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Real logarithmic models for real analytic foliations in the plane

Nuria CORRAL · Fernando SANZ

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Abstract Let S be a germ of a holomorphic curve at $(\mathbb{C}^2, 0)$ with finitely many branches $S_1, \ldots, S_r$ and let $\mathcal{I} = (I_1, \ldots, I_r) \in \mathbb{C}^r$. We show that there exists a non-dicritical holomorphic foliation of logarithmic type at $0 \in \mathbb{C}^2$ whose set of separatrices is S and having index I_i along S_i in the sense of [14] if the following (necessary) condition holds: after a reduction of singularities $\pi : M \to (\mathbb{C}^2, 0)$ of S, the vector $\mathcal{I}$ gives rise, by the usual rules of transformation of indices by blowing-ups, to systems of indices along components of the total transform $\bar{S}$ of S at points of the divisor $E = \pi^{-1}(0)$ satisfying: a) at any singular point of $\bar{S}$ the two indices along the branches of $\bar{S}$ do not belong to $\mathbb{Q}_{\geq 0}$ and they are mutually inverse; b) the sum of the indices along a component D of E for all points in D is equal to the self-intersection of D in M. This construction is used to show the existence of logarithmic models of real analytic foliations which are real generalized curves. Applications to real center-focus foliations are considered.

Keywords Singular holomorphic foliation · logarithmic foliations · generalized curves · center-focus plane vector fields

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1 Preliminaries

Let $\mathcal{F}$ be a germ of singular holomorphic foliation at $(\mathbb{C}^2, 0)$ given by the differential 1-form $\omega = a(x, y)dx + b(x, y)dy$. Recall that a *separatrix* of $\mathcal{F}$ at 0 is a germ of irreducible analytic curve Γ through 0 such that $\Gamma \setminus \{0\}$ is a leaf of $\mathcal{F}$. Denote by $Sep_0(\mathcal{F})$ the set of separatrices. It is a non-empty set [4] and $\mathcal{F}$ is called *non-dicritical* iff it is finite. In this case, the union of the separatrices is a plane curve S with a finite number of branches.

We will consider the *index* $I_0(\mathcal{F}, \Gamma) \in \mathbb{C}$ associated to each separatrix Γ defined by Lins Neto in [14]. The main properties of the index are the following ones:

1) Let $\pi : M \longrightarrow (\mathbb{C}^2, 0)$ be the blowing-up of the origin and let $\mathcal{F}'$ be the strict transform of $\mathcal{F}$ by π. If $P = \Gamma' \cap \pi^{-1}(0)$, with Γ' the strict transform of Γ, then
$$I_P(\mathcal{F}', \Gamma') = I_0(\mathcal{F}, \Gamma) - m_0(\Gamma)^2$$
where $m_0(\Gamma)$ is the multiplicity of Γ at the origin (see [3,22]).
2) Let $\mathcal{F}$ be a holomorphic foliation on a complex surface M and let D be a non-singular complex compact curve in M invariant by $\mathcal{F}$. Then
$$\sum_{P \in D} I_P(\mathcal{F}, D) = D \cdot D$$
where $D \cdot D$ is the self-intersection of D (see [4,14]).
3) Assume that $\Gamma = \{y = 0\}$ and $\omega = y\{(\lambda + \varphi)dx - (\mu x + \psi)dy/y\}$, where $\varphi(0) = 0$ and $\nu(\psi) \geq 2$. If $\mu \neq 0$, then $I_0(\mathcal{F}, \Gamma) = \lambda/\mu$.

Recall that the origin is a *simple singularity* of $\mathcal{F}$ if ω has the form $\omega = (\lambda y dx - \mu x dy) + \omega_1$ where the coefficients of ω_1 have order ≥ 2 and λ, μ are complex numbers such that one of them, say μ, is non zero and $\lambda/\mu \notin \mathbb{Q}_{>0}$. In the *hyperbolic* case, $\lambda\mu \neq 0$, the set of separatrices $Sep_0(\mathcal{F})$ consists of exactly two non singular branches S_1, S_2 with transversal tangent lines at 0. If $\lambda = 0$, the origin is a *saddle-node* singularity: we have two formal non singular separatrices at 0 but, in general, only one of them is convergent [2].

