Topics on Complex Variables

Course Notes

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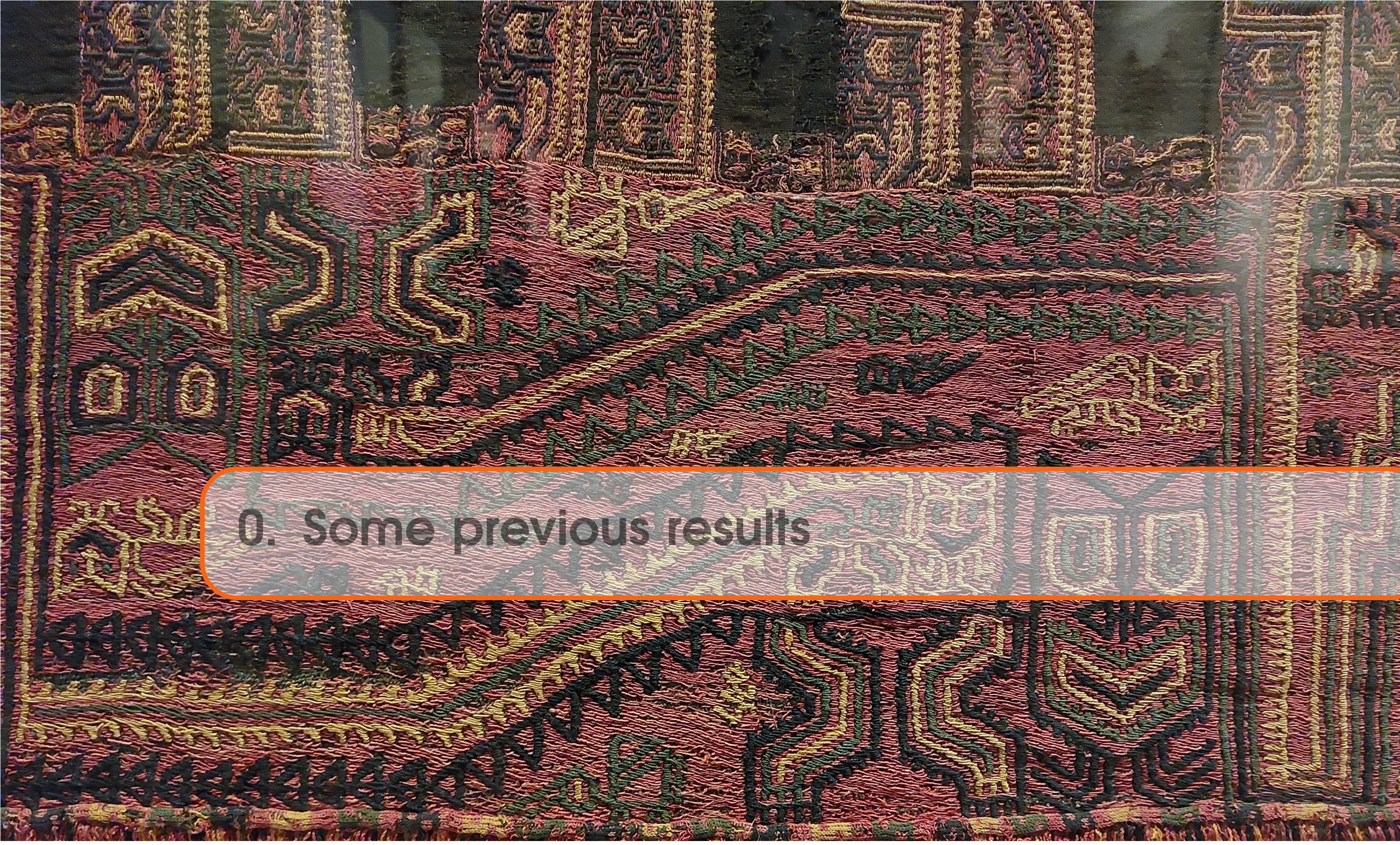
Introduction

These notes are originated in the CIMPA Course on Complex Analysis, delivered and recorded at Nesin Mathematical Village in Turkey, in March 2021, and freely available in `https://www.youtube.com/playlist?list=PLLxG_np9o-2eWZDgjRsY1b_TsTwl1pn6P`. They cover some of the topics of the subject Ampliación de Teoría de Funciones, belonging to the Máster en Matemáticas of Valladolid University. They are in evolution, and will be updated from time to time.

The selection of the subjects is quite personal, and this text does not have the intention to replace any existing text (some of them are mentioned in the references). This is only to be taken as Course Notes. Of course, many mistakes are expected, so I will appreciate, if you find some, to communicate them to me, so I can correct them for future versions of these Notes.

I would like to thank the people at Nesin Mathematical Village in Turkey, for their welcome, and the facilities I had to deliver and record the course. Specially I thank Aslı and Aycan for their attentions, and of course, many thanks to Tuğçe for the recordings.



0. Some previous results

We are going to recall some results from classical complex variables, which may be used sometimes in the notes. They can be found in any textbook in complex analysis.

Theorem 0.0.1 — Argument principle. Let $U \subseteq \mathbb{C}$ be a domain, $f \in \mathscr{O}(U)$, not identically zero, and $\gamma : [a,b] \to U \setminus Z(f)$ a loop, 0-homologous on U. Then,

$$\eta(f \circ \gamma, 0) = \sum_{z \in Z(f)} \eta(\gamma, z) \cdot m(f,z).$$

Here, and throughout the text, $\eta(\gamma, z)$ will denote the index of γ around the point z and $m(f,z)$ the multiplicity of z as a zero of f.

Proof.

$$\begin{aligned}\eta(f \circ \gamma, 0) &= \frac{1}{2\pi i}\int_{f\circ\gamma} \frac{1}{z} dz = \frac{1}{2\pi i}\int_a^b \frac{f'(\gamma(t))\gamma'(t)}{f(\gamma(t))} dt = \frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)} dz \\ &= \sum_{z \in U\setminus\gamma} \eta(\gamma, z)\,\mathrm{Res}(f'/f; z).\end{aligned}$$

A simple computation shows that $\mathrm{Res}(f'/f; z) = m(f,z)$. ■

Theorem 0.0.2 — Rouché. Let f, $g \in \mathscr{O}(U)$, U a domain, not identically zero, and γ a loop on U, 0-homologous on U. Assume that $|f(z) - g(z)| < |f(z)|$ on γ. Then,

$$\sum_{z \in Z(f)} \eta(\gamma, z) \cdot m(f,z) = \sum_{z \in Z(g)} \eta(\gamma, z) \cdot m(g,z).$$

Proof. Let us observe that neither $f(z)$ nor $g(z)$ have zeros on γ^*. Then, on γ^* we have $\left|1 - \dfrac{g(z)}{f(z)}\right| < 1.$

So, the loop $\Gamma(t) = \dfrac{g(\gamma(t))}{f(\gamma(t))}$ is totally contained on $D(1;1)$. Hence,

$$0 = \eta\left(\frac{g(\gamma)}{f(\gamma)}, 0\right) = \eta(g \circ \gamma, 0) - \eta(f \circ \gamma, 0),$$

and the result follows from Argument Principle 0.0.1. ■

Corollary 0.0.3 In the particular case that γ is a simple closed path, we deduce that f and g have the same number of zeros inside γ.

■ **Example 0.1** Take $f(z) = z^4 + 6z + 3$. All its zeros are in $|z| < 2$, and three of them in $1 < |z| < 2$. Indeed, if $g(z) = z^4$, and $h(z) = 6z$, we have that $|f(z) - g(z)| = |6z + 3| \leq 15 < 16 = \left|z^4\right|$ on $|z| = 2$, and $|f(z) - h(z)| = \left|z^4 + 3\right| \leq 4 < 6 = |6z|$ on $|z| = 1$. ■

Some consequences of these results are the following ones.

Proposition 0.0.4 Let U be a domain, $f \in \mathscr{O}(U)$ non constant, $z_0 \in U$, $f(z_0) = w_0$. Assume that f takes at z_0 the value w_0 with multiplicity $k \geq 1$. Then, there exists neighbourhoods V of z_0, W of w_0, such that if $w \in W \setminus \{w_0\}$, f takes on V the value w exactly k times, all of them with multiplicity 1.

Proof. There exists $g \in \mathscr{O}(U)$, $g(z_0) \neq 0$, such that $f(z) = w_0 + (z - z_0)^k g(z)$. As $f(z) - w_0$, $f'(z)$ are not zero, there exists $r > 0$ such that $\bar{D}(z_0;r) \subseteq U$, and such that $f(z) - w_0$ and $f'(z)$ have no zeros on $\bar{D}(z_0;r) \setminus \{z_0\}$. Let γ be the boundary of this disk, and $\alpha = f \circ \gamma$. α is a loop and w_0 is not on α^*. Let W be the connected component of $\mathbb{C} \setminus \alpha^*$ containing w_0 and $V = f^{-1}(W) \cap D(z_0;r)$, a neighbourhood of z_0. By Argument Principle we have

$$\begin{aligned} k &= \eta(f \circ \gamma, w_0) = \eta(f \circ \gamma, w) = \eta((f - w) \circ \gamma, 0) \\ &= \sum_{f(z) = w} \eta(\gamma, z) m(f - w, z) = \sum_{\substack{f(z) = w \\ z \in V}} m(f - w, z). \end{aligned}$$

Then, $f(z)$ takes the value w in k points of V, counted with multiplicities. As $f'(z) \neq 0$ away from z_0, all these multiplicities are 1, and so, these k points are different. ■

Corollary 0.0.5 — Open Mapping Theorem. If U is a domain, and $f \in \mathscr{O}(U)$ is not constant, then f is open.

Theorem 0.0.6 — Hurwitz. Let $\{f_n\}_{n=1}^{\infty} \subseteq \mathscr{O}(U)$ be a sequence of functions converging uniformly on compact sets to $f \in \mathscr{O}(U)$. Let $z_0 \in U$, $\bar{D}(z_0;r) \subseteq U$, and let us assume that f does not vanish on $|z - z_0| = r$. Then, there exists $N \in \mathbb{N}$ such that if $n \geq N$, f_n and f have the same number of zeroes on $D(z_0;r)$.

Proof. Let $m = \min\{|f(z)| \mid |z - z_0| = r\} > 0$. There exists $N \in \mathbb{N}$ such that if $n \geq N$, then, on the compact set $C(z_0;r)$ we have

$$|f_n(z) - f(z)| \leq \frac{m}{2} < |f(z)|.$$

We conclude using Rouché's Theorem 0.0.2. ■

Theorem 0.0.7 Let U be a domain, $f_n \in \mathscr{O}(U)$, converging uniformly in compact sets to $f \in \mathscr{O}(U)$. If the f_n don't vanish, then, either f does not vanish, or it is identically zero.

Proof. Let $z_0 \in U$ be an isolated zero of f and $\bar{D}(z_0;r) \subseteq U$ a disk without further zeros. By Hurwitz Theorem 0.0.6, f_n must have some zero on $D(z_0;r)$, for some $n \geq N$. ■

Theorem 0.0.8 Under previous conditions, if all f_n are injective, then either f is injective or it is constant.

Proof. Let $a \neq b$, such that $f(a) = f(b)$. The sequence of functions $f_n(z) - f_n(a)$ converges to $f(z) - f(a)$ on $V = U \setminus \{a\}$. As $f_n(z) - f_n(a)$ has no zeros on V, then Theorem 0.0.7 allows to conclude that, as $f(b) - f(a) = 0$, $f(z) - f(a)$ must be identically zero. ■

1. Conformal Mappings

The main objective of this chapter will be the study of the biholomorphic maps $f : U \to U$, where U is an open set, either of $\mathbb{C}$ or of the Riemann Sphere. We will be mainly interested in the case where U is simply connected. In order to consider the point at infinity, we shall introduce first the stereographic projection.

1.1 Stereographic projection

Let $\mathbb{S}^2 \subseteq \mathbb{R}^3$ be the unit sphere, defined by the real equation $x_1^2 + x_2^2 + x_3^2 = 1$. The North Pole will be the point $N = (0,0,1)$. We shall project $\mathbb{S}^2 \setminus \{N\}$ over the plane $x_3 = 0$, which we shall identify with the complex plane $\mathbb{C}$ as $z = x_1 + ix_2$.

Let $P = (x_1, x_2, x_3) \in \mathbb{S}^2 \setminus \{N\}$. The straight line through P and N is parameterized as $(tx_1, tx_2, 1 + t(x_3 - 1))$. It cuts $\mathbb{C}$ where $1 + t(x_3 - 1) = 0$, i.e., $t = \frac{1}{1-x_3}$. The intersection will be

$$\frac{1}{1-x_3}(x_1 + ix_2).$$

Let us denote $\pi : \mathbb{S}^2 \setminus \{N\} \longrightarrow \mathbb{C}$ the map previously defined, which will be called **stereographic projection**. From $z = x + iy \in \mathbb{C}$, we can find its inverse image , that will be the point

$$\pi^{-1}(z) = \frac{1}{|z|^2 + 1}(2x, 2y, |z|^2 - 1).$$

The map π defines a homeomorphism between $\mathbb{S}^2 \setminus \{N\}$ and $\mathbb{C}$. It allows to consider a new point, called infinity (∞), that will be added to $\mathbb{C}$, and define $\pi(N) = \infty$. The resulting space will be denoted as $\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$. A topology is defined on $\overline{\mathbb{C}}$ making it homeomorphic to $\mathbb{S}^2$. We shall call $\overline{\mathbb{C}}$ the **Riemann Sphere**.

R The space $\overline{\mathbb{C}}$ is in fact the Alexandroff compactification of the plane $\mathbb{C}$. In geometric terms, it is the complex projective line $\mathbb{P}^1_{\mathbb{C}}$.

It is often convenient to consider the point ∞ at the origin, which can be achieved by means of the transformation $J : \mathbb{C} \setminus \{0\} \longrightarrow \mathbb{C} \setminus \{0\}$ defined by $J(z) = \frac{1}{z}$. Over $\mathbb{S}^2$, J can be seen as $\bar{J}(x_1, x_2, x_3) = (x_1, -x_2, -x_3)$, and this map verifies $J \circ \pi = \pi \circ \bar{J}$. Let us remark that $\bar{J}(N) = (0, 0, -1) =: S$, where S is the South Pole of the sphere. So, J can be extended as a homeomorphism of the Riemann Sphere, that we continue denoting $\frac{1}{z}$. We can write, then, that $\frac{1}{0} = \infty$ and $\frac{1}{\infty} = 0$.

If $U \subseteq \overline{\mathbb{C}}$ is an open set containing ∞, a map f defined on U will be called holomorphic at infinity if $\tilde{f}(z) = f\left(\frac{1}{z}\right)$ is holomorphic at 0. We also have that

$$\lim_{z \to \infty} f(z) = \lim_{z \to 0} \tilde{f}(z).$$

1.2 Automorphisms of $\overline{\mathbb{C}}$. Möbius transformations.

We shall start our study of biholomorphic maps with the Riemann Sphere $\overline{\mathbb{C}}$. A holomorphic map $\overline{\mathbb{C}} \longrightarrow \overline{\mathbb{C}}$ is, precisely, a meromorphic map on $\overline{\mathbb{C}}$, having also (perhaps) a pole at infinity.

Proposition 1.2.1 Every meromorphic function on $\overline{\mathbb{C}}$ is rational.

Proof. Let $z_1, z_2, \ldots, z_s$ be the poles, with multiplicities $\alpha_1, \alpha_2, \ldots, \alpha_s$. Let $g(z) = (z - z_1)^{\alpha_1} \cdots (z - z_s)^{\alpha_s} f(z)$ be an entire function with a pole at infinity, so $g\left(\frac{1}{z}\right)$ has a pole at 0. There exists a polynomial $P(z)$, with $P(0) = 0$, such that $g\left(\frac{1}{z}\right) - P\left(\frac{1}{z}\right)$ is entire. We have that

$$\lim_{z \to \infty} g\left(\frac{1}{z}\right) - P\left(\frac{1}{z}\right) = \lim_{z \to 0} g(z) - P(z) = g(0).$$

So, $g(z) = P(z) + C$, for some constant C, and

$$f(z) = \frac{P(z) + C}{(z - z_1)^{\alpha_1} \cdots (z - z_s)^{\alpha_s}}.$$

■

If $f(z) = \dfrac{P(z)}{Q(z)}$, call $d = \max\{\deg(P), \deg(Q)\}$ the degree of f. $f(z)$ takes the value ∞ in $\deg(Q)$ complex points. Moreover, if $d = \deg(P)$,

$$f\left(\frac{1}{z}\right) = \frac{z^d P\left(\frac{1}{z}\right)}{z^{d - \deg(Q)} z^{\deg(Q)} Q\left(\frac{1}{z}\right)}.$$

So 0 is a pole of order $d - \deg(Q)$ for $f\left(\frac{1}{z}\right)$, which means that $f(z)$ takes the value ∞ d times.

If $w_0 \in \mathbb{C}$, the rational function

$$g(z) = \frac{1}{f(z) - w_0} = \frac{Q(z)}{P(z) - w_0 Q(z)}$$

has degree d and takes the value ∞ where $f(z)$ takes the value w_0. As a consequence, $f(z)$ takes each complex value exactly d times, counting multiplicities.

So, if $f(z)$ is an automorphism of $\overline{\mathbb{C}}$, it turns out that $d = 1$, and so,

$$f(z) = \frac{az + b}{cz + d},$$

with

$$D = \det \begin{pmatrix} a & b \\ c & d \end{pmatrix} \neq 0.$$

Definition 1.2.1 A Möbius transformation, linear transformation, or homography, is a rational map

$$f(z) = \frac{az+b}{cz+d}, \text{ with } ad-bc \neq 0.$$

Möbius transformations form a group under composition.

The coefficients of a Möbius transformation are "almost" determined: if $\lambda \neq 0$, the map $\dfrac{\lambda az+\lambda b}{\lambda cz+\lambda d}$ represents the same transformation. We can always choose λ in such a way as $1 = \lambda^2 D$, and suppose that $D = 1$: this is the normal form of the Möbius transformation, determined modulo sign. The map

$$\begin{array}{ccc} GL(2,\mathbb{C}) & \xrightarrow{\theta} & Aut(\overline{\mathbb{C}}) \\ \begin{pmatrix} a & b \\ c & d \end{pmatrix} & \longmapsto & \dfrac{az+b}{cz+d} \end{array}$$

is a surjective group morphism: $\theta(A \cdot B) = \theta(A) \circ \theta(B)$. So we have that $Aut(\overline{\mathbb{C}}) \cong GL(2,\mathbb{C})/\ker\theta$, where $\ker\theta$ is the subgroup of scalar matrices $\begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}$, isomorphic to $\mathbb{C}^*$.

The group of Möbius transformations is, then, the group of projective linear transformations, $PGL(1,\mathbb{C})$, group of Staudt projectivities of the complex projective line, $\mathbb{P}^1_{\mathbb{C}}$, with the identity as field automorphism. If $SL(2,\mathbb{C})$ is the special linear group, elements of $GL(2,\mathbb{C})$ with determinant equal to 1, and $H = \{I, -I\}$, we also have

$$SL(2,\mathbb{C})/H \cong PGL(1,\mathbb{C}) \cong Aut(\overline{\mathbb{C}}).$$

This group can be generated by simple transformations:

Proposition 1.2.2 $\mathrm{Aut}(2,\overline{\mathbb{C}})$ is generated by:

1. Translations $T_a(z) = z+b$, where $b \in \mathbb{C}$.
2. Homotheties $H_a(z) = az$, where $a \in \mathbb{C}^*$.
3. Inversion $J(z) = \frac{1}{z}$.

Proof. If $f(\infty) = \infty$, $f(z) = az+b = H_a \circ T_{b/a}$. If $f(z_0) = \infty$, $g(z) = f \circ T_{z_0} \circ J$ sends ∞ to ∞, and $f = g \circ J \circ T_{-z_0}$. ∎

1.2.1 Fixed points of a Möbius transformation. Transitivity

We are goint to study the set $Fix(T)$ of fixed points of a Möbius transformation $T(z) = \frac{az+b}{cz+d}$ different from identity. The point at infinity is fixed if and only if $c = 0$. Moreover,

$$\frac{az+b}{cz+d} = z \text{ implies that } cz^2 + (d-a)z - b = 0.$$

Two cases may appear:

Case 1.- If $c \neq 0$, there are two complex fixed points, counted with multiplicity. They coincide if and only if $(d-a)^2 + 4bc = 0$.

Case 2.- If $c = 0$, there is one complex fixed point when $d-a \neq 0$. If $d-a = 0$, the point at infinity is a double fixed point.

In any of the previous cases, $(d-a)^2 + 4bc = 0$ is the condition that must be satisfied in order to have only one fixed point. This condition can be written as $\mathrm{tr}(A)^2 - 4\det(A) = 0$. If $T(z)$ is in normal form, there is only one fixed point if and only if $\mathrm{tr}(A) = \pm 2$.

Möbius transformations are 3-transitive. This means:

Proposition 1.2.3 Given three different points $\{z_1, z_2, z_3\} \subseteq \overline{\mathbb{C}}$, there exists only one $T(z) \in \mathrm{Aut}(2, \overline{\mathbb{C}})$ such that $T(z_1) = 0$, $T(z_2) = 1$, $T(z_3) = \infty$.

Proof. The map

$$T(z) = \frac{\dfrac{z - z_1}{z - z_3}}{\dfrac{z_2 - z_1}{z_2 - z_3}}$$

verifies the statement. If T_1, T_2 are two such Möbius transformations, $T_2^{-1} \circ T_1$ fixes 3 points, so it is the identity map. ■

Corollary 1.2.4 Given two triples $\{z_1, z_2, z_3\}$, $\{w_1, w_2, w_3\}$ of points of $\overline{\mathbb{C}}$, there exists a unique $T(z) \in \mathrm{Aut}(\overline{\mathbb{C}})$ with $T(z_i) = w_i$.

Proof. Inmediate composing two transformations as in the previous proposition. ■

1.2.2 Circles

A circle $\mathscr{C}$ on the unit sphere $\mathbb{S}^2$ is the intersection of the sphere with a plane. We will study the stereographic projection $\pi(\mathscr{C})$ of a circle. If $\mathscr{C} = \mathbb{S}^2 \cap \Pi$, where $\Pi \equiv \alpha x_1 + \beta x_2 + \gamma x_3 = \delta$, a point $z = x + iy$ belongs to $\pi(\mathscr{C})$ if and only if

$$\frac{1}{|z|^2 + 1}(2x, 2y, |z|^2 - 1) \in \mathscr{C},$$

i.e., if

$$\frac{2\alpha x + 2\beta y + \gamma(|z|^2 - 1)}{|z|^2 + 1} = \delta.$$

This is equivalent to

$$a|z|^2 + \omega \bar{z} + \bar{\omega} z + c = 0, \tag{1.1}$$

where $a = \gamma - \delta$, $\omega = \alpha + i\beta$, $c = -(\gamma + \delta)$. Note that the plane Π cuts the sphere in a "true" circle if and only if $\delta^2 < \alpha^2 + \beta^2 + \gamma^2$, i.e. if $|\omega|^2 > ac$.

We may consider two cases:

Case 1.- $a = 0$. This occurs if $N \in \Pi$. The projection $\pi(\mathscr{C})$ is a straight line.

Case 2.- $a \neq 0$. Equation (1.1) can be written as

$$\left(x + \frac{\alpha}{a}\right)^2 + \left(y + \frac{\beta}{a}\right)^2 = \frac{|\omega|^2 - ac}{a^2}.$$

From this equation it is easy to show:

Proposition 1.2.5 A Möbius transformation sends circles into circles.

Proof. It is clear for translations and homotheties. For the inversion $J(z) = \frac{1}{z}$, applied to (1.1) we obtain

$$a \cdot \frac{1}{|z|^2} + \frac{\omega}{\bar{z}} + \frac{\bar{\omega}}{z} + c = 0.$$

So the image is the circle

$$a + \omega z + \bar{\omega}\bar{z} + c\,|z|^2 = 0.$$

■

1.2.3 Cross-ratio

The notion of cross-ratio we shall introduce here allows to study some properties of circles. Given z_1, z_2, z_3, z_4 points in $\overline{\mathbb{C}}$, consider $T(z)$ the unique Möbius transformation such that $T(z_2) = 0$, $T(z_3) = 1$, $T(z_4) = \infty$, which exists by Proposition 1.2.3. The cross-ratio of z_1, z_2, z_3, z_4 will be defined as $T(z_1)$. It will be denoted by $[z_1, z_2, z_3, z_4]$. We have that

$$[z_1, z_2, z_3, z_4] = \frac{(z_1 - z_2)(z_3 - z_4)}{(z_1 - z_4)(z_3 - z_2)}.$$

If S is another Möbius transformation, $T \circ S^{-1}$ sends $S(z_2)$, $S(z_3)$, $S(z_4)$ in 0, 1, ∞, and so,

$$[S(z_1), S(z_2), S(z_3), S(z_4)] = T \circ S^{-1}(S(z_1)) = T(z_1) = [z_1, z_2, z_3, z_4].$$

Cross-ratio is preserved under Möbius transformations.

Proposition 1.2.6 Four points $\{z_1, z_2, z_3, z_4\}$ lie on a circle if and only if $[z_1, z_2, z_3, z_4] \in \mathbb{R} \cup \{\infty\}$.

Proof. Four points lie on a circle if and only if there exists a Möbius transformation such that their images are on a real projective line, for instance, on $\mathbb{R} \cup \{\infty\}$. Its cross-ratio is then real. ■

1.2.4 Inversion

Given is $\mathscr{C} = \mathscr{C}(p, R)$, circumference centered at p with radius R. We define the inversion of a point $q \neq p$ with respect to $\mathscr{C}$ as follows: consider the half-line with origin at p, passing through q. Over this half-line, we can choose a unique point q' such that $|q' - p|\,|q - p| = R^2$. We define the inversion as $I_{\mathscr{C}}(q) = q'$. In coordinates,

$$I_{\mathscr{C}}(q) = p + \frac{R^2}{\bar{q} - \bar{p}}.$$

Inversion is a transformation very useful in classical geometry, as it has very interesting properties: it transforms circles into circles, and it conserves angles. In fact, if we consider the circumference $a\,|z|^2 + \omega\bar{z} + \bar{\omega} z + c = 0$, centered at $-\frac{\omega}{a}$ and with radius $\frac{\sqrt{|\omega|^2 - ac}}{a}$, if $q = z$ we then have

$$I_{\mathscr{C}}(z) = -\frac{\omega}{a} + \frac{\dfrac{|\omega|^2 - ac}{a^2}}{\bar{z} + \dfrac{\bar{\omega}}{a}} = -\frac{\omega\bar{z} + c}{a\bar{z} + \bar{\omega}}.$$

If $a = 0$, previous expression reduces to

$$I_{\mathscr{C}}(z) = -\frac{\omega\bar{z} + c}{\bar{\omega}}.$$

A simple computation shows in this case that

$$|I_{\mathscr{C}}(z) - q| = |z - q| \text{ if } q \in \mathscr{C}.$$

The transformation is the symmetry with respect to the straight line $\omega\bar{z} + \bar{\omega}z + c = 0$.

The inversion is not a Möbius transformation, as it is anti-holomorphic. But Möbius transformations preserve the inversion. In fact, if T is a Möbius transformation such that $T(\mathscr{C}_1) = \mathscr{C}_2$, with $\mathscr{C}_1$, $\mathscr{C}_2$ circles in $\overline{\mathbb{C}}$, the composition $T \circ I_{\mathscr{C}_1} \circ T^{-1} \circ I_{\mathscr{C}_2}$ is a Möbius transformation fixing every point in $\mathscr{C}_2$, so it is the identity map. If $I_{\mathscr{C}_1}(z) = z'$, $I_{\mathscr{C}_2}T(z) = T \circ I_{\mathscr{C}_1}(z) = T(z')$. So, Möbius transformations send inverse points on inverse points.

This fact can be shown geometrically. Two points z_1, z_2 are symmetric with respect to a straight line L if and only if every circle through them cuts the straight line orthogonally. Take a Möbius transformation T which transforms L in a circle $\mathscr{C}$, and denote $w_i = T(z_i)$, $i = 1, 2$. Every circle through w_1 and w_2 comes from a circle trough z_1 and z_2, and thanks to the conformality of Möbius transformations, it mus cut orthogonally $T(\mathscr{C})$. If $T(\mathscr{C})$ is a straight line, this means that w_1, w_2 are symmetric. If it is a circumference, the points are inverse each other with respect to $T(\mathscr{C})$.

1.3 Dynamics of Möbius transformations

Consider $T \in \mathrm{Aut}(\overline{\mathbb{C}})$, and its matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ in normal form. We will say that two Möbius transformations T_1, T_2 are conjugated if there exists another $S \in \mathrm{Aut}(\overline{\mathbb{C}})$ such that $S \circ T_1 \circ S^{-1} = T_2$. In this case, $\mathrm{tr}(T_1) = \mathrm{tr}(T_2)$. Also, it this is the case, and p is a fixed point for T_1, $q = S(p)$ is a fixed point for T_2. We shall call parabolic the Möbius transformations with a unique fixed point, i.e., such that $\mathrm{tr}(T)^2 = 4$.

Given a parabolic Möbius transformation, let us conjugate it through S, to another Möbius transformation with ∞ as the only fixed point, obtaining $T_2(z) = z + a$. A new conjugation through the homotethy H_b gives

$$H_b \circ T_2 \circ H_b^{-1}(z) = z + ab.$$

Choosing $b = 1/a$, we see that every parabolic transformation is conjugated to the translation $T(z) = z + 1$. We see that $T^n(z) = z + n$, so $\lim_{n\to\infty} T^n(z) = \infty$. This means that by iteration, every point tends to the fixed point, which is an attractor.

If T has two fixed points, we can send them by a conjugation to 0, ∞, and we then have $T = H_\lambda$, $\lambda \in \mathbb{C} \setminus \{0, 1\}$.

$$\mathrm{tr}(T) = \sqrt{\lambda} + \frac{1}{\sqrt{\lambda}}, \text{ and so } \mathrm{tr}(T)^2 = \lambda + \frac{1}{\lambda} + 2.$$

The equality $\lambda + \frac{1}{\lambda} + 2 = \mu + \frac{1}{\mu} + 2$ implies that, either $\lambda = \mu$ or $\lambda = \frac{1}{\mu}$, so H_λ and $H_{1/\lambda}$ are the unique conjugated homotheties.

As a conclusion, two Möbius transformations in normal form are conjugated if and only if $\mathrm{tr}(T_1)^2 = \mathrm{tr}(T_2)^2$. We shall study now the dynamics of H_λ. If $|\lambda| \neq 1$, consider $z_0 \in \mathbb{C} \setminus \{0\}$, and distinguish two cases:

- If $|\lambda| > 1$, $\lim_{n\to\infty} H_\lambda^n(z_0) = \infty$.
- If $|\lambda| < 1$, $\lim_{n\to\infty} H_\lambda^n(z_0) = 0$.

One of the two fixed points is an attractor, and the other, a repulsor. When $|\lambda| = 1$, $\lambda = e^{i\theta}$, and

$$\lambda + \frac{1}{\lambda} + 2 = 2(1 + \cos\theta) \in [0, 4).$$

Conversely, if $\lambda + \frac{1}{\lambda} + 2 = r \in [0,4)$, from the equality

$$\lambda + \frac{1}{\lambda} + 2 = \bar{\lambda} + \frac{1}{\bar{\lambda}} + 2$$

we deduce that, either $\lambda \in \mathbb{R}$ or $\lambda = \frac{1}{\bar{\lambda}}$, and then $|\lambda| = 1$.

In the former case, $s = \lambda + \frac{1}{\lambda} \in [-2,2)$. But from the AM-GM inequality,

$$\left|\lambda + \frac{1}{\lambda}\right| \geq 2\sqrt{\lambda \cdot \frac{1}{\lambda}} = 2,$$

which can only occur if $\lambda = -1$. In the latter case, the Möbius transformation is called elliptic. As $H_\lambda^n = H_{\lambda^n}$, it is periodic if and only if λ is a root of unity.

1.4 Rotations

If $T \in \mathrm{Aut}(\mathbb{C})$, and $\pi : \mathbb{S}^2 \longrightarrow \mathbb{C}$ is the stereographic projection, the composition $\pi^{-1} \circ T \circ \pi$ is a homeomorphism of $\mathbb{S}^2$. Let us study which Möbius transformation correspond to direct rigid motions, i.e., with rotations of the sphere. They will contitute a group, denoted $Rot(\mathbb{C})$

If $P \in \mathbb{S}^2$, let us denote $\bar{P}$ its antipodal point. If $\pi(P) = z$, then $\pi(\bar{P}) = -\frac{1}{\bar{z}}$. As rotations preserve antipodal points, if $T(z) = \frac{az+b}{cz+d} \in Rot(\overline{\mathbb{C}})$, we have

$$T\left(-\frac{1}{\bar{z}}\right) = -\frac{1}{\overline{T(z)}}, \text{ so } \frac{-\frac{a}{\bar{z}} + b}{-\frac{c}{\bar{z}} + d} = -\frac{\bar{c}\bar{z} + \bar{d}}{\bar{a}\bar{z} + \bar{b}}.$$

So, there exists $\lambda \in \mathbb{C}^*$ such that

$$\begin{pmatrix} b & -a \\ d & -c \end{pmatrix} = \lambda \begin{pmatrix} -\bar{c} & -\bar{d} \\ \bar{a} & \bar{b} \end{pmatrix}.$$

Assuming $ad - bc = 1$, we obtain that $\lambda^2 = 1$. If $\lambda = -1$, we would have that $-|d|^2 - |c|^2 = 1$, which is impossible. So, $\lambda = 1$, and then,

$$T(z) = \frac{az+b}{-\bar{b}z + \bar{a}}, \text{ with } |a|^2 + |b|^2 = 1.$$

The group of rotations agrees with $PSU(2,\mathbb{C})$, unitary matrices of determinant 1. Indeed, if $T \in PSU(2,\mathbb{C})$, and $b = 0$ ($T(0) = 0$), the map is $T(z) = a^2 z$, $|a| = 1$, a rotation. In the general case, compose with a rotation R such that $R \circ T(0) = 0$ to conclude.

1.5 Automorphisms of $\mathbb{C}$

Once we have introduced Möbius transformations as the only automorphisms of the Riemann Sphere $\overline{\mathbb{C}}$, we study the group of automorphisms of the complex plane $\mathbb{C}$. If $f : \mathbb{C} \longrightarrow \mathbb{C}$ is such an automorphism, let us consider the singularity at ∞. If it is an essential singularity, the set

$$f(\{z \in \mathbb{C} |\ |z| > 1\})$$

must be dense in $\mathbb{C}$, and then

$$f(\{z \in \mathbb{C} |\ |z| > 1\}) \cap f(D(0;1)) \neq \emptyset,$$

which is impossible. So, ∞ must be at most a pole for f, and then, $f(z)$ is a polynomial of degree d. The equation $f(z) = w_0$ has generically d different solutions, so $d = 1$ and $f(z) = az + b$.

As a conclusion, $\mathrm{Aut}(\mathbb{C}) < \mathrm{Aut}(\overline{\mathbb{C}})$ is the subgroup of Möbius transformation having the point at infinity as a fixed point.

1.6 Automorphisms of the unit disk

In this section, D will denote the unit disk $D(0;1)$. A very useful result is:

Theorem 1.6.1 — Schwarz's Lemma. Let $f : D \longrightarrow D$ be a holomorphic function, such that $f(0) = 0$. Then, $\forall z \in D$, $|f(z)| \leq |z|$ and $|f'(0)| \leq 1$.

Moreover, if for some point $z \neq 0$ we have that $|f(z)| = |z|$, or if $|f'(0)| = 1$, $f(z) = az$ with $|a| = 1$.

Proof. The function $h(z) = \frac{f(z)}{z}$ extends to 0 as a holomorphic function, with $h(0) = f'(0)$. If $|z| = r < 1$, $|f(z)| \leq \dfrac{1}{r}$. By maximum modulus principle, $|h(z)| \leq \dfrac{1}{r}$ if $|z| \leq r$. When $r \to 1^-$, we conclude that $|h(z)| \leq 1$, as requested.

If for some point $|h(z_0)| = 1$, $h(z)$ would be a constant a, and necessarily, $|a| = 1$. ■

We want to study the group $\mathrm{Aut}(D)$. Given $a \in D$, the holomorphic function:

$$\varphi_a(z) = \frac{z-a}{1-\bar{a}z}$$

is an element of $\mathrm{Aut}(D)$ that sends a to 0. To see this, let us note that if $|z| = 1$,

$$|\varphi_a(z)| = \left|\frac{z-a}{1-\bar{a}z}\right| = \left|\frac{z-a}{z\bar{z}-\bar{a}z}\right| = \left|\frac{z-a}{z(\bar{z}-\bar{a})}\right| = 1.$$

Some properties of this function are:

1. $\varphi_a^{-1} = \varphi_{-a}$.
2. $\varphi_a'(z) = \dfrac{1-|a|^2}{(1-\bar{a}z)^2}$, so $\varphi_a'(0) = 1-|a|^2$ and $\varphi_a'(a) = \dfrac{1}{1-|a|^2}$.

Let $f(z) \in \mathrm{Aut}(D)$, such that $f(a) = b$. The composite function

$$g = \varphi_b \circ f \circ \varphi_{-a}$$

is an element of $\mathrm{Aut}(D)$ sending 0 to 0. By Schwarz's Lemma 1.6.1, $|g(z)| \leq |z|$, and then,

$$|\varphi_b \circ f(z)| \leq |\varphi_a(z)|\,, \text{ or } \left|\frac{f(z)-f(a)}{1-\bar{f(a)}f(z)}\right| \leq \left|\frac{z-a}{1-\bar{a}z}\right|.$$

This last inequality is often known as **Schwarz-Pick's Lemma**.

Moreover, as $|g'(0)| \leq 1$,

$$g'(0) = \frac{1-|a|^2}{1-|f(a)|^2} f'(a), \text{ and then } |f'(a)| \leq \frac{1-|f(a)|^2}{1-|a|^2}.$$

If equality holds for some point, $g(z) = \lambda z$, with $|\lambda| = 1$, and then

$$f(z) = \varphi_b(\lambda \varphi_a(z)).$$

If $f(z) \in \mathrm{Aut}(D)$, with $f(a) = 0$, we have that

$$|f'(a)| \leq \frac{1}{1-|a|^2}.$$

As $f^{-1} \in \mathrm{Aut}(D)$, $|(f^{-1})'(0)| \leq 1-|a|^2$. So,

$$1 = |f'(a) \cdot (f^{-1})'(0)| \leq \frac{1}{1-|a|^2} \cdot (1-|a|^2) = 1.$$

We deduce that $f(z) = \lambda \varphi_a(z)$, for some $a \in D$ and $|\lambda| = 1$. Every element of $\mathrm{Aut}(D)$ is also a Möbius transformation.

