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NON-INTERLACED SOLUTIONS OF 2-DIMENSIONAL SYSTEMS OF LINEAR ORDINARY DIFFERENTIAL EQUATIONS

O. LE GAL, F. SANZ, AND P. SPEISSEGGER

ABSTRACT. We consider a two-dimensional system of linear ordinary differential equations whose coefficients are definable in an o-minimal structure $\mathcal{R}$. We prove that either every pair of solutions at 0 of the system is interlaced, or the expansion of $\mathcal{R}$ by all solutions at 0 of the system is o-minimal. We also show that, if the coefficients of the system have a Taylor development of sufficiently large finite order, then the question of which of the two cases holds can be effectively determined in terms of the coefficients of this Taylor development.

INTRODUCTION

We consider a system of n ordinary differential equations of the form

$$(\mathrm{S}_G) \qquad \frac{dY}{dx} = G(x, Y), \quad 0 < x < a,$$

where $a > 0$, $G : (0, a) \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is of class C^1 and $Y = (Y_1, \ldots, Y_n)$. For the purposes of this paper, a **solution at 0 of** (S_G) is a C^1 map $Y : (0, \varepsilon) \longrightarrow \mathbb{R}^n$ satisfying (S_G) with $0 < \varepsilon < a$. We are interested in the following vague questions:

(a) What is the relative behavior between distinct solutions at 0 of (S_G)?
(b) What finiteness properties, relative to a given family of sets, does a solution Y at 0 of (S_G) have?

As is often the case, a considerable effort is needed to make these questions precise. Our approach here is inspired by the ways this was done (and some answers were given) by Rosenlicht [9] and Boshernitzan [1] in the Hardy field setting, and by Cano, Moussu and Sanz [2, 3] and Rolin, Sanz and Schäfke [10] in the setting of real analytic vector fields. Here we consider cases where either $n = 1$, or $n = 2$ and G is linear in the dependent variables Y and definable in some o-minimal structure. In these cases, we show how elementary methods from the field of ordinary differential equations combine naturally with a result from o-minimality to state and answer some of the strongest known versions of Questions (a) and (b). As the linearity assumption on G allows us to avoid the more subtle phenomena encountered in [2, 3] and [10], this paper can also serve as an introduction to the study of Questions (a) and (b). Finally, let us point out that, in the case where n is arbitrary and $G(x, Y) = AY$ with A constant, a complete classification regarding Questions (a) and (b) can be given in model-theoretic terms, see Miller [7] and Tychonievic, theorem 7 in [12].

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To make Question (a) precise in the case $n = 2$, we consider the following notion adapted from [3]: let $Y, Z : (0, \varepsilon) \longrightarrow \mathbb{R}$ be two distinct solutions at 0 of (S_G), and denote by $\theta(x) \in \mathbb{R}/2\pi\mathbb{Z}$ the angle between $(Y - Z)(x)$ and the point $(1, 0)$, for $x \in (0, \varepsilon)$. We say that Y and Z are **interlaced** if any continuous lifting $\tilde{\theta} : (0, \varepsilon) \longrightarrow \mathbb{R}$ of θ to $\mathbb{R}$ tends to $-\infty$ or to $+\infty$ as x approaches 0. (Intuitively speaking, Y and Z are interlaced if they twist around each other infinitely often as x approaches 0. Related definitions can also be found in Comte and Yomdin [4].) Question (a) can then be stated as follows: does (S_G) have two interlaced solutions at 0? Or: are any two distinct solutions at 0 of (S_G) interlaced?

Example 1. The general solution at 0 of the system

$$\frac{dY}{dx} = \begin{pmatrix} 0 & \frac{1}{x^2} \\ -\frac{1}{x^2} & 0 \end{pmatrix} Y$$

is of the form $Y(x) = (A\cos(1/x) + B\sin(1/x),\, A\sin(1/x) - B\cos(1/x))$, with $(A, B) \in \mathbb{R}^2$, so any two distinct solutions at 0 are interlaced.

In the cases of (S_G) considered below with $n = 2$, we shall see that if there exist two distinct solutions at 0 of (S_G) that are *not* interlaced, then *all* solutions at 0 of (S_G) have very strong finiteness properties. To formulate these finiteness properties and corresponding precisions of Question (b), we use some model-theoretic terminology: throughout this paper, we call "o-minimal structure" an o-minimal expansion of the real field, and we call a set "definable" if it is definable with real parameters. We refer the reader to van den Dries and Miller [5] for an introduction to o-minimality from a geometric point of view and for further general references on this topic. For example, the structure $\overline{\mathbb{R}}$ of the real field is o-minimal, and its definable sets are exactly the *semialgebraic* sets; see [5, Example 2.5(3)].

