$ of $\overline{\xi}$ accumulating to $\mathcal{C}$ is contained in Ω'.
Assume $\overline{\gamma}$ is parameterized by $t\in\mathbb{R}_{\geq 0}$ and consider the real function

$$h = h_{\overline{\gamma}} : \mathbb{R}_{\geq 0}\to\mathbb{R},\;\; t\mapsto h(t) = H(\overline{\gamma}(t)).$$

The function h is C^1. Let $p\in\mathcal{C}\setminus\Omega'$ and let $a := H(p)$. We just have to show that a cannot be an accumulation value of h when $t\to+\infty$. The function $|\partial_\varphi H|$ is bounded below on the compact set $(H|_{\mathcal{C}})^{-1}(a)$: there exists $c>0$ such that $|\partial_\varphi H| \geq 2c > 0$ on a given neighborhood $\mathcal{V}$ of $(H|_{\mathcal{C}})^{-1}(a)$ in $\mathbb{S}^1\times[0,\varepsilon]$. Since $\overline{\gamma}(t) = (\varphi(t), r(t))$ satisfies Equations (9), if $\overline{\gamma}(t)\in\mathcal{V}$ then, up to shrinking $\mathcal{V}$ (taking into account Remark 13), we find

$$\dot h(t) = \frac{\partial H}{\partial r}(\overline{\gamma}(t))\,\dot r(t) + \frac{\partial H}{\partial\varphi}(\overline{\gamma}(t))\,\dot\varphi(t) > c.$$

For $\varepsilon'>0$ sufficiently small, we assume that $H^{-1}(a)\cap\mathbb{S}^1\times[0,\varepsilon']\subset\mathcal{V}$. Thus, there exists some $\delta>0$ such that

$$h(t)\in]a-\delta, a+\delta[\Rightarrow \overline{\gamma}(t)\in\mathcal{V}.$$

From all these properties, for $t\in h^{-1}(]a-\delta,a+\delta[)$ large enough, $\dot h(t)\geq c/2 > 0$. Thus when $t\to+\infty$, the value a cannot be an accumulation value of h. $\square$

Remark 23. The non-monodromic situation (b) described in Proposition 22 can be generalized as follows:
Assume that the given analytic vector field ξ_0 does not vanish in S_0 and that the foliation $\overline{\mathcal{F}}$ on $\mathbb{S}^1\times]0,\varepsilon]$ induced by $\overline{\xi} = (\beta\circ\Phi)^*\xi_0$ extends continuously to $\mathcal{C}\setminus\Omega$ such that $\mathcal{C}$ is invariant. Assume there exist, two distinct points q_1, q_2 of $\mathcal{C}\setminus\Omega$ where $\overline{\mathcal{F}}$ is not singular, and two continuous germs of vector fields ξ_1 at q_1 and ξ_2 at q_2, which are local generators of the foliation $\overline{\mathcal{F}}$, positively co-linear to $\overline{\xi}$ in the common domain of definition and "pointing in different directions": if φ denotes a global coordinate on $\mathcal{C}\simeq\mathbb{S}^1$, writing $\xi_i(q_i) = c_i\partial_\varphi$, then $c_1c_2<0$. Then any trajectory of ξ_0 accumulating to the origin is non-oscillating.

6. Proof of the main result

This section is devoted to the proof of the main result of this paper, Theorem 1.

The next Lemma shows that the only case requiring work is when both $f_0|_{S_0}$ and $\nabla_{\mathbf{h}} f_0$ do not vanish on S_0.

Lemma 24. *If either $f_0|_{S_0}$ or $\nabla_{\mathbf{h}} f_0$ vanishes in any neighborhood of $\mathbf{0}$ in S_0, then any trajectory of the restricted gradient $\nabla_{\mathbf{h}} f_0$ accumulating to the origin is non-oscillating.*

Proof. First, note that $\nabla_{\mathbf{h}} f_0$ extends to a continuous subanalytic mapping $\nabla_{\mathbf{h}} f_0 : \mathrm{clos}(S_0) \to T\mathbb{R}^n$ by $\nabla_{\mathbf{h}} f_0(\mathbf{0}) = \mathbf{0}$, and that f_0 vanishes on any connected component of the zero locus of $\nabla_{\mathbf{h}} f_0$ containing $\mathbf{0}$ in its closure. The subanalytic Curve Selection Lemma guarantees there exists a subanalytic (thus semi-analytic) curve $\Gamma \subset S_0$ such that $\mathbf{0} \in \mathrm{clos}(\Gamma)$ and $\Gamma \subset f_0^{-1}(0)$. Let γ be a non-trivial trajectory of the restricted gradient $\nabla_{\mathbf{h}} f_0$ accumulating to $\mathbf{0}$. The function $t \mapsto f_0(\gamma(t))$ is increasing and tends to 0 as $t \to \infty$. Thus $f_0(\gamma(t)) < 0$ for any t and γ does not cut Γ. Apply now Proposition 21. $\square$

Assume from now on that there exists a neighborhood $\mathcal{V}$ of $\mathbf{0}$ in X, such that $f_0|_{S_0}$ and $\nabla_{\mathbf{h}} f_0$ do not vanish in $\mathcal{V} \cap S_0$.