1.7 Automorphisms of the upper half plane

Denote $\mathbb{H}$ the open upper half plane. We are going to study, using previous results, the set $\mathrm{Aut}(\mathbb{H})$. The transformation $\varphi : \mathbb{H} \longrightarrow D$ defined by

$$\varphi(z) = \frac{z-i}{z+i}$$

is called Cayley transformation. As it is a Möbius transformation defining a biholomorphic mapping between $\mathbb{H}$ and D, we deduce from Section 1.6 that $\mathrm{Aut}(\mathbb{H})$ is a subgroup of $\mathrm{Aut}(\overline{\mathbb{C}})$. The inverse of the Cayley transformation is

$$\varphi^{-1}(z) = i \cdot \frac{1+z}{1-z}.$$

Let $f : \mathbb{H} \longrightarrow D$ be a biholomorphic function. As $f \circ \varphi^{-1} \in \mathrm{Aut}(D)$, we have that

$$f \circ \varphi^{-1}(z) = \lambda \varphi_a(z),$$

for some $a \in D$ and $|\lambda| = 1$, and then

$$f(z) = \lambda \varphi_a \circ \varphi(z) = \lambda \frac{1-a}{1-\bar{a}} \cdot \frac{z - i\frac{1+a}{1-a}}{z + i\frac{1+\bar{a}}{1-\bar{a}}} = \mu \frac{z-\alpha}{z-\bar{\alpha}},$$

for some $|\mu| = 1$ and α verifying

$$\mathrm{Im}(\alpha) = \mathrm{Im}\left(i\frac{1+a}{1-a}\right) = \frac{1-|a|^2}{|1-a|^2} > 0.$$

Conversely, every Möbius transformation of this form defines a biholomorphism between $\mathbb{H}$ and D.

Now, let $f(z) \in \mathrm{Aut}(\mathbb{H})$. If $f(\infty) = \infty$, $f(z) = az+b$, with a, $b \in \mathbb{R}$, $a > 0$. In the general case, $g = \varphi \circ f$ must be of the form

$$g(z) = \mu \frac{z-\alpha}{z-\bar{\alpha}},$$

as stated. So,

$$f(z) = \varphi^{-1} \circ g(z) = i \cdot \frac{(1+\mu)z - (\bar{\alpha} + \mu\alpha)}{(1-\mu)z - (\bar{\alpha} - \mu\alpha)},$$

which in matrix form is

$$\begin{pmatrix} i(1+\mu) & -i(\bar{\alpha}+\mu\alpha) \\ 1-\mu & -(\bar{\alpha}-\mu\alpha) \end{pmatrix} \sim \begin{pmatrix} 1 & -\frac{\bar{\alpha}+\mu\alpha}{1+\mu} \\ -i\frac{1-\mu}{1+\mu} & i\frac{\bar{\alpha}-\mu\alpha}{1+\mu} \end{pmatrix} \in GL(2,\mathbb{R}),$$

with determinant

$$\frac{4\,\mathrm{Im}(\alpha)}{|1+\mu|^2} > 0.$$

Conversely, every matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL(2,\mathbb{R})$ with positive determinant defines an automorphism of $\mathbb{H}$. So, $\mathrm{Aut}(\mathbb{H}) \cong GL_+(2,\mathbb{R})$.

Alternatively, using the results stated in 1.2.4, if a Möbius transformation T sends $\mathbb{R}$ on $\mathbb{R}$, it must verify $T(\bar{z}) = \overline{T(z)}$. If $T(\infty) = a \in \mathbb{R}$, we have that $T(z) = \frac{az+b}{z+d}$ and

$$\frac{a\bar{z}+b}{\bar{z}+d} = \frac{a\bar{z}+\bar{b}}{\bar{z}+\bar{d}}.$$

We conclude that b, $d \in \mathbb{R}$, and $ad - b > 0$ (evaluating in i).

1.8 Riemannian geometry and Schwarz Lemma

The introduction of a metric in the complex plane allows to understand in a different way some of the results of the previous sections. Let us sketch this fact in this section, ending with a generalization due to Ahlfors of Schwarz Lemma.

Definition 1.8.1 Let $U \subseteq \mathbb{C}$ be an open set. A (hermitian) metric on U is a $\mathscr{C}^2-$function $\rho : U \longrightarrow \mathbb{R}$, strictly positive. Sometimes it will be allowed that ρ vanishes in a discrete set. We will call pseudometric to such ρ.

■ **Example 1.1**
1. $\rho(z) \equiv 1$ on $\mathbb{C}$. This is the Euclidean matric.
2. $\sigma(z) = \dfrac{1}{1-|z|^2}$ on the unit disk D. It defines Poincaré metric. Let us note that the metric becomes bigger when approaching the boundary of D.

■

Given a metric ρ on U, and $z_0 \in U$, the norm at z_0 of a vector $\xi \in T_{z_0}U$ will be $||\xi||_{\rho,z_0} = \rho(z_0)\,|\xi|$. If $\gamma\colon [a,b] \longrightarrow U$ is a continuous path, piecewise C^1, we define its length as

$$l_\rho(\gamma) = \int_a^b ||\gamma'(t)||_{\rho,\gamma(t)}\,dt = \int_a^b \rho(\gamma(t))\,|\gamma'(t)|\,dt.$$

■ **Example 1.2** On the unit disk D provided with Poincaré metric σ, the length of the segment joining 0 with $b \in (0,1)$ is

$$\int_0^b ||1||_{\sigma,t}\,dt = \int_0^b \frac{1}{1-t^2}dt = \operatorname{argth}(b) = \frac{1}{2}\log\left(\frac{1+b}{1-b}\right) \xrightarrow{b\to 1^-} +\infty.$$

Let us consider another path joining 0 and b, $\gamma(t) = x(t)+iy(t)$. $\alpha(t) = x(t)$ also defines a path with same end points, and

$$l_\sigma(\alpha) = \int_0^1 \frac{|x'(t)|}{1-x(t)^2}dt \le \int_0^1 \frac{|\gamma'(t)|}{1-|\gamma(t)^2|}dt = l_\sigma(\gamma).$$

As a consequence, a shortest distance path between these two points is real. If $x(t)$ is not monotonic, there will be a shortest path after eliminating some parts of the path. Then, the shortest path is real and increasing: it is the segment $[0,b]$. ■

Let $f : U \longrightarrow U'$ be analytic and conformal ($f'(z) \neq 0$ for every $z \in U$). If ρ is a metric on U', the pull-back of ρ is defined as the following metric on U:

$$(f^*\rho)(z) = \rho(f(z)) \cdot |f'(z)|.$$

Analogously, the pull-back can be defined if f is differentiable and $\left|\frac{\partial f}{\partial z}\right| \neq 0$ on U. If γ is a path between z_0 and z_1 on U, we have

$$\begin{aligned} l_{f^*\rho}(\gamma) &= \int_0^1 \rho(f(\gamma(t)))\,|f'(\gamma(t))|\,|\gamma'(t)|\,dt \\ &= \int_0^1 \rho(f(\gamma(t)))\left|\frac{d}{dt}(f\circ\gamma)(t)\right|dt = l_\rho(f\circ\gamma). \end{aligned}$$

So, the image $f\circ\gamma$ of a path γ is another path of same length.

Given an open set U and a metric ρ on U, the distance between two points P and Q is

$$d_\rho(P,Q) = \inf\left\{l_\rho(\gamma) \mid \gamma\colon [0,1] \longrightarrow U \text{ is a } C^1 \text{ path from } P \text{ to } Q\right\}.$$

Previous computations show that if $f : U \longrightarrow U'$ is C^1 and ρ is a metric on U', given points P, $Q \in U$, we have

$$d_\rho(f(P), f(Q)) \leq d_{f^*\rho}(P, Q).$$

Equality is not guaranteed as not every path on U' from $f(P)$ to $f(Q)$ is the image of path on U.

If (U_1, ρ_1) and (U_2, ρ_2) are two open sets provided with metrics, a map $f : U_1 \longrightarrow U_2$ is an isometry if it is C^1, bijective, and preserves distances. This happens, for instance, when $f^*\rho_2 = \rho_1$.

Proposition 1.8.1 Every element of $\mathrm{Aut}(D)$ is an isometry taking Poincaré metric.

Proof. By Schwarz-Pick's Lemma, such an automorphisms verifies

$$|f'(z)| = \frac{1 - |f(z)|^2}{1 - |z|^2},$$

so,

$$\sigma(z) = \frac{1}{1 - |z|^2} = \frac{|f'(z)|}{1 - |f(z)|^2} = \sigma(f(z)) \cdot |f'(z)|.$$

■

As a consequence, $\mathrm{Aut}(D)$ preserves Poincaré distance. This fact allows to compute easily the distance from P to Q. Assume $P = 0$. After a rotation we can assume that Q is real positive. Then,

$$d_\sigma(0, Q) = d_\sigma(0, |Q|) = \frac{1}{2} \cdot \log\left(\frac{1 + |Q|}{1 - |Q|}\right).$$

In general, the Möbius transformation $\varphi_P(z) = \frac{z - P}{1 - \bar{P}z}$ sends P to 0. Then,

$$d_\sigma(P, Q) = d_\sigma(0, \varphi_P(Q)) = d_\sigma\left(0, \frac{Q - P}{1 - \bar{P}Q}\right) = \frac{1}{2} \cdot \log\left(\frac{1 + \frac{|P-Q|}{|1-\bar{P}Q|}}{1 - \frac{|P-Q|}{|1-\bar{P}Q|}}\right).$$

Another consequence is that geodesics in D are arcs meeting $\mathbb{S}^1$ orthogonally.

Proposition 1.8.2 The topology induced on the unit disk D by the Poincaré metric is the usual Euclidean topology. D turns out to be a complete metric space.

Proof. As every point can be taken to 0 by a Möbius transformation, preserving σ, it is enough to verify that the neighbourhoods of 0 agree in both topologies. Let us observe that

$$d_\sigma(z, 0) < r \Longleftrightarrow \frac{1}{2}\log\left(\frac{1 + |z|}{1 - |z|}\right) < r \Longleftrightarrow |z| < \frac{e^{2r} - 1}{e^{2r} + 1},$$

so $B_\sigma(0 : r)$ is an open set in the usual metric. Analogously,

$$|z| < r \text{ implies that } d_\sigma(0, z) < \frac{1}{2}\log\left(\frac{1 + r}{1 - r}\right),$$

which proves the first statement.

If $\{z_n\}_{n=1}^\infty$ is a Cauchy sequence on D for the metric σ, it is bounded. There exists $R > 0$ such that $d_\sigma(0, z_n) \leq R$, or equivalently, $|z_n| \leq \dfrac{e^{2R} - 1}{e^{2R} + 1} = R' < 1$. Then, $\{z_n\}_{n=1}^\infty$ is a sequence on the compact set $\bar{D}(0, R')$, having then a convergent subsequence. As the original sequence is Cauchy, then it converges. ■

Poincaré metric plays an important role in the study of the self-mappings of the unit disk. There is a converse of Proposition 1.8.1:

Proposition 1.8.3 Let $f: D \longrightarrow D$ be an isometry for the Poincaré metric σ. Then, either $f(z)$ or $\bar{f}(z)$ belongs to $\mathrm{Aut}(D)$.

Proof. Take $g \in \mathrm{Aut}(D)$ such that $(g \circ f)(0) = 0$. As $g \circ f$ sends the circles $|z| = r$ into themshelves, we can assume, after a rotation, that $(g \circ f)(1/2) = 1/2$. Isometries preserve geodesics, so all points of the interval $(-1,1)$ are now fixed by $g \circ f$.

The points $i/2$ and $-i/2$ are equidistant from $1/2$ and $-1/2$, and are the only ones at these distances of these two points. Hence, $(g \circ f)(i/2) = \pm i/2$. After a conjugation, assume that $(g \circ f)(i/2) = i/2$. A similar argument considering distances show that, in this situation, $g \circ f$ is the identity map, which ends the proof. ■

Let us consider now a holomorphic function $f: D \longrightarrow D$. By Schwarz-Pick's Lemma we know that

$$\left|f'(z)\right| \leq \frac{1-|f(z)|^2}{1-|z|^2}.$$

This means that $f^*\sigma \leq \sigma$: f is distance decreasing. From this fact, other results can be obtained.

Definition 1.8.2 Let $\rho(z)$ be a metric on U. The curvature of ρ is

$$\kappa_{\rho,U}(z) = -\frac{\Delta(\log(\rho(z)))}{\rho(z)^2},$$

where $\Delta f = \dfrac{\partial^2 f}{\partial x^2} + \dfrac{\partial^2 f}{\partial y^2} = 4\dfrac{\partial^2 f}{\partial z \partial \bar{z}}$ is the Laplacian.

Proposition 1.8.4 The curvature is preserved under conformal holomorphic functions. That is, if $f: U \longrightarrow U'$ is such a map, and ρ is a metric on U', then

$$\kappa_{f^*\rho}(z) = \kappa_\rho(f(z)).$$

Proof. If $f: U \longrightarrow U'$ be holomorphic, and $h: U' \longrightarrow \mathbb{R}$ is C^2, we have that $\Delta(h \circ f) = (\Delta h \circ f) \cdot |f'(z)|^2$. Indeed,

$$\frac{\partial}{\partial z}(h \circ f(z)) = \left(\frac{\partial h}{\partial z} \circ f\right) \cdot \frac{\partial f}{\partial z} + \left(\frac{\partial h}{\partial \bar{z}} \circ f\right) \cdot \frac{\partial \bar{f}}{\partial z} = \left(\frac{\partial h}{\partial z} \circ f\right) \cdot \frac{\partial f}{\partial z}, \text{ and}$$

$$\frac{\partial}{\partial \bar{z}}\left(\frac{\partial}{\partial z}(h \circ f)(z)\right) = \left[\left(\frac{\partial^2 h}{\partial z^2} \circ f\right) \cdot \frac{\partial f}{\partial \bar{z}} + \left(\frac{\partial^2 h}{\partial z \partial \bar{z}} \circ f\right) \cdot \frac{\partial \bar{f}}{\partial \bar{z}}\right] \cdot \frac{\partial f}{\partial z} = \left(\frac{\partial^2 h}{\partial z \partial \bar{z}} \circ f\right) \cdot \left|f'(z)\right|^2.$$

Hence,

$$\begin{aligned} &-\frac{\Delta(\log((\rho \circ f)(z)\,|f'(z)|))}{(\rho \circ f)(z)^2 \cdot |f'(z)|^2} = -\frac{\Delta(\log((\rho \circ f)(z)))}{(\rho \circ f)(z)^2 \cdot |f'(z)|^2} \\ &= -\frac{(\Delta(\log \rho) \circ f)(z) \cdot |f'(z)|^2}{(\rho \circ f)(z)^2 \cdot |f'(z)|^2} = -\frac{\Delta(\log \rho)(z)}{\rho(z)^2} \circ f(z) = \kappa_\rho(f(z)). \end{aligned}$$

■

■ **Example 1.3** Let us compute the curvature of Poincaré metric:

$$-\Delta(\log\sigma(z)) = \Delta(\log(1-z\bar{z})) = 4\cdot\frac{\partial}{\partial\bar{z}}\left(\frac{-\bar{z}}{1-|z|^2}\right) = -\frac{4}{(1-|z|^2)^2}.$$

Then,

$$\kappa_\sigma(z) = -\frac{4}{(1-|z|^2)^2}\cdot(1-|z|^2)^2 = -4.$$

■

The notion of curvature allows to rewrite Schwarz Lemma, as follows:

Theorem 1.8.5 — Ahlfors-Schwarz Lemma. Let $f : D \longrightarrow U$ be a holomorphic function, ρ a metric on U with curvature $\kappa_\rho(z) \le -4$, $\forall z \in U$. Then $(f^*\rho)(z) \le \sigma(z)$ on D.

Proof. Let $0 < r < 1$, and $\tau_r : D(0;r) \longrightarrow D(0;1)$ the biholomorphism $\tau_r(z) = \frac{z}{r}$. A simple computation shows that

$$\tau_r^*\sigma(z) = \frac{r}{r^2-|z|^2}.$$

Let us denote σ_r this metric on $D(0;r)$, with curvature -4 by Proposition 1.8.4. On $D(0;r)$, let us denote

$$\eta_r(z) = \frac{(f^*\rho)(z)}{\sigma_r(z)}.$$

The numerator, being continuous, is bounded on $D(0;r)$, and so, $\lim_{|z|\to r^-}\eta_r(z) = 0$. Let $z_0 \in D(0;r)$ be a local maximum for $\eta_r(z)$. If $\eta_r(z_0) = 0$, the result is obvious. Assume $\eta_r(z_0) > 0$. It is a local maximum for $\log\eta_r$, so $\Delta(\log(\eta_r))(z_0) \le 0$. We have:

$$\begin{aligned}
0 &\ge \Delta(\log(\eta_r))(z_0) = \Delta(\log(f^*\rho))(z_0) - \Delta(\log(\sigma_r))(z_0)\\
&= -\kappa_{f^*\rho}(z_0)\cdot(f^*\rho)(z_0)^2 + \kappa_{\sigma_r}(z_0)\cdot\sigma_r(z_0)^2\\
&\ge 4(f^*\rho)(z_0)^2 - 4\sigma_r(z_0)^2 = 4((f^*\rho)(z_0)^2 - \sigma_r(z_0)^2).
\end{aligned}$$

Hence, $(f^*\rho)(z_0)^2 \le \sigma_r(z_0)^2$. So, $\eta_r(z_0) \le 1$ and by the maximum property of z_0, $\eta_r(z) \le 1$ for every $z \in D(0;r)$. Taking limits when $r \to 1^-$, $\eta_r(z) \le 1$. ■

Corollary 1.8.6 If ρ is a metric on U with $\kappa_\rho(z) \le -A < 0$, and $f : D(0;r) \longrightarrow U$ is a holomorphic function, then

$$(f^*\rho)(z) \le \frac{2r}{\sqrt{A}}\cdot\frac{1}{r^2-|z|^2}.$$

Proof. Note that if $\lambda > 0$, $\kappa_{\lambda\rho}(z) = \dfrac{\kappa_\rho(z)}{\lambda^2}$, so, taking $\lambda = \frac{\sqrt{A}}{2}$ we have that $\kappa_{\lambda\rho}(z) \le -4$. Denote $\rho_1(z) = \frac{\sqrt{A}}{2}\cdot\rho(z)$. Consider now $\alpha_r : D(0;1) \longrightarrow D(0;r)$ defined by $\alpha_r(z) = rz$. Applying Ahlfors-Schwarz Lemma 1.8.5, $(f\circ\alpha_r)^*(\rho_1)(z) = \alpha_r^*(f^*\rho_1)(z) \le \sigma(z)$ on $D(0;1)$. This means that $(f^*\rho_1)(rz)\cdot r \le \frac{1}{1-|z|^2}$, or $\dfrac{r\sqrt{A}}{2}\cdot(f^*\rho)(rz) \le \dfrac{1}{1-|z|^2}$. Replacing z by z/r, we obtain

$$(f^*\rho)(z) \le \frac{1}{1-\frac{|z|^2}{r^2}}\cdot\frac{2}{r\sqrt{A}} = \frac{2r}{\sqrt{A}}\cdot\frac{1}{r^2-|z|^2},$$

as stated.

■

Many consequences may be obtained from previous result. Let us mention a couple of them:

1. Schwarz Lemma follows from Ahlfors-Schwarz Lemma: If $f : D(0;1) \longrightarrow D(0;1)$ sends 0 to a, taking $\rho = \sigma$ we have
$$(f^*\sigma)(z) \le \sigma(z),$$
which implies $\sigma(f(z)) \cdot |f'(z)| \le \sigma(z)$. In particular, $|f'(0)| \le 1 - |a|^2 \le 1$.
2. Let $U \subseteq \mathbb{C}$ be an open set and $\rho(z)$ a metric on U with curvature $\kappa_\rho(z) \le -A < 0$. Let $f : \mathbb{C} \longrightarrow U$ be holomorphic. If $r > 0$ $f|_{D(0;r)}: D(0;r) \longrightarrow U$ verifies
$$(f^*\rho)(z) \le \frac{2r}{\sqrt{A}} \cdot \frac{1}{r^2 - |z|^2}.$$
Taking limits when $r \longrightarrow \infty$, $(f^*\rho)(z) \longrightarrow 0$, and then, $(f^*\rho)(z) = 0$. This implies that $f'(z) = 0$ for every z and so, f is constant. So, Liouville's Theorem turns out to be a consequence of Ahlfors-Schwarz Lemma, taking as U a disk on $\mathbb{C}$.

Some other consequences, regarding the behaviour of entire functions, shall be stated in Chapter 4.

1.9 Families of analytic functions. Montel's Theorem

In this Section we shall study the spaces of holomorphic functions defined on an open set $U \subseteq \overline{\mathbb{C}}$. The main result of this section is Montel's Theorem, that provides a characterization of the compact families of analytic functions, and will be used in the proof of Riemann mapping Theorem in Section 1.10.2.

The first result will be the construction of a topology on $\mathscr{O}(U)$.

Proposition 1.9.1 There exists a sequence of compact sets $\{K_n\}_{n=1}^{\infty}$, $K_n \subseteq U$, such that:

1. $K_n \subseteq \mathring{K}_{n+1}$.
2. $\bigcup_{n=1}^{\infty} K_n = U$.

Proof. Define $K_n = \bar{D}(0;n) \cap \{z \in \mathbb{C} | d(z, \mathbb{C} \setminus U) \ge \frac{1}{n}\}$. K_n is compact, as it is closed and bounded. Moreover
$$K_n \subseteq D(0;n+1) \cap \{z \in \mathbb{C} | d(z, \mathbb{C} \setminus U) > \frac{1}{n+1}\},$$
the latter being an open set contained in K_{n+1}. So, $K_n \subseteq \mathring{K}_{n+1}$. From the construction, it becomes clear that $\bigcup_{n=1}^{\infty} K_n = U$. ■

R A sequence of compacts as in Proposition 1.9.1 is called an **exhaustion** of U. Given such an exhaustion, and a compact $K \subseteq U$, as $\{\mathring{K}_n\}_{n=1}^{\infty}$ is an open covering of K, it turns out that for some $n_0 \in \mathbb{N}$, $K \subseteq \mathring{K}_{n_0} \subseteq K_{n_0}$.

Given an exhaustion $\{K_n\}_{n=1}^{\infty}$ of U, and $f \in \mathscr{C}(U)$, denoting
$$||f||_K := \sup\{|f(z)| \,|\, z \in K\},$$
we can define, for $f, g \in \mathscr{C}(U)$
$$d(f,g) := \sum_{n=1}^{\infty} \frac{1}{2^n} \cdot \frac{||f-g||_{K_n}}{1 + ||f-g||_{K_n}}.$$

Proposition 1.9.2 d is a distance in $\mathscr{C}(U)$.

Proof. The only non trivial property to prove is the triangular inequality. If $0 \le t < s$, $\dfrac{t}{1+t} < \dfrac{s}{1+s}$. So, given f, g, $h \in \mathscr{C}(U)$, from the inequality

$$||f-g||_{K_n} \le ||f-h||_{K_n} + ||h-g||_{K_n}$$

we obtain that

$$\frac{||f-g||_{K_n}}{1+||f-g||_{K_n}} \le \frac{||f-h||_{K_n}+||h-g||_{K_n}}{1+||f-h||_{K_n}+||h-g||_{K_n}} \le \frac{||f-h||_{K_n}}{1+||f-h||_{K_n}} + \frac{||h-g||_{K_n}}{1+||h-g||_{K_n}}.$$

The inequality $d(f,g) \le d(f,h) + d(h,g)$ follows easily. ■

The topology defined by the previous distance is independent of the exhaustion chosen. In fact:

Proposition 1.9.3 The sequence $\{f_n\}_{n=1}^{\infty} \subseteq \mathscr{C}(U)$ converges to a function f in the distance d if and only if it converges uniformly in compact sets to f.

Proof. If $\lim_{n\to\infty} f_n = f$ in d, given $\varepsilon > 0$ and a compact set $K \subseteq U$, take $n_0 \in \mathbb{N}$ such that $K \subseteq K_{n_0}$, and $j_0 \in \mathbb{N}$ such that if $j \ge j_0$, $d(f, f_j) < \dfrac{\varepsilon}{2^{n_0}}$. We then have

$$\frac{1}{2^{n_0}} \cdot \frac{||f-f_j||_{K_{n_0}}}{1+||f-f_j||_{K_{n_0}}} \le \frac{\varepsilon}{2^{n_0}}, \text{ so } \frac{||f-f_j||_{K_{n_0}}}{1+||f-f_j||_{K_{n_0}}} \le \varepsilon.$$

Therefore,

$$||f-f_j||_K \le ||f-f_j||_{K_{n_0}} \le \frac{\varepsilon}{1-\varepsilon}.$$

Conversely, given $\varepsilon > 0$ and $m \in \mathbb{N}$ such that $\dfrac{1}{2^m} = \sum_{n=m+1}^{\infty} \frac{1}{2^n} < \dfrac{\varepsilon}{2}$, we have

$$\begin{aligned} d(f,f_j) &= \sum_{n=1}^{m} \frac{1}{2^n} \cdot \frac{||f-f_j||_{K_n}}{1+||f-f_j||_{K_n}} + \sum_{n=m+1}^{\infty} \frac{1}{2^n} \cdot \frac{||f-f_j||_{K_n}}{1+||f-f_j||_{K_n}} \\ &\le \sum_{n=1}^{m} \frac{1}{2^n} \cdot \frac{||f-f_j||_{K_n}}{1+||f-f_j||_{K_n}} + \frac{\varepsilon}{2} \le \sum_{n=1}^{m} \frac{1}{2^n} \cdot \frac{||f-f_j||_{K_m}}{1+||f-f_j||_{K_m}} + \frac{\varepsilon}{2} \\ &\le \frac{||f-f_j||_{K_m}}{1+||f-f_j||_{K_m}} + \frac{\varepsilon}{2} \le ||f-f_j||_{K_m} + \frac{\varepsilon}{2}. \end{aligned}$$

For some $j_0 \in \mathbb{N}$, if $j \ge j_0$, $||f-f_j||_{K_m} \le \dfrac{\varepsilon}{2}$, and this ends the proof. ■

1.9.1 Relatively compact families

The objective of this section will be to state and prove Montel's Theorem about relatively compact families of holomorphic functions. This result is strongly based on Arzelà-Ascoli Theorem, a result from function theory which will be established here in a more general way that we need. So, let us consider a metric space (E,d) and a Banach space $(F, ||\cdot||)$.

Definition 1.9.1 Let $\mathscr{F} \subseteq \mathscr{C}(E,F)$.

1. $\mathscr{F}$ is pointwise bounded if, for every $z_0 \in E$, the set
$$\{f(z_0) \mid f \in \mathscr{F}\}$$
is relatively compact.
2. $\mathscr{F}$ is equicontinuous at a point $z_0 \in E$ if, $\forall \varepsilon > 0$, there exists $\delta > 0$ such that if $f \in \mathscr{F}$ and $d(z,z_0) < \delta$, then $||f(z) - f(z_0)|| < \varepsilon$.

Assume that the metric space E has the following property:

There exists a sequence $\{K_n\}_{n=1}^{\infty}$ of compact sets such that $K_n \subseteq \mathring{K}_{n+1}$ and $\bigcup_{n=1}^{\infty} K_n = E$.

In particular, E turns out to be a separable space, and locally compact. Let $D \subseteq E$ be a countable dense subset of E. In order to state Arzelà-Ascoli Thorem, we will establish first a series of Lemmas.

Lemma 1.9.4 Assume that $\{f_n\}_{n=1}^{\infty} \subseteq \mathscr{C}(E,F)$ is a pointwise bounded sequence of functions. There exists a subsequence that converges in the points of D.

Proof. $D = \{z_k\}_{k=1}^{\infty}$. As $\{f_n(z_1)\}_{n=1}^{\infty}$ is relatively compact, it has a convergent subsequence $\{f_{1n}(z_1)\}_{n=1}^{\infty}$. The sequence $\{f_{1n}(z_2)\}_{n=1}^{\infty}$ is relatively compact, so it has again a convergent subsequence $\{f_{2n}(z_2)\}_{n=1}^{\infty}$. Recursively we construct subsequences $\{f_{mn}\}_{n=1}^{\infty}$, which are convergent in $\{z_1, z_2, \dots, z_m\}$. Denote $g_n = f_{nn}$. It is the subsequence we are looking for. ■

Lemma 1.9.5 Assume that $\{f_n\}_{n=1}^{\infty} \subseteq \mathscr{C}(E,F)$ is a sequence of functions, equicontinuous at every point of E, that converges in a countable and dense subset D of E. Then, it converges in every point of E.

Proof. Let $z \in E$. Our goal will be to show that $\{f_n(z)\}_{n=1}^{\infty}$ is a Cauchy sequence. Take $\varepsilon > 0$, and $\delta > 0$ satisfying the equicontinuity condition: if $d(x,z) < \delta$ and $n \in \mathbb{N}$, then $||f_n(x) - f_n(z)|| < \varepsilon$.

Take $x \in B(z,\delta) \cap D$, and $n_0 \in \mathbb{N}$ such that if n, $m \geq n_0$, then $||f_n(x) - f_m(x)|| < \varepsilon$, which is possible as $\{f_n(x)\}_{n=1}^{\infty}$ is a Cauchy sequence. Then,
$$||f_n(z) - f_m(z)|| \leq ||f_n(z) - f_n(x)|| + ||f_n(x) - f_m(x)|| + ||f_m(x) - f_m(z)|| \leq \varepsilon + \varepsilon + \varepsilon = 3\varepsilon.$$
So, $\{f_n(z)\}_{n=1}^{\infty}$ is Cauchy, hence convergent. ■

Lemma 1.9.6 Assume that $\{f_n\}_{n=1}^{\infty}$ is a equicontinuous sequence of functions in $\mathscr{C}(E,F)$ converging pointwise to a function f. Then, it converges uniformly in the compact sets of E.

Proof. Let $K \subseteq E$ be a compact set, and $\varepsilon > 0$. If $z \in K$, there exists $\delta_z > 0$ such that if $d(x,z) < \delta_z$, then $||f_n(x) - f_n(z)|| < \varepsilon$, and taking limits, $||f(x) - f(z)|| \leq \varepsilon$.

The family $\{B(z,\delta_z) \mid z \in K\}$ covers K, so there exists a finite subcovering $K \subseteq B(z_1,\delta_{z_1}) \cup \cdots \cup B(z_m,\delta_{z_m})$. Let $n_0 \in \mathbb{N}$ be such that if $n \geq n_0$ and $i \in \{1,2,\dots,m\}$ then $||f_n(z_i) - f(z_i)|| < \varepsilon$. Take now $z \in K$, $n \geq n_0$ and i such that $z \in B(z_i,\delta_{z_i})$. So,
$$||f_n(z) - f(z)|| \leq ||f_n(z) - f_n(z_i)|| + ||f_n(z_i) - f(z_i)|| + ||f(z_i) - f(z)|| \leq 3\varepsilon.$$
First and third inequalities stem from equicontinuity condition, and the second one from the choice of z_i. Previous bound is independent of $z \in K$, and so, the result follows. ■

Now, we are in the conditions to prove:

Theorem 1.9.7 — Arzelà-Ascoli. Under previous conditions on E and F, let $\mathscr{F} \subseteq \mathscr{C}(E,F)$. The following are equivalent:

1. $\mathscr{F}$ is relatively compact.
2. $\mathscr{F}$ is equicontinuous in each point, and pointwise bounded.

Proof. (1) $\Rightarrow$ (2). Take $z_0 \in E$. The hypothesis on E imply that E is locally compact. So, consider $K = \bar{B}(z_0;R)$ a compact neighbourhood of z_0. By restriction, $\mathscr{F}$ induces a family $\mathscr{F} \subseteq \mathscr{C}(K,F)$ (abuse of notation), again relatively compact.

As a relatively compact space is precompact, given $\varepsilon > 0$, there exists a finite family $\{f_1, f_2, \dots, f_m\} \subseteq \mathscr{F}$ such that if $f \in \mathscr{F}$, $||f - f_i||_K < \varepsilon/3$, for some $i \in \{1,2,\dots,m\}$. Take $0 < \delta < R$ such that if $d(z,z_0) < \delta$ then $||f_i(z) - f_i(z_0)|| < \varepsilon/3$ for every i. If $f \in \mathscr{F}$, we have:

$$||f(z) - f(z_0)|| \leq ||f(z) - f_i(z)|| + ||f_i(z) - f_i(z_0)|| + ||f_i(z_0) - f(z_0)|| \leq \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon.$$

So, equicontinuity of $\mathscr{F}$ follows.

Take again $z_0 \in E$. To show that $\{f(z_0) \mid f \in \mathscr{F}\}$ is relatively compact, consider again a compact neighbourhood K of z_0, and a finite number of functions $\{f_1, f_2, \dots, f_m\}$ such that if $f \in \mathscr{F}$, $||f - f_i||_K \leq \varepsilon/3$ for some i. In particular, $||f(z_0) - f_i(z_0)||_K \leq \varepsilon/3$, and so, $\{f(z_0) \mid f \in \mathscr{F}\}$ is precompact, then relatively compact, F being complete.

(2) $\Rightarrow$ (1). Take $f_1, f_2, \dots, f_n$ a sequence of elements of $\mathscr{C}(E,F)$. As E is separable, there exists a countable dense set $D \subseteq E$. A subsequence of $f_1, f_2, \dots, f_n$ converges on D (Lemma 1.9.4), hence on E (Lemma 1.9.5). The convergence is uniform in compact sets (Lemma 1.9.6), which ends the proof. ■

R A subset A of a metric space E is called precompact if, for every $\varepsilon > 0$, there exists a finite family of elements $\{a_1, a_2, \dots, a_n\} \subseteq A$ such that if $a \in A$, there exists i with $d(a, a_i) < \varepsilon$. A relatively compact set is precompact, and in a complete metric space, both notions are equivalent. For a metric space E, compactness is equivalent to be precompact and complete. The interested reader is invited to go to [9] for further information.

In this text, and in frequent applications, F is either $\mathbb{C}$ or $\mathbb{R}$. Here, to be precompact, or relatively compact, is the same that to be bounded, thanks to Bolzano-Weierstrass Theorem. Most previous statements may be simplified.

We come back to holomorphic functions.

Definition 1.9.2 A family $\mathscr{F} \subseteq \mathscr{C}(E,\mathbb{C})$ is locally bounded if, for every compact set $K \subseteq E$, the set

$$\{||f||_K \mid f \in \mathscr{F}\}$$

is relatively compact (i.e., bounded).

Theorem 1.9.8 If $\mathscr{F} \subseteq \mathscr{O}(U)$ is locally bounded, it is equicontinuous at every point $z_0 \in U$.

Proof. Let $z_0 \in U$, and $\bar{D}(z_0;r) \subseteq U$. If $|z - z_0| < r/2$ and $f \in \mathscr{F}$,

$$\begin{aligned} |f(z) - f(z_0)| &= \left| \frac{1}{2\pi i} \int_\gamma \left(\frac{f(\omega)}{\omega - z} - \frac{f(\omega)}{\omega - z_0} \right) d\omega \right| = \left| \frac{1}{2\pi i} \int_\gamma f(\omega) \frac{z - z_0}{(\omega - z)(\omega - z_0)} d\omega \right| \\ &\leq \frac{1}{2\pi} M_\gamma 2\pi r \frac{|z - z_0|}{r/2 \cdot r} = \frac{2}{r} M_\gamma |z - z_0|, \end{aligned}$$

where γ is the circle with center z_0 and radius r, and M_γ a bound of the elements of $\mathscr{F}$ on γ. So, $\mathscr{F}$ is equicontinuous at z_0. ■

Theorem 1.9.9 — Montel's Theorem. If $\mathscr{F} \subseteq \mathscr{O}(U)$ is locally bounded, it is relatively compact.

Proof. By 1.9.8, $\mathscr{F}$ is equicontinuous. By 1.9.7, it is relatively compact. ■

Let us state some consequences of Montel's Theorem.

Theorem 1.9.10 — Vitali. Let U be a domain, and $\{f_n(z)\}_{n\in\mathbb{N}}$ a sequence of holomorphic functions on U, uniformly bounded in compact sets. Assume that this sequence converges in every point of $S \subseteq U$, S being a set having at least an accumulation point on U. Then, the sequence converges uniformly in the compact sets of U.

Proof. If the assertion is false, there exists a compact set $K \subseteq U$ such that $\{f_n(z)\}$ is not a Cauchy sequence for the topology of the uniform convergence on K. There exists $\varepsilon > 0$ and two interwinned sequences $n_1 < m_1 < n_2 < m_2 < \cdots$ of natural numbers such that $||f_{n_k} - f_{m_k}||_K \geq \varepsilon$. Take subsequences $\{f_{n'_k}\}$, $\{f_{m'_k}\}$ converging to $f(z)$, $g(z)$ respectively. As $f(z)$ and $g(z)$ coincide in S, they coincide everywhere in U, and this is a contradiction because we must have $||f(z) - g(z)||_K \geq \varepsilon$. ■

Theorem 1.9.11 — Osgood. Assume that the sequence $\{f_n\}_{n=1}^{\infty} \subseteq \mathscr{O}(U)$ converges pointwise to f. There exists a dense open set $V \subseteq U$ such that f_n converges uniformly in the compact sets of V to f.

Proof. Denote

$$A_n = \{z \in U | \forall k \in \mathbb{N},\ |f_k(z)| \leq n\}.$$

As for every $z \in U$, the set $\{f_k(z)\}_{k=1}^{\infty}$ is bounded, we have that $\cup_{n=1}^{\infty} A_n = U$. By Baire Category Theorem, at least one A_n has non empty interior, so a disk D exists such that $D \subseteq A_n$. Over D, by Vitali's Theorem 1.9.10, the sequence f_n converges uniformly in the compacts. Let V be the union of all those disks D: V is dense as for any open set $W \subseteq U$, the same argument applies to find a disk $D \subseteq V$. ■

1.10 Riemann conformal mapping Theorem

In this section we will prove one of most important and striking results of the theory of holomorphic functions in one variable, that is Riemann's Theorem. It states that every simply connected open set of $\mathbb{C}$, different from $\mathbb{C}$ itself, is biholomorphic to the unit disk D. As holomorphic functions are quite rigid, such a general result is surprising. It has implications even of topological nature: every simply connecetd open subset of $\mathbb{R}^2$ is homeomorphic to a disk.