We now fix an o-minimal structure $\mathcal{R}$ and assume that G is definable in $\mathcal{R}$. For a solution Y at 0 of (S_G), we denote by $(\mathcal{R}, Y)$ the expansion of $\mathcal{R}$ by the graph of Y. (As is customary, we simply say in this situation that "$(\mathcal{R}, Y)$ is the expansion of $\mathcal{R}$ by Y".) The following represents a precise version of Question (b): is $(\mathcal{R}, Y)$ again o-minimal? The facts that every set definable in an o-minimal structure has finitely many connected components and that (by definition) structures are closed under first-order definability makes the statement "$(\mathcal{R}, Y)$ is o-minimal" one of the strongest finiteness properties we can hope to hold for Y. Of course, this question has a negative answer in general: for the solution $Y(x) = (\cos(1/x), \sin(1/x))$ of the system in Example 1, the structure $\left(\overline{\mathbb{R}}, Y\right)$ is not o-minimal.

The following stronger version of Question (b) is linked to our precision of Question (a) above: let Y and Z be two distinct solutions at 0 of (S_G), and denote by $(\mathcal{R}, Y, Z)$ the expansion of $\mathcal{R}$ by both Y and Z. Is $(\mathcal{R}, Y, Z)$ o-minimal? If Y and Z are interlaced, the answer to this question is negative. However, the situation can be quite subtle, as illustrated by the following example:

Example 2 (see example 5 in [10])**.** Consider the system

$$\frac{dY}{dx} = \begin{pmatrix} \frac{1}{x^2} & \frac{1}{x^2} \\ -\frac{1}{x^2} & \frac{1}{x^2} \end{pmatrix} Y + \begin{pmatrix} \frac{1}{x} \\ 0 \end{pmatrix}.$$

For every solution Y at 0 of this system, the structure $\left(\overline{\mathbb{R}}, Y\right)$ is o-minimal, but any two distinct solutions at 0 of this system are interlaced.

In this paper, we do not focus on Question (b) when (S_G) has interlaced solutions at 0. Instead, we establish several equivalent criteria for the existence of two interlaced solutions at 0, one of which is given in terms of an answer to the following version of Question (b): we let $\mathcal{R}_G$ be the expansion of $\mathcal{R}$ by *all* solutions at 0 of (S_G); we then ask whether $\mathcal{R}_G$ is o-minimal. This question has a positive answer in the case $n = 1$: note that in this case, the system (S_G) is a pfaffian system as defined in Example 1.3 of Speissegger [11]. In this situation, we let $\mathcal{R}'$ be the expansion of $\mathcal{R}$ by *all* solutions at 0 of (S_G), for *all* $G : (0, a) \times \mathbb{R} \longrightarrow \mathbb{R}$ of class C^1 and definable in $\mathcal{R}$ (with a depending on G). The following is then a consequence of the main Theorem and Example 1.3 of [11]:

Fact 3. *The structure $\mathcal{R}'$ is o-minimal.*

It follows that $\mathcal{R}_G$ is o-minimal if $n = 1$. Not much is known for $n > 1$ in this generality; here we answer the above questions under the following additional assumption:

(L) $n = 2$ and the map G is linear in Y, that is, $G(x, Y) = A(x)Y + B(x)$ with $A : (0, a) \longrightarrow M_{2\times2}(\mathbb{R})$ and $B : (0, a) \longrightarrow M_{2\times1}(\mathbb{R})$ definable in $\mathcal{R}$ and C^1.

Note that, under the assumption (L), since G is Lipschitz in Y on any compact subset of $(0, a)$ there exists, for any given initial condition $(x_0, Y_0) \in (0, a) \times \mathbb{R}^2$, a unique function $f : (0, a) \longrightarrow \mathbb{R}^2$ such that $f(x_0) = Y_0$ and, for $\varepsilon \in (0, a)$, the restriction of f to $(0, \varepsilon)$ is a solution at 0 of (S_G).

To state our main result, under the assumption (L), we associate to (S_G) the Riccati equation

$$(\mathrm{R}_G) \qquad \frac{dy}{dx} = -a_{1,2}(x)y^2 + (a_{2,2}(x) - a_{1,1}(x))y + a_{2,1}(x),$$

where the $a_{i,j}$ are the entries of A. Note that (R_G) is, in particular, a system of the form (S_H) with $n = 1$, for a certain definable H; we use lower case letters for the solutions of (R_G) to distinguish them from the solutions of (S_G).

Theorem 4. *Assume that* (L) *holds. Then the following are equivalent:*

(1) *The system* (S_G) *has two distinct non-interlaced solutions at 0.*
(2) *No two distinct solutions at 0 of* (S_G) *are interlaced.*
(3) *All solutions at 0 of* (S_G) *are definable in the o-minimal structure* $(\mathcal{R}')'$.
(4) *The Riccati equation* (R_G) *has a solution at 0.*

Condition (3) of Theorem 4 implies, in particular, that $\mathcal{R}_G$ is o-minimal.