The sketch of the proof of Theorem 1 is as follows. We first open the surface S_0 by means of a suitable opening blow-up mapping $\beta : M \times \mathbb{R}_{\geq 0} \to \mathbb{R}^n$ as defined in Proposition 8. Then we take a suitable resolution $\mathcal{R} = (\widetilde{S}, \widetilde{E}, \sigma)$ of the surface $S = \beta^{-1}(S_0)$ as in Theorem 9. We then pick a parametrization $\widetilde{\Phi} : \mathbb{S}^1 \times [0, \varepsilon] \to \widetilde{S}$ associated to $\mathcal{R}$. Writing $\Phi = \sigma \circ \widetilde{\Phi}$, the mapping $\beta \circ \Phi$ is a diffeomorphism from the open cylinder $\mathbb{S}^1 \times]0, \varepsilon]$ onto S_0. Thus, the pull-back $\widetilde{\mathbf{h}} := (\beta \circ \Phi)^* \mathbf{h}$ of the metric $\mathbf{h}$ is an analytic Riemannian metric on the open cylinder. If f^Φ denotes the composition $f_0 \circ \beta \circ \Phi$, then the pull-back $\bar{\xi} := (\beta \circ \Phi)^* \nabla_{\mathbf{h}} f_0$ is just the gradient vector field of f^Φ with respect to $\widetilde{\mathbf{h}}$, that is,

$$\bar{\xi} = (\beta \circ \Phi)^* \nabla_{\mathbf{h}} f_0 = \nabla_{\widetilde{\mathbf{h}}} f^\Phi.$$

The proof will be finished, using Proposition 22, once we have proved that $\overline{\xi}$ satisfies one of the two situations described there: either (a), dicritical or (b), non-monodromic.

Our proof will only deal with the metric $\mathbf{g}$ on $\mathbb{R}^n$ be the Euclidean metric. We can reduce to this case using Cartan-Janet's Theorem [15, 7]: an analytic Riemannian manifold can be locally isometrically embedded into an Euclidean space as an analytic submanifold equipped with the induced Riemannian structure.

Notation. Let Ω be a finite subset of $\mathcal{C}$ (such as for instance the exceptional set of a parameterization as in Proposition 14). In Definition 15 was introduced the notion Ω-u-a-r-a function on the open cylinder $\mathbb{S}^1 \times]0, \varepsilon]$. For any rational number $\nu \geq 0$, let $\mathcal{A}_{\geq \nu}$ be the real algebra of all the Ω-u-a-r-a functions $\psi : \mathbb{S}^1 \times]0, \varepsilon] \to \mathbb{R}$ for which the function $r^{-\nu}\psi$ is also an Ω-u-a-r-a function along $\mathcal{C}$. Let $\mathcal{A}_{>\nu} := \cap_{\mu > \nu} \mathcal{A}_{\geq \mu}$ be the ideal of $\mathcal{A}_{\geq \nu}$ of the functions

ψ such that the function $r^{-\nu}\psi$ vanishes identically on $\mathcal{C} \setminus \Omega$. In particular $\psi \in \mathcal{A}_{>\nu}$ means there exists a rational number $\nu' > \nu$ such that $\psi \in \mathcal{A}_{\geq\nu'}$.

We are dealing first with the CTC case in rather detailed fashion. It requires much more work than the OTC case, and this latter will follow from exactly the same arguments as those used in the CTC case.

Cuspidal case.
Assume that the tangent cone of S_0 at the origin is reduced to a single point. Take linear coordinates $(x_1, \ldots, x_{n-1}, z)$ at $\mathbf{0}$ adapted to S_0 which are also orthonormal coordinates for the Euclidean metric.

We consider an opening blow-up mapping of the form

$$\beta : M \times \mathbb{R}_{\geq 0} \to \mathbb{R}^n, \;\; (\mathbf{y}, z) \to (z^{eN}\mathbf{y} + \theta(z), z^N),$$

where $M = \mathbb{R}^{n-1}$ and e, N, θ are defined as in Proposition 8. We recall that $z \mapsto (\theta(z), z) = (\theta_1(z), \ldots, \theta_{n-1}(z), z^N)$ is a parametrization of an analytic half-branch in S_0. Let $m+1 \in \mathbb{N}_{\geq 1}$ be the minimum order at 0 with respect to z of the components θ_j. The cuspidal nature of the surface S_0 implies

$$m \geq N.$$

We define the following functions on the open cylinder

$$R = eN(y_1^2 + \cdots + y_{n-1}^2) \circ \Phi \;\; \text{and} \;\; U = \textstyle\sum_j ((y_j \circ \Phi)_\varphi)^2,$$

where the subscript φ stands for partial derivative with respect to the angular variable φ. Again R and U depend on the resolution and on the associated parameterization Φ considered, but both are u-a-r-a functions with respect to the resulting exceptional set Ω.

Lemma 25. *There is a non-empty open arc J of $\mathcal{C} \setminus \Omega$ with non empty interior in $\mathcal{C} \setminus \Omega$ along which the restricted function $U|_J$ is positive.*

Proof. We just have to show that the function U does not vanish on the whole of $\mathcal{C}\setminus\Omega$. If $U|_{\mathcal{C}\setminus\Omega} \equiv 0$, by definition of U, each $y_j \circ \Phi$ is locally constant when restricted to $\mathcal{C}\setminus\Omega$, and thus constant on $\mathcal{C}$ by continuity. Using (iii) of Proposition 14, this would imply the constancy of the coordinates y_j along $E = \sigma(\widetilde{\Phi}(\mathcal{C}))$, which is impossible since by construction $\dim E = 1$. $\square$

Using the coordinates (φ, r) in the open cylinder, the metric $\widetilde{\mathbf{h}}$ writes

$$\widetilde{\mathbf{h}} = (\beta \circ \Phi)^*\mathbf{g} = A(r,\varphi)dr^2 + 2B(r,\varphi)drd\varphi + C(r,\varphi)d\varphi^2. \tag{11}$$

The following lemma describes the coefficients of the metric $\widetilde{\mathbf{h}}$. The dominant part of each term of interest is explicit. It is important to remark that the statement neither deals with a fixed resolution $\mathcal{R}$, nor an associated parametrization. It is very useful and necessary to obtain the needed conclusions up to dominating resolutions of a given one as we will see.