A *reduction of singularities* ([21], see also [15]) of a singular foliation $\mathcal{F}$ is a morphism $\sigma : M \to (\mathbb{C}^2, 0)$ obtained as a finite composition of blowing-ups at points such that all the singular points $Sing(\mathcal{F}')$ of the transformed foliation $\mathcal{F}'$ on M are simple ones. In the case that $\mathcal{F}$ is non dicritical, each component of the divisor $E = \pi^{-1}(0)$ is invariant by $\mathcal{F}'$. We say that $\mathcal{F}$ is a *complex generalized curve* (CGC) if in one (and hence in any) reduction of singularities there are no saddle-nodes (see [5]). This is equivalent to say that all the Lins Neto indices at the simple singularities are non-zero. A reduction of singularities of $\mathcal{F}$ gives a reduction of singularities of the curve S of separatrices of $\mathcal{F}$ at 0 in the sense that $\sigma^{-1}(S)$ is a normal crossing divisor. The reciprocal is true for a non dicritical CGC.

The simplest example of a CGC is the foliation $\{df = 0\}$ for a holomorphic function f. A more general example is a *logarithmic foliation* $\mathcal{L}_{\underline{\lambda},F}$ given by $\omega_{\underline{\lambda},F} = 0$ where

$$\omega_{\underline{\lambda},F} = f_1 \cdots f_r \sum_{j=1}^{r} \lambda_j \frac{df_j}{f_j}.$$

Here $F = (f_1, \ldots, f_r)$ is a r-uple of relatively prime irreducible germs of holomorphic functions with $f_j(0) = 0$ and $\underline{\lambda} = (\lambda_1, \ldots, \lambda_r) \in (\mathbb{C}^*)^r$. We will also write $\mathcal{L}_{\underline{\lambda},F} = \{d(f_1^{\lambda_1} \cdots f_r^{\lambda_r}) = 0\}$.

In this article we will only consider the most elementary properties of logarithmic foliations (see [8,16,17] for other results):

1. The curves $S_j = \{f_j = 0\}$, $j = 1, \ldots, r$ are separatrices of $\mathcal{L}_{\underline{\lambda},F}$.
2. The foliations $\mathcal{L}_{\underline{\lambda},F}$ and $\mathcal{L}_{\underline{\mu},F}$ coincide iff $[\underline{\lambda}] = [\underline{\mu}] \in \mathbf{P}^{r-1}_{\mathbb{C}}$.
3. We have

$$I_0(\mathcal{L}_{\underline{\lambda},F}, S_j) = -\sum_{\substack{k=1 \\ k \neq j}}^{r} \frac{\lambda_k}{\lambda_j} (S_k, S_j)_0, \qquad j = 1, \ldots, r. \tag{1}$$

where $(S_k, S_j)_0$ is the intersection number of the branches S_k and S_j at the origin. (Do a similar calculation as in [14]; see also [20]).

The above properties are true for $\mathcal{L}_{\underline{\lambda},F}$ dicritical or not. Now let us remark some statements and properties concerning dicriticalness.

We say that the logarithmic foliation $\mathcal{L} = \mathcal{L}_{\underline{\lambda},F}$ has the *main resonance* if

$$\sum_{j=1}^{r} \lambda_j m_0(S_j) = 0$$

where $m_0(S_j)$ is the multiplicity of S_j at the origin.