To prove it, we will use the fact that in a simply connected open subset of $\mathbb{C}$, every zero-free holomorphic function has a holomorphic square root. In fact, we will show that if an open set $U \neq \mathbb{C}$ has this property (which we will call *square root property*), it is biholomorphic to D.

Lemma 1.10.1 If $U \subsetneq \mathbb{C}$ is an open subset with the square root property, there exists $f : U \longrightarrow D$ holomorphic and injective.

Proof. Take $a \in \mathbb{C} \setminus U$, and $h(z) \in \mathscr{O}(U)$ such that $h(z)^2 = z - a$. We have:

1. $h(z)$ is injective.
2. $0 \notin h(U)$.
3. $h(U)$ and $-h(U)$ are disjoint open sets.

Take $D(w;r) \subseteq -h(U)$. As $h(U) \cap D(w;r) = \emptyset$, the function $f(z) = \dfrac{1}{h(z) - w} \in \mathscr{O}(U)$ and $|f(z)| \leq \frac{1}{r}$. The function $rf(z)$ verifies the statement. ■

Theorem 1.10.2 — Riemann. Let $U \subseteq \mathbb{C}$ be an open subset with the square root property. There exists a biholomorphic $f : U \longrightarrow D(0;1)$.

Proof. Take an injective map $f_0 : U \longrightarrow D$ (which exists by Lemma 1.10.1), and $z_0 \in U$. Let us consider

$$\mathscr{F} = \left\{ f : U \longrightarrow D \text{ injective} \mid \left|f'(z_0)\right| \geq \left|f_0'(z_0)\right| \right\}.$$

$\mathscr{F} \neq \emptyset$, as $f_0 \in \mathscr{F}$, and it is locally bounded. If $\{f_n\}_{n=1}^{\infty}$ is a sequence in $\mathscr{F}$ that converges uniformly in compact sets to f, then $|f'(z_0)| \geq |f_0'(z_0)| > 0$. So, f is not constant, and so, it is itself injective: $f \in \mathscr{F}$, hence $\mathscr{F}$ is closed, and by 1.9.7, it is compact. The map

$$\begin{array}{ccc} \mathscr{F} & \longrightarrow & \mathbb{C} \\ f & \longmapsto & \left|f'(z_0)\right| \end{array}$$

takes a maximum, say at $g \in \mathscr{F}$. Hence, $|g'(z_0)| \geq |f'(z_0)|$, for every $f \in \mathscr{F}$. If g is not surjective, there exists $a \in D(0;1) \setminus g(U)$, If $\varphi_a(z) = \dfrac{z-a}{1-\bar{a}z}$, $\varphi_a \circ g$ has no zeros in U, so there exists $h(z) \in \mathscr{O}(U)$ such that $h(z)^2 = \varphi_a \circ g(z)$, and h is injective. If $h(z_0) = b$, take $f(z) = \varphi_b \circ h(z)$, $f(z_0) = 0$.

$g(z) = (\varphi_{-a} \circ \varphi_{-b}^2) \circ f(z)$, and then $g'(z_0) = (\varphi_{-a} \circ \varphi_{-b}^2)'(0) f'(z_0)$. As $\varphi_{-a} \circ \varphi_{-b}^2 : D \longrightarrow D$ is not injective, it is a consequence of the properties that follow Schwarz's Lemma 1.6.1 that $\left|(\varphi_{-a} \circ \varphi_{-b}^2)'(0)\right| < 1 - \left|\varphi_{-a} \circ \varphi_{-b}^2(0)\right|^2$, and so, $|g'(z_0)| < |f'(z_0)|$, in contradiction with the hypothesis. ■

R This proof is not constructive. A somewhat more constructive proof may be read in [11].

The map given by Riemann's Theorem 1.10.2 is not unique. Nevertheless something can be said.

1. Given biholomorphisms $f, g : U \longrightarrow D$ with $f(z_0) = g(z_0)$ and $f'(z_0) = g'(z_0)$, then $f = g$.

   Proof. Indeed, if $f(z_0) = g(z_0) = 0$, then $g \circ f^{-1} : D \longrightarrow D$ is such that $(g \circ f^{-1})(0) = 0$ and biholomorphic. So $g(z) = \lambda f(z)$ for some $|\lambda| = 1$. As $g'(z_0) = \lambda f'(z_0)$, then $\lambda = 1$. If $f(z_0) = g(z_0) = a \neq 0$, consider $\varphi_a \circ f(z)$ and $\varphi_a \circ g$. ■

2. The function $g(z)$ given in the proof of Riemann's Theorem sends z_0 to 0. Indeed, if $g(z_0) = a \neq 0$, $h(z) = \varphi_a \circ g(z) : U \longrightarrow D$ sends z_0 to 0 and $h'(z_0) = \dfrac{1}{1-|a|^2} g'(z_0)$, so we arrive to the contradiction that $|h'(z_0)| > |g'(z_0)|$.
3. Given a holomorphic function $f : U \longrightarrow D$ with $f(z_0) = 0$, then $|f'(z_0)| \leq |g'(z_0)|$. Equality holds if and only if $f(z) = \lambda g(z)$ for some $|\lambda| = 1$.

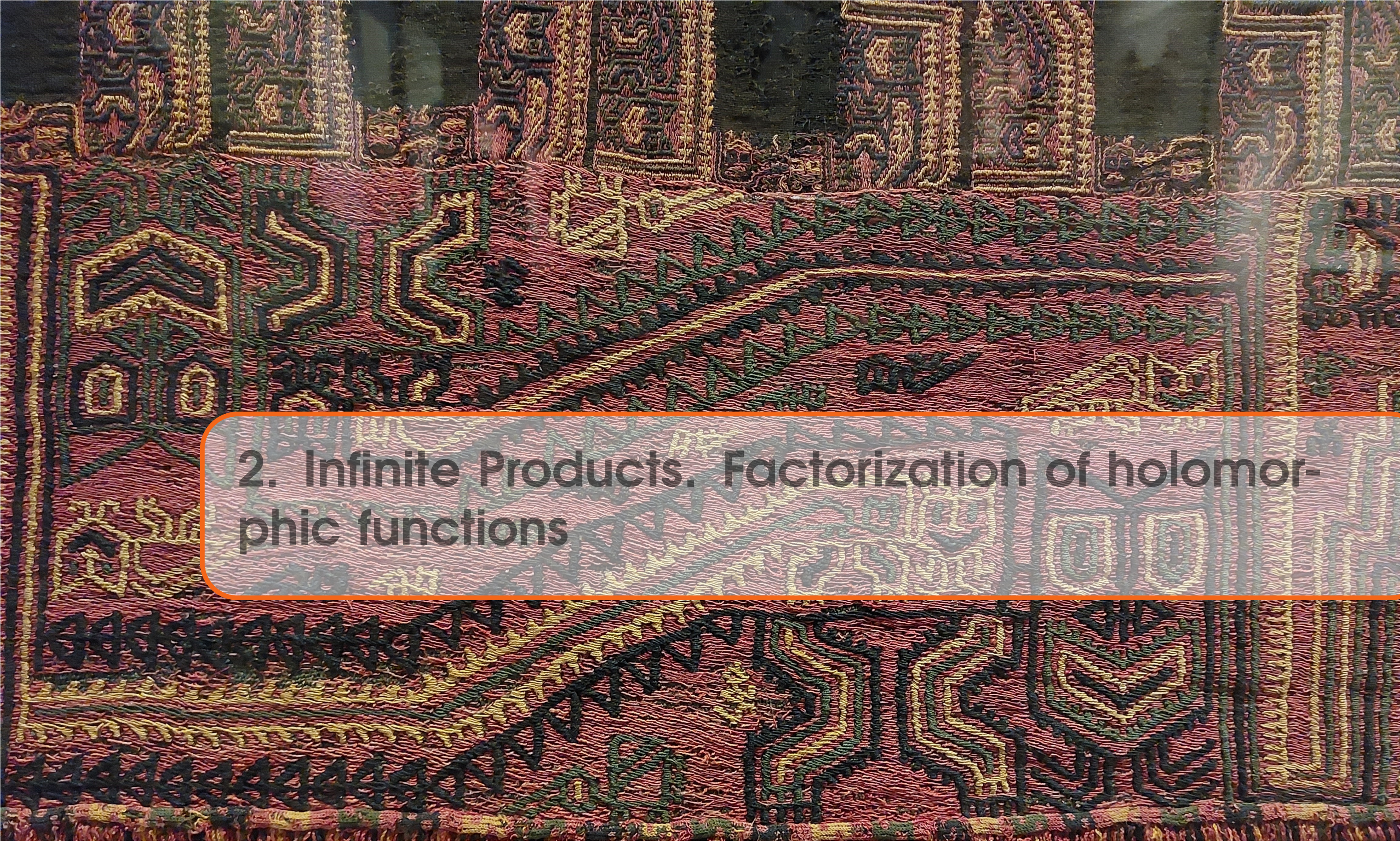
   Proof. $f \circ g^{-1} : D \longrightarrow D$ sends 0 to 0, so, by Schwarz's Lemma 1.6.1, $|f'(z_0)| \cdot \dfrac{1}{|g'(z_0)|} \leq 1$. Equality holds exactly when $f(z) = \lambda g(z)$ for some $|\lambda| = 1$. ■

4. There exists a unique $g : U \longrightarrow D$ biholomorphic such that $g(z_0) = 0$, $g'(z_0) \in \mathbb{R}_+$.
5. Given two simply connected open sets U_1, U_2 different from $\mathbb{C}$, and $z_i \in U_i$, there exists a unique biholomorphism $f : U_1 \longrightarrow U_2$ with $f(z_1) = z_2$ and $f'(z_1) \in \mathbb{R}_+$.

1.11 Exercises

Some of the exercises listed here are related with the previous results stated in Chapter 0.

1. Let U be an open set, and $\bar{D} \subseteq U$. Assume that $f \in \mathscr{O}(U)$ verifies that $f(\bar{D}) \subseteq D$. How many solutions has the equation $z = f(z)$?
   Will the solution be different if we only assume that $f(D) \subseteq D$?
2. Let $f(z)$, $g(z)$ be entire functions such that $|f(z)| \leq |g(z)|$ for every $z \in \mathbb{C}$. What can be said about them?
3. Using the properties of the inversion, construct a Möbius transformation which transforms the upper half plane in the interior of the circle $|z| = R$, and takes a point α into 0.
4. Show that it doesn't exist a holomorphic and injective function $f : D(0;1) \longrightarrow \mathbb{C}$ such that $f(D(0;1)) = \mathbb{C}$. .
5. Show that the map $f(z) = z + \dfrac{1}{z}$ defines a biholomorphism between $D(0;1) \setminus \{0\}$ and $\mathbb{C} \setminus [-2,2]$.
6. Let $\mathscr{C}$ be a circle on $\overline{\mathbb{C}}$, and $I_{\mathscr{C}}$ be the inversion with respect to $\mathscr{C}$. Let z_1, z_2, z_3 be diferent points on $\mathscr{C}$. Show that $I_{\mathscr{C}}(z) = w$ if and only if
   $$[z_1, z_2, z_3, w] = \overline{[z_1, z_2, z_3, z]}.$$
7. Let T, S be two Möbius transformations with the same fixed points. Show that $TS = ST$.
8. Suppose that T is an analytic function from the unit disk $D(0;1)$ to itself, that is not the identity map. Show that T has at most one fixed point.
9. Let $f : \mathbb{H} \longrightarrow \mathbb{C}$ be a holomorphic function such that $|f(z)| \leq 1$ and $f(i) = 0$. Show that
   $$|f(z)| \leq \left| \frac{z-i}{z+i} \right|.$$
10. Let $f : D(0;1) \longrightarrow D(0;1)$ be a holomorphic function such that $f(0) = f'(0) = \cdots = f^{(n-1)}(0) = 0$, $f^{(n)}(0) \neq 0$. Show that $|f(z)| \leq |z^n|$ and $\left|f^{(n)}(0)\right| \leq n!$.
11. (Gauss-Lucas Theorem)
    (a) Let $P(z)$ be a degree n polynomial, with roots $\omega_1, \omega_2, \ldots, \omega_n$. Assume that $\mathrm{Re}(\omega_1) \geq 0$, for every i. Show that, if z_0 is a root of $P'(z)$, then $\mathrm{Re}(z_0) \geq 0$.
    (b) Do the same with any closed half plane: if $P(z)$ has all its roots in a half plane $\mathrm{Re}(cw) \geq 0$, all the roots of $P'(z)$ are in the sane half plane.
    (c) Conclude that any root of $P'(z)$ is contained in the convex hull of the roots of $P(z)$.
12. Let $f \in \mathscr{O}(D(0;1))$ such that $f(0) = 1$, $f'(0) = 2i$ and $\mathrm{Re}\, f(z) \geq 0$ for every z. Determine f.
13. Let $\mathscr{F} \subseteq \mathscr{O}(D(0;1))$. Show that $\mathscr{F}$ is relatively compact if and only if there exists a sequence of positive numbers $\{M_n\}_{n=1}^{\infty}$ such that $\limsup_{n \longrightarrow \infty} M_n^{1/n} \leq 1$ and for every $f \in \mathscr{F}$, $\left| \dfrac{f^{(n)}(0)}{n!} \right| \leq M_n$.
14. Let $\mathscr{F} \subseteq \mathscr{O}(D(0;1))$. Assume that the set $\mathscr{F}' = \{f'(z) | f \in \mathscr{F}\}$ is relatively compact, and that the set $\{f(0) | f \in \mathscr{F}\}$ is bounded. Show that $\mathscr{F}$ is relatively compact.



2. Infinite Products. Factorization of holomorphic functions

2.1 Infinite products of complex numbers

The so called Basel's Problem asked to evaluate the sum of the series

$$\sum_{n=1}^{\infty} \frac{1}{n^2}.$$

This value was computed first by L. Euler, who showed that it equals $\frac{\pi^2}{6}$. Different ways exist today to make this computation. The most popular, that can be read in virtually any book in Complex Analysis, uses Residue Theorem. The objective is to compute the integral

$$\int_{C_N} \frac{\pi \cot \pi z}{z^2} dz, \tag{2.1}$$

where C_N is the square whose vertices are $\left(N+\frac{1}{2}\right)(\pm 1 \pm i)$. Poles of the integrand inside C_N are located in

$$\{n \in \mathbb{Z} | -N \leq n \leq N\}.$$

At those points, the value of the residues are:

- If $n \neq 0$,

$$\lim_{z \to n} (z-n) \frac{\pi \cos \pi z}{z^2 \sin \pi z} = \frac{1}{n^2}.$$

- If $n = 0$,

$$\operatorname{Res}\left(\frac{\pi \cot \pi z}{z^2}; 0\right) = -\frac{\pi^2}{3}.$$

Taking limits when N tends to infinity, the integral (2.1) tends to 0. As

$$-\frac{\pi^2}{3}+2\sum_{n=1}^{N}\frac{1}{n^2}\longrightarrow 0 \text{ si } N\to\infty,$$

it follows that

$$\sum_{n=1}^{\infty}\frac{1}{n^2}=\frac{\pi^2}{6}.$$

Euler did not compute this sum in this way. Instead, he considered the function $\frac{\sin z}{z}$, and treated it as if it were a polynomial. Let us observe that a polynomial $P(z)$ with $P(0)=1$ and non-zero roots $\alpha_1,\alpha_2,\dots,\alpha_k$ can be written as

$$P(z)=\prod_{i=1}^{k}\left(1-\frac{z}{\alpha_i}\right).$$

More generally every polynomial $P(z)\in\mathbb{C}[z]$ is

$$P(z)=cz^m\prod_{i=1}^{k}\left(1-\frac{z}{\alpha_i}\right).$$

Euler, with an apparent lack of rigor, assumed that a similar expression could be valid for any entire function, and so he wrote

$$\frac{\sin z}{z}=c\prod_{\substack{k\in\mathbb{Z}\\ k\neq 0}}\left(1-\frac{z}{k\pi}\right)=c\prod_{k=1}^{\infty}\left(1-\frac{z^2}{k^2\pi^2}\right).$$

As $\lim_{z\to 0}\frac{\sin z}{z}=1$, $c=1$ in previous expression. After some computation,

$$\prod_{k=1}^{\infty}\left(1-\frac{z^2}{k^2\pi^2}\right)=1-\left(\sum_{k=1}^{\infty}\frac{1}{k^2\pi^2}\right)z^2+\left(\sum_{i<j}\frac{1}{i^2j^2\pi^4}\right)z^4+\cdots$$

But $\frac{\sin z}{z}=1-\frac{1}{3!}z^2+\frac{1}{5!}z^4+\cdots$. So, identifying coefficients,

$$-\frac{1}{6}=-\sum_{k=1}^{\infty}\frac{1}{k^2\pi^2},\text{ hence }\sum_{k=1}^{\infty}\frac{1}{k^2}=\frac{\pi^2}{6}.$$

In the same way, as

$$\sum_{i<j}\frac{1}{i^2j^2}=\frac{\pi^4}{120},$$

we obtain

$$\frac{\pi^4}{36}=\left(\sum_{k=1}^{\infty}\frac{1}{k^2}\right)^2=\sum_{k=1}^{\infty}\frac{1}{k^4}+2\sum_{i<j}\frac{1}{i^2j^2},$$

and consequently,

$$\sum_{k=1}^{\infty}\frac{1}{k^4}=\frac{\pi^4}{36}-\frac{\pi^4}{60}=\frac{\pi^4}{90}.$$

Euler succeeded in this way in computing the values $\zeta(2n)$, when $n\in\mathbb{N}$, $\zeta(z)$ being Riemann's zeta function

$$\zeta(z)=\sum_{n=1}^{\infty}\frac{1}{n^z},$$

that will be extensively treated in Section 3.3 The objective of this chapter will be to justify previous computations, as the expression obtained by Euler of $\frac{\sin z}{z}$ as an infinite product. First, we shall study the basic theory of infinite products of complex numbers, $\prod_{n=1}^{\infty} z_n$, where $z_n\in\mathbb{C}$. The possible definition of $\prod_{n=1}^{\infty} z_n$ as $\lim_{N\to\infty}\prod_{n=1}^{N} z_n$ is not fully satisfactory, because if some factor z_k vanishes, the infinite product would vanish, despite any condition of, for instance, growth of the terms. This would made rather unnatural to state convergence results. Moreover, it may happen that $z_n\neq 0$, but

$$\lim_{N\to\infty}\prod_{n=1}^{N} z_n=0.$$

If we admit this definition, this would lead us to lose some of the basic properties that we would like a product to have: if a product vanishes, at least one factor does.

Due to previous considerations, we shall adopt the following definition:

Definition 2.1.1
1. Let $\{z_n\}_{n\in\mathbb{N}}$ be a sequence of non-zero complex numbers. We will say that the infinite product $\prod_{n=1}^{\infty} z_n$ converges to a limit $l\neq 0$, if and only if the following limit exists

$$\lim_{N\to\infty}\prod_{n=1}^{N} z_n$$

and takes the value l.
2. Let $\{z_n\}_{n\in\mathbb{N}}$ be a sequence of complex numbers, with a finite number of zeros. We shall say that the infinite product $\prod_{n=1}^{\infty} z_n$ converges to 0 if, given $n_0\in\mathbb{N}$ such that $\forall n\geq n_0,\ z_n\neq 0$, the infinite product

$$\prod_{n=n_0}^{\infty} z_n$$

converges in the sense of previous item.

R

1. The second definition is independent of the chosen number n_0.
2. Some texts, as [4] or [7] give a slightly different definition. [6] gives the definition we have followed here.

In the sequel, we shall assume that $\forall n\in\mathbb{N}$, $z_n\neq 0$.

Lemma 2.1.1 If $\prod_{n=1}^{\infty} z_n$ is convergent, then $\lim_{n\to\infty} z_n = 1$.

Proof. Indeed, for $n > 1$,

$$z_n = \frac{\prod_{k=1}^{n} z_k}{\prod_{k=1}^{n-1} z_k} \overset{n\to\infty}{\longrightarrow} \frac{l}{l} = 1.$$

■

It is often convenient to write an infinite product as $\prod_{n=1}^{\infty}(1+z_n)$. With this notation, a necessary (and not sufficient) condition for the convergence would be $\lim_{n\to\infty} z_n = 0$.

■ **Example 2.1** 1. The infinite product $\displaystyle\prod_{n=1}^{\infty} \frac{n+1}{n}$ does not converge. Indeed,

$$\prod_{n=1}^{N} \frac{n+1}{n} = \frac{2}{1}\cdot\frac{3}{2}\cdots\frac{N+1}{N} = N+1 \longrightarrow +\infty.$$

2.

$$\prod_{n\geq 2}\left(1-\frac{1}{n^2}\right) = \prod_{n\geq 2}\frac{(n-1)(n+1)}{n^2} = \lim_{N\to\infty}\frac{N+1}{2N} = \frac{1}{2}.$$

3.

$$\prod_{n=1}^{\infty}\left(1-\frac{3}{(n+2)(n+4)}\right) = \prod_{n=1}^{\infty}\frac{(n+1)(n+5)}{(n+2)(n+4)}.$$

It follows that

$$\prod_{n=1}^{N}\left(1-\frac{3}{(n+2)(n+4)}\right) = \frac{2\cdot 6}{3\cdot 5}\cdot\frac{3\cdot 7}{4\cdot 6}\cdots\frac{(N+1)(N+5)}{(N+2)(N+4)} = \frac{2}{5}\frac{N+5}{N+2} \overset{N\to\infty}{\longrightarrow} \frac{2}{5}.$$

4. $\displaystyle\prod_{n=2}^{\infty}\left(1-\frac{(-1)^n}{n}\right)$. As

$$\prod_{n=2}^{N}\left(1-\frac{(-1)^n}{n}\right) = \begin{cases} \frac{N+1}{2N} & \text{if } N \text{ is odd,} \\ \frac{1}{2} & \text{if } N \text{ is even,} \end{cases}$$

the value of the infinite product is $\frac{1}{2}$.

■

We shall study now absolute convergence. Again, a straightforward modification of the usual definition of absolute convergence for series is not a good idea. For instance, if $z_n = -1$ we have that the infinite product

$$\prod_{n=1}^{\infty}|z_n| = \prod_{n=1}^{\infty} 1 = 1$$

converges, while

$$\prod_{n=1}^{N} z_n = (-1)^N,$$

does not converge. We will adopt the following definition:

Definition 2.1.2 The infinite product $\prod_{n=1}^{\infty}(1+z_n)$ converges absolutely if $\prod_{n=1}^{\infty}(1+|z_n|)$ converges.

Lemma 2.1.2 Let

$$p_N = \prod_{n=1}^{N}(1+z_n), \qquad p_N^* = \prod_{n=1}^{N}(1+|z_n|).$$

Then:

1. $p_N^* \leq \exp(|z_1|+\cdots+|z_N|)$.
2. $|p_N - 1| \leq p_N^* - 1$.

Proof. 1. Recall that $1+x \leq e^x$ if $x \in \mathbb{R}$. Hence, $1+|z| \leq e^{|z|}$, and the first inequality follows.

2. We shall use induction. $|p_1 - 1| = |z_1| = p_1^* - 1$. If the inequality is correct up to N,

$$\begin{aligned}|p_{N+1}-1| &= |p_N(1+z_{N+1})-1| = |(p_N-1)(1+z_{N+1})+z_{N+1}| \\ &\leq (p_N^*-1)(1+|z_{N+1}|)+|z_{N+1}| = p_{N+1}^* - 1 - |z_{N+1}| + |z_{N+1}| = p_{N+1}^* - 1.\end{aligned}$$

■

The following result relates the convergence of an infinite product with convergence of a numerical series. Let us denote $\operatorname{Log} z$ the principal branch of the logarithm, i.e., that branch taking arguments in $[-\pi,\pi)$. Let us assume that $\forall n \in \mathbb{N}$, $z_n \neq 1$.

Proposition 2.1.3 The following are equivalent:

1. The infinite product $\prod_{n=1}^{\infty}(1+z_n)$ converges.
2. The series $\sum_{n=1}^{\infty}\operatorname{Log}(1+z_n)$ converges.

Proof. We shall denote

$$S_N = \sum_{n=1}^{N}\operatorname{Log}(1+z_n), \qquad P_N = \prod_{n=1}^{N}(1+z_n).$$

As $e^{S_N} = P_N$, if the series $\sum_{n=1}^{\infty}\operatorname{Log}(1+z_n)$ converges to $S = \lim_{N\to\infty} S_N$, then $\lim_{N\to\infty} P_N = e^S \neq 0$.

Conversely, asssume that $\prod_{n=1}^{\infty}(1+z_n)$ converges to $P \neq 0$. Take a continuous logarithm $\log z$, defined in a neighbourhood of P. As $\lim_{N\to\infty} P_N = P$, then $\lim_{N\to\infty}\log P_N = \log P$. Now, as $e^{S_N} = P_N$, we will have $S_N = \log P_N + 2\pi i l_N$, for some $l_N \in \mathbb{Z}$.

$$S_{N+1} - S_N = \log(P_{N+1}) - \log(P_N) + 2\pi i(l_{N+1} - l_N) = \operatorname{Log}(1+z_{N+1}) \overset{N\to\infty}{\longrightarrow} 0,$$

because $\lim_{n\to\infty} z_n = 0$. As

$$\lim_{N\to\infty}\log(P_{N+1}) - \log(P_N) = \log P - \log P = 0,$$

we deduce that $l_{N+1} - l_N$ tends to 0. As a consequence, there exists a natural number n_0 such that if $n \geq n_0$, $l_n = l \in \mathbb{Z}$, fixed: $S_N = \log(P_N) + 2\pi i l$ tends to $\log P + 2\pi i l$. ■

Proposition 2.1.4 If, for every $n \in \mathbb{N}$, $a_n \geq 0$ ($a_n \in \mathbb{R}$), then

$$\prod_{n=1}^{\infty}(1+a_n) \text{ converges if and only if } \sum_{n=1}^{\infty} a_n \text{ converges.}$$

Proof. It becomes clear from the estimates

$$1+a_1+\cdots+a_N \leq \prod_{n=1}^{\infty}(1+a_n) \leq \exp(a_1+\cdots+a_N),$$

and from the fact that, in the conditions of the statement, the sequences $\{S_N\}_{N\in\mathbb{N}}$ y $\{P_N\}_{N\in\mathbb{N}}$ are monotonically increasing. ■

Proposition 2.1.5 If $\prod_{n=1}^{\infty}(1+z_n)$ converges absolutely, then it converges.

Proof. If it converges absolutely, we have that $\sum_{n=1}^{\infty}|z_n| < +\infty$, by Proposition 2.1.4. Let us assume, then, that $|z_n| < 1$. If $|z| < 1$, then $\operatorname{Log}(1+z) = zh(z)$, for certain $h(z)$ with $\lim_{z\to 0} h(z) = 1$. It follows that

$$\left|\sum_{n=m}^{p} \operatorname{Log}(1+z_n)\right| \leq \sum_{n=m}^{p} |z_n|\,|h(z_n)| \leq M \sum_{n=m}^{p} |z_n|.$$

If $m \in \mathbb{N}$ is big enough, given $\varepsilon > 0$ we have that $\sum_{n=m}^{p}|z_n| < \varepsilon$. By Cauchy convergence criterium, $\sum_{n=1}^{\infty} \operatorname{Log}(1+z_n)$ converges. Hence, the infinite product $\prod_{n=1}^{\infty}(1+z_n)$ is also convergent. ■

Without additional hypotheses, it is not true that the convergence of the infinite product $\prod_{n=1}^{\infty}(1+z_n)$ is related with the convergence of the series $\sum_{n=1}^{\infty} z_n$. Let us see a criterium relating both convergences in a particular case.

Lemma 2.1.6 If $|x| < 1$, then $-\ln(1-x) = x + g(x)x^2$, where $g(x)$ verifies that $\lim_{x\to 0} g(x) = \frac{1}{2}$. Consequently, if $a_n \in \mathbb{R}$, and the series $\sum_{n=1}^{\infty} a_n$ converges, we have that $\prod_{n=1}^{\infty}(1-a_n)$ converges if and only if the series $\sum_{n=1}^{\infty} a_n^2$ converges.

Proof. It is left as an exercise to the reader. ■

■ **Example 2.2** 1. Take the infinite product $\prod_{n=1}^{\infty}\left(1-\frac{(-1)^n}{\sqrt{n}}\right)$. The series $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^n}{\sqrt{n}}$ converges, but $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n}$ diverges, so

$$\prod_{n=1}^{\infty}\left(1-\frac{(-1)^n}{\sqrt{n}}\right) \text{ is divergent.}$$

2. [21, p. 17, Example (v)]. Define

$$a_{2n-1} = -\frac{1}{\sqrt{n+1}}; \qquad a_{2n} = \frac{1}{\sqrt{n+1}} + \frac{1}{n+1} + \frac{1}{(n+1)\sqrt{n+1}}.$$

Firstly, we will see that the series $\sum_{n=1}^{\infty} a_n$ diverges. Indeed, the partial sum $\sum_{n=1}^{2N}$ takes the value

$$\sum_{n=1}^{2N} a_n = \sum_{k=1}^{N}(a_{2k-1}+a_{2k}) = \sum_{k=1}^{N}\left(\frac{1}{k+1}+\frac{1}{(k+1)\sqrt{k+1}}\right),$$

and is, then, divergent. On the other hand,

$$1+a_{2n-1} = 1-\frac{1}{\sqrt{n+1}} = \frac{\sqrt{n+1}-1}{\sqrt{n+1}} = \frac{n}{(\sqrt{n+1}+1)\sqrt{n+1}},$$
$$1+a_{2n} = \frac{(n+1)\sqrt{n+1}+n+1+\sqrt{n+1}+1}{(n+1)\sqrt{n+1}} = \frac{(n+2)(\sqrt{n+1}+1)}{(n+1)\sqrt{n+1}}.$$

Thus,

$$\prod_{k=1}^{2N} = \prod_{k=1}^{N}(1+a_{2k-1})(1+a_{2k}) = \prod_{k=1}^{N}\frac{k}{(\sqrt{k+1}+1)\sqrt{k+1}}\cdot\frac{(k+2)(\sqrt{k+1}+1)}{(k+1)\sqrt{k+1}}$$
$$= \prod_{k=1}^{N}\frac{k(k+2)}{(k+1)^2} = \frac{N+2}{2(N+1)} \overset{N\to\infty}{\longrightarrow} \frac{1}{2}.$$

In addition,

$$\prod_{k=1}^{2N-1}(1+a_n) = \frac{N+1}{2N}\frac{N}{(\sqrt{N+1}+1)\sqrt{N+1}} = \frac{N+1}{2(N+1+\sqrt{N+1})} \overset{N\to\infty}{\longrightarrow} \frac{1}{2}.$$

We conclude that $\prod_{n=1}^{\infty}(1+a_n)$ converges, while $\sum_{n=1}^{\infty} a_n$ does not converge.

■

2.2 Infinite products of holomorphic functions

Let $U \subseteq \mathbb{C}$ be an open set, and $\{f_n(z)\}_{n\in\mathbb{N}} \subseteq \mathscr{O}(U)$. We are going to study the infinite product

$$\prod_{n=1}^{\infty}(1+f_n(z)).$$

Theorem 2.2.1 If the functions $f_n(z)$ are bounded in $S \subseteq \mathbb{C}$, and the series $\sum_{n=1}^{\infty}|f_n(z)|$ converges uniformly to $u(z)$ in S, the the infinite product $\prod_{n=1}^{\infty}(1+f_n(z))$ converges uniformly to a function $f(z)$ in S. As a result, $f(z_0) = 0$ if and only if there exists $n_0 \in \mathbb{N}$ such that $1+f_n(z_0) = 0$.

In particular, if the series of functions $\sum_{n=1}^{\infty}|f_n(z)|$ converges uniformly in the compact sets of U, the infinite product $\prod_{n=1}^{\infty}(1+f_n(z))$ converges uniformly in the compact sets of U to $f(z) \in \mathscr{O}(U)$, and we have the equality

$$\frac{f'(z)}{f(z)} = \sum_{n=1}^{\infty}\frac{f_n'(z)}{1+f_n(z)}.$$

Proof. Every function $|f_n(z)|$ is bounded in S by a certain constant M_n, and the series $\sum_{n=1}^{\infty}|f_n(z)|$ converges uniformly in S. As a result, the limit $u(z)$ is bounded on S. Indeed, there exists $n_0 \in \mathbb{N}$ such that

$$\left|u(z) - \sum_{n=1}^{n_0}|f_n(z)|\right| < 1 \text{ in } S,$$

and then

$$|u(z)| \leq 1 + \sum_{n=1}^{n_0}|f_n(z)| \leq 1 + \sum_{n=1}^{n_0} M_n = L.$$

Let us recall here that it is a general fact that if all the functions in a sequence are bounded, and they converge uniformly, the limit is itself bounded.

Let us now denote $p_N(z)=\prod_{n=1}^{N}(1+f_n(z))$. As the series $\sum_{n=1}^{\infty}|f_n(z)|$ converges, the infinite product $\prod_{n=1}^{\infty}(1+f_n(z))$ converges absolutely, and then it converges in S. Let us remark that, if $N\in\mathbb{N}$,

$$|P_N(z)|\le\prod_{n=1}^{N}(1+|f_n(z)|)\le e^{\sum_{n=1}^{N}|f_n(z)|}\le e^L=M.$$

If $\tilde{N}>N$,

$$\begin{aligned}|p_{\tilde{N}}(z)-p_N(z)|&=|p_N(z)|\cdot\left|\prod_{n=N+1}^{\tilde{N}}(1+f_n(z))-1\right|\le|p_N(z)|\cdot\left|\prod_{n=N+1}^{\tilde{N}}(1+|f_n(z)|)-1\right|\\&\le|p_N(z)|\cdot\left(\exp\left(\sum_{n=N+1}^{\tilde{N}}|f_n(z)|\right)-1\right).\end{aligned}$$

Let $\varepsilon>0$. Take $N\in\mathbb{N}$ such that $\sum_{k=N}^{\infty}|f_k(z)|<\varepsilon$. Thus, $|p_{\tilde{N}}(z)-p_N(z)|\le M(e^{\varepsilon}-1)$. As this last expression tends to 0 when ε tends to 0, we obtain that the sequence of functions $\{p_N(z)\}$ is a uniform Cauchy sequence in S, and then, it converges uniformly.

As a result, if $\sum_{n=1}^{\infty}|f_n(z)|$ converges uniformly in the compact sets of U, using the fact that every function $|f_N(z)|$ is locally bounded, we have that the infinite product $\prod_{n=1}^{\infty}(1+f_n(z))$ converges uniformly in compact sets to a function $f(z)\in\mathscr{O}(U)$. We then get that the sequence of the derivatives $\{p_N'(z)\}$ also converges uniformly in compact sets to $f'(z)$.

Now, let $K\subseteq U$ be a compact set. There exists $n_0\in\mathbb{N}$ such that if $n\ge n_0$, $|f_n(z)|<\frac{1}{2}$ in K, and then $|1+f_n(z)|\ge\frac{1}{2}$. In particular, $1+f_n(z)\neq0$. The zeros of $f_n(z)$ in K are, then, the union of the zeros of the functions $1+f_1(z),\dots,1+f_{n_0-1}(z)$. For $N\ge n_0$ we can write

$$P_N(z)=(1+f_1(z))\cdots(1+f_{n_0-1}(z))\prod_{n=n_0}^{N}(1+f_n(z)).$$

Denote $\Phi_N(z)$ the infinite product of the last term of previous formula. As $\Phi_N(z)$ does not vanish in K, the infinite product $\Phi(z)=\prod_{n=n_0}^{\infty}(1+f_n(z))$ neither does. If $m=\min\{|\Phi(z)|,|\,z\in K\}$, $|\Phi_N|\ge\frac{m}{2}$ for N big enough. As a consequence, given that $\lim_{N\to\infty}\Phi_N(z)=\Phi(z)$ and $\lim_{N\to\infty}\Phi_N'(z)=\Phi'(z)$, we obtain that

$$\lim_{n\to\infty}\frac{\Phi_N'(z)}{\Phi_N(z)}=\frac{\Phi'(z)}{\Phi(z)}\text{ uniformly in compact sets.}$$

In other words, we have the identity

$$\frac{\Phi'(z)}{\Phi(z)}=\sum_{n=n_0}^{\infty}\frac{f_n'(z)}{1+f_n(z)}.$$

As a result,

$$\frac{f'(z)}{f(z)}=\sum_{n=1}^{n_0-1}\frac{f_n'(z)}{1+f_n(z)}+\frac{\Phi'(z)}{\Phi(z)}=\sum_{n=1}^{\infty}\frac{f_n'(z)}{1+f_n(z)}.$$

The same arguments shows that if $K\subseteq U\setminus f^{-1}(0)$ is compact, the convergence of previous series is uniform in K. ■

2.3 Weierstrass factorization

As we have said in the introduction of this chapter, we want to represent an entire function as a product of certain elementary factors, in a similar way as we normally do with polynomials. With this objective in mind, define

$$E_0(z) = 1 - z$$
$$\text{If } p \in \mathbb{N},\; E_p(z) = (1-z)\exp\left(z + \frac{z^2}{2} + \cdots + \frac{z^p}{p}\right).$$

These functions are called **Weierstrass' elementary factors**. They own the following properties:

Lemma 2.3.1 1. For every p, $E_p(0) = 1$.
2. $|1 - E_p(z)| \le \left|z^{p+1}\right|$, when $|z| \le 1$.

Proof. Indeed, if $p > 0$, then $-E_p'(z) = z^p \exp\left(z + \frac{z^2}{2} + \ldots + \frac{z^p}{p}\right)$. As $-E_p'(z)$ has an order p zero at 0, $1 - E_p(z)$ has a zero of order $p+1$. The expansion in power series at the origin is

$$\frac{1 - E_p(z)}{z^{p+1}} = \sum_{n=0}^{\infty} a_n z^n, \text{ where } a_n \ge 0.$$

Then, if $|z| \le 1$,

$$\left|\frac{1 - E_p(z)}{z^{p+1}}\right| \le 1 - E_p(1) = 1.$$

■

Theorem 2.3.2 Let $\{z_n\}_{n=1}^{\infty}$ be a sequence of non-zero complex numbers, with $\lim_{n\to\infty} z_n = \infty$. Let $p_n \in \mathbb{N}_0$ be such that for every $r > 0$,

$$\sum_{n=1}^{\infty} \left(\frac{r}{r_n}\right)^{1+p_n} < \infty,$$

where we have denoted $r_n = |z_n|$. Then the infinite product $\prod_{n=1}^{\infty} E_{p_n}\left(\frac{z}{z_n}\right)$ converges, defining an entire function with zeros precisely at the points z_n.