Lemma 26. *There exists a resolution $\mathcal{R}^0$ of S such that, for any other resolution $\mathcal{R} \succeq \mathcal{R}^0$ and parameterization Φ associated to $\mathcal{R}$, we have the following description: There exist $s \in \mathbb{Q}_{\geq 0} \cup \{+\infty\}$ with*

$$s \geq eN + m, \tag{12}$$

an analytic power series $\psi(r)$ with $\psi(0) = 0$ and an u-a-r-a function H on $\mathbb{S}^1 \times]0, \varepsilon]$ which extends continuously to the circle $\mathcal{C}$ in such a way that $H|_C$ is not constant, such that we obtain the following expressions for the coefficients of $\widetilde{\mathbf{h}}$ in (11):

$$\begin{cases} A &= r^{2N-2}[N^2 + \psi(r)] + r^{eN+m-1}\overline{A_1} + r^{2eN-2}\overline{A_2} \\ B &= r^s H_\varphi + r^{2eN-1}R_\varphi + \overline{B} \\ C &= r^{2eN}U \end{cases} \tag{13}$$

where $\overline{A}_1, \overline{A}_2 \in \mathcal{A}_{\geq 0}$ and $\overline{B} \in \mathcal{A}_{>2eN-1}$ and with the convention that the term $r^s H_\varphi \equiv 0$ if $s = \infty$. Moreover, s and $\psi(r)$ depend neither on $\mathcal{R} \succeq \mathcal{R}^0$ nor on the parameterization Φ.

Proof. We start with a given resolution $\mathcal{R}^1$ of the surface S and adopt the notations above. Let $\mathbf{w} = \mathbf{y} \circ \Phi = (w_1, \ldots, w_{n-1})$ and $\theta(z) = z^{m+1}\overline{\theta}(z)$ with $\overline{\theta}(0) \neq \mathbf{0}$. Let $\lambda(r) := (1+m)\overline{\theta}(r) + r\overline{\theta}'(r) = r^{-m}\theta'(r)$ where the prime denotes the usual derivative. It is an analytic mapping which does not vanish at $r = 0$. From the expression of β, we deduce

$$\begin{aligned} (\beta \circ \Phi)^* \mathrm{d}x_n &= N r^{N-1} \mathrm{d}r \\ (\beta \circ \Phi)^* \mathrm{d}x_j &= [r^{eN-1}(eNw_j + r(w_j)_r) + r^m \lambda_j]\mathrm{d}r + r^{eN}(w_j)_\varphi \mathrm{d}\varphi. \end{aligned}$$

Note that each w_j is u-a-r-a and extends continuously on the whole bottom circle $\mathcal{C}$. But $(w_j)_r$ could even be unbounded. However, by Remark 13, each function $r(w_j)_r$ is u-a-r-a and belongs to $\mathcal{A}_{>0}$. Taking this property into account and since $\mathbf{g}$ is the Euclidean metric, we obtain

$$\begin{aligned} \widetilde{\mathbf{h}} &= (\beta \circ \Phi)^*(\mathrm{d}x_1^2 + \cdots + \mathrm{d}x_n^2) \\ &= [N^2 r^{2N-2} + r^{2m} \textstyle\sum_j \lambda_j^2 + r^{eN+m-1}(\sum_j eN\lambda_j w_j + \cdots) \\ &\quad + r^{2eN-2}(\textstyle\sum_j e^2N^2 w_j^2 + \cdots)]\mathrm{d}r^2 + \\ &\quad 2[r^{eN+m} \textstyle\sum_j \lambda_j (w_j)_\varphi + r^{2eN-1}(\sum_j eNw_j(w_j)_\varphi + \cdots)]\mathrm{d}r\mathrm{d}\varphi \\ &\quad + [r^{2eN} \textstyle\sum_j (w_j)_\varphi^2]\mathrm{d}\varphi^2, \end{aligned} \tag{14}$$

where $\cdots$ stands for an element of $\mathcal{A}_{>0}$. Let $\psi(r) := r^{2(m-N)+2} \sum_j \lambda_j^2(r)$. Since $m \geq N$ and $e > 1$ we can define $\overline{A}_1, \overline{A}_2 \in \mathcal{A}_{\geq 0}$ so that A, the coefficient of $\mathrm{d}r^2$ in (14) writes as in (13). Notice that $\psi(r)$ does not depend on the resolution or the parameterization.

On the other hand, the second summand of the coefficient of $\mathrm{d}r\mathrm{d}\varphi$ in (14) is given by $r^{2eN-1}R_\varphi + \overline{B}$ where $\overline{B} \in \mathcal{A}_{>2eN-1}$, while the coefficient of $\mathrm{d}\varphi^2$

in (14) is just $r^{2eN}U$.
In order to complete the expressions of (13), let us have a look at the first summand of the coefficient of $\mathrm{d}r\mathrm{d}\varphi$ in (14). Consider the function

$$h(\mathbf{y},z) = z^{eN+m}\sum_j \lambda_j(z)y_j,$$

defined and analytic in a neighborhood of E in $M \times \mathbb{R}$. Applying Proposition 16 to h, there exists a resolution $\mathcal{R}^0$ of the surface S so that, given any other resolution $\mathcal{R} \succeq \mathcal{R}^0$ and any associated parametrization Φ, the composition $h^\Phi = h \circ \Phi$ on the open cylinder $\mathbb{S}^1\times]0,\varepsilon]$ writes

$$h^\Phi(\varphi,r) = P(r) + r^s H(\varphi,r) \tag{15}$$

where $s \in \mathbb{Q}_{\geq 0}$, $P(r)$ is a $\mathbb{Q}$-generalized polynomial and H is an u-a-r-a function that extends continuously to the bottom circle $\mathcal{C}$ such that $H|_\mathcal{C}$ is either not constant if $s < +\infty$ or, for $s = \infty$, H is identically zero and $P(r)$ is a convergent Puiseux series. The first summand of the coefficient of $\mathrm{d}r\mathrm{d}\varphi$ in (14) is just the partial derivative $(h^\Phi)_\varphi$ (identically zero if $s = \infty$), equal to $r^s H_\varphi$ by (15). Since $H \in \mathcal{A}_{\geq 0}$, we get Inequality (12). Moreover, Proposition 16 also ensures that the exponent s does not depend on the given resolution $\mathcal{R}$ (or on the parametrization Φ) as long as it dominates $\mathcal{R}^0$. This completes the proof of the lemma. $\square$