Proposition 1 *Suppose that $\mathcal{L} = \mathcal{L}_{\underline{\lambda},F}$ does not have the main resonance. Let $\pi : M \to (\mathbb{C}^2, 0)$ be the blowing-up at the origin and consider $\mathcal{L}'$ the strict transform of $\mathcal{L}$ by π. Then the exceptional divisor $E = \pi^{-1}(0)$ is invariant by $\mathcal{L}'$ and the tangent cone $C_0(\mathcal{L}) = Sing(\mathcal{L}') \cap E$ of $\mathcal{L}$ at 0 corresponds exactly to the tangents of the separatrices S_j, $j = 1, \ldots, r$.*

Proof Let $\nu + 1 = \sum_{j=1}^{r} m_0(S_j)$ and write $\omega = \omega_{\underline{\lambda},F} = a(x,y)dx + b(x,y)dy$. One can show that $xa + yb$ has order $\nu + 1$ and that the zeroes of the homogeneous term $(xa + yb)_{\nu+1}$ of degree $\nu + 1$ are precisely the tangents of the separatrices S_j at the origin. Thus ν is the minimum of the orders of a or b and the tangent cone $xa_\nu + yb_\nu = 0$ of ω is equal to $(xa + yb)_{\nu+1} = 0$. The conclusion follows.

2 Foliations with prescribed indices

In this paragraph we refine the arguments in [9,10] to construct logarithmic foliations with prescribed indices along the branches of a given curve.

Let us consider pairs $\mathcal{A} = (M \stackrel{\sigma}{\to} (\mathbb{C}^2, 0), S)$, where $S \subset (\mathbb{C}^2, 0)$ is an analytic curve with finitely many branches $S = \cup_{j=1}^r S_j$ and $\sigma : M \to (\mathbb{C}^2, 0)$ is the composition of a finite sequence of punctual blowing-ups. Denote by $S^\sigma \subset M$ the strict transform of S and by $E = \sigma^{-1}(0)$ the exceptional divisor. A blowing-up $\pi : M' \to M$ with center $P \in E$ is called *irredundant* for $\mathcal{A}$ iff $E \cup S^\sigma$ does not have normal crossings at P. There is a number $n(\mathcal{A})$ such that after exactly $n(\mathcal{A})$ irredundant blowing-ups we get a new pair $\widetilde{\mathcal{A}} = (\tilde{\sigma} : \tilde{M} \to (\mathbb{C}^2, 0), S)$ such that $\tilde{E} \cup S^{\tilde{\sigma}}$ has normal crossings at each point.

A *system of indices* $\mathcal{I} = \mathcal{I}_{\mathcal{A}}$ is a family $\{I_P\}_{P \in E}$ where I_P is a function that assigns a complex number $I_P(\Gamma) \in \mathbb{C}$ to each irreducible component Γ of the germ of $E \cup S^\sigma$ at P. Given a blowing-up $\pi : M' \to M$ with center $P \in E$ and a system of indices $\mathcal{I}'$ on $\mathcal{A}' = (M' \stackrel{\sigma'}{\to} (\mathbb{C}^2, 0), S)$, where $\sigma' = \sigma \circ \pi$, we can *blow-down* $\mathcal{I}'$ as follows. For a component Γ of $E \cup S^\sigma$ at Q we define

$$I_Q(\Gamma) = \begin{cases} I'_{Q'}(\Gamma') & \text{if } Q \neq P \\ I'_{Q'}(\Gamma') + [m_Q(\Gamma)]^2 & \text{if } Q = P \end{cases}$$

where Γ' is the strict transform of Γ by π, $Q' = \Gamma' \cap E'$ and $m_Q(\Gamma)$ is the multiplicity of Γ at Q.

Definition 1 Let $\mathcal{I}$ be a system of indices on $\mathcal{A}$ and put $n = n(\mathcal{A})$. If $n = 0$, we say that $\mathcal{I}$ is *Lins-compatible* iff the following properties are satisfied:

a) If P is a non singular point of $E \cup S^\sigma$, then $I_P(E) = 0$.
b) If D is an irreducible component of E, then $\sum_{P \in D} I_P(D) = D \cdot D$.
c) If P is a singular point of $E \cup S^\sigma$ and Γ_1, Γ_2 are the two branches of $E \cup S^\sigma$ at P, then $I_P(\Gamma_1) \cdot I_P(\Gamma_2) = 1$ and $I_P(\Gamma_i) \notin \mathbb{Q}_{\geq 0}$, $i = 1, 2$.