Proof. . Fix $r > 0$. There exists $n_0 \in \mathbb{N}$ such that if $n \ge n_0$, $r_n \ge r$. Hence, if $|z| < r$,

$$\left|1 - E_{p_n}\left(\frac{z}{z_n}\right)\right| \le \left|\frac{z}{z_n}\right|^{p_n+1} \le \left(\frac{r}{r_n}\right)^{p_n+1}.$$

As $\sum_{n=1}^{\infty} \left(\frac{r}{r_n}\right)^{p_n+1} < +\infty$, we have uniform convergence in $D(0;1)$. As a result, the infinite product converges uniformly in the compact sets of $\mathbb{C}$. ■

Corollary 2.3.3 If $p_n = n-1$, we are under previous conditions. Thus, the infinite product

$$\prod_{n=1}^{\infty} E_{n-1}\left(\frac{z}{z_n}\right)$$

converges.

Proof. Let us consider the series

$$\sum_{n=1}^{\infty}\left(\frac{r}{r_n}\right)^n.$$

From the fact that

$$\lim_{n\to\infty}\sqrt[n]{\left(\frac{r}{r_n}\right)^n}=\lim_{n\to\infty}\frac{r}{r_n}=0,$$

we deduce the convergence of the series. ■

Corollary 2.3.4 If $\sum_{n=1}^{\infty}\frac{1}{r_n}<+\infty$, we can take $p_n=0$ in previous expression.

Proof.

$$\sum_{n=1}^{\infty}\frac{r}{r_n}=r\cdot\sum_{n=1}^{\infty}\frac{1}{r_n}<\infty.$$

■

Theorem 2.3.5 — Weierstrass' Factorization Theorem. Let $f(z)\in\mathscr{O}(\mathbb{C})$, with $f(0)\neq 0$, and zeros in $\{z_n\}_{n=1}^{\infty}$, repeated as many times as its multiplicity. There exists another entire function $g(z)$ such that

$$f(z)=e^{g(z)}\prod_{n=1}^{\infty}E_{p_n}\left(\frac{z}{z_n}\right).$$

Proof. Consider the function

$$F(z)=\frac{f(z)}{\prod_{n=1}^{\infty}E_{p_n}\left(\frac{z}{z_n}\right)},$$

whose singularities are removable, and then, defines a zero-free entire function. So, there exists $g(z)\in\mathscr{O}(\mathbb{C})$ such that $F(z)=e^{g(z)}$. ■

Corollary 2.3.6 If $f(z)\in\mathscr{O}(\mathbb{C})$ has a zero of order m at 0, there exists $g(z)\in\mathscr{O}(\mathbb{C})$ such that

$$f(z)=e^{g(z)}z^m\prod_{n=1}^{\infty}E_{p_n}\left(\frac{z}{z_n}\right).$$

Proof. It is enough to apply 2.3.5 to $\dfrac{f(z)}{z^m}$. ■

Weierstrass' elementary factors are also useful in order to constract entire functions with prescribed zeros. Ths only restriction on the zeros will be that they don't have cluster points, as if it were the case, the holomorphic function might be identically zero. More precisely,

Theorem 2.3.7 Let $U \subseteq \overline{\mathbb{C}}$ be an open set, $A = \{z_n\}_{n=1}^{\infty} \subseteq U$ a set without cluster points in U, and $\{m_n\}_{n=1}^{\infty} \subseteq \mathbb{N}$. There exists $f(z) \in \mathscr{O}(U)$ with zeros exactly on A, with multiplicities $\{m_n\}_{n=1}^{\infty}$.

Proof. Assume that $\infty \in U \setminus A$ (after a Mespacio vectorialöbius transformation we can always assume that this is the case).

If $U = \overline{\mathbb{C}}$, A must be finite, $A = \{z_1, z_2, \dots, z_n\}$. We can construct a meromorphic function on U, that must have at least one pole (or being constant). If we accept that the point at infinity is the only pole, the holomorphic function

$$f(z) = (z - z_1)^{m_1} \cdots (z - z_n)^{m_n}$$

verifies the statement.

If $U \neq \overline{\mathbb{C}}$ but A is still finite, take $b \in \mathbb{C} \setminus U$, and consider the function

$$f(z) = \frac{(z - z_1)^{m_1} \cdots (z - z_n)^{m_n}}{(z - b)^{m_1 + \cdots + m_n}}.$$

Let us now deal with the general case, assuming that A is an infinite set. Consider, for every $n \in \mathbb{N}$, $w_n \in \overline{\mathbb{C}} \setminus U$ such that $|z_n - w_n| = d(z_n, \overline{\mathbb{C}} \setminus U)$. With this choice, $\lim_{n \to \infty} |z_n - w_n| = 0$. If not, there exists $\varepsilon > 0$ and a subsequence $\{z_{n_k}\}_{k=1}^{\infty}$ such that $\left|z_{n_k} - w_{n_q}\right| \geq \varepsilon$. Taking, if necessary, a subsequence, we can assume that $\{z_{n_k}\}_{k=1}^{\infty}$ converges to $a \in U$, which is not possible.

Next, define

$$f(z) = \prod_{n=1}^{\infty} E_n\left(\frac{z_n - w_n}{z - w_n}\right).$$

Take $K \subseteq U$ compact, and $r = d(K, \overline{\mathbb{C}} \setminus U)$. There exists $n_0 \in \mathbb{N}$ such that if $n \geq n_0$, $|z_n - w_n| < r/2$. Thus,

$$\left|\frac{z_n - w_n}{z - w_n}\right| \leq \frac{r/2}{r} = \frac{1}{2} \text{ on } K.$$

As a result,

$$\left|1 - E_n\left(\frac{z_n - w_n}{z - w_n}\right)\right| \leq \left|\frac{z_n - w_n}{z - w_n}\right|^{n+1} \leq \left(\frac{1}{2}\right)^{n+1}$$

on K, if $n \geq n_0$, so $f(z)$ defines a holomorphic function on U, with the prescribed zeros.

Finally, let us observe that if $z \longrightarrow \infty$, then $\dfrac{z_n - w_n}{z - w_n} \longrightarrow 0$, hence $E_n(0) = 1$. So, $f(\infty) = 1$. ■

■ **Example 2.3** Let us develop here an important example, that will be revisited several times throughout these notes. The function

$$f(z) = \begin{cases} \frac{\sin z}{z} & \text{if } z \neq 0, \\ 1 & \text{if } z = 0 \end{cases}$$

has zeros in $k\pi$, where $k \in \mathbb{Z} \setminus \{0\}$. As

$$\sum_{k=1}^{\infty} \frac{1}{k^2 \pi^2} < \infty,$$

we can take $p_n = 1$, according to 2.3.2 If $z_k = k\pi$,

$$E_1\left(\frac{z}{z_k}\right)E_1\left(\frac{z}{z_{-k}}\right) = \left(1-\frac{z}{k\pi}\right)\left(1+\frac{z}{k\pi}\right)\exp\left(\frac{z}{k\pi}\right)\exp\left(-\frac{z}{k\pi}\right) = 1-\frac{z^2}{k^2\pi^2}.$$

Hence, we can represent

$$f(z) = \frac{\sin z}{z} = e^{g(z)}\prod_{k=1}^{\infty}\left(1-\frac{z^2}{k^2\pi^2},\right) \tag{2.2}$$

for some entire function $g(z)$, that we shall determine in the sequel. For this, let us observe that

$$\frac{f'(z)}{f(z)} = \cot z - \frac{1}{z} = g'(z) + \sum_{k=1}^{\infty}\frac{2z}{z^2-k^2\pi^2}.$$

We need to represent the cotangent function as an infinite sum. Let us consider

$$F(z) = \frac{\cot \pi z}{z^2-\omega^2},$$

where $\omega \in \mathbb{C}\setminus\mathbb{Z}$. It has poles in the points $z_n = n \in \mathbb{Z}$, and in $\pm\omega$. The residues are:

$$\operatorname{Res}(F(z);n) = \frac{1}{\pi}\cdot\frac{1}{n^2-\omega^2},$$
$$\operatorname{Res}(F(z);\pm\omega) = \frac{\cot\pi\omega}{2\omega}.$$

Let γ_n be a rectangular path, with vertices in $\left(n+\frac{1}{2}\right)(\pm 1 \pm i)$, where $n > |\omega|$. We have that

$$\frac{1}{2\pi i}\int_{\gamma_n}\frac{\cot \pi z}{z^2-\omega^2}dz = \frac{1}{\pi}\sum_{k=-n}^{n}\frac{1}{k^2-\omega^2} + \frac{1}{\omega}\cot(\pi\omega).$$

Letting $n \longrightarrow \infty$, it is not difficult to verify that the integral tends to zero. Indeed, in vertical sides $\pm\left(n+\frac{1}{2}\right)+iy$ we have

$$|\cot\pi z| = \left|\frac{e^{\pi y}-e^{-\pi y}}{e^{\pi y}+e^{-\pi y}}\right| \leq 1,$$

while in the horizontal sides $x \pm i\left(n+\frac{1}{2}\right)$,

$$|\cot \pi z| \leq \frac{1+e^{-2\pi\left(n+\frac{1}{2}\right)}}{1-e^{-2\pi\left(n+\frac{1}{2}\right)}} \leq \frac{2}{1-e^{-\pi}}.$$

So, $\cot\pi z$ is bounded on γ_n by a constant M independent of n. So, on γ_n,

$$\left|\frac{\cot\pi z}{z^2-\omega^2}\right| \leq \frac{M}{\left(n+\frac{1}{2}\right)^2-|\omega|^2},$$

which proves the statement.

So we have

$$\frac{1}{\pi}\sum_{n=-\infty}^{\infty}\frac{1}{n^2-\omega^2} + \frac{1}{\omega}\cot(\pi\omega) = 0,$$

and then,

$$\pi\cot(\pi\omega) = \frac{1}{\omega} + \sum_{n=1}^{\infty} \frac{2\omega}{\omega^2 - n^2}. \tag{2.3}$$

From (2.2), replacing z by πz and taking logarithmic derivatives, we have

$$\pi\cot(\pi z) - \frac{1}{z} = \pi\frac{g'(\pi z)}{g(\pi z)} + \sum_{n=1}^{\infty} \frac{2z}{z^2 - n^2}.$$

Comparing this with (2.3), $g'(\pi z) = 0$, so $g(z)$ is a constant C, and

$$\frac{\sin z}{z} = e^C \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{n^2\pi^2}\right).$$

Taking $z = 0$, it turns out that $e^C = 1$, and then,

$$\sin z = z\prod_{n=1}^{\infty} \left(1 - \frac{z^2}{n^2\pi^2}\right).$$

■

2.4 Mittag-Leffler Theorem

Several consequences of previous results can be deduced. Particularly we are going to construct holomorphic and meromorphic functions satisfying certain properties.

Theorem 2.4.1 — Mittag-Leffler. Let $\{z_n\}_{n=1}^{\infty}$ be a sequence of complex numbers, with $\lim_{n\to\infty} z_n = \infty$. For every $n \in \mathbb{N}$, let $m_n \in \mathbb{N}$, and let us define

$$q_n(z) = \sum_{\nu=1}^{m_n} \frac{c_{n\nu}}{(z - z_n)^\nu}.$$

Then, there exists $f(z)$ a meromorphic function on $\mathbb{C}$, with poles at the points z_n, such that the principal parts in every pole are $q_n(z)$.

Proof. Reordering the poles, if necessary, we can assume that for every $n \in \mathbb{N}$, $|z_n| \leq |z_{n+1}|$. Let us also assume that $z_n \neq 0$. So, $q_n(z) \in \mathscr{O}(D(0;|z_n|))$, and it admits a power series expansion at the origin

$$q_n(z) = \sum_{k=0}^{\infty} a_{nk} z^k. \tag{2.4}$$

Take $1/2 < q < 1$: on the closed disk $K_n = \bar{D}(0;q|z_n|)$, the series (2.4) converges uniformly, so there exists $\nu_n \in \mathbb{N}$ such that if $P_n(z) = \sum_{k=0}^{\nu_n} a_{nk} z^k$, then $|q_n(z) - P_n(z)| < \frac{1}{2^n}$ on K_n.

Consider the series

$$f(z) = \sum_{n=1}^{\infty} (q_n(z) - P_n(z)).$$

If $K \subseteq \mathbb{C}$ is a compact, $K \subseteq \bar{D}(0;R)$ for some $R > 0$. Take $n_0 \in \mathbb{N}$ such that $|z_n| \geq 2R$ if $n \geq n_0$. On K,

$$\sum_{n\geq n_0} |q_n(z) - P_n(z)| \leq \sum_{n\geq n_0} \frac{1}{2^n},$$

so this series converges uniformly. Then, $f(z)$ defines a meromorphic function on $D(0;R)$, with poles on the set $\{z_1, \ldots, z_{n_0}\}$. As R can be taken arbitrarily large, $f(z)$ defines a meromorphic function on $\mathbb{C}$ satisfying the statement. ■

The above result allows to construct meromorphic functions defined on $\mathbb{C}$ with fixed poles and principal parts. We shall adapt it in order to construct such a meromorphic function on an arbitrary open set $U \subseteq \mathbb{C}$.

Theorem 2.4.2 Let $U \subseteq \mathbb{C}$ be an open set, $A = \{z_n \mid n \in \mathbb{N}\} \subseteq U$ a set without cluster points on U. Define, for every $n \in \mathbb{N}$,

$$q_n(z) = \sum_{\nu=1}^{m_n} \frac{c_{n\nu}}{(z-z_n)^\nu}. \tag{2.5}$$

There exists $f \in \mathscr{M}(U)$, with poles on A with principal parts $q_n(z)$, and without other poles.

Proof. Assume again that $\infty \in U \setminus A$, and take $\omega_n \in \mathbb{C} \setminus U$ such that $d_n = |\omega_n - z_n| = d(z_n; \overline{\mathbb{C}} \setminus U)$. Develop $q_n(z)$ as a Laurent series on the annulus $\mathbb{C} \setminus \bar{D}(\omega_n; d_n)$, the convergence being uniform in compact sets: there exists $N_n \in \mathbb{N}$ such that if

$$P_n(z) = \sum_{k=1}^{N_n} \frac{p_{nk}}{(z-\omega_n)^k},$$

then $|q_n(z) - P_n(z)| \leq \frac{1}{2^n}$ on $\mathbb{C} \setminus \bar{D}(\omega_n; 2d_n)$.

Consider, as before, the sum

$$f(z) = \sum_{n=1}^{\infty} (q_n(z) - P_n(z)).$$

If $z_0 \in U \setminus A$, there exists $r > 0$ such that $\bar{D}(z_0; r) \subseteq U \setminus A$, and $n_0 \in \mathbb{N}$ such that if $n \geq n_0$, $2d_n < d(\bar{D}(z_0; r), \mathbb{C} \setminus U)$. We have $\bar{D}(z_0; r) \subseteq \overline{\mathbb{C}} \setminus \bar{D}(\omega_n; 2d_n)$. Using the same arguments than in Theorem 2.4.1, the series

$$\sum_{n \geq n_0} |q_n(z) - P_n(z)|$$

converges uniformly in $\bar{D}(z_0; r)$. A similar argument works in appropriate disks $\bar{D}(z_j; r) \subseteq U \setminus (A \setminus \{z_n\}_{n=1}^{\infty})$. Finally we obtain a function on U with poles and principal parts as stated. ∎

A combination of Weierstrass and Mittag-Leffler Theorems give:

Theorem 2.4.3 Let $U \subseteq \mathbb{C}$ be an open set, $A = \{z_n\}_{n=1}^{\infty} \subseteq U$ without cluster points. For every $n \in \mathbb{N}$, let us consider

$$P_n(z) = \sum_{k=1}^{m_n} c_{nk}(z-z_n)^k \in \mathbb{C}[z-z_n].$$

Then, there exists $f(z) \in \mathscr{O}(U)$ such that the Taylor expansion of order m_n of $f(z)$ at z_n is $T_{m_n}(f(z); z_n) = P_n(z)$.

Proof. Take $g(z) \in \mathscr{O}(U)$ with a zero of order $m_n + 1$ in z_n. Let us write

$$\frac{P_n(z)}{g(z)} = Q_n(z) + g_n(z),$$

where $Q_n(z)$ is the principal part of this quotient around z_n, and $g_n(z)$ is a holomorphic function at z_n. Using Theorem 2.4.2, construct $h(z) \in \mathscr{M}(U)$ having $Q_n(z)$ as principal part at z_n. Then $P_n(z) = g \cdot Q_n + g \cdot g_n = g \cdot h + g \cdot (Q_n - h) + g \cdot g_n$. The function $g(z) \cdot h(z)$ satisfies the statement, as $g \cdot h = P_n + g \cdot (h - Q_n - g_n)$, and tle second term has a zero of order $m_n + 1$ at z_n. ∎

2.4.1 "Practical" application of Mittag-Leffler's Theorem

Let $\{z_n\}_{n=1}^{\infty} \subseteq \mathbb{C} \setminus \{0\}$ be a sequence of complex numbers, with $\lim_{n\to\infty} |z_n| = \infty$. Let us construct a function $f(z) \in \mathscr{M}(\mathbb{C})$, with simple poles in each z_n and fixed residues α_n. Following previous constructions, if the principal part at z_n is $\frac{\alpha}{z-z_n}$, the expansion of this fraction at 0 is

$$\frac{\alpha_n}{z-z_n} = -\frac{\alpha_n}{z_n}\left[\sum_{k=0}^{m-1}\left(\frac{z}{z_n}\right)^k + \frac{z^m}{z_n^m}\cdot\frac{z_n}{z_n-z}\right].$$

So, the previous construction asks to guarantee the convergence of

$$\sum_{n=1}^{\infty} \frac{z^m}{z_n^m}\cdot\frac{\alpha_n}{z-z_n}$$

on $\mathbb{C}$.

Proposition 2.4.4 If

$$\sum_{n=1}^{\infty} \frac{|\alpha_n|}{|z_n|^{m+1}} < +\infty,$$

then the expression

$$\sum_{n=1}^{\infty} \frac{z^m}{z_n^m}\cdot\frac{\alpha_n}{z-z_n}$$

defines a holomorphic function on $\mathbb{C} \setminus \{z_n\}_{n=1}^{\infty}$, with simple poles on each z_n and residues α_n.

Proof. Indeed, if $R > 0$, there exists $n_0 \in \mathbb{N}$ such that if $n \geq n_0$, then $|z_n| \geq 2R$. We have, then,

$$\left|\sum_{n=n_0}^{\infty} \frac{z^m}{z_n^m}\cdot\frac{\alpha_n}{z-z_n}\right| \leq \sum_{n=n_0}^{\infty} \frac{R^m|\alpha_n|}{|z_n|^m|z-z_n|} \underset{\substack{\uparrow \\ (*)}}{\leq} 2R^m \sum_{n=n_0}^{\infty} \frac{|\alpha_n|}{|z_n|^{m+1}} < +\infty,$$

where $(*)$ is due to the fact that $|z-z_n| \geq |z_n| - |z| \geq |z_n| - \frac{|z_n|}{2} = \frac{|z_n|}{2}$. ■

Corollary 2.4.5 If $f(z) \in \mathscr{M}(\mathbb{C})$, with simple poles on $\{z_n\}_{n=1}^{\infty}$ and residues $\{\alpha_n\}_{n=1}^{\infty}$, such that

$$\sum_{n=1}^{\infty} \frac{|\alpha_n|}{|z_n|^{m+1}} < \infty, \tag{2.6}$$

then there exists $h(z) \in \mathscr{O}(\mathbb{C})$ such that

$$f(z) = \sum_{n=1}^{\infty} \frac{z^m}{z_n^m}\cdot\frac{\alpha_n}{z-z_n} + h(z).$$

■ **Example 2.4** Let us consider again the function $f(z) = \pi\cot\pi z - \frac{1}{z}$, with $\operatorname{Res}(f(z);n) = 1$ if $n \in \mathbb{Z}\setminus\{0\}$. As $\sum_{\substack{n\in\mathbb{Z}\\ n\neq 0}} \frac{1}{n^2} < \infty$, we can take $m = 1$ in (2.6). So,

$$\sum_{n=1}^{\infty} \frac{z}{n}\cdot\frac{1}{z-n} + \sum_{n=1}^{\infty} \frac{z}{-n}\cdot\frac{1}{z+n} = \sum_{n=1}^{\infty} \frac{z}{n}\left(\frac{1}{z-n} - \frac{1}{z+n}\right) = \sum_{n=1}^{\infty} \frac{z}{n}\cdot\frac{2n}{z*2-n^2} = \sum_{n=1}^{\infty} \frac{2z}{z^2-n^2}.$$

As a result, there exists $h(z) \in \mathscr{O}(\mathbb{C})$ such that

$$\pi \cot \pi z = \frac{1}{z} + \sum_{n=1}^{\infty} \frac{2z}{z^2 - n^2} + h(z).$$

■

Let us determine $h(z)$ in previous example. We shall make use of the following result:

Theorem 2.4.6 — Cauchy simple fractions Theorem. Let $f(z) \in \mathscr{M}(\mathbb{C})$ be a meromorphic function with simple poles on $\{z_n\}_{n=1}^{\infty}$, $z_n \neq 0$. Assume the poles ordered in such a way as $|z_n| \leq |z_{n+1}|$, and $\lim_{n\to\infty} |z_n| = \infty$. Let $\alpha_n = \operatorname{Res}(f(z); z_n)$. Let us assume that a sequence of closed and simple curves C_n exist such that

1. $C_m^* \cap \{z_n\}_{n=1}^{\infty} = \emptyset$.
2. 0 is in the bounded connected component of the complement $\mathbb{C} \setminus C_n^*$.
3. If $R_m = d(0, C_m^*)$, then there exists $K > 0$ such that the length of C_m, denoted $l(C_m)$, is bounded above by KR_m.
4. $$\lim_{n\to\infty} \frac{||f||_{C_m}}{R_m^{p+1}} = 0,$$
   for some $p \in \mathbb{N}$.

Then,

$$f(z) = \sum_{k=0}^{p} \frac{f^{(k)}(0)}{k!} z^k + \underbrace{\sum_{n=1}^{\infty} \alpha_n \left(\frac{1}{z - z_n} + \frac{1}{z_n} + \frac{z}{z_n^2} + \cdots + \frac{z^p}{z_n^{p+1}} \right)}_{\sum_{n=1}^{\infty} \frac{\alpha_n}{z - z_n} \left(\frac{z}{z_n}\right)^{p+1}}.$$

Proof. Let us compute the integral

$$\frac{1}{2\pi i} \int_{C_m} \frac{f(w)}{w^{p+1}(w - z)} dw. \tag{2.7}$$

The integrand has poles at 0, z, and at some of the points z_n. Residues are:

1. At 0,
   $$-\frac{1}{z} \left(\frac{f(0)}{z^p} + \frac{f'(0)}{1! z^{p-1}} + \cdots + \frac{f^{(p)}(0)}{p!} \right).$$
2. At z, $\dfrac{f(z)}{z^{p+1}}$.
3. At z_n, $\dfrac{\alpha_n}{z_n^{p+1}(z_n - z)}$.

The integral (2.7) can be bounded as

$$\frac{1}{2\pi} ||f||_{C_m} \frac{1}{R_m^{p+1}} \cdot \frac{1}{R_m - |z|} l(C_m) \leq \frac{1}{2\pi} ||f||_{C_m} \frac{KR_m}{R_m^{p+1}(R_m - |z|)} = \frac{1}{2\pi} ||f||_{C_m} \frac{K}{R_m^p (R_m - |z|)} \xrightarrow{m\to\infty} 0.$$

The curves ultimately encircle all points z_n, and so we obtain

$$-\frac{1}{z} \left(\frac{f(0)}{z^p} + \frac{f'(0)}{1! z^{p-1}} + \cdots + \frac{f^{(p)}(0)}{p!} \right) + \frac{f(z)}{z^{p+1}} + \sum_{n=1}^{\infty} \frac{\alpha_n}{z_n^{p+1}(z_n - z)} = 0.$$

Solving for $f(z)$ we end at

$$f(z)=\sum_{k=0}^{p}\frac{f^{(k)}(0)}{k!}z^p+\sum_{n=1}^{\infty}\alpha_n\frac{z^{p+1}}{z_n^{p+1}}\frac{1}{z-z_n}=\sum_{k=0}^{p}\frac{f^{(k)}(0)}{k!}z^p+\sum_{n=1}^{\infty}\alpha_n\left(\frac{1}{z-z_n}+\frac{1}{z_n}\sum_{k=1}^{p}\frac{z^k}{z_n^k}\right).$$

■

Corollary 2.4.7 If $f(z)$ is uniformly bounded on C_n^*, we can take $p=0$ and then

$$f(z)=f(0)+\sum_{n=1}^{\infty}\alpha_n\left(\frac{1}{z-z_n}+\frac{1}{z_n}\right).$$

■ **Example 2.5** Let us complete Example 2.4, where $f(z)=\pi\cot(\pi z)-\dfrac{1}{z}$, and let C_n be the square of vertices $\left(n+\frac{1}{2}\right)(\pm1\pm i)$. We know from Example 2.3 that a constant $K>0$ independent of m exists such that $|\cot\pi z|\leq K$ on C_m. As $R_m=m+\frac{1}{2}$, we can take $m=0$ in Theorem 2.4.6, according to Corollary 2.4.7. Consequently,

$$\pi\cot\pi z-\frac{1}{z}=f(0)+\sum_{n=1}^{\infty}\left(\frac{1}{z-n}+\frac{1}{n}\right)+\sum_{n=1}^{\infty}\left(\frac{1}{z-n}-\frac{1}{n}\right)=\sum_{n=1}^{\infty}\left(\frac{1}{z-n}+\frac{1}{z+n}\right)=\sum_{n=1}^{\infty}\frac{2z}{z^2-n^2},$$

so we arrive at the (already known) formula

$$\pi\cot\pi z=\frac{1}{z}+\sum_{n=1}^{\infty}\frac{2z}{z^2-n^2}.$$

■

2.4.2 Complement: Wallis' formula

From the expressions

$$\pi\cot\pi z=\frac{1}{z}+\sum_{n=1}^{\infty}\frac{2z}{z^2-n^2},$$
$$\sin z=\prod_{n=1}^{\infty}\left(1-\frac{z^2}{n^2\pi^2}\right),$$

which have been obtained using several methods, other formulas can be deduced. As an example, if $z=\dfrac{\pi}{2}$ in second formula, we have

$$\frac{2}{\pi}=\prod_{n=1}^{\infty}\left(1-\frac{1}{4n^2}\right)=\prod_{n=1}^{\infty}\frac{(2n-1)(2n+1)}{(2n)^2}=\frac{1}{2}\cdot\frac{3}{2}\cdot\frac{3}{4}\cdot\frac{5}{4}\cdot\frac{5}{6}\cdot\frac{7}{6}\cdots,\tag{2.8}$$

which is known as Wallis' formula. Let us note that this formula can also been obtained using "elementary" methods. Indeed, following [8], set

$$I_n=\int_0^{\pi/2}\sin^n x dx.$$

Integration by parts give $I_n=\dfrac{n-1}{n}I_{n-2}$ if $n\geq2$. As $I_0=\frac{\pi}{2}$ and $I_1=1$, we have

$$\frac{\pi}{2}=\frac{2\cdot2\cdot4\cdot4\cdot6\cdot6\cdots2m\cdot2m}{1\cdot3\cdot3\cdot5\cdot5\cdots(2m-1)(2m-1)(2m+1)}\cdot\frac{I_{2m}}{I_{2m+1}}.$$

Using the inequalities

$$\sin^{2m+1} x \le \sin^{2m} x \le \sin^{2m-1} x,$$

we have

$$1 \le \frac{I_{2m}}{I_{2m+1}} \le \frac{I_{2m-1}}{I_{2m+1}} = \frac{2m+1}{2m}$$

and then, $\lim\limits_{n\to\infty} \dfrac{I_{2m}}{I_{2m+1}} = 1$. This gives the formula.

2.5 Exercises

1. Compute the value of the following infinite products:
   (a) $\displaystyle\prod_{n=2}^{\infty} e\left(1-\frac{1}{n^2}\right)^{n^2}$.
   (b) $\displaystyle\prod_{n=2}^{\infty} \left(1-\frac{(-1)^n}{n}\right)$.
2. Consider the infinite product

$$\prod_{n=1}^{\infty} \frac{(n-a_1)\cdots(n-a_r)}{(n-b_1)\cdots(n-b_s)},$$

   where $a_1,\dots,a_r,b_1,\dots,b_s$ are complex numbers. Show that it converges if and only if $r=s$ and $a_1+\cdots+a_r = b_1+\cdots+b_s$.
3. Show that if $|x|<1$, $-\ln(1-x) = x+g(x)x^2$, where $g(x) \xrightarrow{x\to 0} 1/2$. Conclude that if $\{a_n\}_{n=1}^{\infty}$ is a sequence of real numbers such that $\sum_{n=1}^{\infty} a_n$ converges, then the infinite product $\displaystyle\prod_{n=1}^{\infty}(1-a_n)$ converges if and only if $\sum_{n=1}^{\infty} a_n^2$ converges.
4. Determine in which domain the following infinite products represent holomorphic functions in z:

$$\prod_{n=1}^{\infty}(1-e^{-nz}); \qquad \prod_{n=1}^{\infty}(1-n^z).$$

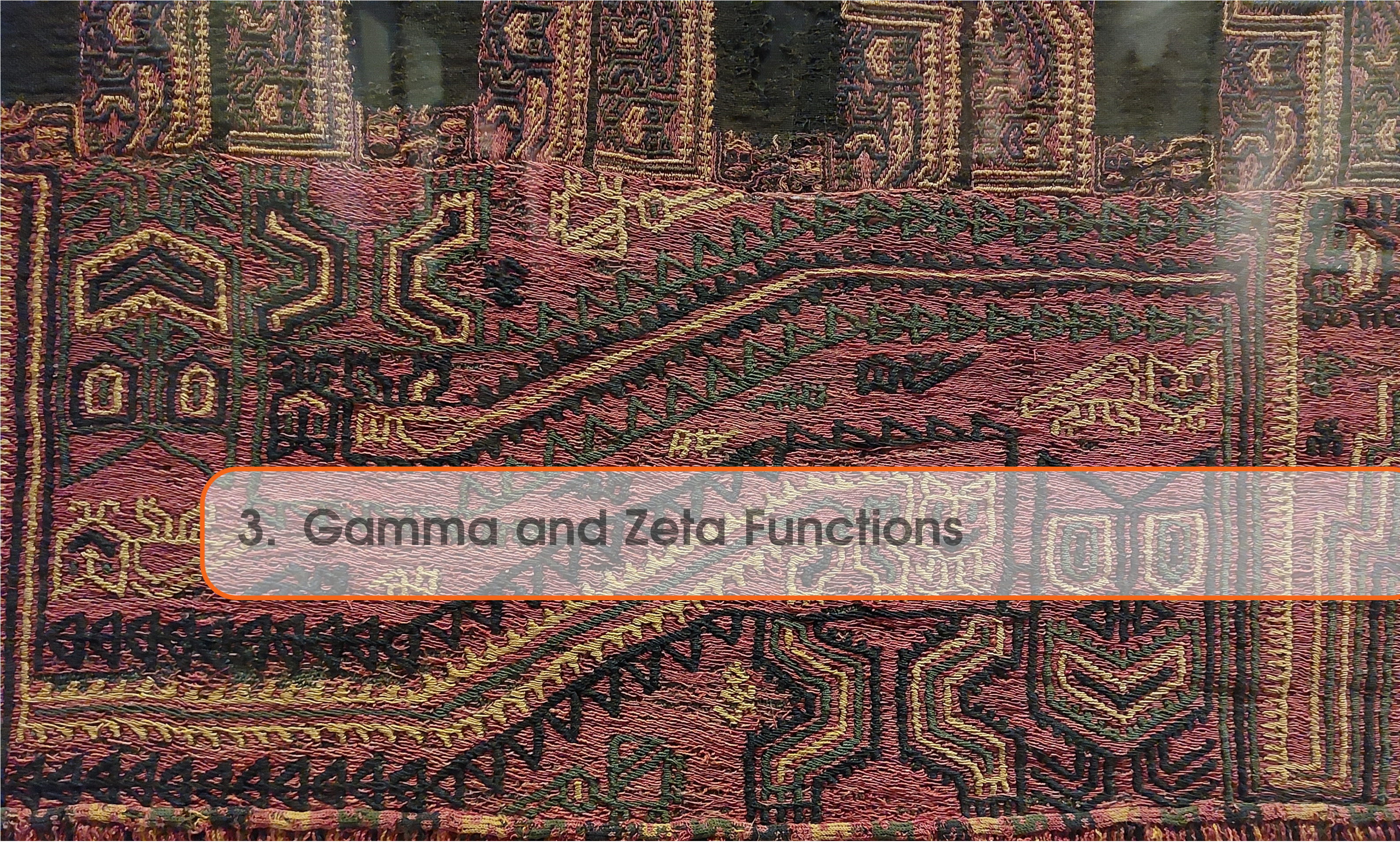
5. (a) Show that, if $z \ne 0$,

$$\prod_{n=0}^{\infty} \cos\left(\frac{z}{2^n}\right) = \frac{\sin 2z}{2z}.$$

   HINT: Use convenient trigonometrical identities.
   (b) Compute the value of Vieta's product:

$$\sqrt{\frac{1}{2}}\cdot\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}}}\cdot\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}}}}\cdots$$

6. Show that every meromorphic function $f(z)$ in $\mathbb{C}$ can be written as g/h, where $g,h \in \mathscr{O}(\mathbb{C})$.
7. Let $U \subseteq \mathbb{C}$ be a domain, and I an ideal of the ring $\mathscr{O}(U)$.
   (a) Show that if I is finitely generated, then I is principal.
   (b) Show an example of a non principal ideal $I \subseteq \mathscr{O}(\mathbb{C})$.
   HINT: Investigate about ideals of holomorphic functions, for instance in the textbooks of Ash [4] or Rudin [18].



3. Gamma and Zeta Functions

3.1 Euler Gamma function

The tradicional definition of Gamma function for real values is as follows:

$$\Gamma(z) = \int_0^\infty t^{z-1} t e^{-t} dt. \tag{3.1}$$

This integral converges when $\mathrm{Re}(z) > 0$. Indeed, split the integral as

$$\int_0^1 t^{z-1} e^{-t} dt + \int_1^\infty t^{z-1} e^{-t} dt. \tag{3.2}$$

When $t \geq 1$ we have bounds

$$\left| t^{z-1} e^{-t} \right| = t^{\mathrm{Re}(z)-1} e^{-t} = t^{\mathrm{Re}(z)+1} e^{-t} t^{-2} \leq M t^{-2},$$

where M is a constant bounding $t^{\mathrm{Re}(z)+1} e^{-t}$, whose existence is guaranteed due to

$$\lim_{t \to \infty} t^{\mathrm{Re}(z)+1} e^{-t} = 0.$$

Assume $t \in [0,1]$. When $\mathrm{Re}(z) \geq 1$, the function inside the integral is continuous at 0, so there is no problem regarding convergence. If $0 < \mathrm{Re}(z) < 1$, take $\alpha < 1$ such that $\mathrm{Re}(z) - 1 + \alpha > 0$. We have

$$\left| t^{\mathrm{Re}(z)+1} e^{-t} \right| = t^{\mathrm{Re}(z)-1+\alpha} e^{-t} \frac{1}{t^\alpha},$$

which integral clearly converges at 0.

Assume K is a compact set contained in the open right half plane, and $0 < C_1 < C_2$ such that $C_1 < \mathrm{Re}(z) < C_2$ if $z \in K$. The second integral in (3.2) can be bounded by

$$\int_1^\infty t^{C_2-1} e^{-t} dt,$$

and is, then uniformly convergent. The first integral, as before, can be bounded by

$$\int_0^1 t^{C_1-1}e^{-t}dt = \int_0^1 t^{C_1-1+\alpha}e^{-t}\frac{1}{t^\alpha}dt,$$

again convergent. This implies continuity of $\Gamma(z)$ in the right half plane U. Derivating the integrand with respect to z we are led to consider the integral

$$\int_0^\infty t^{z-1}(\log t)e^{-t}dt,$$

which converges uniformly in the compact sets of the right half plane U using the same arguments as before. As a consequence, $\Gamma(z) \in \mathscr{O}(U)$.

Integrating by parts,

$$\Gamma(z+1) = \int_0^\infty t^z e^{-t}dt = -t^z e^{-t}\Big|_0^\infty + z\int_0^\infty t^{z-1}e^{-t}dt = z\Gamma(z).$$

Moreover, a simple computation shows that $\Gamma(1) = 1$. Consequently:

1. $\Gamma(n+1) = n!$ if $n \in \mathbb{N}$.
2. $\Gamma(z+n) = (z+n-1)(z+n-2)\cdots z\Gamma(z)$ if $\mathrm{Re}(z) > 0$.
3. Define, inspired by previous equality, a function

$$\Gamma_n(z) = \frac{\Gamma(z+n)}{(z+n-1)(z+n-2)\cdots z},$$

meromorphic when $\mathrm{Re}(z) > -n$, with poles in $0, -1, \ldots, -(n-1)$. The function $\Gamma_n(z)$ agrees with $\Gamma(z)$ on U, so it provides a meromorphic extension to this bigger set. Poles anr simple and

$$\mathrm{Res}(\Gamma_n(z); -(n-1)) = \frac{\Gamma(1)}{(-1)^{n-1}(n-1)!} = \frac{(-1)^{n-1}}{(n-1)!}.$$

As this can be performed for every n we have found that $\Gamma(z)$ can be meromorphically continued to $\mathbb{C}$, with simple poles over $\mathbb{Z} \setminus \mathbb{N}$ and residues as before.

Let us propose an alternative definition of Gamma function, using the results about infinite products we have stated in Chapter 2. Poles of $\Gamma(z)$ are in the non positive integers, and they are simple, so its inverse, $\frac{1}{\Gamma(z)}$ must have simple zeros in these points. As

$$\sum_{n=1}^\infty \left(\frac{r}{n}\right)^2 < \infty,$$

using Weierstrass' elementary factors with $p = 1$ we can define the entire function

$$H(z) = z\prod_{n=1}^\infty \left(1+\frac{z}{n}\right)e^{-z/n}.$$

A simple computation shows that

$$H(z)H(-z) = -z^2\prod_{n=1}^\infty \left(1-\frac{z^2}{n^2}\right) = -\frac{z}{\pi}\sin \pi z.$$

Also,

$$\log H(1) = \lim_{N\to\infty}\sum_{n=1}^N \left(\log(n+1) - \log n - \frac{1}{n}\right) = \lim_{N\to\infty}\left(\log(N+1) - \sum_{n=1}^N \frac{1}{n}\right) = -\gamma,$$

where γ is Euler's constant. So, $H(1)=e^{-\gamma}$. Define

$$\Delta(z)=e^{\gamma z}H(z)=ze^{\gamma z}\prod_{n=1}^{\infty}\left(1+\frac{z}{n}\right)e^{-z/n},\ \Delta(1)=1.$$

This is the Weierstrass' Δ function. Let us deduce some of its properties.