Now consider the function $f = f_0 \circ \beta$ which is an analytic function in a neighborhood of E in $M \times \mathbb{R}$. We can assume that $f_0(\mathbf{0}) = 0$ so that $f|_E \equiv 0$. Applying Proposition 16 to f, there exists a resolution $\mathcal{R}$ of S and an associated parametrization Φ such that expression (5) is valid: We can write either $f^\Phi = f^\Phi(r)$ as a convergent Puiseux series only depending on r or else

$$f^\Phi = a_0 r^{\alpha_0} + \ldots + a_m r^{\alpha_m} + r^\alpha F(\varphi,r) = P(r) + r^\alpha F(\varphi,r), \tag{16}$$

where $a_j \in \mathbb{R}\setminus\{0\}$, $0 \leq \alpha_0 < \cdots < \alpha_m < \alpha$ are non-negative rational numbers and F is u-a-r-a and extends continuously to the whole cylinder and the restriction $F|_\mathcal{C}$ is not a constant function. Recall moreover that $P(r)$, as well as the exponent α, are independent of the parameterization and of any resolution $\mathcal{R}'$ which dominates $\mathcal{R}$. Therefore, we can suppose that $\mathcal{R}$ and Φ are chosen such that the expressions (13) for the coefficients of the transformed metric in Lemma 26 also hold.

In what follows, we treat the degenerate case when f^Φ only depends on r as the case (16) with α as big as we want but without requiring that $F|_C$ is not constant.

Up to a multiplication by a function that does not vanish on the open cylinder, the differential equation provided by the transformed vector field

$\bar{\xi}$ writes:

$$\left\{\begin{array}{rcl}\dot{r} & = & [P'(r)+(r^{\alpha}F)_r]C \;-\; r^{\alpha}F_{\varphi}B \\ \dot{\varphi} & = & [P'(r)+(r^{\alpha}F)_r]B \;+r^{\alpha}F_{\varphi}A\end{array}\right. \tag{17}$$

We have several cases to deal with.

Case (1): $\widetilde{\alpha} = 2N-2+\alpha < \min\{s+\alpha_0-1, 2eN+\alpha_0-2\}$.
From the expression of $\dot{\varphi}$ in (17) we obtain

$$\dot{\varphi} = r^{\widetilde{\alpha}}(N^2F_{\varphi}+\Delta), \text{ whith } \Delta \in \mathcal{A}_{>0}.$$

On the other hand, from

$$2eN+\alpha-1 \geq 2eN+\alpha_0-1 > \widetilde{\alpha}+1 \quad \text{and} \quad s+\alpha \geq s+\alpha_0 > \widetilde{\alpha}+1$$

we deduce $\dot{r} \in \mathcal{A}_{>\widetilde{\alpha}+1}$. Eventually $\bar{\xi}$ is in the non-monodromic case (9) of Proposition 22.

Case (2): $\widetilde{\alpha} = 2N-2+\alpha \geq \min\{s+\alpha_0-1, 2eN+\alpha_0-2\}$.
Using (12) in this case, we find $\alpha > \alpha_0$ and thus $P \neq 0$ and $a_0\alpha_0 \neq 0$. We distinguish two sub-cases:
Case (2a): $2eN+\alpha_0-2 < s+\alpha_0-1$.
We deduce first that $\dot{\varphi} = r^{2eN+\alpha_0-2}(G_{\varphi}+\Delta)$ where

$$G = \left\{\begin{array}{ll}\alpha_0 a_0 R, & \text{if } 2eN+\alpha_0-2 < \widetilde{\alpha}; \\ \alpha_0 a_0 R + N^2 F & \text{if } 2eN+\alpha_0-2 = \widetilde{\alpha}.\end{array}\right. \tag{18}$$

and $\Delta \in \mathcal{A}_{>0}$. We observe that the function $r^{2eN+\alpha_0-2}G$ is of the form $g^{\Phi} = g\circ\Phi$ for some ramified-analytic function g on a neighborhood of E in $M\times\mathbb{R}_{\geq 0}$. Thus the function G is continuous on the cylinder and u-a-r-a.
On the other hand, the term $r^{\alpha}F_{\varphi}B$ in the expression for $\dot{r}$ in (17) belongs to $\mathcal{A}_{>2eN+\alpha_0-1}$. Thus $\dot{r} = r^{2eN+\alpha_0-1}(a_0\alpha_0U+\Upsilon)$ for $\Upsilon \in \mathcal{A}_{>0}$.
If $G|_{\mathcal{C}}$ is not constant, our situation is non-monodromic in the sense of Proposition 22. Otherwise, thanks to Lemma 25, it is the dicritical case of Equation (8) with $\mu = 2eN+\alpha_0-1$.

Case (2b): $s+\alpha_0-1 \leq 2eN+\alpha_0-2$.
This case is the most difficult since several of the terms involved in the expression of $\dot{\varphi}$ may be of the same order with respect to r so that all the "initial parts" which are derivatives with respect to φ of a function may cancel.

From Equation (12) and $m \geq N$, we first find

$$\alpha+2eN-1 \geq \alpha+s \geq 2eN+\alpha_0+1. \tag{19}$$

Using Equations (13) and (19) we get $r^{\alpha}F_{\varphi}B \in \mathcal{A}_{\geq 2eN+\alpha_0+1}$. Thus, in (17), we obtain

$$\dot{r} = r^{2eN+\alpha_0-1}(a_0\alpha_0U+\Upsilon), \quad \text{for } \Upsilon \in \mathcal{A}_{>0}. \tag{20}$$

Using again (19) we find the following estimates on the order of some terms in the expression of $\dot\varphi$:

$$\begin{aligned} P'(r)\overline{B} &\in \mathcal{A}_{>2eN+\alpha_0-2},\\ (r^\alpha F_\varphi)r^{eN+m-1}\overline{A}_1 &\in \mathcal{A}_{>2eN+\alpha_0},\\ (r^\alpha F_\varphi)r^{2eN-2}\overline{A}_2 &\in \mathcal{A}_{\geq 2eN+\alpha_0}\\ (r^\alpha F)_\varphi B &\in \mathcal{A}_{\geq 2eN+\alpha_0}. \end{aligned}$$