If $n > 0$, we say that $\mathcal{I}$ is Lins-compatible iff there is an irredundant blowing-up $M' \stackrel{\pi}{\to} M$ and a Lins-compatible system of indices $\mathcal{I}'$ on $\mathcal{A}' = (M' \stackrel{\sigma'}{\to} (\mathbb{C}^2, 0), S)$, where $\sigma' = \sigma \circ \pi$, such that $\mathcal{I}$ is the blowing-down of $\mathcal{I}'$.

Proposition 2 *The indices on the branches of S^σ determine the indices over the divisor in a Lins-compatible system of indices. Moreover a Lins-compatible system of indices can be blown-up and blown-down in a unique way.*

Proof Note that the blowing-down of a system of indices is uniquely defined and that an irredundant blowing-down transforms Lins-compatible systems of indices into Lins-compatible systems. Consider a Lins-compatible system of indices $\mathcal{I}$ on $\mathcal{A}$ with $n = n(\mathcal{A})$ and let us first prove the following statements by induction on n:

$A(n)$: The indices on E are determined by the indices on the branches of S^σ.

$B(n)$: Given $\pi : M' \to M$ an irredundant blowing-up with center $P \in E$, there is a unique Lins-compatible system of indices $\mathcal{I}' = \pi^*\mathcal{I}$ on $\mathcal{A}' = (M' \stackrel{\sigma'}{\to} (\mathbb{C}^2, 0), S)$, where $\sigma' = \sigma \circ \pi$, such that $\mathcal{I}$ is the blowing-down of $\mathcal{I}'$.

If $n = 0$, then $B(0)$ is trivial since there are no irredundant blowing-ups. To prove $A(0)$ it is enough to use the fact that the dual graph of E is connected. The fact that $A(n-1)$, $B(n-1)$ implies $A(n)$ is evident from the definition of a Lins-compatible system of indices: just take the irredundant blowing-up given in the definition. Let us prove $B(n)$. First, note that the uniqueness stated in $B(n)$ is consequence of the definition of blowing-down of Lins-compatible indices and the induction hypothesis $A(n-1)$ applied to $\mathcal{A}'$. If π is the blowing-up given in the definition, then $B(n)$ is a consequence of $A(n-1)$ applied to $\mathcal{A}' = (M' \stackrel{\sigma'}{\to} (\mathbb{C}^2, 0), S)$. Otherwise, let $\pi_1 : M_1 \to M$ be the blowing-up of the definition, with center $Q \neq P$, and let $\mathcal{I}_1$ be the Lins-compatible system of indices on $\mathcal{A}_1 = (M_1 \stackrel{\sigma_1}{\to} (\mathbb{C}^2, 0), S)$, $\sigma_1 = \sigma \circ \pi_1$, that projects to $\mathcal{I}$ by π_1. Consider the following diagram of irredundant blowing-ups

$$\begin{array}{ccc} M' & \xrightarrow{\ \pi\ } & M \\ {\scriptstyle \tilde{\pi}_1}\uparrow & & \uparrow{\scriptstyle \pi_1} \\ M_1' & \xrightarrow{\ \tilde{\pi}\ } & M_1 \end{array}$$

where $P_1 = \pi_1^{-1}(P)$ is the center of $\tilde{\pi}$ and $Q' = \pi^{-1}(Q)$ is the center of $\tilde{\pi}_1$. By the induction hypothesis $B(n-1)$ applied to $\mathcal{A}_1$ we have a unique Lins-compatible system of indices $\mathcal{I}_1' = \tilde{\pi}^*\mathcal{I}_1$ on $\mathcal{A}_1' = (M_1' \stackrel{\sigma_1 \circ \tilde{\pi}}{\longrightarrow} (\mathbb{C}^2, 0), S)$. Since $\tilde{\pi}_1$ is irredundant then, by definition, the blowing-down $\mathcal{I}'$ of $\mathcal{I}_1'$ over $\mathcal{A}'$ is a Lins-compatible system, that obviously gives $\mathcal{I}$ by projection.