1. From the expression for $H(z)$ we have that

$$\begin{aligned}H(z)&=\lim_{n\to\infty}\frac{z(z+1)\cdots(z+n)}{n!}\cdot\exp\left(-z\sum_{k=1}^{n}\frac{1}{k}\right)\\&=\lim_{n\to\infty}\frac{z(z+1)\cdots(z+n)}{n!n^z}\cdot\exp\left(z\left(\log n\sum_{k=1}^{n}\frac{1}{k}\right)\right)=e^{-\gamma z}\lim_{n\to\infty}\frac{z(z+1)\cdots(z+n)}{n!n^z},\end{aligned}$$

and then

$$\Delta(z)=\lim_{n\to\infty}\frac{z(z+1)\cdots(z+n)}{n!n^z}.$$

A simple computation shows that

$$\Delta(z+1)=\lim_{n\to\infty}\frac{(z+1)(z+2)\cdots(z+n+1)}{n!n^{z+1}}=\frac{1}{z}\lim_{n\to\infty}\frac{z(z+1)\cdots(z+n)}{n!n^z}\cdot\frac{z+n+1}{n}=\frac{1}{z}\Delta(z).$$

So, $\Delta(z)=z\Delta(z+1)$.

2. $\Delta(z)\Delta(1-z)=\Delta(z)\cdot\frac{-\Delta(-z)}{z}=\frac{1}{\pi}\sin\pi z$

Let us define $\Gamma(z)=\dfrac{1}{\Delta(z)}$. Previous properties can be written as:

1. $\Gamma(1)=1$.
2. $\Gamma(z)=\lim_{n\to\infty}\dfrac{n!n^z}{z(z+1)\cdots(z+n)}$.
3. $\Gamma(z+1)=z\Gamma(z)$.
4.
$$\Gamma(z)\Gamma(1-z)=\frac{\pi}{\sin\pi z}. \tag{3.3}$$
5. $\Gamma(z)=\dfrac{1}{z}e^{-\gamma z}\prod_{n=1}^{\infty}\left(1+\dfrac{z}{n}\right)^{-1}e^{z/n}$.
6. As $(z+n)\Gamma(z)=\dfrac{\Gamma(z+n+1)}{z(z+1)\cdots(z+n-1)}$, we have that $\lim_{z\to-n}(z+n)\Gamma(z)=\dfrac{(-1)^n}{n!}$.
7. The logarithmic derivative of Gamma function is

$$\frac{\Gamma'(z)}{\Gamma(z)}=-\frac{1}{z}-\gamma+\sum_{n=1}^{\infty}\left(\frac{1}{n}-\frac{1}{n+z}\right)=-\frac{1}{z}-\gamma+\sum_{n=1}^{\infty}\frac{z}{n(n+z)}.$$

Taking derivatives again we obtain

$$(\log\Gamma(z))''=\left(\frac{\Gamma'(z)}{\Gamma(z)}\right)'=\frac{1}{z^2}+\sum_{n=1}^{\infty}\frac{1}{(n+z)^2}=\sum_{n=0}^{\infty}\frac{1}{(n+z)^2}.$$

In particular, for real values of z, $\log\Gamma(z)$ is a convex function.

8. $\Gamma\left(\frac{1}{2}\right)^2=\dfrac{\pi}{\sin\pi/2}=\pi$, so $\Gamma(1/2)=\sqrt{\pi}$.

9. Let us consider the functions $\Gamma(2z)$ and $\Gamma(z)\Gamma(z+1/2)$, both with simples poles in $0,-1,-2,\dots,$ $-\frac{1}{2},-\frac{3}{2},\dots$. Its quotient is an entire function. Let us compute it:

$$\left(\frac{\Gamma'(z)}{\Gamma(z)}\right)'+\left(\frac{\Gamma'(z+1/2)}{\Gamma(z+1/2)}\right)'=\sum_{n=0}^{\infty}\frac{1}{(n+z)^2}+\sum_{n=0}^{\infty}\frac{1}{(n+z+\frac{1}{2})^2}=4\sum_{n=0}^{\infty}\frac{1}{(2z+n)^2}=2\left(\frac{\Gamma'(2z)}{\Gamma(2z)}\right)'.$$

As a consequence, there exists a constant A such that

$$\frac{\Gamma'(z)}{\Gamma(z)}+\frac{\Gamma'(z+1/2)}{\Gamma(z+1/2)}=2\frac{\Gamma'(2z)}{\Gamma(2z)}+A,$$

and another constant B giving the equality

$$\log\Gamma(2z)=\log\left(\Gamma(z)\Gamma\left(z+\frac{1}{2}\right)\right)-Az-B,\text{ or }\Gamma(2z)e^{Az+B}=\Gamma(z)\Gamma\left(z+\frac{1}{2}\right).$$

Giving values $z=1$ and $z=1/2$ we can compute the values of A and B: $A=-2\log 2$, $B=\log(2\sqrt{\pi})$. Then,

$$\Gamma(2z)\sqrt{\pi}=2^{2z-1}\Gamma(z)\Gamma\left(z+\frac{1}{2}\right) \tag{3.4}$$

This is known as *Lagrange's duplication formula*.

We shall see now that Gamma function, defined as $\frac{1}{\Delta(z)}$, agrees with the function $\Gamma(z)$ defined by (3.1) over the right half plane. It is enough to show that both expressions agree for $x\in\mathbb{R}$, $x>1$, by Identity principle. Let us define

$$f_n(t)=\begin{cases}t^{x-1}\left(1-\frac{t}{n}\right)^n & \text{if } t\in[0,n],\\ 0 & \text{if } t>n.\end{cases}$$

Denoting $\Phi_n(t)=\left(1-\dfrac{t}{n}\right)^n$, let us show that $\Phi_n(t)\le\Phi_{n+1}(t)$ on $[0,n]$, using Bernoulli inequality. Indeed,

$$\frac{\Phi_{n+1}(t)}{\Phi_n(t)}=\frac{n-t}{n}\left(\frac{(n+1-t)n}{(n+1)(n-t)}\right)^{n+1}=\frac{n-t}{n}\left(1+\frac{t}{(n+1)(n-t)}\right)^{n+1}\ge\frac{n-t}{n}\left(1+\frac{t}{n-t}\right)=1.$$

So $\{f_n(t)\}_{t=1}^{\infty}$ is an increasing sequence of functions, and $\lim_{n\to\infty}f_n(t)=t^{x-1}e^{-t}$ on $\mathbb{R}$. Monotone convergence Theorem implies that

$$\int_0^{\infty}t^{x-1}e^{-t}dt=\lim_{n\to\infty}\int_0^n\left(1-\frac{t}{n}\right)^n t^{x-1}dt=\lim_{n\to\infty}n^x\int_0^1(1-s)^n s^{x-1}ds=\lim_{n\to\infty}\frac{n^x n!}{x(x+1)\cdots(x+n)}=\Gamma(x).$$

3.2 Stirling's formula

One of the most interesting results related with Gamma function is Stirling's formula, which describes the asymptotic behaviour at infinity of factorial numbers and Gamma function. We shall propose here several different approaches to this result.

3.2.1 Stirling's formula for factorial numbers

The approach we follow here uses a simplified version of Euler-MacLaurin's summation formula, that we shall develop later for a general case. If $k>1$ we have

$$\int_{k-1}^{k}\log t\,dt=\left(t-k+\frac{1}{2}\right)\log t\Big|_{k-1}^{k}-\int_{k-1}^{k}\frac{t-k+\frac{1}{2}}{t}dt=\frac{1}{2}(\log(k-1)+\log k)-\int_{k-1}^{k}\frac{\{t\}-\frac{1}{2}}{t}dt,$$

where $\{t\}$ denotes the fractional part of the real number t. Denote $\varphi(x)=\frac{1}{2}(x^2-x)$ on $(0,1)$, extended to $\mathbb{R}$ by periodicity. We have $\varphi'(x)=\{x\}-\frac{1}{2}$ if $x\notin\mathbb{Z}$. Then,

$$\int_{k-1}^{k}\frac{\{t\}-1/2}{t}dt=\int_{k-1}^{k}\frac{\varphi(t)}{t^2}dt,\ \text{and so}$$

$$\int_{k-1}^{k}\log t\,dt=\frac{1}{2}(\log(k-1)+\log k)-\int_{k-1}^{k}\frac{\varphi(t)}{t^2}dt.$$

Summing from $k=2$ to $k=n$ we obtain

$$n\log n-n+1+\frac{1}{2}\log n=\log n!-\int_{1}^{n}\frac{\varphi(t)}{t^2}dt=\log n!-\int_{1}^{n}\frac{\{t\}-1/2}{t}dt.$$

As $\varphi(x)$ is bounded, the last integral converges, so it can be written as

$$\int_{1}^{n}\frac{\varphi(t)}{t^2}dt=\int_{1}^{\infty}\frac{\varphi(t)}{t^2}dt-\int_{n}^{\infty}\frac{\varphi(t)}{t^2}dt,$$

hence

$$\log n!-\left(n+\frac{1}{2}\right)\log n+n=\alpha-\int_{n}^{\infty}\frac{\varphi(t)}{t^2}dt,$$

where

$$\alpha=1+\int_{1}^{\infty}\frac{\varphi(t)}{t^2}dt,\ \text{and}\ \int_{n}^{\infty}\frac{\varphi(t)}{t^2}dt\overset{n\to\infty}{\longrightarrow}0.$$

Let us determine the constant α with the help of Wallis' formula (2.8). Setting

$$x_n=\frac{n!}{n^{n+1/2}e^{-n}e^{\alpha}},$$

we have that

$$\frac{x_n^2}{x_{2n}}=\frac{(n!)^2 2^{2n}}{(2n)!\sqrt{n}}\cdot\frac{\sqrt{2}}{e^{\alpha}}\overset{n\to\infty}{\longrightarrow}\frac{\sqrt{2\pi}}{e^{\alpha}}=1,$$

so $e^{\alpha}=\sqrt{2\pi}$ and then

$$n!\sim n^{n+1/2}e^{-n}\sqrt{2\pi}.$$

Moreover, we obtain that

$$\int_{1}^{\infty}\frac{\{t\}-1/2}{t}dt=\int_{1}^{\infty}\frac{\varphi(t)}{t^2}dt=\log(\sqrt{2\pi})-1. \tag{3.5}$$

3.2.2 Stirling's formula for real values

Let us propose here a different proof, valid in the real case. The objective will be to show that

$$\lim_{x\to\infty} \frac{\Gamma(x+1)}{e^{-x}x^x\sqrt{2\pi x}} = 1.$$

A change of variable $t = x + s\sqrt{x}$ in the expression

$$\Gamma(x+1) = \int_0^\infty t^x e^{-t} dt$$

leads to

$$\Gamma(x+1) = \int_{-\sqrt{x}}^{\infty} (x+s\sqrt{x})^x e^{-(x+s\sqrt{x})} \sqrt{s} ds = x^x e^{-x} \sqrt{x} \int_{-\sqrt{x}}^{\infty} \left(1+\frac{s}{\sqrt{x}}\right)^x e^{-s\sqrt{x}} ds.$$

Take an increasing sequence $\{x_n\}_{n=1}^{\infty}$, with $\lim_{n\to\infty} x_n = +\infty$, and consider

$$f_n(s) = \exp\left(-s\sqrt{x_n} + x_n \log\left(1+\frac{s}{\sqrt{x_n}}\right)\right) \chi_{[-\sqrt{x_n},+\infty)}.$$

Write

$$s\sqrt{x_n} - x_n \log\left(1+\frac{s}{\sqrt{x_n}}\right) = x_n \int_0^{s/\sqrt{x_n}} \frac{udu}{1+u}.$$

If $s \geq 0$,

$$x_n \int_0^{s/\sqrt{x_n}} \frac{udu}{1+u} \geq x_n \int_0^{s/\sqrt{x_n}} \frac{udu}{1+\frac{s}{\sqrt{x_n}}} = \frac{s^2}{2\left(1+\frac{s}{\sqrt{x_n}}\right)} \geq \frac{s^2}{2(1+s)},$$

when $x_n \geq 1$. The integral

$$\int_0^\infty \exp\left(-\frac{s^2}{2(1+s)}\right) ds \text{ converges.}$$

When $s \in [-\sqrt{x_n}, 0]$,

$$x_n \int_0^{s/\sqrt{x_n}} \frac{udu}{1+u} = x_n \int_{s/\sqrt{x_n}}^0 \frac{-udu}{1+u} \geq x_n \int_{s/\sqrt{x_n}}^0 -udu = \frac{s^2}{2},$$

and $\exp(-s^2/2)$ is integrable in $(-\infty, 0]$.

Using dominated convergence theorem,

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} f_n(s) ds = \int_{-\infty}^{\infty} \lim_{n\to\infty} f_n(s) ds = \int_{-\infty}^{\infty} e^{-s^2/2} ds = \sqrt{2\pi},$$

and the result follows.

3.2.3 Stirling's formula in the complex case. Bernoulli polynomials

Let $z \notin \mathbb{R}_-$, and denote $\Gamma_n(z) = n^z \dfrac{n!}{z(z+1)\cdots(z+n)}$. We have that

$$\log\Gamma_n(z) = z\log n - \log z + \sum_{k=1}^{n} \log\left(\frac{k}{z+k}\right).$$

We want to estimate the last sum. Let us suppose that $f : [0,\infty) \longrightarrow \mathbb{C}$ is a $\mathscr{C}^1$–function. We have that

$$\begin{aligned}
\int_k^{k+1} f(t)dt &= f(t)\left(t-k-\frac{1}{2}\right)\Big|_k^{k+1} - \int_k^{k+1}\left(t-k-\frac{1}{2}\right)f'(t)dt \\
&= \frac{1}{2}(f(k)+f(k+1)) - \int_k^{k+1}\left(t-k-\frac{1}{2}\right)f'(t)dt.
\end{aligned}$$

Let us denote $B_1(t)$ the function defined as $t-k-\frac{1}{2} = \{t\}-\frac{1}{2}$ on $[k,k+1)$ (i.e. it is the function $t-\frac{1}{2}$ on $[0,1)$, extended by 1-periodicity). We have

$$\int_1^n f(t)dt = (f(1)+\cdots+f(n)) - \frac{1}{2}(f(1)+f(n)) - \int_1^n B_1(t)f'(t)dt. \tag{3.6}$$

Let us apply (3.6) to $f(t) = \log\left(\frac{t}{z+t}\right)$. We have:

$$\begin{aligned}
\int_1^n \log\left(\frac{t}{z+t}\right)dt &= n\log n - (z+n)\log(z+n) + (z+1)\log(z+1), \\
f(1)+f(n) &= -\log(z+1) + \log n - \log(z+n).
\end{aligned}$$

Consequently,

$$\begin{aligned}
\log\Gamma_n(z) &= z\log n - \log z + n\log n - (z+n)\log(z+n) + (z+1)\log(z+1) \\
&\quad - \frac{1}{2}\log(z+1) + \frac{1}{2}\log n - \frac{1}{2}\log(z+n) + \int_1^n B_1(t)f'(t)dt \\
&= \left(z+n+\frac{1}{2}\right)\log\left(\frac{n}{z+n}\right) - \log(z) + \left(z+\frac{1}{2}\right)\log(z+1) + \int_1^n B_1(t)\left(\frac{1}{t}-\frac{1}{z+t}\right)dt.
\end{aligned}$$

When n tends to infinity, we obtain

$$\begin{aligned}
\log\Gamma(z) &= -z - \log z + \left(z+\frac{1}{2}\right)\log(z+1) + \log(\sqrt{2\pi}) - 1 - \int_1^\infty \frac{B_1(t)}{z+t}dt \\
&= -z + \left(z-\frac{1}{2}\right)\log z + \log(\sqrt{2\pi}) - \int_0^\infty \frac{B_1(t)}{z+t}dt,
\end{aligned}$$

as

$$\int_0^1 \frac{B_1(t)}{z+t}dt = 1 + \left(z+\frac{1}{2}\right)\log\left(\frac{z}{z+1}\right).$$

Denoting by $\varphi(t)$ the function on $[0,1)$ defined by $\frac{1}{2}(t^2-t)$, extended to $\mathbb{R}$ by periodicity, we have

$$\int_0^\infty \frac{B_1(t)}{z+t}dt = \int_0^\infty \frac{\varphi(t)}{(z+t)^2}dt.$$

Assume now that $z = |z|\,e^{i\theta} \notin \mathbb{R}_-$. Then, if $t \geq 0$,

$$\begin{aligned}|z+t|^2 &= (|z|\cos\theta + t)^2 + (|z|\sin\theta)^2 = |z|^2 + 2t\,|z|\cos\theta + t^2\\ &= (|z|+t)^2 - 4t\,|z|\sin^2\theta/2 = (|z|+t)^2\cos^2\theta/2 + (|z|-t)^2\sin^2\theta/2 \geq (|z|+t)^2\cos^2\theta/2,\end{aligned}$$

and, hence, if $\arg z \in [-\pi+\delta, \pi-\delta]$,

$$\left|\int_0^\infty \frac{\varphi(t)}{(z+t)^2}dt\right| \leq \frac{M}{\sin^2\delta/2}\int_0^\infty \frac{1}{(|z|+t)^2}dt = \frac{M}{\sin^2\delta/2}\cdot\frac{1}{|z|}.$$

Consequently, this integral tends to zero uniformly in the abovementioned sector, and then

$$\frac{\Gamma(z)}{e^{-z}z^{z-1/2}\sqrt{2\pi}} \overset{|z|\to\infty}{\longrightarrow} 1$$

uniformly. This is the simplest version of complex Stirling's formula.

These asymptotics may be improved by means of the so-called **Euler-McLaurin formula**, that we shall explain here at least in the case we will need. Before developing it, we need to introduce Bernoulli numbers and polynomials. Expand

$$\frac{z}{e^z-1} = \sum_{k=0}^\infty \frac{B_k}{k!}z^k.$$

The coefficients B_k are called **Bernoulli numbers**. From the identity

$$z = \left(\sum_{k=1}^\infty \frac{z^k}{k!}\right)\cdot\left(\sum_{k=0}^\infty \frac{B_k}{k!}z^k\right),$$

we deduce that

$$B_0 = 1;\ B_1 = -\frac{1}{2};\ B_2 = \frac{1}{6},\ \text{and the relation}\ \sum_{k=0}^{n-1}\binom{n}{k}B_k = 0\ \text{if}\ n\geq 2,$$

which allows to compute these numbers by recurrence. As

$$\frac{z}{e^z-1} + \frac{z}{2} = \frac{z}{2}\cdot\frac{e^z+1}{e^z-1}$$

is an even function, we deduce that $B_{2k+1} = 0$ if $k \geq 1$.

As an application, we can compute the values $\displaystyle\sum_{n=1}^\infty \frac{1}{n^{2k}} = \zeta(2k)$. Indeed,

$$1 + \sum_{k=2}^\infty \frac{B_k}{k!}(2\pi i z)^k = \pi z\cot\pi z.$$

As

$$\pi z\cot\pi z = 1 + \sum_{n=1}^\infty \frac{2z^2}{z^2-n^2} = 1 - 2\sum_{n=1}^\infty \frac{z^2}{n^2}\cdot\sum_{k=1}^\infty \frac{z^{2k}}{n^{2k}} = 1 - 2\sum_{k=1}^\infty \zeta(2k)z^{2k},$$

we deduce that

$$-2\zeta(2k) = \frac{B_{2k}}{(2k)!}2^{2k}\pi^{2k}(-1)^k,$$

and hence

$$\zeta(2k) = (-1)^{k+1} 2^{2k-1} \pi^{2k} \frac{B_{2k}}{(2k)!}.$$

For instance, as $B_4 = -\frac{1}{30}$, and $B_6 = \frac{1}{42}$, we have

$$\zeta(2) = \frac{\pi^2}{6};\ \zeta(4) = \frac{\pi^4}{90};\ \zeta(6) = \frac{\pi^6}{945}.$$

Define now Bernoulli polynomials by

$$\Phi(x,z) = \frac{ze^{xz}}{e^z - 1} = \sum_{k=0}^{\infty} \frac{B_k(x)}{k!} z^k.$$

We collect in the sequel some of the properties of these polynomials that are relevant for us. As $\Phi(0,z) = \dfrac{z}{e^z-1}$, we have that $B_k(0) = B_k$. In particular, $B_{2k+1}(0) = 0$ if $k \geq 1$. Derivating we can see that $B'_{k+1}(x) = (k+1)B_k(x)$. As $\Phi(1-x,z) = \Phi(x,-z)$ we obtain $B_k(1) = B_k(0)$ if $k \geq 2$. Hence

$$0 = B_k(1) - B_k(0) = \int_0^1 B'_k(t)dt = k\int_0^1 B_{k-1}(t)dt,$$

so

$$\int_0^1 B_k(t)dt = 0 \text{ if } k \geq 1.$$

$B_n(x)$ can be written explicitely as $B_n(x) = \sum_{k=0}^{\infty} \binom{n}{k} B_k x^{n-k}$. It is monic, and characterized by previous properties.

We shall denote in the sequel as $B_k(x)$ the function defined as the corresponding Bernoulli polynomial on $[0,1)$, extended periodically to $\mathbb{R}$. Observe that $B_k(x)$ is continuous if $k \geq 2$ and in fact, $B_k(x)$ is a $\mathscr{C}^{k-2}-$function.

Using these polynomial functions, for any infinitely differentiable function $f(t)$ we have that

$$\begin{aligned}\int_1^n B_1(t)f'(t)dt &= \frac{B_2}{2}(f'(n) - f'(1)) - \frac{1}{2}\int_1^n B_2(t)f''(t)dt = \cdots \\ &= \sum_{k=2}^{N} \frac{B_k}{k!}(-1)^k (f^{(k-1)}(n) - f^{(k-1)}(1)) + (-1)^{N+1}\frac{1}{N!}\int_1^n B_N(t)f^{(N)}(t)dt.\end{aligned}$$

Replacing in (3.6) and using that $B_{2k+1} = 0$, we finally get

$$\begin{aligned}f(1) + \cdots + f(n) = \int_1^n f(t)dt + \frac{1}{2}(f(1) + f(n)) + \sum_{k=1}^{N-1} \frac{B_{2k}}{(2k)!}(f^{(2k-1)}(n) - f^{(2k-1)}(1)) \\ - \frac{1}{(2N)!}\int_1^n (B_{2N}(t) - B_{2N}) f^{(2N)}(t)dt.\end{aligned}$$

Let us apply previous formula to the function $f(x) = \log x$, to obtain

$$\log n! = n\log n - n + 1 + \frac{1}{2}\log n + \sum_{k=1}^{N-1} \frac{B_{2k}}{2k(2k-1)}\left[\frac{1}{n^{2k-1}} - 1\right] + \frac{1}{2N}\int_1^n (B_{2N}(t) - B_{2N})\frac{1}{t^{2N}}dt. \quad (3.7)$$

As $B_k(t) - B_k$ are bounded, the integral $\displaystyle\int_1^\infty (B_k(t) - B_k) \cdot \frac{1}{t^k} dt$ converges absolutely when $k \geq 2$ and we can rewrite (3.7) as follows:

$$\log n! - \left(n + \frac{1}{2}\right) \log n + n - \sum_{k=1}^{N-1} \frac{B_{2k}}{2k(2k-1)} \frac{1}{n^{2k-1}}$$
$$= \left[1 - \sum_{k=1}^{N-1} \frac{B_{2k}}{2k(2k-1)} + \frac{1}{2N} \int_1^\infty \frac{B_{2N}(t) - B_{2N}}{t^{2N}} dt\right] - \frac{1}{2N} \int_n^\infty (B_{2N}(t) - B_{2N}) \frac{1}{t^{2N}} dt.$$

Denote C_N the number in brackets, and a_n the last term (the remainder). It is clear that there exists bounds

$$|a_n| \leq \frac{M_N}{n^{2N-1}}, \; M_N \text{ a constant.}$$

Concerning C_N, compute the difference

$$C_{N+1} - C_N = -\frac{B_{2N}}{2N(2N-1)} + \frac{1}{2N+2} \int_1^\infty \frac{B_{2N+2}(t) - B_{2N+2}}{t^{2N+2}} dt - \frac{1}{2N} \int_1^\infty \frac{B_{2N}(t) - B_{2N}}{t^{2N}} dt.$$

As $\displaystyle\frac{1}{2N} \int_1^\infty \frac{B_{2N}}{t^{2N}} dt = \frac{B_{2N}}{2N(2N-1)}$, and integrating by parts twice,

$$\frac{1}{2N} \int_1^\infty \frac{B_{2N}(t)}{t^{2N}} dt = -\frac{B_{2N+2}}{(2N+2)(2N+1)} + \frac{1}{2N+2} \int_1^\infty \frac{B_{2N+2}(t)}{t^{2N+2}} dt,$$

we get that $C_{N+1} - C_N = 0$, so $C_N = C$, constant, for $N \geq 1$. In fact, when $N = 1$ we have, by (3.5),

$$1 + \frac{1}{2} \int_1^\infty \frac{B_2(t) - B_2}{t^2} dt = \log(\sqrt{2\pi}).$$

Collecting everything together, we get the following improved version of Stirling's formula:

$$\log n! - \left(n + \frac{1}{2}\right) \log n + n - \log(\sqrt{2\pi}) - \sum_{k=1}^{N-1} \frac{B_{2k}}{2k(2k-1)} \frac{1}{n^{2k}} = a_n = O\left(\frac{1}{n^{2N-1}}\right).$$

As an example, when $N = 4$ this formula gives

$$\log n! = \left(n + \frac{1}{2}\right) \log n - n + \log(\sqrt{2\pi}) + \frac{1}{12} \frac{1}{n} - \frac{1}{360} \frac{1}{n^3} + \frac{1}{1260} \frac{1}{n^5} + O\left(\frac{1}{n^7}\right).$$

As a final remark, this formula can be adapted to fit in the complex case, as we did for standard Stirling's formula, and get an asymptotic expression for the complex Gamma function.

3.3 Riemann's zeta function

Let us define Riemann's zeta function by the sum

$$\zeta(z) = \sum_{n=1}^\infty \frac{1}{n^z},$$

for those values of z such that previous series converges. If $K \subseteq \mathbb{C}$ is a compact set such that there exists $\delta > 1$ with $\text{Re}(z) \geq \delta$ in K, we have bounds $\left|\dfrac{1}{n^z}\right| \leq \dfrac{1}{n^\delta}$, and previous sum converges uniformly in K. So, it defines a holomorphic function on $H = \{z \in \mathbb{C} | \operatorname{Re}(z) > 1\}$. We shall continue analytically this function to $\mathbb{C}$. To get this continuation, let us relate $\zeta(z)$ and $\Gamma(z)$ using the integral

$$\frac{1}{n^z}\Gamma(z) = \frac{1}{n^z}\int_0^\infty t^{z-1}e^{-t}dt = \int_0^\infty u^{z-1}e^{-nu}du.$$

We want to compute the sum $\sum_{n=1}^\infty \int_0^\infty t^{z-1}e^{-nt}dt$. Let us observe that

$$\sum_{n=1}^\infty t^{z-1}e^{-nt} = \frac{t^{z-1}}{e^t-1}.$$

This function is integrable in $(0,+\infty)$ if $\operatorname{Re}(z) > 1$. Indeed,

$$\lim_{t\to\infty}\frac{t^{z-1}e^{t/2}}{e^t-1} = 0,$$

so a constant $C > 0$ exists such that $\left|\frac{t^{z-1}}{e^t-1}\right| < Ce^{-t/2}$ on $[1,+\infty)$. As

$$\left|\frac{t^{z-1}}{e^t-1}\right| = \left|\frac{t}{e^t-1}t^{z-2}\right| = \frac{t}{e^t-1}t^{\operatorname{Re}(z)-2}, \text{ and } \lim_{t\to 0}\frac{t}{e^t-1} = 1,$$

the function is integrable at 0 provided that $\operatorname{Re}(z)-2 > -1$. So, the series of positive terms

$$\sum_{n=1}^\infty t^{\operatorname{Re}(z)-1}e^{-nt}$$

converges to the integrable function $\frac{t^{\operatorname{Re}(z)-1}}{e^t-1}$, and monotone convergence theorem allows to exchange sum and integral, and then

$$\sum_{n=1}^\infty \int_0^\infty t^{z-1}e^{-nt}dt = \int_0^\infty \frac{t^{z-1}}{e^t-1}dt.$$

Consequently,

$$\int_0^\infty \frac{t^{z-1}}{e^t-1}dt = \sum_{n=1}^\infty \frac{1}{n^z}\cdot\Gamma(z) = \zeta(z)\Gamma(z).$$

Our next goal will be to extend this integral, in order to find an analytic continuation of $\zeta(z)$. The problem is in the integral near 0, so we will avoid the origin, considering the integral

$$H_\varepsilon(z) = \int_{\gamma_\varepsilon}\frac{t^{z-1}}{e^t-1}dt,$$

where $0 < \varepsilon < 2\pi$, and the path is as in Figure 3.1.

The integral is divided in three. Let $\log t$ be a determination of the logarithm in $\mathbb{C}\setminus\mathbb{R}_+$, where $\arg(t) \in (0,2\pi)$. We have:

$$I_1 = \int_\infty^{\tilde{\varepsilon}} \frac{t^{z-1}}{e^t-1}dt,$$
$$I_3 = \int_{\tilde{\varepsilon}}^\infty \frac{\exp((z-1)\log t + 2\pi i)}{e^t-1}dt,$$
$$I_2 = \int_\delta^{2\pi-\delta}\frac{\exp((z-1)(\log\varepsilon + i\theta))}{e^{\varepsilon e^{i\theta}}-1}\varepsilon i e^{i\theta}d\theta = \int_\delta^{2\pi-\delta}\frac{\varepsilon^z\exp(i\theta(z-1))}{e^{\varepsilon e^{i\theta}}-1}ie^{i\theta}d\theta.$$

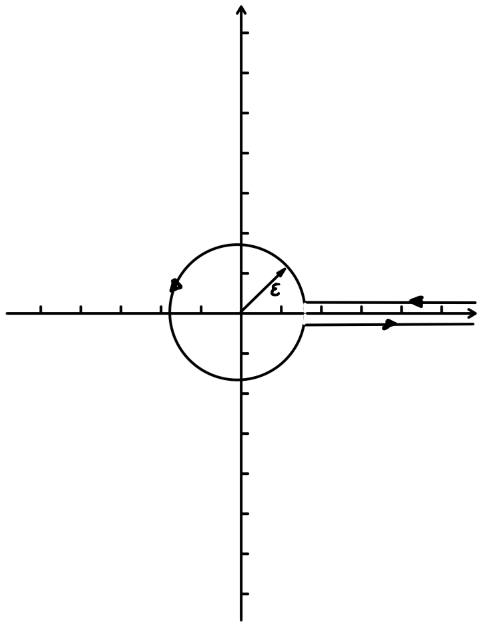


Figure 3.1: Path of integration.

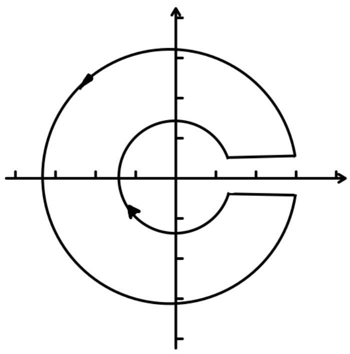


Figure 3.2: Path $\gamma_{\varepsilon,\varepsilon'}$.

We can bound the integral of I_2 as

$$\left|\frac{\varepsilon e^{i\theta}}{e^{\varepsilon e^{i\theta}}-1}\cdot\frac{1}{\varepsilon e^{i\theta}}\cdot\varepsilon^z e^{i\theta z}e^{-i\theta}ie^{i\theta}\right| \le \left|\frac{\varepsilon e^{i\theta}}{e^{\varepsilon e^{i\theta}}-1}\cdot\varepsilon^{\mathrm{Re}(z)-1}e^{2\pi|\mathrm{Im}(z)|}\right| \xrightarrow{\varepsilon\to 0} 0 \text{ if } \mathrm{Re}(z)>1.$$

On the other hand, if both ε, ε' are small enough, we have

$$H_\varepsilon(z)-H_{\varepsilon'}(z)=\int_{\gamma_{\varepsilon,\varepsilon'}}\frac{t^{z-1}}{e^t-1}dt=0,$$

where $\gamma_{\varepsilon,\varepsilon'}$ is a path as in Figure 3.2, which does not encircle any singularity of the integrand.

As a result, we have

$$H(z)=H_\varepsilon(z)=\lim_{\varepsilon\to 0}H_\varepsilon(z)=\int_0^\infty\frac{(e^{2\pi iz}-1)t^{z-1}}{e^t-1}dt=(e^{2\pi iz}-1)\zeta(z)\Gamma(z).$$

The function $H(z)$ being an entire function allows us to define

$$\zeta(z)=\frac{1}{e^{2\pi iz}-1}\cdot\frac{1}{\Gamma(z)}H(z)$$

on $\mathbb{C}$. It is a meromorphic function eventually with order one poles over the integers. When $k=0,-1,-2,\ldots$, $\dfrac{1}{\Gamma(z)}$ has an order one zero, so there is no singularity in these points. We already know that $\zeta(z)$ is well defined on $k=2,3,\ldots$. So, the only possible singularity remaining is $z=1$. As

$$H(1)=H_\varepsilon(1)=\int_{\gamma_\varepsilon}\frac{dt}{e^t-1}=2\pi i\,\mathrm{Res}\left(\frac{1}{e^t-1};0\right)=2\pi i,\text{ and }\lim_{z\to 1}\frac{z-1}{e^{2\pi iz}-1}=\frac{1}{2\pi i},$$

we can conclude that

$$\lim_{z\to 1}(z-1)\zeta(z)\frac{1}{2\pi i}\cdot 1\cdot 2\pi i=1,$$

and $\zeta(z)$ has a pole of order 1 in $z=1$ such that $\mathrm{Res}(\zeta(z);1)=1$.

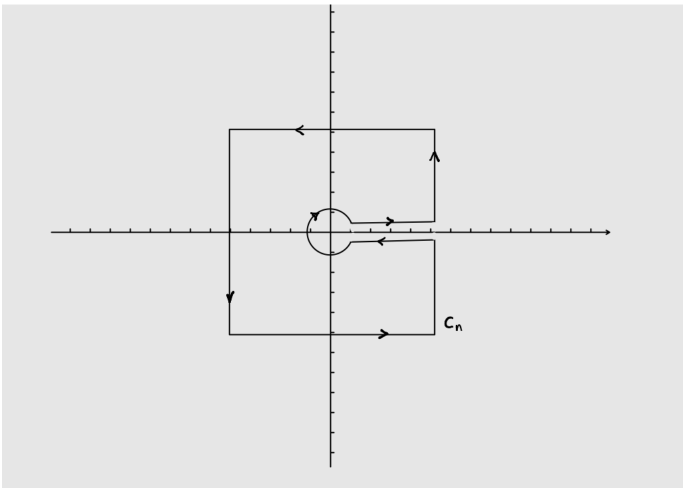


Figure 3.3: Path Γ_n.

R For many applications, it is not necessary to have an extension of $\zeta(z)$ to the whole complex plane, but only for some half-plane bigger that $\mathrm{Re}(z) > 1$. For instance, if $\mathrm{Re}(z) > 1$, integrating by parts we get

$$\int_n^{n+1} \frac{1}{t^z} dt = \frac{1}{(n+1)^z} + z \int_n^{n+1} \frac{\{t\}}{t^{z+1}} dt,$$

and so,

$$\int_1^N \frac{1}{t^z} dt = \sum_{n=1}^{N-1} \int_n^{n+1} \frac{1}{t^z} dt = \frac{1}{z-1}\left(1 - \frac{1}{N^{z-1}}\right) = \frac{1}{2^z} + \cdots + \frac{1}{N^z} + z \int_1^N \frac{\{t\}}{t^{z+1}} dt.$$

We deduce that

$$1 + \frac{1}{2^z} + \cdots + \frac{1}{N^z} + z \int_1^N \frac{\{t\}}{t^{z+1}} dt = \frac{z}{z-1} - \frac{1}{z-1}\frac{1}{N^{z-1}}.$$

Taking limits when $N \longrightarrow \infty$, we get the formula

$$\zeta(z) = \frac{z}{z-1} - z \int_1^\infty \frac{\{t\}}{t^{z+1}} dt.$$

As last integral converges when $\mathrm{Re}(z) > 0$, it provides an analytic expression of the continuation of $\zeta(z)$ to this half-plane. It also shows that it has a pole at $z = 1$ and that $\mathrm{Res}(\zeta(z); 1) = 1$.

3.4 Functional equation for $\zeta(z)$ and $\Gamma(z)$

Let us establish an important relation between $\Gamma(z)$ and $\zeta(z)$. To do this, let us consider the path C_n defined as follows: C_n comes from infinity, through the positive real line, to the point $(2n+1)\pi$, then follows counterclockwise the square centered at the origin with vertices $(2n+1)\pi(\pm 1 \pm i)$, and then, returns to infinity. Denote $I_n(z) = \int_{C_n} \frac{t^{z-1}}{e^t - 1} dt$. The difference $I_n(z) - H(z)$ is equal to the integral $\int_{\Gamma_n} \frac{t^{z-1}}{e^t - 1} dt$, where Γ_n is a path as in Figure 3.3. By Residue Theorem, we have that

$$I_n(z) - H(z) = 2\pi i \sum_{\substack{-n \le k \le n \\ k \ne 0}} \mathrm{Res}\left(\frac{t^{z-1}}{e^t - 1}; 2k\pi i\right),$$

where

$$\operatorname{Res}\left(\frac{t^{z-1}}{e^t-1};2k\pi i\right) = (2k\pi)^{z-1}\exp\left(i\frac{\pi}{2}(z-1)\right),$$
$$\operatorname{Res}\left(\frac{t^{z-1}}{e^t-1};-2k\pi i\right) = (2k\pi)^{z-1}\exp\left(i\frac{3\pi}{2}(z-1)\right).$$

Summing everything,

$$I_n(z)-H(z) = 2\pi i\sum_{k=1}^{n}(2k\pi)^{z-1}e^{i\pi(z-1)}2\cos\left(\frac{\pi}{2}(z-1)\right).$$

On the square C_n, $\dfrac{1}{e^t-1}$ is bounded. Indeed, on the vertical right side we have

$$|e^z-1| \geq |e^z|-1 = e^{(2n+1)\pi}-1 \geq e^{3\pi}-1,$$

and similarly on the left one. On horizontal sides,

$$|e^z-1| = e^{\operatorname{Re}(z)}+1 > 1.$$

On the other hand, $\left|t^{z-1}\right| = O(|t|^{\operatorname{Re}(z)-1})$. So, on C_n the integral I_n is $O(|t|^{\operatorname{Re}(z)})$, which tends to zero when $\operatorname{Re}(z)<0$. On this set we have, taking limits, that

$$H(z) = (e^{2\pi iz}-1)\Gamma(z)\zeta(z) = -2\pi ie^{i\pi(z-1)}2\sin\left(\frac{\pi z}{2}\right)(2\pi)^{z-1}\zeta(1-z).$$

After some computation, using formulas (3.3) and (3.4), defining $\xi(z) = \dfrac{\zeta(z)\Gamma(z/2)}{\pi^{z/2}}$, we obtain that $\xi(z)=\xi(1-z)$. This is the functional equation relating Gamma and zeta functions.