This allow us to write $\dot\varphi$ as

$$\dot\varphi = -P'(r)[r^s H_\varphi + r^{2eN-1}R_\varphi] + r^{\widetilde{\alpha}}(N^2+\psi(r))F_\varphi + \Delta = G_\varphi + \Delta, \tag{21}$$

where $\Delta \in \mathcal{A}_{>2eN+\alpha_0-2}$ and

$$G = -P'(r)[r^s H + r^{2eN-1}R] + r^{\widetilde{\alpha}}(N^2+\psi(r))F.$$

Once again $G = g^\Phi = g\circ\Phi$ for some function g in a neighborhood of E in $M\times\mathbb{R}_{\geq 0}$ which is ramified analytic along E. From the definition of R and Remark 18 for H and F, the function g depends only on f and β but not on the resolution $\mathcal{R}$ or on an associated parameterization Φ. So, up to further finitely many blowing-ups and applying Proposition 16, we can assume that

$$G(\varphi, r) = Q(r) + r^\rho \widetilde{G}(\varphi, r)$$

where Q is a $\mathbb{Q}$-generalized polynomial, $\rho\in\mathbb{Q}_{>0}$ and $\widetilde{G}$ is an u-a-r-a function which extends continuously to the bottom circle $\mathcal{C}$ with, either $\widetilde{G}\mid_{\mathcal{C}}$ is not constant or ρ can be chosen as large as we want (we just need $\rho > 2eN+\alpha_0-2$).
Two cases are to be considered:
- If $\rho\leq 2eN+\alpha_0-2$, Equation (21) writes $\dot\varphi = r^\rho(\widetilde{G}_\varphi + \widetilde{\Delta})$ for $\widetilde{\Delta}\in\mathcal{A}_{>0}$. Combined with (20), we find a non-monodromic situation (9).
- If $\rho > 2eN+\alpha_0-2$ then we are in the dicritical situation (8) with $\mu = 2eN+\alpha_0-1$ thanks to Lemma 25.

This finishes the proof of the main theorem in the CTC case.

Open tangent cone case.
Let us have a quick look at the open tangent case (OTC).

We first choose orthonormal coordinates $x = (x_1,\ldots,x_n)$. Consider the opening blow-up mapping $\beta : M\times\mathbb{R}_{\geq 0}\to\mathbb{R}^n$, $(\mathbf{y}, z)\to z\mathbf{y}$, where $M = \mathbb{S}^{n-1}$, as in (2), and let $S = \beta^{-1}(S_0)$. Let $\mathcal{R} = (\widetilde{S}, \widetilde{E}, \sigma)$ be a (S,z)-resolution of S as in Theorem 9 and pick an associated parameterization $\Phi : \mathbb{S}^1\times[0,\varepsilon]\to S$ satisfying the conditions of Proposition 14.

Let $\widetilde{\mathbf{h}} = (\beta\circ\Phi)^*\mathbf{h}$ be the pull-back metric in the open cylinder. With computations similar to those done in the proof of Lemma 26, and using Remark 13, we can write

$$\widetilde{\mathbf{h}} = (1+\overline{A})\mathrm{d}r^2 + 2r\overline{B}\mathrm{d}r\mathrm{d}\varphi + r^2 U\mathrm{d}\varphi^2,$$

where $U = \sum_i (w_i)_\varphi^2$, $\overline{A} = r^2 \sum_i (w_i)_r^2$ and $\overline{B} = r \sum_i (w_i)_r (w_i)_\varphi$, since $\sum_i w_i^2 = 1$. We note that $\overline{A}, \overline{B} \in \mathcal{A}_{>0}$.

Writing $f^\Phi = f_0 \circ \beta \circ \Phi$, the pull-back $\bar{\xi} = (\beta \circ \Phi)^* \nabla_{\mathbf{h}} f_0$ of the restricted gradient vector field has the following associated system of differential equations (up to the multiplication by the determinant of the metric $\widetilde{\mathbf{h}}$):

$$\left\{ \begin{array}{l} \dot{r} = r^2 U (f^\Phi)_r - r\overline{B}(f^\Phi)_\varphi \\ \dot{\varphi} = -r\overline{B}(f^\Phi)_r + (1+\overline{A})(f^\Phi)_\varphi . \end{array} \right. \tag{22}$$

We consider cases (a) or (b) of Proposition 16 for the function $f := f_0 \circ \beta$.

In case (a) the function f depends only on z and thus $(f^\Phi)_\varphi \equiv 0$. Dividing (22) by $r(f^\Phi)_r$, which does not vanish on the open cylinder, we obtain the dicritical situation of Proposition 22.

In case (b), we assume that the resolution $\mathcal{R}$ is such that

$$f^\Phi(\varphi, r) = P(r) + r^\alpha F(\varphi, r),$$

where $P(r) = a_0 r^{\alpha_0} + \cdots + a_m r^{\alpha_m}$, $\alpha_m < \alpha$ if $a_0 \neq 0$, and F extends continuously to the bottom circle $\mathcal{C}$ and its restriction $F|_\mathcal{C}$ is not constant. If $\alpha \leq 1$, Equations (22) become

$$\dot{r} \in \mathcal{A}_{\geq \alpha+1} \text{ and } \dot{\varphi} = r^\alpha [F_\varphi + \Delta],$$

where $\Delta \in \mathcal{A}_{>0}$. We have a non-monodromic situation as in (9) and we are done.
If $\alpha > 1$, we have two sub-cases:
- Case $\alpha = \alpha_0$. This means that $P \equiv 0$. Thus

$$r(f^\Phi)_r = r^\alpha(\alpha F + rF_r) \text{ and } (f^\Phi)_\varphi = r^\alpha F_\varphi.$$

Using Remark 13 and Equation (22), we find $\dot{\varphi} = r^\alpha(F_\varphi + \Delta)$ where $\Delta \in \mathcal{A}_{>0}$. We still have $\dot{r} \in \mathcal{A}_{\geq\alpha+1}$ and thus we obtain a non-monodromic situation.
Case $\alpha > \alpha_0$. We deduce

$$\dot{r} = r^{\alpha_0+1}(a_0\alpha_0 U + \Upsilon) \quad \text{and} \quad \dot{\varphi} \in \mathcal{A}_{>\alpha_0}$$

with $\Upsilon \in \mathcal{A}_{>0}$. We obtain the dicritical situation (8) with $\mu = \alpha_0 + 1$ thanks to Lemma 25.