Each of the four conditions of Theorem 4 is difficult to verify for any given G. To obtain a more effectively verifiable, equivalent condition, we make the following more precise assumption: there exists $k \in \mathbb{N}$ such that

$(\mathrm{LT})_k$ condition (L) holds, and there exist a nonzero $d \in \mathbb{N}$ and a definable C^1 map $A_1 : [0, a) \longrightarrow M_{2\times2}(\mathbb{R})$ such that A_1 has a Taylor development of order $2k + 1$ at 0 and $A(x) = A_1(x^{1/d})/x^{k/d}$ for $x \in (0, a)$.

Remark 5. If $\mathcal{R} = \mathbb{R}_{\mathrm{an}}$, the o-minimal structure whose definable sets are exactly the *globally subanalytic* sets [5, Example 2.5(4)], then [5, 5.1(2)] shows that condition (L) implies condition $(\mathrm{LT})_k$, for some $k \in \mathbb{N}$ depending on A. The same is true for certain larger o-minimal structures; see for instance [6, Theorem 7.6].

The next proposition shows that, under the assumption $(\mathrm{LT})_k$, each of the conditions of Theorem 4 can be effectively verified in terms of the given Taylor development of A_1:

Proposition 6. *For each $k \in \mathbb{N}$, there is a semialgebraic set $E_k \subseteq \mathbb{R}^{8k+8}$, defined without parameters, such that whenever the system (S_G) satisfies condition $(\mathrm{LT})_k$ and $\mathbf{a}$ is the $(8k+8)$-tuple of all coefficients of the Taylor expansion of order $2k+1$ of A_1, then each of the conditions of Theorem 4 is equivalent to the condition that $\mathbf{a}$ belongs to E_k.*

1. Proof of Theorem 4

We prove (4) $\Rightarrow$ (3) and (1) $\Rightarrow$ (4); the implications (3) $\Rightarrow$ (2) $\Rightarrow$ (1) are obvious.

Proof of (4) $\Rightarrow$ (3). Assume that condition (4) holds. We claim that it suffices to prove the following:

(3') for any solution at 0 Y of (S_G), there exists $\delta > 0$ such that the restriction of Y to $(0, \delta)$ is definable in $(\mathcal{R}')'$.

Indeed, for $x_0 \in (0, a)$, we let $G^+_{x_0}(x, Y) := G(x_0 + x, Y)$ and $G^-_{x_0}(x, Y) := G(x_0 - x, Y)$; then (4) is satisfied for the systems $(\mathrm{S}_{G^+_{x_0}})$ and $(\mathrm{S}_{G^-_{x_0}})$, because G is of class C^1. Thus, if $Y : (0, \varepsilon) \mapsto \mathbb{R}^2$ is a solution at 0 of (S_G) then, by (3'), for any $x_0 \in [0, \varepsilon]$ there is an open subinterval I of $[0, \varepsilon)$ containing x_0 such that the restriction of Y to I is definable in $(\mathcal{R}')'$. By compactness, Y is therefore definable in $(\mathcal{R}')'$, which proves (3).

To prove (3'), we suppose that (R_G) has a solution y at 0. First, we claim that there exists a second solution z at 0 of (R_G) distinct from y, and that we may assume y never vanishes: indeed, since (R_G) is 1-dimensional, every solution at 0 of (R_G) is definable in $\mathcal{R}'$ by Fact 3. Moreover, the equation

$$u' = (-2a_{1,2}y + a_{2,2} - a_{1,1})u - a_{1,2} \tag{1.1}$$

is linear and therefore admits a solution $u : (0, \varepsilon) \longrightarrow \mathbb{R}$ satisfying any given initial condition (x_0, u_0) with $x_0 \in (0, \varepsilon)$. By Fact 3, with $\mathcal{R}'$ in place of $\mathcal{R}$, each of these solutions is definable in the o-minimal structure $(\mathcal{R}')'$ as well. Thus, at most one of these solutions vanishes identically at 0, and we now fix a solution u at 0 of (1.1) that does not vanish identically at 0. Shrinking ε if necessary, we may assume that $u(x) \neq 0$ for $x \in (0, \varepsilon)$. Then the function $z : (0, \varepsilon) \longrightarrow \mathbb{R}$ defined by $z(x) := y(x) + 1/u(x)$ is solution at 0 of (R_G) distinct from y. Arguing as before we may assume, after shrinking ε again and switching y and z if necessary, that y never vanishes.

Next, we claim that (S_G) can be diagonalized as follows: there exist C^1 maps $T, D : (0, \delta) \longrightarrow M_{2\times 2}(\mathbb{R})$ and $V : (0, \delta) \longrightarrow M_{2\times 1}(\mathbb{R})$, definable in $\mathcal{R}'$, such that $T(x)$ is invertible and $D(x)$ is diagonal for each x, and such that $Z : (0, \delta) \longrightarrow \mathbb{R}$ is a solution at 0 of

$$Z'(x) = D(x)Z(x) + V(x), \tag{1.2}$$

if and only if $Y := TZ$ is a solution at 0 of (S_G).