3.5 The Prime Number Theorem

The objective of this section is the study of some properties regarding the distribution of primes, and the mail goal will be to prove Prime Number Theorem, that asserts, roughly, that for big values of N, the number $\pi(N)$ of primes not greater than N is asymptotically $\dfrac{N}{\log N}$. The main tool will be the fact that $\zeta(z)$ does not have zeros on the line $\operatorname{Re}(z)=1$. We have to connect primes with $\zeta(z)$.

Theorem 3.5.1 — Euler. If $\operatorname{Re}(z)>1$, we have the identity

$$\zeta(z) = \prod_{n=1}^{\infty}\left(\frac{1}{1-p_n^{-z}}\right), \tag{3.8}$$

where $\{p_n\}_{n=1}^{\infty}$ is the sequence of prime numbers.

Proof. If $\operatorname{Re}(z)>1$, $|p_n^{-z}| < \dfrac{1}{p_n} < 1$, and then,

$$\frac{1}{1-p_n^{-z}} = \sum_{k=0}^{\infty}p_n^{-kz},$$

the convergence being uniform in the compacts contained in the halfplane $\operatorname{Re}(z) > 1$. Denote

$$P_m(z) = \prod_{n=1}^{m} \frac{1}{1-p_n^{-z}} = \prod_{n=1}^{m} \left(1 + \frac{1}{p_n^z} + \frac{1}{p_n^{2z}} + \cdots\right).$$

The convergence being absolute, we can compute the Cauchy product of the series and write $P_m(z) = \sum_{n\in\mathscr{P}_m} \dfrac{1}{n^z}$, where we have denoted

$$\mathscr{P}_m = \{1\} \cup \{n \in \mathbb{N} |\text{ the only prime factors of } n \text{ are } p_1, p_2, \ldots, p_m\}.$$

We have

$$\prod_{n=1}^{\infty} \frac{1}{1-p_n^{-z}} = \lim_{m\to\infty} P_m(z) = \zeta(z).$$

■

Corollary 3.5.2 $\zeta(z) \neq 0$ if $\operatorname{Re}(z) > 1$.

From Euler's identity (3.8) we can deduce that the series of the inverses of the primes diverges:

Theorem 3.5.3 Let $\{p_n\}_{n=1}^{\infty}$ be the sequence of primes. Then

$$\sum_{n=1}^{\infty} \frac{1}{p_n} \text{ diverges.}$$

Proof. If $s \in \mathbb{R}$, $s > 1$, and $m \in \mathbb{N}$, denote, as in Euler's Theorem 3.5.1, $P_m(s) = \prod_{n=1}^{m} \left(1 - \frac{1}{p_n^s}\right)^{-1}$. From Theorem 3.8, we know that $\lim_{n\to\infty} P_m(s) = \zeta(s)$.

$$\begin{aligned}\log P_m(s) &= -\sum_{n=1}^{m} \log\left(1 - \frac{1}{p_n^s}\right) = \sum_{n=1}^{m}\sum_{k=1}^{\infty} \frac{1}{k}\frac{1}{p_n^{ks}} \leq \sum_{n=1}^{m}\left[\frac{1}{p_n^s} + \sum_{k=2}^{\infty}\frac{1}{p_n^{ks}}\right] \\ &\leq \sum_{n=1}^{m}\left[\frac{1}{p_n^s} + \frac{1}{p_n^s(p_n^s-1)}\right] \leq \sum_{n=1}^{m}\frac{1}{p_n^s} + \sum_{n=1}^{m}\frac{1}{(p_n^s-1)^2}.\end{aligned}$$

So,

$$\log P_m(s) = \log\left(\sum_{n\in\mathscr{P}_m} \frac{1}{n^s}\right) \leq \sum_{n=1}^{m}\frac{1}{p_n^s} + \sum_{n=1}^{m}\frac{1}{(p_n^s-1)^2},$$

and this inequality also holds for $s = 1$. Taking limits when $m \longrightarrow \infty$ we get

$$\log\left(\sum_{n=1}^{\infty}\frac{1}{n}\right) \leq \sum_{n=1}^{\infty}\frac{1}{p_n} + \sum_{n=1}^{\infty}\frac{1}{(p_n-1)^2},$$

and hence, $\sum_{n=1}^{\infty} \dfrac{1}{p_n}$ diverges. ■

There are many different proofs of this beautiful result. Let us collect here some of them.

Proof taken from [3]. Assume $\sum_{n=1}^{\infty} \frac{1}{p_n} < \infty$, and let $N \in \mathbb{N}$ be such that $\sum_{n>N} \frac{1}{p_n} < \frac{1}{2}$. Denote $Q = p_1 \cdots p_N$. For every $n \in \mathbb{N}$, none of the primes $p_1, \ldots, p_N$ appear as factors of $1+nQ$. Then,

$$\sum_{n=1}^{r} \frac{1}{1+nQ} \leq \sum_{t=1}^{\infty} \left(\sum_{n>N} \frac{1}{p_n} \right)^t \leq \sum_{t=1}^{\infty} \frac{1}{2^t} < \infty,$$

which contradicts the fact that $\sum_{n=1}^{\infty} \frac{1}{1+nQ}$ diverges. ∎

Proof from I. Niven [15]. Denote $\sum' \frac{1}{k}$ the sum of the inverses of the square-free natural numbers. As

$$\left(\sum{}' \frac{1}{k} \right) \left(\sum_{m=1}^{\infty} \frac{1}{m^2} \right) \geq \sum_{n=1}^{\infty} \frac{1}{n},$$

then $\sum' \frac{1}{k}$ diverges. If $\sum_{n=1}^{\infty} \frac{1}{p_n}$ converges to a limit l, and $N \in \mathbb{N}$, we have

$$e^l > \exp\left(\prod_{p_n<N} \frac{1}{p_n} \right) = \prod_{p_n<N} e^{1/p_n} > \prod_{p_n<N} \left(1 + \frac{1}{p_n} \right) \geq \sum_{k<N}{}' \frac{1}{k},$$

which is absurd. ∎

Proof from Erdös [2]. Assume that our series converges, and let $m \in \mathbb{N}$ be such that $\sum_{n>m} \frac{1}{p_n} < \frac{1}{2}$. Call $p_1, \ldots, p_m$ small primes, and big primes the others. Let us observe that for every $N \in \mathbb{N}$,

$$\sum_{n>m} \frac{N}{p_n} < \frac{N}{2}.$$

Define

$$N_b = \#\{n \in \mathbb{N} \,|\, n \leq N \text{ and } n \text{ has some big prime factor}\}$$
$$N_s = \#\{n \in \mathbb{N} \,|\, n \leq N \text{ and } n \text{ all prime factors of } N \text{ are small }\}.$$

We have that

$$N_b \leq \sum_{n>m} \left[\frac{N}{p_n} \right] < \frac{N}{2}.$$

If N has only small prime factors, and we decompose $n = a_n b_n^2$, where a_n has no multiple prime factors, a_n can take at most 2^m possible different values, and b_n no more than $\sqrt{N}$. Then,

$$N_s \leq 2^m \sqrt{N} < \frac{N}{2} \text{ if } N > 2^{2m+2},$$

and we would have $N_s + N_b < N$, which is impossible.

∎

The key fact to show Prime Number Theorem will be the non existence of zeros of $\zeta(z)$ on the line $\mathrm{Re}(z) = 1$. We shall show it now.

Proposition 3.5.4 $\zeta(z)$ does not vanish on the line $\mathrm{Re}(z)=1$.

Proof. Fix $y\in\mathbb{R}\setminus\{0\}$, and define, for $x>1$,

$$h(x)=\zeta(x)^3\zeta(x+iy)^4\zeta(x+2iy).$$

We have

$$\log|\zeta(z)| = -\sum_{j=1}^{\infty}\log\left|1-p_j^{-z}\right| = -\mathrm{Re}\left(\sum_{j=1}^{\infty}\log(1-p_j^{-z})\right) = \mathrm{Re}\left(\sum_{j=1}^{\infty}\sum_{n=1}^{\infty}\frac{1}{n}p_j^{-nz}\right).$$

Then,

$$\begin{aligned}
\log|h(x)| &= 3\log|\zeta(x)|+4\log|\zeta(x+iy)|+\log|\zeta(x+2iy)|\\
&= 3\,\mathrm{Re}\left(\sum_{j=1}^{\infty}\sum_{n=1}^{\infty}\frac{1}{n}p_j^{-nx}\right)+4\,\mathrm{Re}\left(\sum_{j=1}^{\infty}\sum_{n=1}^{\infty}\frac{1}{n}p_j^{-nx-iny}\right)+\mathrm{Re}\left(\sum_{j=1}^{\infty}\sum_{n=1}^{\infty}\frac{1}{n}p_j^{-nx-2iny}\right)\\
&= \sum_{j=1}^{\infty}\sum_{n=1}^{\infty}\frac{1}{n}p_j^{-nx}\,\mathrm{Re}(3+4p_j^{-iny}+p_j^{-2iny})\\
&= \sum_{j=1}^{\infty}\sum_{n=1}^{\infty}\frac{1}{n}p_j^{-nx}\cdot(3+4\cos(ny\log p_j)+\cos(2ny\log p_j)).
\end{aligned}$$

As $3+4\cos\theta+\cos 2\theta = 2(1+\cos\theta)^2\geq 0$, $\log|h(x)|\geq 0$, and so, $|h(x)|\geq 1$. If $x>1$,

$$\frac{|h(x)|}{x-1}=|(x-1)\zeta(x)|^3\left|\frac{\zeta(x+iy)}{x-1}\right|^4\cdot|\zeta(x+2iy)|\geq\frac{1}{x-1}.$$

When $x\longrightarrow 1^+$, assuming that $\zeta(1+iy)=0$, last expression takes the value

$$1\cdot\left|\zeta'(1+iy)\right|^4\cdot|\zeta(1+2iy)|,$$

which is absurd. This gives the result. ■

In the theory of numbers, arithmetical functions are very often used. An arithmetic function is no more than a sequence, while sometimes we will use the notation $f(n), g(n),\dots$ instead of f_n, g_n. Given an arithmetic function $f(n)$, its sum $F:[1,+\infty)\to\mathbb{R}$ will be the function $F(x)=\sum_{n\leq x} f(n)$. It is a piecewise constant function, with discontinuities at (some of the) the natural numbers. An example of arithmetic functions we will use is Mangoldt function, defined as

$$\Lambda(n)=\begin{cases}\log p & \text{if } n=p^m,\ p \text{ a prime number},\\ 0 & \text{otherwise}.\end{cases}$$

Its sum is known as Chebyshev function,

$$\Psi(x)=\sum_{n\leq x}\Lambda(n)=\sum_{p\leq x}\left[\frac{\log x}{\log p}\right]\log p\leq \pi(x)\log x, \tag{3.9}$$

where

$$\pi(x)=\#\{p \text{ prime}\mid p\leq x\}.$$

Let us observe that $\pi(x)$ is also the sum of the arithmetic function that takes the value 1 when n is a prime and 0 otherwise.

Arithmetic functions and their sums satisfy a relation that we will call *Abel summation formula*, which reminds also integration by parts.

Theorem 3.5.5 — Abel summation. Let $f(n)$ be an arithmetic function, $F(x)$ its sum and $\varphi(x) \in \mathscr{C}^1([1,+\infty))$. Then

$$\sum_{n\leq x} f(n)\varphi(n) = F(x)\varphi(x) - \int_1^x F(t)\varphi'(t)dt.$$

Proof.

$$\begin{aligned}\sum_{n\leq x} f(n)(\varphi(x)-\varphi(n)) &= \sum_{n\leq x} f(n)\int_n^x \varphi'(t)dt \\ &= f(1)\int_1^x \varphi'(t)dt + f(2)\int_2^x \varphi'(t)dt + \cdots + f([x])\int_{[x]}^x \varphi'(t)dt \\ &= F(1)\int_1^2 \varphi'(t)dt + F(2)\int_2^3 \varphi'(t)dt + \cdots + F([x]-1)\int_{[x]-1}^{x-1} \varphi'(t)dt + F([x])\int_{[x]}^x \varphi'(t)dt \\ &= \int_1^x F(t)\varphi'(t)dt.\end{aligned}$$

■

Theorem 3.5.6 If $\mathrm{Re}(z) > 1$,

$$-\frac{\zeta'(z)}{\zeta(z)} = z\int_1^\infty \Psi(t)t^{-z-1}dt.$$

Proof. From the identity (3.8) we have

$$\begin{aligned}-\frac{\zeta'(z)}{\zeta(z)} &= \sum_{p \text{ prime}} \frac{\log p}{p^z-1} = \sum_{p \text{ prime}} \frac{p^{-z}\log p}{1-p^{-z}} = \sum_{p \text{ prime}} \log p \sum_{j=1}^\infty p^{-jz} \\ &= \sum_{j=1}^\infty \sum_{p \text{ prime}} \log p \cdot p^{-jz} = \sum_{n=1}^\infty \Lambda(n)n^{-z}\end{aligned}$$

As

$$\sum_{n=1}^M \Lambda(n)n^{-z} = \Psi(M)M^{-z} - \int_1^M \Psi(t)\cdot(-z)\cdot t^{-z-1}dt = \Psi(M)M^{-z} + z\int_1^M \Psi(t)t^{-z-1}dt.$$

If $M \longrightarrow +\infty$ and $\mathrm{Re}(z) > 1$,

$$\left|\frac{\Psi(z)}{M^z}\right| \leq \frac{M\log M}{M^{\mathrm{Re}(z)}} \longrightarrow 0,$$

and the result follows. ■

Lemma 3.5.7 The following are equivalent:

1. Prime Number Theorem, i.e.,

$$\lim_{x\to\infty} \frac{\pi(x)\log x}{x} = 1.$$

2. $\lim_{x\to\infty} \frac{\Psi(x)}{x} = 1$.

Proof. From (3.9) we obtain the inequality

$$\frac{\Psi(x)}{x} \le \frac{\pi(x)\log x}{x}.$$

If $1 < y < x$,

$$\pi(x) = \pi(y) + \sum_{y<p\le x} 1 \le \pi(y) + \sum_{y<p\le x} \frac{\log p}{\log y} \le \pi(y) + \frac{1}{\log y}(\Psi(x) - \Psi(y)) \le y + \frac{\Psi(x)}{\log y}.$$

Take $y = \frac{x}{(\log x)^2}$:

$$\pi(x) \le \frac{x}{(\log x)^2} + \frac{\Psi(x)}{\log x - 2\log\log x},$$

and then,

$$\frac{\pi(x)\log x}{x} \le \frac{1}{\log x} + \frac{\Psi(x)}{x} \cdot \frac{\log x}{\log x - 2\log\log x},$$

and the result follows taking limits, as

$$\limsup \pi(x)\frac{\log x}{x} \le \liminf \frac{\Psi(x)}{x} \le \limsup \frac{\Psi(x)}{x} \le \liminf \pi(x)\frac{\log x}{x}.$$

■

Lemma 3.5.8 $\frac{\Psi(x)}{x}$ is bounded above.

Proof. If $p \in (n, 2n]$, p a prime number, p divides $\binom{2n}{n}$. Then, we have that

$$\prod_{n<p\le 2n} p \le \binom{2n}{n} \le 2^{2n},$$

and then,

$$\sum_{n<p\le 2n} \log p \le 2n\log 2.$$

As a consequence,

$$\sum_{p\le 2^m} \log p = \sum_{k=1}^{m} \left(\sum_{2^{k-1}<p\le 2^k} \log p \right) \le \sum_{k=1}^{m} 2^k \log 2 \le 2^{m+1}\log 2.$$

When $\left[\frac{\log x}{\log p}\right] \ge 2$, $p \le \sqrt{x}$. The contribution of these terms to the sum defining $\Psi(x)$ is

$$\sum_{p\le\sqrt{x}} \left[\frac{\log x}{\log p}\right] \log p \le \pi(\sqrt{x})\log x.$$

For the rest of the terms, $\left[\dfrac{\log x}{\log p}\right] = 1$. Then,

$$\begin{aligned}\Psi(x) = \sum_{p\leq x}\left[\frac{\log x}{\log p}\right]\log p &\leq \pi(\sqrt{x})\log x + \sum_{p\leq 2^{m+1}}\log p \\ &\leq \pi(\sqrt{x})\log x + 2^{m+2}\log 2 \leq \pi(\sqrt{x})\log x + 4x\log 2 \leq \sqrt{x}\log x + 4x\log 2,\end{aligned}$$

having chosen $m\in\mathbb{N}$ such that $2^m < x \leq 2^{m+1}$. Dividing by x, we obtain

$$\frac{\Psi(x)}{x} \leq \frac{\log x}{\sqrt{x}} + 4\log 2,$$

which is bounded. ■

The following is a general result concerning Laplace transforms that will allow, applied to the integral representation of Theorem (3.5.6) to show the Prime Number Theorem.

Proposition 3.5.9 Let $f:[0,+\infty)\to\mathbb{C}$ be be a bounded, piecewise continuous function. Assume that the function

$$G(z) = \int_0^\infty f(t)e^{-tz}dt$$

can be analytically continued to a holomorphic function on an open set U containing the line $\mathrm{Re}(z) = 0$. Then, the integral

$$\int_0^\infty f(t)dt$$

exists and takes the value $G(0)$.

Proof. Let $\lambda > 0$, and $G_\lambda(z) = \int_0^\lambda f(t)e^{-tz}dt$. Assume that $|f(t)| \leq M$. We must show that $\lim_{\lambda\to\infty} G_\lambda(0) = G(0)$. Denote γ_R $(R > 0)$ the path of the Figure 3.4, where the vertical segment is on the line $\mathrm{Re}(z) = -\delta(R)$, and totally contained in U. Denote γ_R^+ the portion of γ_R contained in the halfplane $\mathrm{Re}(z) > 0$, and γ_R^- the rest. Write

$$G(0) - G_\lambda(0) = \frac{1}{2\pi i}\int_{\gamma_R}(G(z) - G_\lambda(z))e^{\lambda z}\left(\frac{1}{z} + \frac{z}{R^2}\right)dz.$$

If $|z| = R$, $\dfrac{1}{z} + \dfrac{z}{R^2} = \dfrac{2\,\mathrm{Re}(z)}{R^2}$.
If $\mathrm{Re}(z) > 0$,

$$|G(z) - G_\lambda(z)| \leq \int_\lambda^\infty Me^{-t\,\mathrm{Re}(z)}dt \leq M\frac{e^{-\lambda\,\mathrm{Re}(z)}}{\mathrm{Re}(z)}.$$

Then, on γ_R^+,

$$\left|(G(z) - G_\lambda(z))e^{\lambda z}\left(\frac{1}{z} + \frac{z}{R^2}\right)\right| \leq M\frac{e^{-\lambda\,\mathrm{Re}(z)}}{\mathrm{Re}(z)}e^{\lambda\,\mathrm{Re}(z)}\frac{2\,\mathrm{Re}(z)}{R^2} = \frac{2M}{R^2}.$$

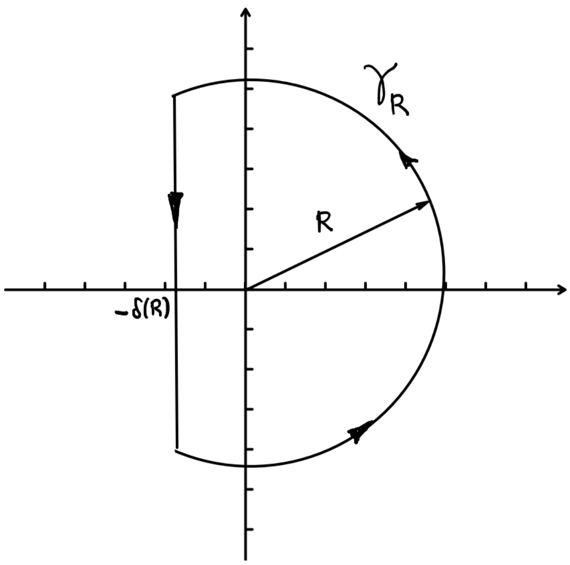


Figure 3.4: Path of integration γ_R

So, the first piece of the integral can be bounded by

$$|I_1| \leq \frac{1}{2\pi}\pi R \frac{2M}{R^2} = \frac{M}{R}.$$

To study the left hand side, let us divide the integral in two:

$$I_1(R) = \frac{1}{2\pi i}\int_{\gamma_R^-} G(z)e^{\lambda z}\left(\frac{1}{z}+\frac{z}{R^2}\right)dz; \qquad I_2(R) = \frac{1}{2\pi i}\int_{\gamma_R^-} G_\lambda(z)e^{\lambda z}\left(\frac{1}{z}+\frac{z}{R^2}\right)dz$$

For $I_2(R)$, as the integrand is an entire function, change the path of integration by the left half circle of radius R:

$$\begin{aligned}\left|G_\lambda(z)e^{\lambda z}\right| &\leq e^{\lambda \operatorname{Re}(z)} M \int_0^\lambda e^{-t\operatorname{Re}(z)}dt = e^{\lambda \operatorname{Re}(z)} M \frac{1-e^{-\lambda \operatorname{Re}(z)}}{\operatorname{Re}(z)} \\ &= M\frac{e^{\lambda \operatorname{Re}(z)}-1}{\operatorname{Re}(z)} = M\frac{1-e^{\lambda \operatorname{Re}(z)}}{|\operatorname{Re}(z)|} \leq \frac{M}{|\operatorname{Re}(z)|}.\end{aligned}$$

Then,

$$|I_2(R)| \leq \left|\frac{1}{2\pi}\int_{C_R^-} G_\lambda(z)e^{\lambda z}\left(\frac{1}{z}+\frac{z}{R^2}\right)dz\right| \leq \frac{1}{2\pi}\pi R\frac{M}{|\operatorname{Re}(z)|}\cdot\frac{2\,|\operatorname{Re}(z)|}{R^2} = \frac{M}{R}.$$

For $I_1(R)$, choose a bound $C(R)$ of $G(z)$ on γ_R^-, and $0 < \delta_1 < \delta(R)$. Consider the integral along the part of γ_R^- that verifies $\operatorname{Re}(z) \leq -\delta_1$. This integral is bounded by

$$\frac{1}{2\pi}C(R)e^{-\lambda\delta_1}\left(\frac{1}{\delta(R)}+\frac{1}{R}\right)\pi R, \tag{3.10}$$

which tends to 0 as λ tends to infinity (with R and δ_1 fixed). Along the remaining part of the path, a bound is

$$\frac{1}{2\pi}C(R)\frac{2\delta_1}{R^2}2R\arcsin\left(\frac{\delta_1}{R}\right). \tag{3.11}$$

Now, fixing $\varepsilon > 0$, take $R = \dfrac{4M}{\varepsilon}$, so $|I_4| \leq \dfrac{\varepsilon}{4}$, $|I_2| \leq \dfrac{\varepsilon}{4}$. Take also δ_1 such that (3.11) is $\leq \dfrac{\varepsilon}{4}$. Finally, take λ_0 such that if $\lambda \geq \lambda_0$, (3.10) is also $\leq \dfrac{\varepsilon}{4}$. The result follows. ■

Corollary 3.5.10 Let $f : [1,\infty) \longrightarrow [0,\infty)$ be a piecewise continuous increasing function, such that $f(x) = O(x)$. Then,

$$g(z) = z\int_1^\infty f(x)\frac{1}{x^{z+1}}dx$$

defines a holomorphic function on $\mathrm{Re}(z) > 1$. If there exists $c > 0$ such that the function $g(z) - \dfrac{c}{z-1}$ extends to a neighbourhood of $\mathrm{Re}(z) = 1$, then we have $\lim\limits_{x\to\infty} \dfrac{f(x)}{x} = c$.

Proof. There exists $K \in \mathbb{R}$ such that $f(x) \leq Kx$. Then,

$$|g(z)| \leq |z| \cdot \int_1^\infty Kx^{-\mathrm{Re}(z)}dx,$$

integral uniformly convergent in the compacts of the halfplane $\mathrm{Re}(z) > 1$. Let us define $F(t) = e^{-t}f(e^t) - c$ on $[0,\infty)$. As $|F(t)| \leq K + |c|$, $F(t)$ satisfies the hypothesis of Proposition 3.5.9 and then we may consider

$$G(z) = \int_0^\infty F(t)e^{-zt}dt,$$

which exists on $\mathrm{Re}(z) > 0$.

$$\begin{aligned} G(z) &= \int_1^\infty \left(\frac{f(x)}{x} - c\right)x^{-z}\frac{dx}{x} = \int_1^\infty \frac{f(x)}{x^{z+2}}dx - c\int_1^\infty \frac{1}{x^{z+1}}dx \\ &= \int_1^\infty \frac{f(x)}{x^{z+2}}dx - \frac{c}{z} = \frac{g(z+1)}{z+1} - \frac{c}{z} = \frac{1}{z+1}\left(g(z+1) - \frac{c}{z} - c\right), \end{aligned}$$

which can be extended by hypothesis to an open set containing $\mathrm{Re}(z) > 0$. As a consequence, the integral

$$\int_1^\infty \left(\frac{f(x)}{x} - c\right)\frac{dx}{x}$$

exists. This implies the result. Indeed, let $\varepsilon > 0$ be given. If there exists $x_0 \in \mathbb{R}$ such that $\dfrac{f(x_0)}{x_0} - c \geq 2\varepsilon$, as f is increasing, for any $x \geq x_0$ we have $f(x) \geq f(x_0) \geq x_0(c + 2\varepsilon) \geq x(c + \varepsilon)$, whenever $x_0 \leq x \leq x_0\dfrac{c+2\varepsilon}{c+\varepsilon}$. Then

$$\int_{x_0}^{x_0\frac{c+2\varepsilon}{c+\varepsilon}} \left(\frac{f(x)}{x} - c\right)\frac{dx}{x} \geq \int_{x_0}^{x_0\frac{c+2\varepsilon}{c+\varepsilon}} \frac{\varepsilon}{x}dx = \varepsilon\log\left(\frac{c+2\varepsilon}{c+\varepsilon}\right).$$

But convergence of the integral implies that

$$\int_{x_0}^{x_0\frac{c+2\varepsilon}{c+\varepsilon}} \left(\frac{f(x)}{x} - c\right)\frac{dx}{x} < \varepsilon\log\left(\frac{c+2\varepsilon}{c+\varepsilon}\right)$$

for x_0 big enough.

Then, we have shown that $\frac{f(x_0)}{x_0} - c \leq 2\varepsilon$ for x_0 big. On the other hand, if $\frac{f(x_0)}{x_0} - c \leq -2\varepsilon$, and $x \leq x_0$, we have $f(x) \leq f(x_0) \leq x_0(c - 2\varepsilon) \leq x(c - \varepsilon)$ if $x_0\frac{c-2\varepsilon}{c-\varepsilon} \leq x \leq x_0$. Then,

$$\int_{x_0\frac{c-2\varepsilon}{c-\varepsilon}}^{x_0} \left(\frac{f(x)}{x} - c\right)\frac{dx}{x} \leq \int_{x_0\frac{c-2\varepsilon}{c-\varepsilon}}^{x_0} -\frac{\varepsilon}{x}dx = \varepsilon\log\left(\frac{c-2\varepsilon}{c-\varepsilon}\right).$$

As before, the absolute value of this integral can be bounded above by

$$\left|\varepsilon\log\left(\frac{c-2\varepsilon}{c-\varepsilon}\right)\right| = \varepsilon\log\left(\frac{c-\varepsilon}{c-2\varepsilon}\right)$$

for x_0 big enough.

Collecting everything, it has been shown that, for big enough values of x_0 we have that

$$-2\varepsilon < \frac{f(x_0)}{x_0} - c < 2\varepsilon,$$

and consequently, $\lim_{x\to\infty}\frac{f(x)}{x} = c$. ■

Theorem 3.5.11 — Prime Number Theorem.

$$\pi(x) \sim \frac{x}{\log x}.$$

Proof. Take $f(x) = \Psi(x)$ in Corollary 3.5.10. As

$$-\frac{\zeta'(z)}{\zeta(z)} = z\int_1^\infty \Psi(x)x^{-z-1}dx,$$

and $-\frac{\zeta'(z)}{\zeta(z)} - \frac{1}{z-1}$ extends to an open set containing $\mathrm{Re}(z) = 1$ (by Proposition (3.5.4)!), the result can be applied and $\lim_{x\to\infty}\frac{\Psi(x)}{x} = 1$, which concludes the Theorem. ■

3.5.1 Complement: Bertrand's postulate

Ths distribution of Prime Numbers in the real line is a subject of great interest. For instance, it is easy to find gaps (intervals on the natural numbers) without primes, as big as we want: none of the numbers $n!+2, n!+3, \dots, n!+n$ is prime. It is unknown, however, if there are infinitely many twin primes (pairs of primes p, $p+2$).

As a complement, we will show here the following result:

Theorem 3.5.12 — Bertrand's postulate. If $n \in \mathbb{N}$, there is always a prime p with $n < p \leq 2n$ (of course, the last inequality is strict if $n \neq 1$).

Proof. We follow here the Erdös proof in [2]. For every $x \in \mathbb{R}$, we have the inequality $\prod_{p\leq x} p \leq 4^{x-1}$. It is enough to check it for prime x. When $x = 2$, the inequality reads $2 \leq 4$. For a prime $2m+1$,

$$\prod_{p\leq 2m+1} p = \prod_{p\leq m+1} p \cdot \prod_{m+1<p\leq 2m+1} p \leq 4^m\binom{2m+1}{m} \leq 4^m 2^{2m} = 4^{2m}.$$

The binomial number $\binom{2n}{n}$ contains the prime factor p exactly

$$\sum_{k\geq 1}\left(\left[\frac{2n}{p^k}\right]-2\left[\frac{n}{p^k}\right]\right) \text{ times.} \tag{3.12}$$

As

$$\left[\frac{2n}{p^k}\right]-2\left[\frac{n}{p^k}\right]<\frac{2n}{p^k}-2\left(\frac{n}{p^k}-1\right)=2,$$

each summand is at most 1. So, the number given by (3.12) is at most $\log_p 2n$. In fact, no prime p with $\frac{2n}{3}<p\leq n$ may divide $\binom{2n}{n}$, as $3p>2n$ implies that the only multiples of p that may appear in the numerator are p and $2p$, and p must appear twice in the denominator.

By induction it is easy to see that $\frac{4^n}{2n}\leq\binom{2n}{n}$. So, if $P(n)$ is the number of primes between n and $2n$, we have

$$\frac{4^n}{2n}\leq\binom{2n}{n}\leq\underbrace{\prod_{p\leq\sqrt{2n}}2n}_{\text{by the bound on (3.12)}}\cdot\prod_{\sqrt{2n}<p\leq\frac{2n}{3}}p\cdot\prod_{n<p\leq 2n}p<(2n)^{\sqrt{2n}}\cdot 4^{2n/3}\cdot(2n)^{P(n)}.$$

Then, $4^{n/3}<(2n)^{\sqrt{2n}+1+P(n)}$, and so,

$$P(n)>\frac{2n}{3\log_2(2n)}-(\sqrt{2n}+1)>\frac{2n-1}{3\log_2(2n)}-\frac{2n-1}{\sqrt{2n}-1}.$$

In order to see that this is strictly positive, it is enough to see that $3\log_2(2n)<\sqrt{2n}-1$. As $31>30$, this holds for $n=2^9$. Comparing the derivatives of $\sqrt{x}-1$ and $3\log_2 x$, we see that the inequality is true for $n\geq 2^9=512$.

It is now enough to verify it for $n\leq 511$, and for this, it is enough to check that the numbers $2,3,5,7,13,23,43,83,163,317,521$ are primes.

■

3.6 Exercises

1. Show that

$$\prod_{n=1}^{\infty}\frac{n(n+a+b)}{(n+a)(n+b)}=\frac{\Gamma(a+1)\Gamma(b+1)}{\Gamma(a+b+1)}.$$

2. Compute

$$\lim_{n\to\infty}\left(1+\frac{1}{3}+\cdots+\frac{1}{2n-1}-\frac{1}{2}\log n\right).$$

3. Show the following formulas:
   (a) If $z\in\mathbb{C}$, $\Gamma(\bar{z})=\overline{\Gamma(z)}$.
   (b) If $y\in\mathbb{R}$, $|\Gamma(iy)|^2=\dfrac{\pi}{y\sinh\pi y}$.
   (c) If $y\in\mathbb{R}$, $\left|\Gamma\left(\frac{1}{2}+iy\right)\right|^2=\dfrac{\pi}{\cosh\pi y}$. Deduce that $\lim_{y\to\infty}\Gamma\left(\frac{1}{2}+iy\right)=0$.

(d) For all x, $y \in \mathbb{R}$, we have that $|\Gamma(x+iy)| \leq |\Gamma(x)|$.

4. Prove the following equalities:
   (a) $\zeta(z)^2 = \sum_{n=1}^{\infty} \frac{d(n)}{n^z}$, where $d(n)$ denotes the number of divisors of n.
   (b) $\zeta(z)\zeta(z-1) = \sum_{n=1}^{\infty} \frac{\sigma(n)}{n^z}$, where $\sigma(n)$ is the sum of all divisors of n.
   (c) $\frac{\zeta(z-1)}{\zeta(z)} = \sum_{n=1}^{\infty} \frac{\varphi(n)}{n^z}$, where φ denotes Euler's phi function, defined as the cardinal of invertible elements in the ring $\mathbb{Z}/(n)$.
5. Use the change of variables $s = tu$ to show that

$$\Gamma(x)\Gamma(y) = \int_0^{\infty}\int_0^{\infty} t^{x-1}s^{y-1}e^{-(s+t)}dtds$$

   is equal to $\Gamma(x+y)B(x,y)$, where $B(x,y)$ is Euler's Beta Function, defined as

$$B(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}dt.$$

6. Evaluate $\int_{-1}^{1}(1-t^2)^{x-1}dt$ in two different ways to give an alternative proof of Lagrange Duplication Formula (3.4).
7. Let $a > 1$. Show that, for n big enough, there always exists at least one prime number p between n and an.

4. Entire Functions

In this chapter, the main properties of entire functions, regarding their growth, will be explained.

4.1 Growth of Entire Functions

Let $f \in \mathscr{O}(\mathbb{C})$. Let us denote $M(f,r) = \max\{|f(z)| \mid |z| = r\}$. We will say that $f(z)$ has growth order less or equal than ρ if there exists $C_1, C_2 > 0$ such that

$$M(f,r) \leq C_1 \exp(C_2 r^\rho).$$

Equivalently, there exists $r_0 > 0$ such that if $r \geq r_0$, $M(f,r) \leq \exp(C_2 r^\rho)$. Indeed, there exists some $A > 0$ such that $C_1 \exp(C_2 r^\rho) = \exp(A + C_2 r^\rho) \leq \exp(2C_2 r^\rho)$, whenever $A \leq C_2 r^\rho$, and this happens if $r \geq \left(\dfrac{A}{C_2}\right)^{1/\rho}$.

Conversely, if $M(f,r) \leq \exp(C_2 r^\rho)$ when $r \geq r_0$, taking

$$M = \max\{1\} \cup \{|f(z)| \mid |z| \leq r_0\},$$

we have that $M(f,r) \leq M \exp(C_2 r^\rho)$ for every $r \geq 0$.

Definition 4.1.1 The growth order of $f(z) \in \mathscr{O}(\mathbb{C})$ is defined as

$$\operatorname{ord}(f) = \inf\{\rho \mid f(z) \text{ has growth order smaller or equal than } \rho\}.$$

Example 4.1

1. $f(z) = e^z$. We have that $|e^z| = e^{\operatorname{Re}(z)} \leq e^r$ if $|z| = r$. Moreover, the bound is reached when $z = r$. So $M(f,r) = e^r$ and $\operatorname{ord}(f) = 1$.
2. $f(z) = \sin z = \dfrac{e^{iz} - e^{-iz}}{2i}$. As $|\sin(z)| \leq \dfrac{e^r + e^r}{2} = e^r$, then $M(f,r) \leq e^r$. But $|\sin(ir)| = \left|\dfrac{e^{-r} - e^r}{2i}\right| \geq \dfrac{e^r - e^{-r}}{2} \geq \dfrac{e^r - 1}{2}$. If we had that $M_f(r) \leq \exp(Cr^\rho)$ for certain $\rho < 1$, then $\dfrac{e^r - 1}{2} \leq e^{Cr^\rho}$ if $r \geq r_0$, which is impossible. So, $\operatorname{ord}(f) = 1$.

■

The growth order may be computed using the following result.

Proposition 4.1.1

$$\operatorname{ord}(f) = \limsup_{r\to\infty} \frac{\log\log M(f,r)}{\log r}.$$

Proof. Indeed, if $\operatorname{ord}(f) \le \rho$, $M_f(r) \le \exp(Cr^\rho)$ for r big enough. Then,

$$\frac{\log\log M(f,r)}{\log r} \le \frac{\log C + \rho\log r}{\log r} \xrightarrow{r\to\infty} \rho,$$

and consequently, $\limsup_{r\to\infty} \frac{\log\log M(f,r)}{\log r} \le \operatorname{ord}(f)$.

Conversely, denote $\rho' = \limsup_{r\to\infty} \frac{\log\log M_f(r)}{\log r}$ and assume $\rho' < \operatorname{ord}(f)$. If $\varepsilon > 0$ is such that $\rho' < \operatorname{ord}(f) - \varepsilon$, then we have that

$$\frac{\log\log M(f,r)}{\log r} \le \operatorname{ord}(f) - \varepsilon, \text{ for } r \ge r_0.$$

This implies that $M(f,r) \le \exp(r^{\operatorname{ord}(f)-\varepsilon})$, which is a contradiction. ■

■ **Example 4.2** Let $f_1, f_2 \in \mathscr{O}(\mathbb{C})$, with orders ρ_1, ρ_2 respectively.