This finishes all the cases and the proof of the Main Theorem 1.

7. Consequences

Now we prove Corollary 2 and Theorem 4 as consequences of our main result, Theorem 1. We also sketch a proof of the more elementary Proposition 3.

Proof of Corollary 2. Suppose that $|\gamma| \subset S_0$, a connected component of $X \setminus \{\mathbf{0}\}$. Let β be an opening blowing-up of S_0 and let $\mathcal{R} = (\widetilde{S}, \widetilde{E}, \sigma)$

be a resolution of the surface $S = \beta^{-1}(S_0)$. Theorem 1 ensures that the lifting $\mathcal{L} = (\beta \circ \sigma)^{-1}(|\gamma|)$ accumulates at a single point p of $\widetilde{E}$. Thus $\mathcal{L}$ is contained in a simply connected semi-analytic open set of the strict transform $S' = \sigma^{-1}(S)$ where the foliation $\mathcal{F}$ has no singularities. Using Haefliger's theorem ([12, 21, 25]), we deduce that $\mathcal{L}$ is a Rolle's leaf of $\mathcal{F}$ and thus a pfaffian set. Its image $|\gamma| = \beta(\sigma(\mathcal{L}))$ is a sub-pfaffian set in $\mathbb{R}^n$ since σ and β are proper mappings. $\square$

Proof of Proposition 3. Let S_0 be a connected component of $X \setminus \{\mathbf{0}\}$, homeomorphic to the semi-open cylinder $\mathbb{S}^1 \times]0, \varepsilon]$ for ε small. Denote by $\mathcal{C}_\varepsilon$ the image of $\mathbb{S}^1 \times \{\varepsilon\}$ by such homeomorphism. We consider two cases.

Case 1: The function $f_0|_{S_0}$ is negative and has no critical point.
Let $a_0 < 0$ be the minimum of the function f_0 restricted to $\mathcal{C}_\varepsilon$. Consider a point $p \in S_0$ for which $a_0 < f_0(p) < 0$ and let γ_p be the trajectory of the restricted gradient vector field $\nabla_{\mathbf{h}} f_0$ starting at p. Since $t \to f_0(\gamma_p(t))$ is increasing, γ_p is defined for all positive t and $\lim_{t\to\infty} \gamma_p(t) = \mathbf{0}$.

Case 2. Suppose $f_0^{-1}(0) \cap S_0 \neq \emptyset$.
Up to taking $-f_0^2$ instead of f_0, we assume that $f_0 \leq 0$ on $\operatorname{clos}(S_0)$ and that $Z_0 = f_0^{-1}(0) \cap \operatorname{clos}(S_0)(= \operatorname{crit}(f_0\,|_{S_0}))$ intersects with S_0.
Let U be a connected component of $S_0 \setminus Z_0$. Since S_0 is topologically a cylinder and Z_0 consist of finitely many analytic half-branches at $\mathbf{0}$ (up to taking ε smaller), the component U is simply connected. In fact, we can take a triangle Σ in the plane with sides $\sigma_1, \sigma_2, \sigma_3$ and a continuous map $\kappa : \Sigma \to \operatorname{clos}(U)$ restricting to a diffeomorphism between $\Sigma \setminus (\sigma_1 \cup \sigma_2)$ and U, sending each of the sides σ_1 or σ_2 homeomorphically to a half-branch of Z_0 and sending the side σ_3 onto $\operatorname{clos}(U) \cap \mathcal{C}_\varepsilon$. Note that, if there are at least two half-branches of Z_0 then κ is a homeomorphism, otherwise $\operatorname{clos}(U) = \operatorname{clos}(S_0)$ and κ is just a quotient map gluing the two sides σ_1, σ_2 together.

Since $\operatorname{clos}(U)$ is invariant, we can carry $\nabla_{\mathbf{h}} f_0$ onto Σ via κ (which is singular along $\sigma_1 \cap \sigma_2$). It will be denoted by χ_0 while we will denote $g_0 = \kappa^* f_0$ and $\mathbf{v} = \sigma_1 \cap \sigma_2 = \kappa^{-1}(\mathbf{0})$. We just have to prove that there exists a trajectory of χ_0 accumulating to $\mathbf{v}$.

We use the following properties:

(1) Up to taking a smaller ε, each point $x \in \sigma_1 \cup \sigma_2 \setminus \{\mathbf{v}\}$ is the accumulation point of a unique trajectory of χ_0.
(2) No non-stationary trajectory of χ_0 can have its α-limit point and its ω-limit point both in $\sigma_1 \cup \sigma_2 \setminus \{\mathbf{v}\}$.

The first property is easy to prove using local coordinates or using the classical Łojasiewicz's retraction map of the gradient (cf. [18]). The second one is a consequence of the fact that $g_0(\sigma_1 \cup \sigma_2) = 0$ and that g_0 increases along trajectories of χ_0.