Assuming that such a diagonalization exists, it has to satisfy the following criteria: differentiating $Y = TZ$ gives

$$Y' = T'Z + TZ' = T'Z + TDZ + TV.$$

Since Y is a solution at 0 of (S_G), we now get

$$A(TZ)+B=T'Z+TDZ+TV,$$

that is,

$$(T'-AT+TD)Z=B-TV.$$

This last equality is, in particular, satisfied whenever

$$T'=AT-TD \tag{1.3}$$

and $V=T^{-1}B$. Thus, it suffices to find T and D as above such that equation (1.3) is satisfied; we then set $V:=T^{-1}B$. We now look for T and D of the form

$$T(x)=\begin{pmatrix}1 & s(x)\\ t(x) & 1\end{pmatrix},\qquad D(x)=\begin{pmatrix}d_1(x) & 0\\ 0 & d_2(x)\end{pmatrix}.$$

With these notations, (1.3) becomes

$$\left\{\begin{array}{rcl} 0 &=& a_{1,1}+a_{1,2}t-d_1\\ 0 &=& a_{2,2}+a_{2,1}s-d_2\\ t' &=& a_{2,1}+a_{2,2}t-d_1t,\\ s' &=& a_{1,2}+a_{1,1}s-d_2s\end{array}\right.$$

that is,

$$\left\{\begin{array}{rcl} d_1 &=& a_{1,1}+a_{1,2}t\\ d_2 &=& a_{2,2}+a_{2,1}s\\ t' &=& -a_{1,2}t^2+(a_{2,2}-a_{1,1})t+a_{2,1}\\ s' &=& -a_{2,1}s^2+(a_{1,1}-a_{2,2})s+a_{1,2}.\end{array}\right. \tag{1.4}$$

Now note that the third equation in (1.4) is (R_G), and if t is a nonvanishing solution at 0 of (R_G), then $s:=1/t$ is a solution at 0 of the fourth equation in (1.4). Thus, we define $t:=z$ and $s:=1/y$; this determines T and shows it is definable in $\mathcal{R}'$. Note that the determinant of $T(x)$ is $1-t(x)s(x)=1-Z(x)/Y(x)$, which does not vanish, so $T(x)$ is invertible for all x. Finally, the first two equations in (1.4) now determine D and show that D is definable in $\mathcal{R}'$. The claim is proved.

Now let Z be a solution at 0 of (1.2). Since D is diagonal, the system (1.2) consists of two 1-dimensional equations of type (S_H) for a certain H definable in $\mathcal{R}'$. Hence both components of Z are definable in $(\mathcal{R}')'$. Since T is definable in $\mathcal{R}'$, it follows that TZ is definable in $(\mathcal{R}')'$. On the other hand, every solution at 0 of (S_G) admits a restriction that is of the form TZ, for some solution Z at 0 of (1.2); so condition (3') follows. □

Proof of (1) $\Rightarrow$ (4). We assume that (R_G) does not admit any solution at 0 and establish the negation of (1). Note that condition (4) is satisfied if the germ at 0 of $a_{1,2}$ vanishes identically. So there exists $\delta\in(0,a)$ such that $a_{1,2}(x)$ has constant nonzero sign on $(0,\delta)$. We fix two distinct solutions $Y,Z:(0,\varepsilon)\longrightarrow\mathbb{R}$ of (S_G) with $\varepsilon\in(0,\delta)$ and a continuous lifting $\theta:(0,\varepsilon)\longrightarrow\mathbb{R}$ of the angle between $(Y-Z)(x)$ and $(1,0)$, and we define $r:(0,\varepsilon)\longrightarrow\mathbb{R}$ by $r(x):=|(Y-Z)(x)|$. Since $Y-Z$ satisfies the homogeneous equation $Y'=AY$, both θ and r are differentiable and satisfy

$$\left\{\begin{array}{rcl} r'\cos(\theta)-r\sin(\theta)\theta' &=& a_{1,1}r\cos(\theta)+a_{1,2}r\sin(\theta),\\ r'\sin(\theta)+r\cos(\theta)\theta' &=& a_{2,1}r\cos(\theta)+a_{2,2}r\sin(\theta),\end{array}\right.$$

from which we obtain

$$\theta'=-a_{1,2}(x)\sin^2(\theta)+(a_{2,2}-a_{1,1})(x)\cos(\theta)\sin(\theta)+a_{2,1}(x)\cos^2(\theta). \tag{1.5}$$