1. $\operatorname{ord}(f_1+f_2) \le \max\{\operatorname{ord}(f_1), \operatorname{ord}(f_2)\}$, and $\operatorname{ord}(f_1\cdot f_2) \le \max\{\operatorname{ord}(f_1), \operatorname{ord}(f_2)\}$. Indeed, if $\rho := \max\{\operatorname{ord}(f_1), \operatorname{ord}(f_2)\}$, we have bounds $M(f_i,r) \le \exp(c_i r^{\rho+\varepsilon})$, for r big enough. If $C = \max\{C_1, C_2\}$, it is clear that
$$\begin{aligned} M(f_1+f_2,r) &\le 2\exp(Cr^{\rho+\varepsilon}), \text{ and} \\ M(f_1f_2,r) &\le \exp(2Cr^{\rho+\varepsilon}). \end{aligned}$$
2. If $\rho_1 < \rho_2$, then $\rho = \operatorname{ord}(f_1+f_2) = \rho_2$. For if $\rho < \rho_2$, as $f_2 = (f_1+f_2) - f_2$ we would have $\operatorname{ord}(f_2) < \rho_2$.

■

The growth order may also be computed from the Taylor expansion.

Proposition 4.1.2 Let $f(z) = \sum_{n=0}^\infty c_n z^n \in \mathscr{O}(\mathbb{C})$. Then, $\operatorname{ord}(f) \le \rho$ if and only if $|c_n| \le \left(\frac{C}{n}\right)^{n/\rho}$ for some $C > 0$.

Proof. As $f(z)$ is an entire function, we have that $\lim_{n\to\infty} \sqrt[n]{|c_n|} = 0$.

Assume that $M(f,r) \le K\exp(Ar^\rho)$. By Cauchy inequalities, we have that

$$|c_n| \le \frac{K\exp(Ar^\rho)}{r^n} = K\exp(Ar^\rho - n\log r).$$

Denote $\varphi(r) = Ar^\rho - n\log r$. It has a minimum at $r_0 = \left(\frac{n}{A\rho}\right)^{1/\rho}$, which verifies $A\rho r_0^\rho = n$. At this point we have that $\varphi(r_0) = \frac{n}{\rho} - \frac{n}{\rho}\log\left(\frac{n}{A\rho}\right)$, which implies

$$|c_n| \le K\exp\varphi(r_0) = K\left(\frac{eA\rho}{n}\right)^{n/\rho}.$$

Assuming $K \geq 1$, we have that

$$|c_n| \leq \left(\frac{eK^\rho A\rho}{n}\right)^{n/\rho}.$$

Conversely, assume that $|c_n| \leq \left(\frac{C}{n}\right)^{n/\rho}$. The we have

$$M(f,r) \leq |c_0| + \sum_{n=1}^{\infty} \left(\frac{C}{n}\right)^{n/\rho} r^n = |c_0| + \sum_{n=1}^{\infty} \left(\frac{C^{1/\rho} r}{n^{1/\rho}}\right)^n = |c_0| + \sum_{n=1}^{\infty} \frac{1}{2^n} \left(\frac{Ar}{n^{1/\rho}}\right)^n,$$

with $A = 2C^{1/\rho}$. The function $\varphi(x) = x \log\left(\frac{Ar}{x^{1/\rho}}\right)$ has a maximum when $x_0 = \frac{(Ar)^\rho}{e}$, and then

$$M(f,r) \leq |c_0| + \exp(\varphi(x_0)) \sum_{n=1}^{\infty} \frac{1}{2^n} = |c_0| + \left(\frac{Ar}{x_0^{1/\rho}}\right)^{x_0} = |c_0| + e^{Kr^\rho},$$

for $K = \frac{1}{\rho}\frac{1}{2}A^\rho$. As a result, $\mathrm{ord}(f) \leq \rho$. ■

Corollary 4.1.3

$$\mathrm{ord}(f) = \limsup_{n \longrightarrow \infty} \frac{n \log n}{\log\left(\frac{1}{|c_n|}\right)}.$$

(If $c_n = 0$, we take 0 as the value in the sequence).

Proof. If the growth order is smaller or equal than ρ, then $|c_n| \leq \left(\frac{C}{n}\right)^{n/\rho}$ for some $C > 0$. So, $\log\left(\frac{1}{|c_n|}\right) \geq \frac{n}{\rho}\log(n/C)$, and then

$$\frac{n \log n}{\log\left(\frac{1}{|c_n|}\right)} \leq \frac{\rho \log n}{\log(n/c)} \overset{n \longrightarrow \infty}{\longrightarrow} \rho.$$

So we deduce the inequality

$$\limsup_{n \to \infty} \frac{n \log n}{\log\left(\frac{1}{|c_n|}\right)} \leq \rho.$$

Conversely, assume that

$$\mu := \limsup_{n \to \infty} \frac{n \log n}{\log\left(\frac{1}{|c_n|}\right)} < \rho.$$

If $\varepsilon > 0$ verifies $\mu < \rho - \varepsilon$, then there exists $n_0 \in \mathbb{N}$ such that if $n \geq n_0$ we have

$$\frac{n \log n}{\log\left(\frac{1}{|c_n|}\right)} \leq \rho - \varepsilon.$$

After a small computation we obtain that

$$|c_n| \leq n^{-\frac{n}{\rho - \varepsilon}} = \left(\frac{1}{n}\right)^{\frac{n}{\rho - \varepsilon}},$$

which implies that $\mathrm{ord}(f) \leq \rho - \varepsilon$, which is a contradiction. ■

■ **Example 4.3** It is easy, using Corollary 4.1.3, to construct entire functions with prescribed growth order. For instance, taking $c_n = n^{-n/\rho}$, the growth order is ρ. The choice of $c_n = \frac{1}{(\log n)^n}$ gives a growth order of ∞, and if $c_n = e^{-n^2}$, it would be 0. ■

The following result will allow us to bound the modulus of a holomorphic function from a bound of its real part.

Theorem 4.1.4 — Borel-Carathéodory. Let $f(z) = \mathcal{O}(D(0;R)) \cap \mathcal{C}(\bar{D}(0;R))$. Denote, if $f(z) = u(z)+iv(z)$, $A(f,r) = \max\{u(z) |\ |z| = r\}$. Then, for $r < R$,

$$M(f,r) \le \frac{2r}{R-r}A(f,R) + \frac{R+r}{R-r}|f(0)|.$$

In particular, if $r \le \frac{R}{2}$, $M(f,r) \le 2A(f,R) + 3|f(0)|$.

Proof. If $f(z)$ is a constant $c = a+ib$, then $M(f,r) = |c|$ and $A(f,r) = a$. Then,

$$\frac{2r}{R-r}A(f,R) + \frac{R+r}{R-r}|f(0)| = \frac{2ra+(R+r)|c|}{R-r} \ge \frac{(R+r)|c|-2r|c|}{R-r} = |c|.$$

Assume now f non constant, but $f(0) = 0$. As $A(f,0) = 0$, then $A(f,r) > 0$ if $r > 0$. Denote $\Phi(z) = \frac{f(z)}{2A(f,R)-f(z)}$. It is well defined, as $\mathrm{Re}(2A(f,R)-f(z))) = 2A(f,R) - \mathrm{Re}(f(z)) \ge A(f,R) > 0$. If $f(z) = u(z)+iv(z)$, then

$$|\Phi(z)|^2 = \frac{u(z)^2+v(z)^2}{(2A(f,R)-u(z))^2+v(z)^2} < 1,$$

because, as $u(z) < 2A(f,R)-u(z)$ and $A(f,R) > 0$, we have

$$u(z)-2A(f,R) < u(z) < 2A(f,R)-u(z), \text{ hence, } u(z)^2 < (2A(f,R)-u(z))^2.$$

By Schwarz's Lemma 1.6.1, $|\Phi(z)| \le \frac{|z|}{R}$, which implies that

$$|f(z)| = \left|\frac{2A(f,R)\Phi(z)}{1+\Phi(z)}\right| \le \frac{2A(f,R)\frac{|z|}{R}}{1-\frac{|z|}{R}} = \frac{2A(f,R)|z|}{R-|z|},$$

and then, $M(f,R) \le \frac{2A(f,R)r}{R-r}$.

Consider now the general case, and let $g(z) = f(z)-f(0)$. Using previous case, we have bounds

$$|f(z)-f(0)| \le \frac{2rA(g,R)}{R-r},$$

and then

$$|f(z)| \le |f(0)| + \frac{2r(A(f,R)+|f(0)|)}{R-r} = \frac{2r}{R-r}A(f,R) + \frac{R+r}{R-r}|f(0)|,$$

as stated. ■

Example 4.4 1. Assume that $f(z) \in \mathscr{O}(\mathbb{C})$ and $|f(z)| \leq |z|^{\mu}$, for certain $\mu > 0$. Let us write the Taylor expansion of $f(z) = \sum_{n=1}^{\infty} c_n z^n$, where, by Cauchy inequalities, $|c_n| \leq \frac{M(f,r)}{r^n} \leq r^{\mu-n}$. If $n > \mu$, making $r \longrightarrow \infty$ we deduce that $c_n = 0$. So, $f(z)$ is a polynomial.

2. Let $P(z) = a_n z^n + \cdots + a_0 \in \mathbb{C}[z]$ be a degree n polynomial, and $f(z) = e^{P(z)}$. We have

$$|\operatorname{Re} P(z)| \leq |P(z)| \leq |a_n z^n| \left[1 + \left|\frac{a_{n-1}}{a_n}\right| \frac{1}{|z|} + \cdots + \left|\frac{a_0}{a_n}\right| \frac{1}{|z^n|}\right].$$

Given $\varepsilon > 0$, there exists $R_0 > 0$ such that if $|z| > R_0$, then $1 + \left|\frac{a_{n-1}}{a_n}\right| \frac{1}{|z|} + \cdots + \left|\frac{a_0}{a_n}\right| \frac{1}{|z^n|} \leq 1 + \varepsilon$.
Then, $|f(z)| \leq e^{\operatorname{Re} P(z)} \leq e^{|a_n|(1+\varepsilon)|z|^n}$, and $\operatorname{ord}(f(z)) \leq n$.
Now, write $a_k = |a_k|\, e^{i\theta_k}$, and

$$\operatorname{Re} P(z) = \operatorname{Re}\left[|a_n|\, e^{i(\theta_n + n\theta)} r^n + |a_{n-1}|\, e^{i(\theta_{n-1} + (n-1)\theta)} r^{n-1} + \cdots + |a_0|\, e^{i\theta_0}\right].$$

Choose z with $\theta = -\frac{\theta_n}{n}$, so we have

$$\begin{aligned} \operatorname{Re} P(z) &= |a_n|\, r^n + |a_{n-1}| \cos(\theta_{n-1} + (n-1)\theta) r^{n-1} + \cdots + |a_0| \cos(\theta_0) \\ &= |a_n|\, r^n \left[1 + \frac{|a_{n-1}|}{|a_n|} \frac{1}{r} \cos(\theta_{n-1} + (n-1)\theta) + \cdots + \frac{|a_0|}{|a_n|} \cos(\theta_0) \frac{1}{r^n}\right] \\ &\geq |a_n|\, r^n (1 - \varepsilon), \text{ for } r \text{ big enough.} \end{aligned}$$

Hence, $M(f,r) \geq \exp(|a_n|\, r^n (1-\varepsilon))$, and we deduce that $n = \operatorname{ord}(f(z))$.

3. Let $f(z) \in \mathscr{O}(\mathbb{C})$ with $u(z) \leq |z|^{\mu}$, so $A(f,r) \leq r^{\mu}$. Then,

$$M(f,r) \leq 2A(f,2r) + 3\,|f(0)| \leq 2^{\mu+1} r^{\mu} + 3\,|f(0)|.$$

As a result, $f(z)$ turns out to be a polynomial of degree at most μ.

4. Assume that $f(z) = e^{g(z)}$ is an entire function of finite order. Then,

$$|f(z)| = e^{\operatorname{Re}(g(z))} \leq e^{Ar^{\mu}}, \text{ for } r \text{ big enough.}$$

As $\operatorname{Re}(g(z)) \leq Ar^{\mu}$, $g(z)$ is a polynomial of degree at most μ.

■

From previous examples, we can obtain some results regarding entire functions of finite order.

Theorem 4.1.5 Let $f(z)$ be an entire function of finite order, which doesn't take the value a. Then, there exists a polynomial $P(z)$ with $f(z) = a + e^{P(z)}$.

Consequently, if $\operatorname{ord}(f(z)) \notin \mathbb{N}$, then $f(\mathbb{C}) = \mathbb{C}$.

Proof. Write $f(z) = a + e^{g(z)}$, for some $g(z) \in \mathscr{O}(\mathbb{C})$. Using Example 4.4, $g(z)$ must be a polynomial of degree n, and then, $\operatorname{ord}(f(z)) = n \in \mathbb{N}$. ■

Assume now that, $f(z)$ being of finite order, $f(z)$ does not take the values a and b. Then, $f(z) = a + e^{P(z)} = b$ has no solutions. This implies that $P(z)$ does not take the value $\log(b-a)$, so $P(z)$ must be constant. We have shown:

Theorem 4.1.6 — Little Picard Theorem, finite order case. A non-constant entire function of finite order takes all complex values with at most one exception.

Even more, we have:

Theorem 4.1.7 If $f(z)$ is an entire function of finite order, and takes the value a a finite number of times, then $f(z) = a + Q(z)e^{P(z)}$, with $P(z)$, $Q(z)$ polynomials.

Proof. If $f^{-1}(a) = \{z_1, z_2, \dots, z_n\}$ and $Q(z) = (z-z_1)^{k_1} \cdots (z-z_n)^{k_n}$, where $k_i = \nu(f(z)-a, z_i)$, then, an entire function $g(z)$ exists such that $f(z) = a + Q(z)e^{g(z)}$. As

$$\left|e^{g(z)}\right| \leq \frac{|f(z)|+|a|}{|z-z_1|^{k_1}\cdots|z-z_n|^{k_n}} \leq |f(z)|+|a|, \text{ if } |z| \geq 1+\max\{|z_i| \mid i=1,\dots,n\},$$

$e^{g(z)}$ has finite order, and then, $g(z) \in \mathbb{C}[z]$. ■

Consider now the situation in which $f(z)$ takes two values a and b a finite number of times.

Lemma 4.1.8 Let $P_1(z)$, $P_2(z)$, $Q_1(z)$, $Q_2(z)$ be polynomials with

$$P_1(z)e^{Q_1(z)} - P_2(z)e^{Q_2(z)} = C \neq 0, \text{ constant.} \tag{4.1}$$

Then, $Q_1(z)$ and $Q_2(z)$ are constants.

Proof. Derivating, $(P_1'(z)+P_1(z)Q_1'(z))e^{Q_1(z)} = (P_2'(z)+P_2(z)Q_2'(z))e^{Q_2(z)}$. If Q_1, Q_2 are not constants, $P_1'(z)+P_1(z)Q_1'(z) \neq 0$ and $P_2'(z)+P_2(z)Q_2'(z) \neq 0$, so $e^{Q_2(z)-Q_1(z)}$ must be a rational function. This implies that $Q_2(z) = Q_1(z)+k$, for certain $k \in \mathbb{C}$. Replacing in (4.1), $e^{Q_1(z)}(P_1(z)-P_2(z)e^K) = C \neq 0$, so $Q_1(z)$ and consequently, $Q_2(z)$, must be constants. ■

Theorem 4.1.9 Si $f(z) \in \mathscr{O}(\mathbb{C})$ has finite order, and takes two complex values $a \neq b$ a finite number of times, then $f(z)$ is a polynomial.

Proof. Indeed, $f(z) = a + Q_1(z)e^{P_1(z)} = b + Q_2(z)e^{P_2(z)}$, and $P_1(z)$, $P_2(z)$ must be constants, so $f(z) \in \mathbb{C}[z]$. ■

4.2 Relation between growth and zeros of the entire functions

In this section, we will state some results relating the growth order of an entire function $f(z)$ and the location of its zeros, if any. We shall begin with a technical result, Jensen's Theorem, which we will use in the sequel.

Assume that $f(z)$ is a holomorphic function defined on a open set containing the closed disk $\bar{D}(0;r)$. If $f(z)$ does not vanish, $\log|f(z)|$ is a harmonic function, so it verifies the mean value property and then,

$$\log|f(0)| = \frac{1}{2\pi}\int_0^{2\pi} \log\left|f(re^{it})\right| dt.$$

Jensen's Theorem provides an expression for $\log|f(0)|$ valid when $f(z)$ vanishes at some points.

Theorem 4.2.1 — Jensen's Theorem. Let $f(z) \in \mathscr{O}(D(0;R))$, $0 < r < R$, and $z_1, z_2, \dots, z_N$ the zeros of $f(z)$ on $\bar{D}(0;r)$, repeated according their multiplicities. If $f(0) \neq 0$, we have

$$\log|f(0)| + \log \prod_{n=1}^{N} \left(\frac{r}{|z_n|}\right) = \frac{1}{2\pi} \int_0^{2\pi} \log\left|f(re^{it})\right| dt.$$

Proof. It is clear that, if the result is true for two functions f_1, f_2 in the conditions of the statement, it is also true for $f_1 f_2$. We can write $f(z) = (z-z_1)\cdots(z-z_N)g(z)$, where $g(z)$ does not vanish. As the result is true for $g(z)$, it is enough to verify it for $z-a$, where $0 < |a| \leq r$.

So, we must show that

$$\log|a| + \log\left(\frac{r}{|a|}\right) = \frac{1}{2\pi}\int_0^{2\pi} \log\left|re^{it} - a\right| dt,$$

which is equivalent to

$$0 = \frac{1}{2\pi}\int_0^{2\pi} \log\left|e^{it} - a\right| dt, \text{ for every } 0 < |a| \leq 1.$$

If $|a| < 1$, take $F(z) = 1 - az = e^{G(z)}$ on $\bar{D}(0,1)$. We have that $\log|1-az| = \mathrm{Re}(G(z))$ is harmonic, and then,

$$0 = \frac{1}{2\pi}\int_0^{2\pi} \log\left|1 - ae^{it}\right| dt = \int_0^{2\pi} \log\left|e^{-it} - a\right| dt = \int_0^{2\pi} \log\left|e^{it} - a\right| dt.$$

Assume now $|a| = 1$. The result will be complete if we show that

$$\int_0^{2\pi} \log\left|1 - e^{it}\right| dt = 0.$$

If γ is the unit circle, this will be shown if we verify that

$$0 = \int_\gamma \frac{\log(1-z)}{z} dz = i\int_0^{2\pi} \log(1-e^{it}) dt,$$

where $\log(1-z)$ is a branch of the logarithm on $D(0,1)$ such that $\log(1) = 0$. Take a path $\Gamma = \gamma_1 + \gamma_2$ as in Figure 4.1. Over $\gamma_2(t) = 1 - \delta e^{it}$, $t \in \left(-\frac{\pi}{2} + \varepsilon, \frac{\pi}{2} - \varepsilon\right)$, we have

$$\left|\frac{\log(1-z)}{z}\right| = \left|\frac{\log(\delta e^{it})}{1-\delta e^{it}}\right| \leq \frac{|\log \delta| + \frac{\pi}{2}}{1-\delta},$$

so

$$\left|\int_{\gamma_2} \frac{\log(1-z)}{z} dz\right| \leq \frac{|\log \delta| + \frac{\pi}{2}}{1-\delta}\pi\delta \overset{\delta\to 0}{\longrightarrow} 0.$$

As $\displaystyle\int_\Gamma \frac{\log(1-z)}{z} dz = 0$, this implies the result, making $\delta \longrightarrow 0$. ■

Consider now a sequence $\{z_n\}_{n=1}^\infty$ of non-zero complex numbers, with $|z_n| \leq |z_{n+1}|$, and $r_n = |z_n|$. We know (Corollary 2.3.3) that, for conveniently chosen $\{p_n\}_{n=1}^\infty$, we have that the series $\displaystyle\sum_{n=1}^\infty \left(\frac{r}{r_n}\right)^{p_n+1}$ converges, so the infinite product $\displaystyle\prod_{n=1}^\infty E_{p_n}\left(\frac{z}{z_n}\right)$ defines an entire function with zeros at the points $\{z_n\}_{n=1}^\infty$. We want to optimise the value of p_n.

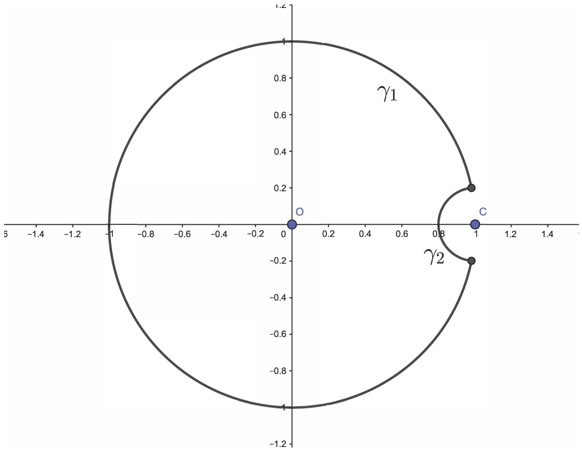


Figure 4.1: Path of integration Γ.

Definition 4.2.1 Let

$$S = \left\{ \lambda > 0 \,\middle|\, \sum_{n=1}^{\infty} \frac{1}{r_n^{\lambda}} < +\infty \right\}.$$

Define

$$\sigma = \begin{cases} \inf S & \text{if } S \neq \emptyset, \\ \infty & \text{if } S = \emptyset. \end{cases}$$

This number σ will be called the *exponent of convergence* of the sequence $\{z_n\}_{n=1}^{\infty}$. If $f \in \mathscr{O}(\mathbb{C})$, $f \neq 0$, the exponent of convergence of f will be the exponent of convergence of its set of zeros, and will be denoted $\sigma(f)$, or σ if no confusion arises.

Proposition 4.2.2

$$\sigma = \limsup_{n \to \infty} \frac{\log n}{\log r_n}.$$

Proof. Let us observe that, as the sequence $\{r_n\}_{n=1}^{\infty}$ tends to infinity, the expression in the limit of the statement make sense at least for n big enough. Denote α the value of the limit of the statement. For positive ε, there exists $n_0 \in \mathbb{N}$ such that if $n \geq n_0$, $\dfrac{\log n}{\log r_n} < \alpha + \varepsilon$, which implies that $\dfrac{1}{r_n^{\lambda}} < \dfrac{1}{n^{\frac{\lambda}{\alpha+\varepsilon}}}$. If $\lambda > \alpha + \varepsilon$, the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{r_n^{\lambda}}$ converges, which means that $\sigma \leq \lambda$. As this happens for every $\varepsilon > 0$, $\sigma \leq \alpha$.

Conversely, take $\lambda \in S$. As $\left\{ \dfrac{1}{r_n^{\lambda}} \right\}_{n \in \mathbb{N}}$ is a decreasing sequence, we have[1] that $\lim_{n \to \infty} \dfrac{n}{r_n^{\lambda}} = 0$, and so, for big enough n, $n < r_n^{\lambda}$. So, $\dfrac{\log}{\log r_n} < \lambda$, $\alpha \leq \lambda$, and then, $\alpha \leq \sigma$. ■

[1] If $\{a_n\}_{n=1}^{\infty}$ is a decreasing sequence, and the series $\sum_{n=1}^{\infty} a_n$ is convergent, then $\lim_{n \to \infty} n a_n = 0$.

Denote now, for $f(z) \in \mathscr{O}(\mathbb{C})$ and $r > 0$,

$$n_f(r) = \#\left(f^{-1}(0) \cap \bar{D}(0;r)\right),$$

each zero of f being counted according its multiplicity.

Theorem 4.2.3 If $f(z) \in \mathscr{O}(\mathbb{C})$ and $f(0) \neq 0$, then, for $r > 0$,

$$\log|f(0)| + \int_0^r \frac{n_f(t)}{t}dt \leq \log M(f,r).$$

Proof. After Jensen's Theorem 4.2.1,

$$\log|f(0)| + \log\prod_{n=1}^{N} \frac{r}{|z_n|} = \frac{1}{2\pi}\int_0^{2\pi} \log\left|f(re^{it})\right| dt$$

where $z_1, z_2, \dots, z_N$ are the zeros of f in $\bar{D}(0;r)$, and $N = n_f(r)$.

$$\begin{aligned}
\log\prod_{n=1}^{N} \frac{r}{|z_n|} &= \log\left[\frac{|z_2|}{|z_1|}\cdot\frac{|z_3|^2}{|z_2|^2}\cdot\frac{|z_4|^3}{|z_3|^3}\cdots\frac{|z_N|^{N-1}}{|z_{N-1}|^{N-1}}\cdot\frac{r^N}{|z_N|^N}\right] \\
&= \log\left|\frac{z_2}{z_1}\right| + 2\log\left|\frac{z_3}{z_2}\right| + \cdots + (N-1)\log\left|\frac{z_N}{z_{N-1}}\right| + N\log\left|\frac{r}{z_N}\right| \\
&= \int_{|z_1|}^{|z_2|}\frac{1}{t}dt + 2\int_{|z_2|}^{|z_3|}\frac{1}{t}dt + \cdots + N\int_{|z_N|}^{r}\frac{1}{t}dt = \int_0^r \frac{n_f(t)}{t}dt.
\end{aligned}$$

Hence,

$$\log|f(0)| + \int_0^r \frac{n_f(t)}{t}dt \leq \log M(f,r).$$

■

Corollary 4.2.4 If $a > 1$, $\log|f(0)| + \log(a)n_f(r) \leq \log M(f,ar)$.

Proof.

$$\log|f(0)| + \int_0^r \frac{n_f(t)}{t}dt + \int_r^{ar}\frac{n_f(t)}{t}dt \leq \log M(f,ar),$$

and

$$\int_r^{ar}\frac{n_f(t)}{t}dt \geq n_f(r)\int_r^{ar}\frac{1}{t}dt = n_f(r)\log(a).$$

Then, $\log|f(0)| + \log(a)n_f(r) \leq \log M(f,ar)$. ■

As a result, if $M(f,r) \leq C\exp(Ar^\rho)$, taking $a = e$ then

$$\log|f(0)| + n_f(r) \leq \log M(f,er) \leq \log C + Ae^\rho r^\rho.$$

Then, for certain constant $K > 0$, $n_f(r) \leq Kr^\rho$. If we define

$$\rho' = \inf\{\lambda \,|\, n_f(r) \leq Kr^\lambda\},$$

then $\rho' \leq \rho$, and hence, $\rho' \leq \operatorname{ord}(f)$.

Theorem 4.2.5 — Hadamard. $\sigma(f) \leq \operatorname{ord}(f)$.

Proof. If $r \gg 0$ and $\beta > \operatorname{ord}(f)$, then $n \leq n_f(r_n) \leq K r_n^\beta$. So, $\frac{1}{r_n^\beta} \leq \frac{K}{n}$. If $\lambda > \beta$, $\frac{1}{r_n^\lambda} \leq \left(\frac{K}{n}\right)^{\lambda/\beta}$, and then, $\sum_{n=1}^{\infty} \frac{1}{r_n^\lambda}$ converges.

Then, $\sigma \leq \lambda$, so $\sigma \leq \beta$, and finally, $\sigma \leq \operatorname{ord}(f)$. ■

R Previous inequality may be strict, as shown by the function e^z.

4.3 Canonical products and Hadamard Factorization Theorem

Let $\{z_n\}_{n=1}^\infty$ be a sequence in $\mathbb{C} \setminus \{0\}$, with $\lim_{n\to\infty} z_n = \infty$, and $|z_n| \leq |z_{n+1}|$. Let σ be the exponent of convergence of this sequence, as defined in 4.2.1. Define an integer p as follows:

$$p := \begin{cases} [\sigma] & \text{if } \sigma \notin \mathbb{Z}, \\ \sigma - 1 & \text{if } \sigma \in \mathbb{Z} \text{ and } \sum_{n=1}^\infty \frac{1}{|z_n|^\sigma} < \infty, \\ \sigma & \text{if } \sigma \in \mathbb{Z} \text{ and } \sum_{n=1}^\infty \frac{1}{|z_n|^\sigma} = \infty. \end{cases} \tag{4.2}$$

So, p is the smallest integer such that $\sum_{n=1}^{\infty} \frac{1}{|z_n|^{p+1}} < \infty$. Moreover, $p \leq \sigma \leq p+1$. Under previous conditions, the infinite product

$$\prod_{n=1}^\infty E_p\left(\frac{z}{z_n}\right)$$

defines an entire function $f(z)$ with zeros at the points of the sequence $\{z_n\}_{n=1}^\infty$, called *canonical product* of the sequence.

Proposition 4.3.1 $\operatorname{ord}(f(z)) = \sigma = \sigma(f)$.

Proof. We have the following bounds:

1. If $|z| \leq 1$, $|E_p(z)| \leq 1 + |z|^{p+1} \leq \exp(|z|^{p+1})$.
2. If $|z| \geq 1$,

$$\begin{aligned} |E_p(z)| &\leq (1+|z|)\exp\left(|z| + \frac{|z|^2}{2} + \cdots + \frac{|z|^p}{p}\right) \\ &\leq \exp(|z|)\exp\left(|z|^p\left(1 + \frac{1}{2} + \cdots + \frac{1}{p}\right)\right) \leq \exp(C_p|z|^p), \end{aligned}$$

where $C_p = 1 + 1 + \frac{1}{2} + \cdots + \frac{1}{p}$.

Let $\lambda \in [\sigma, p+1]$ be such that $\sum_{n=1}^\infty \frac{1}{|z_n|^\lambda} < \infty$. Using previous bounds, as $|z|^{p+1} \leq |z|^\lambda$ for $|z| \leq 1$ and $|z|^p \leq |z|^\lambda$ for $|z| \geq 1$, we have $|E_p(z)| \leq \exp(C_p|z|^\lambda)$. Then,

$$\left|\prod_{n=1}^\infty E_p\left(\frac{z}{z_n}\right)\right| \leq \prod_{n=1}^\infty \exp\left(C_p\left|\frac{z}{z_n}\right|^\lambda\right) = \exp\left(C_p\left(\sum_{n=1}^\infty \frac{1}{|z_n|^\lambda}\right)|z|^\lambda\right),$$

and $\operatorname{ord}(f) \leq \lambda$. By the choice of λ, $\operatorname{ord}(f) \leq \sigma$, and the result follows from Theorem 4.2.5. ■

Take now any entire function $f(z)$ of finite order $\operatorname{ord}(f)$, with zeros at the points $\{z_n\}_{n=1}^{\infty}$, $|z_n| \leq |z_{n+1}|$, and let $\sigma(f) \leq \operatorname{ord}(f)$ be the exponent of convergence of its zeros. Assume $f(0) \neq 0$. Defining p as in (4.2), we know from Theorem 2.3.5 that there exists $g(z) \in \mathscr{O}(\mathbb{C})$ such that

$$f(z) = e^{g(z)} \prod_{n=1}^{\infty} E_p\left(\frac{z}{z_n}\right). \tag{4.3}$$

We want to see that $g(z) \in \mathbb{C}[z]$ and to bound its degree. The following Lemma will allow us to bound from below Weierstrass' elementary factors.

Lemma 4.3.2
1. If $|z| \leq \frac{1}{2}$, then $|E_p(z)| \geq \exp(-2|z|^{p+1})$.
2. If $|z| \geq 2$, then $|E_p(z)| \geq \exp(-2|z|^p)$.

Proof. If $|z| \leq \frac{1}{2}$, we have

$$|E_p(z)| = |1 + (E_p(z) - 1)| \geq 1 - |z|^{p+1} \geq \exp(-2|z|^{p+1}), \tag{4.4}$$

because $1 - x \geq e^{-2x}$ if $x \in \left[0, \frac{1}{2}\right]$.

If $|z| \geq 2$, we have

$$\begin{aligned}
|E_p(z)| &\geq |1 - z| \exp\left(-|z| - \frac{|z|^2}{2} - \cdots - \frac{|z|^p}{p}\right) \\
&\geq (|z| - 1) \exp\left(-|z|^p \left(\frac{1}{p} + \frac{1}{(p-1)|z|} + \cdots + \frac{1}{|z|^{p-1}}\right)\right) \\
&\geq \exp\left(-|z|^p \left(1 + \frac{1}{2} + \cdots + \frac{1}{2^{p-1}}\right)\right) \geq \exp(-2|z|^p).
\end{aligned}$$

■

Theorem 4.3.3 — Hadamard's Factorization Theorem. In the expression (4.3), $g(z)$ is a polynomial of degree $q \leq [\operatorname{ord}(f)]$.

Proof. We have to bound $e^{g(z)}$. Take $R > 0$ and denote

$$F_R(z) = \prod_{|z_n| \leq 2R} E_p\left(\frac{z}{z_n}\right), \quad G_R(z) = \prod_{|z_n| > 2R} E_p\left(\frac{z}{z_n}\right).$$

Take z such that $|z| = R$. Using Lemma 4.3.2,

$$|G_R(z)| \geq \prod_{|z_n| > 2R} \exp\left(-2\left|\frac{z}{z_n}\right|^{p+1}\right) = \exp\left(-2 \sum_{|z_n| > 2R} \left(\frac{R}{|z_n|}\right)^{p+1}\right).$$

Now, $\dfrac{f(z)}{F_R(z)}$ is an entire function without zeros in $|z| \leq 2R$. If $|z| = R$,

$$\left|\frac{f(z)}{F_R(z)}\right| \leq \max\left\{\left|\frac{f(z)}{F_R(z)}\right| ; |z| = 4R\right\} \leq \frac{M(f, 4R)}{m(F_R, 4R)} \leq \frac{C\exp(A(4R)^{\rho+\varepsilon})}{m(F_R, 4R)}, \tag{4.5}$$

where $\rho = \operatorname{ord}(f)$, $\varepsilon > 0$ and

$$m(F_R, 4R) = \min\{|F_R(z)| \mid |z| = 4R\}.$$

When $|z| = 4R$,

$$|F_R(z)| = \left|\prod_{|z_n|\le 2R} E_p\left(\frac{z}{z_n}\right)\right| \ge \prod_{|z_n|\le 2R} \exp\left(-2\left|\frac{z}{z_n}\right|^p\right) = \exp\left(-2\sum_{|z_n|\le 2R}\left(\frac{4R}{|z_n|}\right)^p\right).$$

Then, (4.5) reads

$$\left|\frac{f(z)}{F_R(z)}\right| \le \frac{C\exp(A(4R)^{\rho+\varepsilon})}{\exp\left(-2\sum_{|z_n|\le 2R}\left(\frac{4R}{|z_n|}\right)^p\right)}.$$

Now, write $e^{g(z)} = \dfrac{f(z)}{F_R(z)} \cdot \dfrac{1}{G_R(z)}$. If $|z| = R$,

$$\left|e^{g(z)}\right| \le C\exp\left(A(4R)^{\rho+\varepsilon} + 2\sum_{|z_n|\le 2R}\left(\frac{4R}{|z_n|}\right)^p + 2\sum_{|z_n|>2R}\left(\frac{R}{|z_n|}\right)^{p+1}\right).$$

Choosing, as before, $\lambda \in [\sigma, p+1]$, we have that

$$\text{If } |z_n| > 2R, \text{ then } \frac{R}{|z_n|} < \frac{1}{2} \text{ and } \left(\frac{R}{|z_n|}\right)^{p+1} \le \left(\frac{R}{|z_n|}\right)^{\lambda},$$

$$\text{If } |z_n| \le 2R, \text{ then } \frac{R}{|z_n|} \ge 2 \text{ and } \left(\frac{4R}{|z_n|}\right)^{\lambda} \le \left(\frac{4R}{|z_n|}\right)^{\lambda}.$$

So, $\left|e^{g(z)}\right| \le \exp(A_1R^{\rho+\varepsilon} + A_2R^{\lambda})$, for certain constants A_1, A_2.

So, $g(z)$ turns out to be a polynomial of degree $q \le \max\{\operatorname{ord}(f)+\varepsilon, \lambda\}$. As $\varepsilon > 0$ is arbitrary and $\lambda \in [\sigma, p+1]$, we obtain that $q \le \max\{\operatorname{ord}(f), \sigma\} = \operatorname{ord}(f)$. As a consequence, $\operatorname{ord}(f) \le \max\{q, \sigma\} \le \operatorname{ord}(f)$, and then, $\operatorname{ord}(f) = \max\{q, \sigma\}$. ■

In a general situation, if $f(z) \in \mathscr{O}(\mathbb{C})$ has finite order, we can always express $f(z)$ as

$$f(z) = z^d e^{Q(z)} \prod_{n=1}^{\infty} E_p\left(\frac{z}{z_n}\right),$$

where $p \le \operatorname{ord}(f)$ and $\deg Q(z) \le \operatorname{ord}(f)$.

Corollary 4.3.4 If $\operatorname{ord}(f(z)) \notin \mathbb{Z}$, then $\sigma = \operatorname{ord}(f)$.

■ **Example 4.5** Let $f(z) = \dfrac{\sin(\pi\sqrt{z})}{\pi\sqrt{z}}$, $\operatorname{ord}(f) = 1/2$, with zeros on $\{n^2\}_{n=1}^{\infty}$. So, $\sigma = 1/2$, $p = 0$ and we can write

$$f(z) = \prod_{n=1}^{\infty}\left(1 - \frac{z}{n^2}\right).$$

■

■ **Example 4.6** If $f(z)$ has order $\rho \notin \mathbb{Z}$, we already know that $f(z)$ must take every complex value a. Under these conditions, the exponent of convergence of the zeros of f, σ, coincides with the order of convergence of its $a-$values, σ_a. Indeed, $\sigma = \operatorname{ord}(f) = \operatorname{ord}(f-a) = \sigma_a$. ■

Another consequence of Hadamard Factorization Theorem 4.3.3 is an improvement of the bound on the order of the product, stated in Example 4.2.

Proposition 4.3.5 Let $f(z),\ g(z) \in \mathscr{O}(\mathbb{C})$ be of finite order, with $\rho(f) \neq \rho(g)$. Then, $\rho(f\cdot g) = \max\{\rho(f), \rho(g)\}$.