Claim. *There exists $t_\varepsilon < 0$ such that for $t \in]t_\varepsilon, 0[$, the fiber $g_0^{-1}(t) \subset \Sigma$ is connected.*
Proof of the Claim. Each connected component of a (non-empty) fiber $g_0^{-1}(t)$ with $t < 0$ is either homeomorphic to a circle or to a closed segment with extremities on σ_3. Since g_0 has no critical points in the interior of the triangle Σ, the first case cannot occur. On the other hand, the restriction $g_0|_{\sigma_3}$ vanishes only at the extremities. If we take t_ε equal to the maximum of the critical values of this restriction, g_0 takes any value $t \in]t_\varepsilon, 0[$ exactly twice along σ_3 and this proves the claim. □

For $i = 1, 2$, choose $x_i \in \sigma_i \setminus \{v\}$ and let γ_i be the trajectory of χ_0 accumulating to x_i. Take t_0 with $t_\varepsilon < t_0 < 0$ such that γ_i cuts the fiber $g_0^{-1}(t_0)$, necessarily in a single point y_i. Let I be the closed segment in $g_0^{-1}(t_0)$ joining y_1 and y_2. Consider the domain Λ in Σ enclosed by the piecewise smooth closed curve formed by the segments $[x_i, \mathbf{v}]$ in σ_i, $[y_i, x_i]$ in γ_i and I. By construction, χ_0 enters Λ only through the segment I and leaves Λ positively invariant.

For each z in one of the semi-open sides $[x_1, \mathbf{v}[$ or $[x_2, \mathbf{v}[$, thanks to properties (1) and (2) above, there exists a unique point $\tau(z) \in I$ such that the trajectory starting at $\tau(z)$ accumulates to z for positive infinite time. Moreover, orienting positively I from y_1 to y_2, we find that $\tau(z) < \tau(w)$ whenever $z \in [x_1, \mathbf{v}[$ and $w \in [x_2, \mathbf{v}[$, or $z \in [x_1, \mathbf{v}[$ and $w \in]z, \mathbf{v}[$, or $w \in [x_2, \mathbf{v}[$ and $z \in]w, \mathbf{v}[$.

Let $a = \sup\{\tau(z)/z \in [x_1, \mathbf{v}[\}$ and $b = \inf\{\tau(z)/z \in [x_2, \mathbf{v}[\}$. Thus $a \leq b$ and for every point $y \in [a, b]$ in the segment I, the trajectory of χ_0 starting at y accumulates to $\mathbf{v}$. □

Proof of Theorem 4. We begin with the definition of *formal asymptotic expansion.* A formal curve $\widehat{\Gamma}$ at the origin of $\mathbb{R}^n$ is a formal Puiseux parameterization $\widehat{\Gamma}(T) = (\widehat{\Gamma}_1(T), \ldots, \widehat{\Gamma}_{n-1}(T), T^N)$, where each $\widehat{\Gamma}_i$ is a formal power series in the single indeterminate T with no constant term. A trajectory γ has an asymptotic expansion $\widehat{\Gamma}$ at the origin if it can be smoothly parameterized as

$$z \to \gamma(z) = (\gamma_1(z), \ldots, \gamma_{n-1}(z), z^N), z > 0$$

and the component γ_i admits $\widehat{\Gamma}_i$ as expansion.

If the critical locus of $f_0|_{S_0}$ is not empty then, each connected component of this locus in a semi-analytic invariant set the restricted gradient, so point (ii) is true.

We assume the restricted gradient of f_0 does not vanish in S_0.
Since any restricted gradient trajectory γ is not oscillating at $\mathbf{0}$, the function $z \circ \gamma(t)$ decreases strictly to 0 as $t \to +\infty$. Thus it admits a continuous parameterization $z \to \gamma(z) = (\gamma_1(z), \ldots, \gamma_{n-1}(z), z)$, for $z \geq 0$, which is analytic for $z > 0$.

Let β be an opening blowing-up of S_0 and $S = \beta^{-1}(S_0)$. Let $\mathcal{R} = (\widetilde{S}, \widetilde{E}, \sigma)$ be a resolution of S and let $\mathcal{R}' = (S', E', \sigma')$ be the strict resolution associated to $\mathcal{R}$ as in Theorem 9. Let $\widetilde{\mathbf{h}} = (\beta \circ \sigma)^* \mathbf{g}$ and $\widetilde{f} = f_0 \circ \beta \circ \sigma$. The metric $\widetilde{\mathbf{h}}$ degenerates along the divisor $\widetilde{E}$, so the gradient vector field $\nabla_{\widetilde{\mathbf{h}}} \widetilde{f}$ is defined only on $\widetilde{S} \setminus \widetilde{E}$. However, we can define a one-dimensional analytic foliation $\widetilde{\mathcal{F}}$ in $\widetilde{S}$ whose singular set $\mathrm{sing}(\widetilde{\mathcal{F}})$ is a finite subset of $\widetilde{E}$ and such that $\nabla_{\widetilde{\mathbf{h}}} \widetilde{f}$ is a local generator of $\widetilde{\mathcal{F}}$ at any point of $\widetilde{S} \setminus \widetilde{E}$.

The reduction of singularities of an analytic foliation on a smooth surface ([26]) and the compactness of $\widetilde{E}$ ensure we can assume that any singularity $p \in \mathrm{sing}(\widetilde{\mathcal{F}})$ is simple: a local generator ξ_p has a non-nilpotent linear part at p with eigenvalues λ and $\mu \neq 0$.

Let $\Sigma := E' \cap \mathrm{sing}(\widetilde{E})$ be the finite set of singular points of the strict divisor E'.

In order to complete the proof, we will show that only situations (1) or (2) below happen and the result holds true in both cases.

(1) *Dicritical situation:* There is a point $p \in E' \setminus \Sigma$, not singular for $\widetilde{\mathcal{F}}$, and such that E' is transverse to $\widetilde{\mathcal{F}}$ at p. The leave $\mathcal{L}_p$ of $\widetilde{\mathcal{F}}$ through p is a non-singular analytic curve and transverse to E' at p. The image $\beta \circ \sigma(\mathcal{L}_p \cap S')$ is an analytic separatrix for the restricted gradient on S_0 accumulating to the origin. In fact through each point $q \in E'$ in a neighborhood of p, there is a unique analytic separatrix through q.