From (1.5), we get that θ' has constant nonzero sign on $\theta^{-1}\left(\frac{\pi}{2}+\pi\mathbb{Z}\right)$; in particular, $\theta^{-1}\left(\frac{\pi}{2}+\pi\mathbb{Z}\right)$ contains only isolated points in $(0,\varepsilon)$. On the other hand, $\theta^{-1}\left(\frac{\pi}{2}+\pi\mathbb{Z}\right)$ meets every right neighbourhood of 0: otherwise, $\tan(\theta)$ would be well defined on some interval $(0,\eta)$. But $y := \tan(\theta)$ is then continuous on this interval and satisfies the Riccati equation (R_G), as can be seen when dividing equation (1.5) by $\cos^2(\theta)$; this contradicts our assumption. Consequently, $\theta^{-1}\left(\frac{\pi}{2}+\pi\mathbb{Z}\right)$ is a decreasing infinite sequence of isolated points $(x_i)_{i\in\mathbb{N}}$ accumulating to 0.

We now fix an arbitrary $i \in \mathbb{N}$ and note that

$$\theta(x) = \theta(x_i) + \int_{x_i}^{x} \theta'(t)dt \quad \text{for } x \in (0, x_i).$$

Since θ is continuous and x_i and x_{i+1} are consecutive points in $\theta^{-1}\left(\frac{\pi}{2}+\pi\mathbb{Z}\right)$, the quantity $|\theta(x_{i+1}) - \theta(x_i)|$ is either 0 or π. Since $\theta'(x_i)$ and $\theta'(x_{i+1})$ have the same nonzero sign, we must have $\theta(x_{i+1}) \neq \theta(x_i)$. Thus, if $\sigma \in \{+,-\}$ is the sign of $a_{1,2}(x)$ on $(0,\varepsilon)$, it follows that $\theta(x_{i+1}) = \theta(x_i) + \sigma\pi$. Therefore, $\theta(x_i) \to \sigma\infty$ as $i \to \infty$. Since

$$\left| \int_{x_i}^{x} \theta'(t)dt \right| \leq \pi$$

for $x \in [x_{i+1}, x_i]$, it follows that $\theta(x) \to \sigma\infty$ as $x \to 0$, that is, Y and Z are interlaced. □

2. Proof of Proposition 6

Let $k \in \mathbb{N}$ and assume that (LT_k) holds, and let $\mathbf{a}$ be the $(8k+8)$-tuple of all coefficients of the Taylor expansion of order $2k+1$ of A_1. Replacing x by x^d, we may assume that $d = 1$. Then there exists maximal $r \leq k$, determined by $\mathbf{a}$, such that the system (S_G) can be written as

$$x^k \frac{dY}{dx} = \big(a(x)I + x^r C(x)\big)Y + B_1(x), \tag{2.1}$$

where $I \in M_{2\times 2}(\mathbb{R})$ is the identity matrix, $a(x)$ is a polynomial of degree strictly less than r, $C : [0,a) \longrightarrow M_{2\times 2}(\mathbb{R})$ and $B_1 : (0,a) \longrightarrow M_{2\times 1}(\mathbb{R})$ are of class C^1 and definable, $C(x)$ has a Taylor development of order $2k+1-r$ at 0, and either $r = k$, or $r < k$ and $C(0)$ is not of the form uI for any $u \in \mathbb{R}$. In this situation, the associated Riccati equation is

$$x^{k-r} y' = -c_{1,2}(x)y^2 + (c_{2,2}(x) - c_{1,1}(x))y + c_{2,1}(x), \tag{2.2}$$

where the $c_{i,j}$ are the coefficients of the matrix C.

We let $E'_{k,r} \subseteq \mathbb{R}^{8k+8}$ be the semialgebraic set consisting of all $\mathbf{b}$ such that every system (S_G) satisfying (LT_k), whose $(8k+8)$-tuple of all coefficients of the Taylor expansion of order $2k+1$ of A_1 is equal to $\mathbf{b}$, is of the form (2.1) with r maximal. Note that each $E'_{k,r}$ is defined without parameters and that the collection $\{E'_{k,r} :\ 0 \leq r \leq k\}$ forms a partition of $\mathbb{R}^{8k+8}$, for each k. It now suffices to prove the following:

Proposition 7. *There exists a semialgebraic set $E_{k,r} \subseteq E'_{k,r}$, defined without parameters and depending only on k and r, such that the Riccati equation* (2.2) *has a solution at 0 if and only if $\mathbf{a} \in E_{k,r}$.*

Proof of Proposition 6 from Proposition 7. We take $E_k := E_{k,0} \cup \cdots \cup E_{k,k}$. □

The following is the key ingredient in the proof of Proposition 7.

Lemma 8. (1) *If $r = k$, then the Riccati equation* (2.2) *has a solution at 0.*
(2) *If $r < k$ and $C(0)$ has two distinct eigenvalues, then the Riccati equation* (2.2) *has a solution at 0 if and only if the eigenvalues of $C(0)$ are real.*

In the proof of this lemma, we use the following remarks: given $c > 0$ and $d \in \mathbb{R}$, we denote by $y_{c,d}$ the germ at c of all solutions $y : (a, b) \longrightarrow \mathbb{R}$ of (2.2) such that $0 \le a < c < b$ and $y(c) = d$.