Proof. Assume that $\rho(f) > \rho(g)$ and $\rho(f\cdot g) < \rho(f)$. Write the canonical decomposition of both functions as

$$f(z) = z^{h_f} e^{P_f(z)} \Pi_f(z); \quad g(z) = z^{h_g} e^{P_g(z)} \Pi_g(z),$$

with P_f, P_g polynomials, Π_f, Π_g canonical products. As $\sigma(f) \le \sigma(f\cdot g) \le \rho(f\cdot g) < \rho(f)$, it turns out that $\rho(f) = \deg P_f(z) \in \mathbb{Z}$, and $\deg P_g(z) \le \rho(g) < \rho(f) = \deg P_f(z)$. We have

$$f\cdot g(z) = z^{h_f+h_g} e^{P_f+P_g} \Pi_f \cdot \Pi_g.$$

$\Pi_f \cdot \Pi_g$ is not a canonical product, but $\sigma(\Pi_f \cdot \Pi_g) = \sigma(f\cdot g) < \rho(f)$ and $\rho(\Pi_f \cdot \Pi_g) \le \max\{\sigma(f), \sigma(g)\} < \rho(f)$. We can decompose

$$\Pi_f(z) \cdot \Pi_g(z) = e^{P_{f\cdot g}(z)} \Pi_{f\cdot g}(z), \text{ and } \rho(\Pi_f \cdot \Pi_g) = \max\{\deg(P_{f\cdot g}), \sigma(f\cdot g)\}.$$

Then, $\deg(P_{f\cdot g}) < \deg P_f$ and

$$f\cdot g(z) = z^{h_f+h_g} e^{P_f+P_g+P_{f\cdot g}} \Pi_{f\cdot g}$$

is the canonical decomposition of $f\cdot g$. So, $\rho(f\cdot g) = \max\{\deg(P_f+P_g+P_{f\cdot g}), \sigma(f\cdot g)\} = \rho(f)$, in contradiction with the assumptions. Hence, $\rho(f\cdot g) = \max\{\rho(f), \rho(g)\}$. ■

4.4 Picard's Theorem

We have shown (Theorem 4.1.6) that every non constant entire function of finite order avoids at most one complex value. We want to extend this result to a general entire function, without the restriction of being of finite order. There are different proofs of this result available in the literature. The most geometric one uses the construction of the universal covering of the open set $\mathbb{C}\setminus\{0,1\}$. It will be sketched in Section 4.5. A direct proof uses Bloch's Theorem, and will not be treated here. A third proof relies on the properties of hermitian metrics, as stated in Section 1.8. We will show it in this section.

The main result will be:

Theorem 4.4.1 The map

$$\mu(z) = \left[\frac{(1+|z|^{1/3})^{1/2}}{|z|^{5/6}}\right] \cdot \left[\frac{(1+|z-1|^{1/3})^{1/2}}{|z-1|^{5/6}}\right]$$

defines a metric on $\mathbb{C}\setminus\{0,1\}$, whose curvature verifies $\kappa_\mu(z) \le -B < 0$.

Proof. Denote $g(z) = \dfrac{(1+|z|^{1/3})^{1/2}}{|z|^{5/6}}$, so that $\mu(z) = g(z)g(z-1)$.

$\log\mu(z) = \log g(z) + \log g(z-1)$, and $\log g(z) = \frac{1}{2}\log(1+|z|^{1/3}) - \frac{5}{6}\log(|z|)$. As $\log|z|$ is harmonic, $\Delta(\log g(z)) = \frac{1}{2}\Delta(\log(1+|z|^{1/3}))$.

$$\frac{\partial}{\partial z}\left[\frac{1}{2}\log(1+|z|^{1/3})\right] = \frac{1}{12}\cdot\frac{z^{-5/6}\bar{z}^{1/6}}{1+|z|^{1/3}},$$

$$\frac{\partial^2}{\partial\bar{z}\partial z}\left[\frac{1}{2}\log(1+|z|^{1/3})\right] = \frac{1}{12}\,\frac{\frac{1}{6}(1+|z|^{1/3})z^{-5/6}\bar{z}^{-5/6} - \frac{1}{6}z^{-5/6}\bar{z}^{1/6}z^{1/6}\bar{z}^{-5/6}}{(1+|z|^{1/3})^2} = \frac{1}{72}\cdot\frac{z^{-5/6}\bar{z}^{-5/6}}{(1+|z|^{1/3})^2}.$$

So,

$$-\Delta(\log g(z)) = -\frac{1}{18}\cdot\frac{|z|^{-5/3}}{(1+|z|^{1/3})^2}.$$

Then,

$$-\frac{\Delta(\log g(z))}{\mu(z)^2} = -\frac{1}{18}\cdot\frac{|z|^{-5/3}}{(1+|z|^{1/3})^2}\cdot\frac{|z|^{5/3}\cdot|z-1|^{5/3}}{(1+|z|^{1/3})(1+|z-1|^{1/3})}$$
$$= -\frac{1}{18}\cdot\frac{|z-1|^{5/3}}{(1+|z|^{1/3})^3\cdot(1+|z-1|^{1/3})} =: \kappa_1(z).$$

Note that $\kappa_1(z)<0$, and $\lim_{z\to 0}\kappa_1(z) = -\frac{1}{36}$, $\lim_{z\to 1}\kappa_1(z)=0$, and $\lim_{z\to\infty}\kappa_1(z) = -\infty$.

Similarly,

$$-\frac{\Delta(\log(g(z-1)))}{\mu(z)^2} = -\frac{1}{18}\cdot\frac{|z|^{5/3}}{(1+|z-1|^{1/3})^3\cdot(1+|z|^{1/3})} =: \kappa_2(z),$$

and $\lim_{z\to 0}\kappa_2(z)=0$, $\lim_{z\to 1}\kappa_2(z) = -\frac{1}{36}$, $\lim_{z\to\infty}\kappa_2(z) = -\infty$. As a result, $\kappa_\mu(z)<0$, $\lim_{z\to 0}\kappa_\mu(z) = \lim_{z\to 1}\kappa_\mu(z) = -\frac{1}{36}$, $\lim_{z\to\infty}\kappa_\mu(z) = -\infty$, which concludes the computation.

■

Theorem 4.4.2 — Picard's Theorem, general case. Let $f\in\mathscr{O}(\mathbb{C})$ a function avoiding at least two values a,b. Then, f is constant.

Proof. Defining $g(z) = \frac{f(z)-a}{b-a}$ we can assume that f avoids $\{0,1\}$. Consequence 2 after Corollary 1.8.6 gives the result. ■

4.5 Construction of a modular function. Picard's Theorem revisited

In this section we will provide (sometimes only sketched) a different proof of Picard's Theorem, constructing a universal covering of $\mathbb{C}\setminus\{0,1\}$. This will be done using what is called a modular function. We shall follow mainly [19] in this Section. For the construction of this object, we need to make use of Schwarz Reflection Principle:

Theorem 4.5.1 — Schwarz Reflection Principle. Let $U^+\subseteq\mathbb{H}$ be a domain, I an open interval on $\mathbb{R}$ such that for every $x_0\in I$, there exists $r(x_0)>0$ with $D(x_0;r(x_0))\cap\mathbb{H}\subseteq U^+$ (in particular, I is contained in the boundary of U^+). Let $f\in\mathscr{O}(U^+)$, continuous on $U^+\cup I$, and taking real values on I. Let U^- be the symmetric of U^+ ($U^- = \{z\in\mathbb{C}\,|\,\bar{z}\in U^+\}$), and $U = U^+\cup I\cup U^-$. Then, there exists $F\in\mathscr{O}(U)$ with $F|_{U^+} = f$, i.e., f can be analytically continued to U.

Sketch of proof. Define $F(z)$ as $f(z)$ on $U^+\cup I$ and $\overline{f(\bar{z})}$ on U^-. It is continuous on U, holomorphic on $U^+\cup U^-$. You can easily check that if T is a triangle contained in U, $\int_{\partial T} F\,dz = 0$, so F is analytic. ■

This result can be generalized to a reflection on a circle. Recall from Section 1.2.4 that if $\mathscr{C}$ is a circle and T a Möbius transformation, $I_{T(\mathscr{C})} \circ T(z) = T \circ I_{\mathscr{C}}(z)$, $I_{\mathscr{C}}$ denoting the inversion with respect to $\mathscr{C}$. So, assume that U^+ is a domain "on one side" of $\mathscr{C}$ (i.e., contained in one connected component H^+ of $\bar{\mathbb{C}} \setminus \mathscr{C}$), I an open set on $\mathscr{C}$, such that for every $x_0 \in I$, $D(x_0; r(x_0)) \cap H^+ \subseteq U^+$, for some $r(x_0) > 0$. Let $U^- = I_{\mathscr{C}}(U^+)$, $U = U^+ \cup I \cup U^-$. Any function $f \in \mathscr{O}(U^+) \cap \mathscr{C}(U^+ \cup I)$, taking real values on I, can be analytically extended to U, by symmetry on $\mathscr{C}$.

Denote now

$$\Omega = \{z \in \mathbb{H} \,|\, 0 < \mathrm{Re}(z) < 1,\ \left|z - \frac{1}{2}\right| > \frac{1}{2}\}.$$

It is a simply connected set, so there is a conformal map $\lambda : \Omega \longrightarrow \mathbb{H}$. It can be shown that λ can be extended continuously to the simple points of the boundary, i.e., to the points of $\partial\Omega \setminus \{0,1\}$, and that it takes real values there[2]. Consider the corners of the boundary of Ω, 0, 1, ∞, and take $F(z) = -\cos \pi z$, which is a homeomorphism between the closed band

$$B = \{\mathrm{Im}(z) \geq 0,\ 0 \leq \mathrm{Re}(z) \leq 1\}$$

and $\bar{\mathbb{H}}$, and a biholomorphism between the interiors of these sets. As $\lim\limits_{\substack{z \longrightarrow \infty \\ z \in B}} F(z) = \infty$, $G(z) = \dfrac{1}{F(z)}$ sends ∞ to 0, and the vertical lines $\mathrm{Re}\, z = 0, 1$ onto the intervals $[-1, 0)$ and $(0, 1]$. A conformal map ξ between $G(\Omega)$ and $\mathbb{H}$ is continuous at 0, 0 being a simple point of the boundary. As λ is a composition $\xi \circ G$, λ turns out to be continuous at ∞. The map $\dfrac{1}{1-z}$ is an automorphism of Ω sending $\{0, 1, \infty\}$ to $\{1, \infty, 0\}$, so, by symmetry, $\lambda(z)$ turns out to be continuous on $\{0, 1, \infty\}$. λ can moreover be taken such that $\lambda(\infty) = 0$, $\lambda(0) = 1$, $\lambda(1) = \infty$. So, finally, $\lambda : \Omega \setminus \{0,1\} \longrightarrow \bar{\mathbb{H}} \setminus \{0,1\}$ is a continuous function, bijective, biholomorphic in the interior ($\lambda'(z) \neq 0$), and has analytic continuation through every side of the boundary, having non-zero derivative there.

Denote now I_0, $I_{1/2}$, I_1 the inversions through the sides $\mathrm{Re}(z) = 0$, $\left|z - \dfrac{1}{2}\right| = \dfrac{1}{2}$, and $\mathrm{Re}(z) = 1$, respectively, of Ω. Our objective is to extend λ to a function on $\mathbb{H}$. Since it is extended to the band $B \cap \mathbb{H}$, and is real on the boundary, it can be extended using the reflection I_1 to $\{z \in \mathbb{H} \,|\, 1 \leq \mathrm{Re}(z) \leq 2\}$, and similarly to the rest of bands filling $\mathbb{H}$. As $I_1 \circ I_0(z) = z + 2$, this extension turns out to be periodic of period 2.

So, the objective is to extend $\lambda(z)$ to the band B. Denote $F_0 = \bar{\Omega} \cap \mathbb{H}$, F_1 the symmetric of F_0 with respect to C_0, the circle $\left|z - \frac{1}{2}\right| = \frac{1}{2}$. It is a "triangle" with sides the upper part of C_0, and the upper half circles $\left|z - \frac{1}{4}\right| = \frac{1}{4}$, $\left|z - \frac{3}{4}\right| = \frac{1}{4}$. Call $C_1 = C_1^{(1)} \cup C_1^{(2)}$ these two sides.

After inversion through $C_1^{(1)}$ and $C_1^{(2)}$, F_1 is transformed in two new smaller triangles $F_2^{(1)} \cup F_2^{(2)} = F_2$ and four new sides $C_2 = C_2^{(1)} \cup C_2^{(2)} \cup C_2^{(3)} \cup C_2^{(4)}$. We can continue this way, creating at each step a new set F_n, union of 2^{n-1} triangles, bounded above by the 2^{n-1} sides C_{n-1} and below by the 2^n sides C_n. If $d_{n-1}$is the diameter of one of the circles in step $n-1$, and d_n of one if its inside circles in step n, after reflection two new circles appear inside this last one, of diameters

$$d_{n+1}^{\pm} = \frac{d_n}{2} \pm \frac{\left(\frac{1}{2}d_n\right)^2}{d_{n-1} - \frac{1}{2}d_n}. \tag{4.6}$$

[2]This result is not shown in these Notes. It can be read in [7, 4, 18], among others

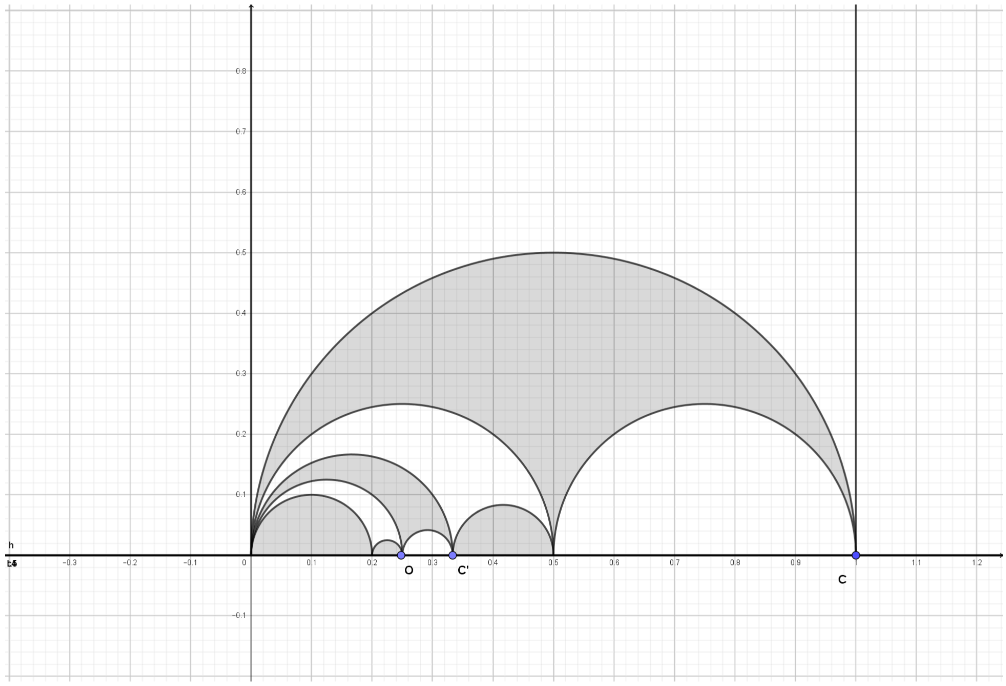


Figure 4.2: Hyperbolic triangles

Indeed, we have that $\overline{OC'} \cdot \overline{OC} = \left(\frac{d_n}{2}\right)^2$ in Figure 4.2. As $\overline{OC} = \dfrac{d_n}{2} + d_{n-1} - d_n$, this gives

$$\overline{OC'} = \frac{\left(\frac{d_n}{2}\right)^2}{\frac{d_n}{2} + d_{n-1} - d_n} = \frac{\left(\frac{1}{2}d_n\right)^2}{d_{n-1} - \frac{1}{2}d_n}.$$

So, $d_{n+1}^{+} = \frac{d_n}{2} + \overline{OC'}$, $d_{n+1}^{-} = \frac{d_n}{2} - \overline{OC'}$, and we get the diameters given by (4.6). The maximal diameter at step n is $\frac{1}{n}$. Indeed, if $n = 1$ we have one half-circle of diameter 1, and if $n = 2$, two half-circles of diameters $\frac{1}{2}$. By induction, if $d_n = \frac{1}{n}$ and $d_{n-1} = \frac{1}{n-1}$, formula (4.6) gives $d_{n+1}^{+} = \frac{1}{n+1}$.

As a result, $\bigcup_{n=1}^{\infty} \bar{F}_n = B$, because $\{z \in B \,|\, \mathrm{Im}(z) \geq \frac{1}{n+1}\} \subseteq B \setminus F_{n+1}$. By succesive reflections, we can extend λ to the band B and hence, to $\mathbb{H}$. We have the a holomorphic function $\lambda : \mathbb{H} \longrightarrow \mathbb{C} \setminus \{0,1\}$, sending F_0 biholomorphically to $\bar{\mathbb{H}} \setminus \{0,1\}$, F_1 to $\overline{-\mathbb{H}} \setminus \{0,1\}$, and in general, each $F_n^{(k)}$ ($k = 1,2,\ldots,2^{n-1}$) to $\overline{(-1)^n\mathbb{H}} \setminus \{0,1\}$.

If $I_{\mathscr{C}}$ is one of the inversions used in the construction, $\lambda \circ I_{\mathscr{C}}(z) = \overline{\lambda(z)}$. Then, if T is a Möbius transformation which is a product of an even number of such inversions, $\lambda \circ T(z) = \lambda(z)$. Denote G the group of those Möbius transformations: it is a subgroup of $\mathrm{Aut}(\mathbb{H})$, so contained in $\mathrm{SL}(2,\mathbb{R})$ (see Section 1.7). Let us remark, for instance, that if $C_1^{(1)}$ is the circle $\left|z - \dfrac{1}{4}\right| = \dfrac{1}{4}$, then $C_1^{(1)} = I_{1/2}(\mathrm{Re}(z) = 0)$ and then, $I_{C_1^{(1)}} = I_{1/2} \circ I_0 \circ I_{1/2}$. Proceeding this way, every inversion $I_{\mathscr{C}}$ turns out to be the composition of an odd number of I_0, $I_{1/2}$, I_1. Then G is a subgroup generated by

$$\sigma(z) = I_1 \circ I_0(z) = z + 2, \text{ and } \tau(z) = I_{1/2} \circ I_0(z) = \frac{z}{2z+1}.$$

Indeed, it is precisely the group

$$G = \left\{ \frac{az+b}{cz+d} \,|\, a,b,c,d \in \mathbb{Z},\ a,b \text{ odd}, b,c \text{ even }, ad - bc = 1 \right\},$$

called the modular group. In [18], more details can be found.

We have then shown:

Theorem 4.5.2 There exists a surjective holomorphic function $\lambda : \mathbb{H} \longrightarrow \mathbb{C} \setminus \{0,1\}$ such that:

1. λ is conformal: $\lambda'(z) \neq 0 \forall z \in \mathbb{H}$.
2. λ is a biholomorphism between each $\overset{\circ}{F_n^{(k)}}$ and $(-1)^n\mathbb{H}$.
3. $\lambda(z_1) = \lambda(z_2)$ if and only if there exists $T \in G$ such that $T(z_1) = z_2$, where G is the modular group described above.

Corollary 4.5.3 $\lambda : \mathbb{H} \longrightarrow \mathbb{C} \setminus \{0,1\}$ is a covering map. In fact, it is the universal covering map of $\mathbb{C} \setminus \{0,1\}$.

With this result in hand, let us provide a different proof of Picard's Little Theorem:

Theorem 4.5.4 — Picard. If $f \in \mathscr{O}(\mathbb{C})$ avoids two points a, b, then it is constant.

Proof. As before, we can suppose that $\{a,b\} = \{0,1\}$, so we have an entire function $f : \mathbb{C} \longrightarrow \mathbb{C} \setminus \{0,1\}$. There is a lifting $\tilde{f} : \mathbb{C} \longrightarrow \mathbb{H}$ of f, such that $\lambda \circ \tilde{f} = f$. By Liouville's Theorem, $\tilde{f}$ is constant, so it is f. ■

Previous construction allows to prove much stronger results.

Definition 4.5.1 A family of holomorphic functions on a open set U is normal if every sequence has, either a subsequence uniformly convergent in compact sets, or a subsequence converging uniformly to infinity.

Theorem 4.5.5 — Montel's Big Theorem. Let $\mathscr{F}$ be a family of meromorphic functions on a domain U, which simultaneously avoid 3 points a,b,c. Then, $\mathscr{F}$ is normal.

Proof. It is enough to show the result when U is a disk, as every open set is a countable union of disks. As before, composing with a Möbius transformation, we can suppose that $\{a,b,c\} = \{0,1,\infty\}$. So, $\mathscr{F} \subseteq \mathscr{O}(U)$, and if $f \in \mathscr{F}$, $f(U) \subseteq \mathbb{C} \setminus \{0,1\}$. Take a sequence $\{f_n\}_{n=1}^{\infty}$ on $\mathscr{F}$, and consider their liftings $\tilde{f}_n : U \longrightarrow \mathbb{H}$, such that $\lambda \circ \tilde{f}_n = f_n$. By Montel's Theorem 1.9.9, there exists a subsequence $\tilde{f}_{n_k}$ converging uniformly in the compact sets of U to $\tilde{f} \in \mathscr{O}(U)$. Then, $\lambda \circ \tilde{f}_{n_k} = f_{n_k}$ converges uniformly in compact sets to $f = \lambda \circ \tilde{f}$. ■

Theorem 4.5.6 — Picard's Big Theorem. Assume that z_0 is an essential singularity for a holomorphic function f. Then, in every neighbourhood of z_0, f takes every complex value with at most one exception.

Proof. By contradiction, after rescaling we can assume that $f \in \mathscr{O}(D \setminus \{0\})$, $D = D(0;1)$, 0 is an essential singularity for f, and f does not take the values $0,1$. Consider the family

$$\{f_n(z) = f\left(\frac{z}{n}\right) \mid n \in \mathbb{N}\}$$

on $D \setminus \{0\}$. By Montel's Big Theorem 4.5.5, $\{f_n\}_{n=1}^{\infty}$ is normal, so it has a subsequence $\{f_{n_k}\}_{k=1}^{\infty}$ which, either converges uniformly in compact sets, or diverges uniformly to infnity.

In the first case, the family $\{f_{n_k}\}_{k=1}^{\infty}$ must be bounded in the compact set $|z| = 1/2$ by a constant M. So $|f(z)| \leq M$ if $|z| = \dfrac{1}{2n_k}$. By the Maximum Modulus Principle, $|f(z)| \leq M$ between two such circles, so $|f(z)| \leq M$ on $D \setminus \{0\}$, which is impossible.

In the second case, the same argument applied to $\dfrac{1}{f}$ shows that $\lim_{z\to 0} \dfrac{1}{f(z)} = 0$, so $f(z)$ must have a pole at 0, and again we have a contradiction. ■

4.6 Exercises

1. Find the zeros of the entire function $f(z) = e^{e^z} - 1$, and compute the exponent of convergence.
2. If $a > 1$, compute the growth order of

$$f(z) = \prod_{n=1}^{\infty} \left(1 + \frac{z}{n^a}\right)$$

3. Show that if $\lambda \neq 0$ and $P(z) \in \mathbb{C}[z]$, then the equation $e^{\lambda z} - P(z) = 0$ has infinity many zeros.
4. Suppose that $f(z)$, $g(z)$ are entire functions or respective orders ρ, ρ', with $\rho' \leq \rho$. Assume that the zeros of g are also zeros of f. Show that $f(z)/g(z)$ is of order at most ρ.
5. (a) Let f, g be entire functions of finite order λ, and suppose that $f(a_n) = g(a_n)$ for a sequence $\{a_n\}_{n=1}^{\infty}$ such that $\displaystyle\sum_{n=1}^{\infty} \frac{1}{|a_n|^{\lambda+\varepsilon}} = \infty$, for some $\varepsilon > 0$. Show that $f = g$.
   (b) Find all entire functions of finite order f such that $f(\log n) = n$.

5. Elliptic Functions

In this chapter we shall give a short introduction to elliptic functions, which turns out to be a subject with many applications in several branches of mathematics.

5.1 Doubly-periodic functions

The main object we shall study is periodic functions. Given a holomorphic or meromorphic function f on $\mathbb{C}$, a period will be a number $\omega \in \mathbb{C} \setminus \{0\}$ such that $f(z+\omega) = f(z)$, for every $z \in \mathbb{C}$. Let us denote Ω the set of periods of f, plus 0.

Lemma 5.1.1 Ω is a $\mathbb{Z}-$submodule of $\mathbb{C}$.

If Ω is not discrete, f must be constant. Indeed, if $\{\omega_n\}_{n=1}^{\infty}$ is a (non constant) sequence in Ω which tends to ω_0, f must take the same value in each of this points, which implies, by the principle of isolated zeroes, that f is constant.

Lemma 5.1.2 If Ω is discrete, it is a free $\mathbb{Z}-$module of dimension at most two. It is generated by two $\mathbb{R}-$linearly independent elements.

Proof. Let $\omega_1 \in \Omega \setminus \{0\}$ of minimum modulus. If $\lambda\omega_1 \in \Omega$, $\lambda \in \mathbb{R}$, we have that $(\lambda - [\lambda])\omega_1 \in \Omega$, and then, by the minimality of $|\omega_1|$, $\lambda \in \mathbb{Z}$. Denote

$$\Omega_1 = \{n\omega_1 \, | n \in \mathbb{Z}\} \subseteq \Omega.$$

If $\Omega_1 \subsetneq \Omega$, take $\omega_2 \in \Omega \setminus \Omega_1$ of minimum modulus, and set

$$\Omega_2 = \{n\omega_1 + m\omega_2 \, | n, m \in \mathbb{Z}\}.$$

If $\Omega_2 \subsetneq \Omega$, take $\omega \in \Omega \setminus \Omega_2$, which can be written as $\omega = \lambda_1\omega_1 + \lambda_2\omega_2$, $\lambda_i \in \mathbb{R} \setminus \{0\}$. Take $n, m \in \mathbb{Z}$ such that $\omega' = \omega - n\omega_1 - m\omega_2 = a\omega_1 + b \in \omega_2$, with $|a|$, $|b| \leq \frac{1}{2}$. Then

$$|\omega'| \underset{\overset{\uparrow}{(1)}}{<} |a\omega_1| + |b\omega_2| \leq \frac{1}{2}(|\omega_1| + |\omega_2|) \leq |\omega_2|,$$

where the strict inequality (1) holds because ω_1 and ω_2 are not aligned. We reach a contradiction. ■

Assume f periodic, with a one-dimensional set of periods generated by ω. As $f\left(\frac{\omega}{2\pi i}\log t\right) = \mathscr{O}(\mathbb{C}^*)$, we can develop

$$f\left(\frac{\omega}{2\pi i}\log t\right) = \sum_{n\in\mathbb{Z}} c_n t^n,$$

and then

$$f(z) = \sum_{n\in\mathbb{Z}} c_n e^{\frac{2\pi i n}{\omega}z}.$$

This is the Fourier expansion of $\log z$. We will be interested in the case where the dimension is two: doubly-periodic functions.

Lemma 5.1.3 A doubly-periodic entire function is constant.

Definition 5.1.1 An elliptic function is a doubly-periodic meromorphic function. Given an elliptic function, with lattice of periods $\Omega =< \omega_1, \omega_2 >$, a fundamental parallelogram will be any set

$$\Delta_\omega = \{t\omega_1 + s\omega_2 \,|\, t, s \in [0,1)\}.$$

The set of generators is not unique. Indeed, any other set can be obtained by an invertible $\mathbb{Z}-$linear map of Ω, which is given by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(\mathbb{Z}),$$

$ad - bc = \pm 1$. This matrix defines a new set of generators as

$$\omega_1' = a\omega_1 + b\omega_2$$
$$\omega_2' = c\omega_1 + d\omega_2.$$

Take one such system, constructed as before. After a sign change we can suppose that $\operatorname{Im}\left(\frac{\omega_2}{\omega_1}\right) > 0$. Denote $\tau = \frac{\omega_2}{\omega_1}$. In fact, we can suppose that this basis satisfies:

1. $\operatorname{Im}(\tau) > 0$.
2. $|\tau| \geq 1$.
3. $-\frac{1}{2} < \operatorname{Re}\tau \leq \frac{1}{2}$.
4. $\operatorname{Re}\tau \geq 0$ if $|\tau| = 1$.

Indeed, from the inequalities $|\omega_2| \leq |\omega_1 \pm \omega_2|$ we obtain $|\operatorname{Re}\tau| \leq \frac{1}{2}$. If $\operatorname{Re}\tau = -\frac{1}{2}$, change ω_2 by $\omega_1 + \omega_2$ to have (iii). If $|\tau| = 1$, change eventually (ω_1, ω_2) by $(-\omega_2, \omega_1)$.

5.2 Liouville's Theorems

Theorem 5.2.1 Let f be an elliptic function. The sum of the residues of f in the poles of a fundamental paralellogram is 0.

Proof. Take a fundamental parallelogram Δ_{z_0}, assuming that $f(z)$ has no poles on $\partial\Delta_{z_0} = C$, and let $\mathscr{P}$ be the set of poles of $f(z)$ inside Δ_{z_0}. We have that

$$\sum_{z_i \in \mathscr{P}} \operatorname{Res}(f; z_i) = \frac{1}{2\pi i} \int_C f(z)dz.$$

Denote $C = \gamma_1 + \gamma_2 - \gamma_3 - \gamma_4$, γ_i the sides of C taken in counterclockwise order beginning in the vertex z_0. Then

$$\int_{\gamma_3} f(z)dz = \int_{\gamma_1} f(z+\omega_2)dz = \int_{\gamma_1} f(z)dz.$$

Similarly, $\int_{\gamma_2} f(z)dz = \int_{\gamma_4} f(z)dz$, so the integral vanishes. ■

Corollary 5.2.2 The order of an elliptic function is at least two.

Proof. Indeed, if it were one, only a simple pole could exist inside Δ_{z_0}, with a non-zero residue. ■

Theorem 5.2.3 The number of zeros inside a fundamental parallelogram is equal to the number of poles, counted with multiplicities. In fact, $f(z)$ takes every value $c \in \bar{\mathbb{C}}$ the same number of times.

Proof. Apply Theorem 5.2.1 to the function $\dfrac{f'(z)}{f(z) - c}$. ■

Theorem 5.2.4 Let $f(z)$ be an elliptic function with zeroes $\alpha_1, \alpha_2, \ldots, \alpha_\nu$ and poles $\beta_1, \beta_2, \ldots, \beta_\nu$ on a fundamental parallelogram. Then,

$$\sum_{k=1}^{\nu} \alpha_k - \sum_{k=1}^{\nu} \beta_k \in \Omega.$$

Proof. With the same notations as before,

$$\sum_{k=1}^{\nu} \alpha_k - \sum_{k=1}^{\nu} \beta_k = \frac{1}{2\pi i} \int_C z \frac{f'(z)}{f(z)} dz.$$

So, on γ_2 we have

$$\frac{1}{2\pi i} \int_{\gamma_2} z \frac{f'(z)}{f(z)} dz = \frac{1}{2\pi i} \int_{\gamma_4} z \frac{f'(z)}{f(z)} dz + \omega_1 \frac{1}{2\pi i} \int_{\gamma_4} \frac{f'(z)}{f(z)} dz.$$

Similarly with the other couple of opposite sides. The proof is finished noticing that

$$\frac{1}{2\pi i} \int_{\gamma_4} \frac{f'(z)}{f(z)} dz$$

is the index of the loop $f \circ \gamma_4$ around 0, which is an integer. ■

5.3 Weierstrass' $\wp$ Function

In this section we shall construct an order 2 elliptic function, with fixed lattice period. The function we are constructing will be "as simple as possible", and we shall see that, in fact, every other elliptic function with the same lattice period will be a rational function of this one (and its derivative).

We suppose that our function has only a order 2 pole in a fundamental parallelogram based at 0, that we shall assume it is the origin. Its principal part, after multiplication by a constant, will be $\frac{1}{z^2}$. If $f(z)$ is such a function, $f(z)-f(-z)$ will be an entire elliptic function, so a constant. As $f(\omega_1/2)-f(-\omega_1/2)=0$, $f(z)$ must be an even function. Adding a constant, the Laurent expansion at 0 it, then, $\dfrac{1}{z^2}+c_2z^2+c_4z^4+\cdots$.

The poles being at the points of Ω, Mittag-Leffler's Theorem suggest to consider

$$\frac{1}{z^2}+\sum{}'\left(\frac{1}{(z-\omega)^2}-\frac{1}{\omega^2}\right),$$

where $\sum'$ will denote the sum extended to the elements in $\Omega\setminus\{0\}$. We need to guarantee the convergence of this series, so we are going to study the convergence of $\displaystyle\sum{}'\frac{1}{|\omega|^\alpha}$, with $\alpha>0$. The elements of $\Omega\setminus\{0\}$ can be ordered in "concentric" layers $\{\Omega_n\}_{n=1}^{\infty}$ around 0. If $\delta=d(\Omega_1,0)$, then $k\delta=d(\Omega_k,0)$, and Ω_k consists in $8k$ points. Hence,

$$\sum{}'\frac{1}{|\omega|^\alpha}=\sum_{k=1}^{\infty}\sum_{\Omega_k}\frac{1}{|\omega|^\alpha}\leq\sum_{k=1}^{\infty}\frac{8k}{(k\delta)^\alpha},$$

which converges if and only if $\alpha>2$. In particular, if $|z|<\dfrac{|\omega|}{2}$,

$$\left|\frac{1}{(z-\omega)^2}-\frac{1}{\omega^2}\right|=\left|\frac{z(2\omega-z)}{(z-\omega)^2\omega^2}\right|\leq\frac{|z|\,(2\,|\omega|+|z|)}{\left(\frac{|\omega|}{2}\right)^2|\omega|^2}\leq\frac{10\,|z|}{|\omega|^3},$$

so the series converges uniformly in the compact sets to a function, that we shall denote $\wp(z)$, called Weierstrass' $\wp$ function associated to the lattice Ω.

Theorem 5.3.1 Weierstrass' $\wp$ function satisfies the following properties:
1. $\wp(z)$ has simple poles in Ω, with residue 0.
2. $\wp$ is an even function.
3. $\wp(z)$ is an elliptic function.

Proof. The first two properties are straightforward from the construction, as $-\Omega=\Omega$. As

$$\wp'(z)=-\frac{2}{z^3}-2\sum{}'\frac{1}{(z-\omega)^3}=-2\sum_{\omega\in\Omega}\frac{1}{(z-\omega)^3},$$

it becomes clear that $\wp'(z)$ is doubly periodic with lattice period Ω. Taking $f(z)=\wp(z+\omega_1)-\wp(z)$, $f'(z)==$, so it is constant. As $f\left(-\frac{\omega_1}{2}\right)=0$, it turns out that ω_1 is a period for $\wp$. Similarly with ω_2, the proof is ended. ∎

5.3.1 Differential equation satisfied by $\wp(z)$

As $\displaystyle\frac{1}{(z-\omega)^2}-\frac{1}{\omega^2}=\sum_{n=1}^{\infty}\frac{(n+1)z^n}{\omega^{n+2}}$, from the expansion at the origin $\wp(z)=\dfrac{1}{z^2}+c_2z^2+c_4z^4+\cdots$ we obtain

$$c_2=3\sum{}'\frac{1}{\omega^4},\qquad c_4=5\sum{}'\frac{1}{\omega^6},\quad\text{and so on.}$$

Computation shows that

$$\mathfrak{p}'(z)^2 - 4\mathfrak{p}(z)^3 = -20c_2\frac{1}{z^2} - 28c_4 + \cdots,$$

so, denoting $g_2 = 20c_2 = 60\sum{}'\frac{1}{\omega^4}$, $g_3 = 28c_4 = 140\sum{}'\frac{1}{\omega^6}$, we have that

$$\mathfrak{p}'(z)^2 - 4\mathfrak{p}(z)^3 + g_2\mathfrak{p}(z) + g_3 = 0,$$

as it is an entire elliptic function which takes the value 0 at the origin. As a consequence, the map $(\mathfrak{p}(z), \mathfrak{p}'(z))$ parametrizes the elliptic curve $Y^2 = 4X^3 - g_2X - g_3$, which is non singular if and only if $g_2^3 - 27g_3^2 \neq 0$. This is the Weierstrass canonical form of an non-singular cubic curve.

Let us study if the roots of $4X^3 - g_2X - g_3$ are different. As $\mathfrak{p}'(z)$ is odd, it turns out that

$$\mathfrak{p}'\left(\frac{\omega_1}{2}\right) = \mathfrak{p}'\left(\frac{\omega_2}{2}\right) = \mathfrak{p}'\left(\frac{\omega_1+\omega_2}{2}\right) = 0.$$

These are the only zeros of $\mathfrak{p}'$ on a fundamental parallelogram, as it has order three. Denote $e_1 = \mathfrak{p}\left(\frac{\omega_1}{2}\right)$, $e_2 = \mathfrak{p}\left(\frac{\omega_2}{2}\right)$, $e_3 = \mathfrak{p}\left(\frac{\omega_1+\omega_2}{2}\right)$. $\mathfrak{p}$ takes each value twice, and as at these points they are taken with multiplicity two, it turns out that they are different. In fact, we can write the elliptic curve as $Y^2 = 4(X-e_1)(X-e_2)(X-e_3)$, and the differential equation satisfied by $\mathfrak{p}(z)$ as

$$\mathfrak{p}'(z)^2 = 4(\mathfrak{p}(z)-e_1)(\mathfrak{p}(z)-e_2)(\mathfrak{p}(z)-e_3).$$

5.3.2 Elliptic functions are rational functions of $\mathfrak{p}(z)$

Let $f(z)$ be an elliptic function, with period lattice Ω, and let $\mathfrak{p}(z)$ the Weierstrass' $\mathfrak{p}$ function associated to this lattice.

Theorem 5.3.2 There exists rational functions $R_1(z)$, $R_2(z)$ such that

$$f(z) = R_1(\mathfrak{p}(z)) + \mathfrak{p}'(z)R_2(\mathfrak{p}(z)).$$

Proof. If $f(z)$ is even, and 0 is neither a zero nor a pole of $f(z)$, associated to a zero a_i of $f(z)$ on the fundamental parallelogram Δ_0, there is another zero $-a_i$, which corresponds to a unique $a_i' \in \Delta_0$ ($a_i + a_i' \in \Omega$). So, zeros appear in couples $a_1, a_1', \ldots, a_\nu, a_\nu'$, and the same happens with poles $b_1, b_1', \ldots, b_\nu, b_\nu'$. The function

$$k(z) = \frac{1}{f(z)} \cdot \prod_{n=1}^{\nu} \frac{\mathfrak{p}(z) - \mathfrak{p}(a_n)}{\mathfrak{p}(z) - \mathfrak{p}(b_n)}$$

is elliptic and entire, hence constant. So we have that $f(z) = R_1(\mathfrak{p}(z))$ in this case.

If $f(z)$ is still even, but with a zero or pole of the origin, of order $2m$, $m \in \mathbb{Z}$, then $f(z)\mathfrak{p}(z)^m$ is in the preceding conditions. If $f(z)$ is odd, $\frac{f(z)}{\mathfrak{p}'(z)}$ is even, so $f(z) = \mathfrak{p}'(z)R(\mathfrak{p}(z))$. Finally, in the general case, $f(z)$ can be decomposed as a sum of an even and an odd function, and the results follows. ■

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