(2) *A non corner singularity:* The strict divisor E' is an invariant set of $\widetilde{\mathcal{F}}$ and there is a point $p \in (E' \setminus \Sigma) \cap \mathrm{sing}(\widetilde{\mathcal{F}})$. A local generator ξ_p of $\widetilde{\mathcal{F}}$ at p has a linear part with two real eigenvalues. One eigen-direction is tangent to E' and the other one is transverse to E'. The theory of local invariant manifolds (see for instance [14]) provides a formal invariant non-singular manifold $\widehat{W}$ at p which is tangent to the transverse eigen-direction[1]. We also get a C^∞ invariant manifold W through p having $\widehat{W}$ as asymptotic expansion at p. The image $\beta \circ \sigma(W \cap S')$ is the desired characteristic trajectory γ of the restricted gradient.

In order to find a contradiction, we assume that neither case (1) or (2) above holds. Thus E' is invariant for $\widetilde{\mathcal{F}}$ and $\mathrm{sing}(\widetilde{\mathcal{F}}) \cap E' = \Sigma$.

Since any point $p \in \Sigma$ is a simple singularity, the two components of $\widetilde{E}$ at p are the two local analytic separatrices of $\widetilde{\mathcal{F}}$ at p.

Let $\{Q_p^j\}_{j=1,2,3,4}$ be the open "quadrants" of $\mathcal{U}_p \setminus \widetilde{E}$ in a small coordinate neighborhood $\mathcal{U}_p$ of p in $\widetilde{S}$. Let $J(p) \subset \{1,2,3,4\}$ be the subset of j for which $Q_p^j \subset S'$ (see part (iv) of Theorem 9). For $j \in J(p)$, the quadrant Q_p^j is either:

- of *saddle type*, if any trajectory of the restricted gradient $\nabla_{\widetilde{\mathbf{h}}} \widetilde{f}$ through a

[1] If the corresponding eigenvalue is non-zero, Briot-Bouquet's theorem guarantees the convergence of $\widehat{W}$

point in Q_p^j escapes from Q_p^j for positive and negative time;
- of *node-source type*, if any trajectory escapes for positive time but accumulates to p for negative time;
- of *node-sink type*, if each trajectory escapes for negative time but accumulates to p for positive time.

We have two possibilities:

(a) Each quadrant Q_p^j is of saddle-type for all $p \in \Sigma$ and all $j \in J(p)$ or there are no singularities at all ($\Sigma = \emptyset$). In this case, classical arguments show that the dynamics is "monodromic" in a neighborhood of E' in S': there exist an analytic half-branch Λ through a point $q \in E' \setminus \Sigma$ contained in S', transverse to $\widetilde{\mathcal{F}}$, a neighborhood Λ_0 of q in Λ and a *Poincaré first return map* $P : \Lambda_0 \to \Lambda$ such that for $x \in \Lambda_0$ given, the leaf $\mathcal{L}_x$ through x cuts again Λ at $P(x)$ after visiting all the quadrants Q_p^j. Since a gradient vector field cannot have closed orbits, P has no fixed points. Poincaré-Bendixson's type arguments imply there are leaves of $\widetilde{\mathcal{F}}$ in S' accumulating to the whole divisor E', thus producing spiraling trajectories of the restricted gradient, which contradicts Theorem 1.

(b) There is a quadrant $Q_{p_0}^{j_0} \subset S'$ of node type for some singularity $p_0 \in \Sigma$ and some $j_0 \in J(p_0)$. Suppose for instance that it is of node-source type (the case of node-sink type is analogous in reversing time). Consider one of the local analytic separatrices of $\widetilde{\mathcal{F}}$ at p_0. There is a connected component of $E' \setminus \Sigma$, say $\mathcal{E}_1$, which meets such a separatrix. Since E' is invariant, $\mathcal{E}_1$ is a leaf of $\widetilde{\mathcal{F}}$. Let $p_1 \in \Sigma$ be the other accumulation point of $\mathcal{E}_1$, different from p_0. The flow-box theorem shows that there is a point $q_0 \in Q_{p_0}^{j_0}$ such that the trajectory γ_0 issued from the point q_0 visits a point in a quadrant $Q_{p_1}^{j_1}$ for some $j_1 \in J(p_1)$.

By definition of a node-source type, the quadrant $Q_{p_1}^{j_1}$ cannot be of node-source type. If $Q_{p_1}^{j_1}$ is of saddle-type, we consider the connected component $\mathcal{E}_2$ of $E' \setminus \Sigma$ meeting the local analytic separatrix at p_1 which is not contained in $\mathcal{E}_1$ and the point $p_2 \in \Sigma$ such that $\mathrm{clos}(\mathcal{E}_2) \setminus \mathcal{E}_2 = \{p_1, p_2\}$. In this case, choosing q_0 in the initial quadrant $Q_{p_0}^{j_0}$ sufficiently close to p_0, we can suppose that the trajectory γ_0 also visits some quadrant $Q_{p_2}^{j_2}$ for some $j_2 \in J(p_2)$.

Continuing this way, if all the visited quadrants are of saddle-type we construct a sequence of singularities $p_1, p_2, \ldots$ different from p_0. Since Σ is finite, we create a cycle $p_l, p_{l+1}, \ldots, p_m = p_l$, with l minimum for this property. Since $\mathcal{E}_m$ is not equal to $\mathcal{E}_l$ (otherwise $p_{m-1} = p_{l-1}$ against the minimality of l) we find three local analytic separatrices through p_l, say $\mathcal{E}_l, \mathcal{E}_{l+1}, \mathcal{E}_m$, which is a contradiction with the fact that p_l is a simple singularity.

Thus, there exist $p_k \in \Sigma$, for some $k \geq 1$ and a quadrant $Q_{p_k}^{j_k}$ of node-sink type, for some $j_k \in J(p_k)$, which intersects the trajectory γ_0. Then p_0 is the α-limit point of γ_0 and p_k a ω-limit point. Its image $\beta \circ \sigma \circ \gamma_0$ is a trajectory of the restricted gradient for which the origin is the α and the ω

limit point which is impossible since the function f_0 increases strictly along this trajectory.

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