Remarks. (1) Let $P = (p_{i,j}) \in M_2(\mathbb{R})$, and denote by $P^*(2.2)$ the pullback of (2.2) via P. Then (2.2) has a solution at 0 if $P^*(2.2)$ has a solution at 0: assuming the latter has a solution y at 0 and arguing as in the proof of (4)⇒(1) of Theorem 4, we may assume that $y \ne -p_{1,1}/p_{1,2}$. Then the function $(p_{2,2}y + p_{2,1})/(p_{1,2}y + p_{1,1})$ is a solution at 0 of (2.2).

(2) Let $\varepsilon, \delta > 0$, $B := (0, \varepsilon)\times(-\delta, \delta)$ and $y : (a, b) \longrightarrow (-\delta, \delta)$, with $0 \le a < b \le \varepsilon$, be a maximal solution of (2.2) inside B. If $a > 0$ then, by the existence and uniqueness theorems for solutions of ODEs, $y(a) := \lim_{x\to a} y(x)$ exists and is equal to $\pm\delta$. In particular, if $y(a) = \delta$ in this case, then the germ $y_{a,\delta}$ cannot be strictly increasing, and if $y(a) = -\delta$, then $y_{a,-\delta}$ cannot be strictly decreasing. Similarly, $y(b) := \lim_{x\to b} y(x)$ exists and belongs to $[-\delta, \delta]$, and if $b < \varepsilon$, then $y(b) = \pm\delta$ as well and similar conclusions hold for $y_{b,\pm\delta}$ in this situation.

Proof of Lemma 8. If $r = k$, equation (2.2) can be divided by x^k, and the resulting equation is nonsingular and C^1 at 0 and therefore has a solution at 0. This proves part (1); for the proof of part (2), we assume that $r < k$ and distinguish two cases.

Case 1: $C(0)$ has real eigenvalues. Pulling back via a suitable $P \in M_2(\mathbb{R})$ and using Remark (1), we may assume that $c_{1,2}(0) = c_{2,1}(0) = 0$. In this situation, we consider the solutions of (2.2) inside a small box $B := (0, \varepsilon) \times (-\delta, \delta)$ with $\varepsilon, \delta > 0$, and we distinguish two subcases.

Subcase 1a: $(c_{2,2}(0) - c_{1,1}(0)) > 0$. Then for all sufficiently small ε (depending on δ) and any $c \in (0, \varepsilon]$, the germ $y_{c,-\delta}$ is strictly decreasing and the germ $y_{c,\delta}$ is strictly increasing. So by Remark (2), if $y : (a, b) \longrightarrow (-\delta, \delta)$ is a maximal solution inside B of (2.2), we must have $a = 0$, so we are done in this subcase.

Subcase 1b: $(c_{2,2}(0) - c_{1,1}(0)) < 0$. Then for all sufficiently small ε (depending on δ) and any $c \in (0, \varepsilon]$, the germ $y_{c,-\delta}$ is strictly increasing and the germ $y_{c,\delta}$ is strictly decreasing. By Remark (2), for every $c \in (0, \varepsilon)$, there are distinct maximal solutions $y_c^1, y_c^2 : (c, \varepsilon) \longrightarrow (-\delta, \delta)$ inside B such that $y_c^1(c) = -\delta$ and $y_c^2(c) = \delta$. Since $y_{\varepsilon,-\delta}$ and $y_{\varepsilon,\delta}$ do not intersect B, we have $y_c^i(\varepsilon) \in (-\delta, \delta)$ for all c and $i = 1, 2$. By the theorem about dependence on initial conditions (see for instance Theorem 1 on p. 80 of Perko [8]), the maps $p_i : (0, \varepsilon) \longrightarrow (-\delta, \delta)$ defined by $p_i(c) := y_c^i(\varepsilon)$ are continuous, and by the uniqueness of solutions of ODEs again, these maps are also injective and open and their images do not intersect. Thus, the map $p : (0, \varepsilon) \times \{-\delta, \delta\} \longrightarrow \{\varepsilon\} \times (-\delta, \delta)$ defined by $p(c, \eta) := p_1(c)$ if $\eta = -\delta$ and $p(c, \eta) := p_2(c)$ if $\eta = \delta$ is a homeomorphism onto its image. Since $(0, \varepsilon) \times \{-\delta, \delta\}$ is not connected, it follows that p is not onto. So choose $d \in (-\delta, \delta)$ such that (ε, d) is not in the image of p. Then the germ $y_{\varepsilon,d}$ gives rise to a maximal solution y of (2.2) inside B and, by Remark (2) again, this y is a solution at 0 of (2.2). This ends the proof of the lemma in Case 1.

Case 2: $C(0)$ has nonreal eigenvalues. Note that the discriminant $\Delta := (c_{2,2}(0) - c_{1,1}(0))^2 + 4c_{1,2}(0)c_{2,1}(0)$ of $C(0)$ is also the discriminant of the right-hand side

$R(x,y)$ of (2.2) at $x=0$. Hence the quadratic polynomial $R(0,y)$ has no root so, by continuity, there are $\mu,\nu>0$ such that either $R(x,y)>\nu$ for every $(x,y)\in[0,\mu)\times\mathbb{R}$, or $R(x,y)<-\nu$ for every $(x,y)\in[0,\mu)\times\mathbb{R}$. Thus, for every solution y at 0 of (2.2), we have either $y'(x)>\nu x^{r-k}$ for all $x\in(0,\mu)$, or $y'(x)<-\nu x^{r-k}$ for all $x\in(0,\mu)$. Since $r-k\le -1$, it follows that $\lim_{x\to 0} y(x)=\pm\infty$.

The same argument applies to the point at infinity of $\{x=0\}$, as can be seen by putting $z=-1/y$: under this change of variables, the equation (2.2) changes to

$$x^{k-r}z' = -c_{1,2}(x) + (c_{2,2}(x)-c_{1,1}(x))z + c_{2,1}(x)z^2.$$

The discriminant of the right-hand side of this equation is also Δ at $x=0$. Thus, for any solution z at 0 of this equation, we have $\lim_{x\to 0} z(x)=\pm\infty$. It follows that equation (2.2) has no solution at 0 in this case. $\square$

Proof of Proposition 7. (Adapted from the proof of Lemma 5.5 in [3].) If $r=k$, we can take $E_{k,k}:=E'_{k,k}$ by Lemma 8(1). So we assume from now on that $r<k$ and proceed by induction on $k-r$. Note that the coefficients of $C(0)$ are given by $\pi_{k,r}(\mathbf{a})$, where $\pi_{k,r}:\mathbb{R}^{8k+8}\longrightarrow\mathbb{R}^4$ is the projection on four particular coordinates independent of $\mathbf{a}\in E'_{k,r}$; we identify $C(0)$ with $\pi_{k,r}(\mathbf{a})$ below. Thus, the sets

$$E'_{k,r,1} := \left\{\mathbf{b}\in E'_{k,r}:\ \text{the char. polynomial of } \pi_{k,r}(\mathbf{b}) \text{ has two distinct roots}\right\}$$

and $E'_{k,r,2}:=E'_{k,r}\setminus E'_{k,r,1}$ are semialgebraic and defined without parameters. We shall define

$$E_{k,r} := E_{k,r,1}\cup E_{k,r,2}, \quad \text{with} \quad E_{k,r,i}\subseteq E'_{k,r,i} \quad \text{for } i=1,2.$$

By Lemma 8(2), we can take

$$E_{k,r,1} := \left\{\mathbf{b}\in E'_{k,r,1}:\ \pi_{k,r}(\mathbf{b}) \text{ has a positive discriminant}\right\};$$

we shall obtain $E_{k,r,2}$ inductively. So we also assume from now on that $\mathbf{a}\in E'_{k,r,2}$, that is, $C(0)$ has only one eigenvalue λ, say. By the Jordan normal form theorem, there is a semialgebraic map $\tau: E'_{k,r,2}\times\mathbb{R}^3\longrightarrow\mathbb{R}^3$ that does not depend on the variable x and is defined without parameters such that, for $\mathbf{b}\in E'_{k,r,2}$, the map $\tau_{\mathbf{b}}:\mathbb{R}^3\longrightarrow\mathbb{R}^3$ defined by $\tau_{\mathbf{b}}(x,Y):=\tau(\mathbf{b},x,Y)$ is a linear isomorphism and the following hold:

(i) the push-forward of the system (2.1) via $\tau_{\mathbf{b}}$ is again of the form (2.1) with k and r unchanged. We denote by $\tau^*(\mathbf{b})$ the resulting $(8k+8)$-tuple of all coefficients of the Taylor expansion of order $2k+1$ of the A_1 as in (LT_k) corresponding to this push-forward;

(ii) we have $\pi_{k,r}(\tau^*(\mathbf{b})) = \begin{pmatrix}\lambda(b) & 1\\ 0 & \lambda(b)\end{pmatrix}$, where $\lambda(\mathbf{b})$ is the single eigenvalue of $\pi_{k,r}(\mathbf{b})$.

Thus, we set $E'_{k,r,3}:=\{\mathbf{b}\in E'_{k,r,2}:\ \pi_{k,r}(\mathbf{b}) \text{ is in Jordan normal form}\}$, so that $\tau^*(\mathbf{b})\in E'_{k,r,3}$ for all $\mathbf{b}\in E'_{k,r,2}$. Note that the map $\tau^*: E'_{k,r,2}\longrightarrow E'_{k,r,3}$ is semialgebraic and defined without parameters. It therefore suffices to find a semialgebraic set $E\subseteq E'_{k,r,3}$, defined without parameters, such that if $\mathbf{a}\in E'_{k,r,3}$, then $\mathbf{a}\in E$ if and only if the Riccati equation (2.2) has a solution at 0; we then set $E_{k,r,2}:=(\tau^*)^{-1}(E)$.

Thus, we may assume that $\mathbf{a} \in E'_{k,r,3}$ and that the Taylor polynomial p_C of order $2k+1-r$ of C at 0 is given by

$$p_C(x) = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix} + \sum_{i=1}^{2k+1-r} \begin{pmatrix} \alpha_i & \beta_i \\ \gamma_i & \delta_i \end{pmatrix} x^i.$$

Note that there is a $\nu \in \{1, \dots, 8k+8\}$, independent of $\mathbf{a} \in E'_{k,r,3}$, such that $\gamma_1 = \pi_\nu(\mathbf{a})$, where $\pi_\nu : \mathbb{R}^{8k+8} \longrightarrow \mathbb{R}$ is the projection on the ν-th coordinate. We now distinguish two more cases: we set

$$E'_{k,r,3,1} := \{\mathbf{b} \in E'_{k,r,3} :\ \pi_\nu(\mathbf{b}) \neq 0\} \quad \text{and} \quad E'_{k,r,3,2} := E'_{k,r,3} \setminus E'_{k,r,3,1},$$

both semialgebraic and defined without parameters. We shall define $E := E_1 \cup E_2$, with $E_i \subseteq E'_{k,r,3,i}$ for $i = 1, 2$:

Case 1: $\mathbf{a} \in E'_{k,r,3,1}$, that is, $\gamma_1 \neq 0$. Let $\sigma_1 : [0,\infty) \times \mathbb{R}^2 \longrightarrow [0,\infty) \times \mathbb{R}^2$ be the semialgebraic map given by $\sigma_1(x, y_1, y_2) := (x^2, y_1, xy_2)$, a combination of a blowing-up and a ramification. The push-forward of system (2.1) via σ_1 is of the form

$$x^{2k-1}\frac{dY}{dx} = \left(\left(a(x^2) + \lambda x^{2r}\right) I + x^{2r+1}\left[\begin{pmatrix} 0 & 1 \\ \gamma_1 & 0 \end{pmatrix} + O(x)\right]\right) Y + B_1(x),$$

which is again of the form (2.1) with $2k-1$ and $2r+1$ in place of k and r. Since the matrix $\begin{pmatrix} 0 & 1 \\ \gamma_1 & 0 \end{pmatrix}$ has two distinct eigenvalues, Lemma 8(2) applies again, so we take

$$E_1 := \left\{\mathbf{b} \in E'_{k,r,3,1} :\ \pi_\nu(\mathbf{b}) > 0\right\}.$$

Case 2: $\mathbf{a} \in E'_{k,r,3,2}$, that is, $\gamma_1 = 0$. Let $\sigma_2 : [0,\infty) \times \mathbb{R}^2 \longrightarrow [0,\infty) \times \mathbb{R}^2$ be the semialgebraic map given by $\sigma_2(x, y_1, y_2) := (x, y_1, xy_2)$, a blowing-up. The push-forward of system (2.1) via σ_2 is of the form

$$x^k \frac{dY}{dx} = \left(\left(a(x) + \lambda x^r\right) I + O\left(x^{r+1}\right)\right) Y + B_1(x),$$

which is again a system of the form (2.1), with the same k but r replaced by some $r' \in \{r+1, \dots, k\}$; in particular, the set $E_{k,r'}$ has been defined by the inductive hypothesis. Proceeding as with τ above, we denote by $\sigma_2^*(\mathbf{a})$ the resulting $(8k+8)$-tuple of all coefficients of the Taylor expansion of order $2k+1$ of the A_1 as in (LT_k) corresponding to this push-forward. The map $\sigma_2^* : E'_{k,r,3,2} \longrightarrow E'_{k,r'}$ is semialgebraic and defined without parameters, so we set $E_2 := (\sigma_2^*)^{-1}(E_{k,r'})$. $\square$

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Université de Savoie, Laboratoire de Mathématiques, Bâtiment Chablais, Campus Scientifique, 73376 Le Bourget-du-Lac Cedex, France

E-mail address: Olivier.Le-Gal@univ-savoie.fr

Universidad de Valladolid, Departamento de Álgebra, Geometría y Topología, Facultad de Ciencias, Prado de la Magdalena, s/n, E-47005 Valladolid, Spain

E-mail address: fsanz@agt.uva.es

Department of Mathematics and Statistics, McMaster University, 1280 Main Street West, Hamilton, Ontario L8S 4K1, Canada

E-mail address: speisseg@math.mcmaster.ca