It remains to prove the existence of the transform of a Lins-compatible system of indices under a *redundant* blowing-up or blowing-down. We proceed again by induction on $n = n(\mathcal{A})$. In order to pass from n to $n-1$ consider a diagram of four blowing-ups as above, where π and $\tilde{\pi}$ are redundant and π_1 and $\tilde{\pi}_1$ are irredundant. Finally, assume that $n = 0$. The first case is that the center of $\pi : M' \to M$ is a non singular point P of $E \cup S^\sigma$. Assume first that $\mathcal{I}$ is a Lins-compatible system of indices given on $\mathcal{A}$. Then we know that $I_P(E) = 0$. Let E^π be the strict transform of E by π and $P' = E^\pi \cap D$, where $D = \pi^{-1}(P)$. We define a system of indices $\mathcal{I}'$ on $\mathcal{A}'$ by $I'_{P'}(E^\pi) = -1 = I'_{P'}(D)$ and $I'_{Q'}(D) = 0$ for $Q' \in D$, $Q' \neq P'$. Verify that $\mathcal{I}'$ is Lins-compatible using the fact that D has self-intersection equal to -1. On the other hand, if $\mathcal{I}'$ is a Lins compatible system of indices on $\mathcal{A}'$, then the only non singular point of $D \cup E^\pi$ in D is P' and hence $I'_{P'}(D) = D \cdot D = -1$, and $I'_P(E^\pi) = -1$ in view of the fact that $I'_{P'}(D) \cdot I'_{P'}(E^\pi) = 1$. Thus, the blowing-down $\mathcal{I}$ of $\mathcal{I}'$ verifies $I_P(E) = 0$. We use the same kind of arguments for the case that P is a singular point of $E \cup S^\sigma$.

Remark 1 Let $\mathcal{I}$ be a Lins-compatible system of indices on $\mathcal{A}$. Then $I_P(E) = 0$ at any non-singular point of $E \cup S^\sigma$ and also $\sum_{P \in D} I_P(D) = D \cdot D$ for any *component D of E. We get these properties by blowing-down the transform of $\mathcal{I}$ under a chain of $n(\mathcal{A})$ irredundant blowing-ups, using the definition of blowing-down a system of indices and the fact that the self-intersection of the strict transform of a connected component D of E by a blowing-up at a point of D decreases by one with respect to the self-intersection of D.*

The above constructions show also that we can *localize* a Lins-compatible system of indices $\mathcal{I}$ at a point $P \in M$. More precisely, the new pair is given by the germs $(M, P) \simeq (\mathbb{C}^2, 0)$ and $(S^\sigma \cup E, P)$. The system of indices will be just $\mathcal{I}_P = \{I_P\}$. To see that $\mathcal{I}_P$ is Lins-compatible the best is to consider a sequence of $n = n(\mathcal{A})$ irredundant blowing-ups

$$M_n \xrightarrow{\pi_n} M_{n-1} \longrightarrow \cdots \xrightarrow{\pi_1} M \xrightarrow{\sigma} (\mathbb{C}^2, 0)$$

and to look $\mathcal{I}_P$ as the direct image of the restriction of $(\pi_1 \circ \cdots \circ \pi_n)^* \mathcal{I}$ to $(\pi_1 \circ \cdots \circ \pi_n)^{-1}(M, P)$.

Example 1 Let $\mathcal{F}$ be a non dicritical CGC in $(\mathbb{C}^2, 0)$ with S as curve of separatrices. Put $I_0(S_i) = I_0(\mathcal{F}, S_i)$. We get a Lins-compatible system of indices $\mathcal{I}(\mathcal{F})$ on $((\mathbb{C}^2, 0), S)$. Moreover, let $\sigma : M \to (\mathbb{C}^2, 0)$ be any sequence of blowing-ups. We can define in the same way the system of indices $\mathcal{I}(\sigma^* \mathcal{F})$ on $\mathcal{A} = (M \overset{\sigma}{\to} (\mathbb{C}^2, 0), S)$. Then we have that

$$\mathcal{I}(\sigma^* \mathcal{F}) = \sigma^* \mathcal{I}(\mathcal{F}).$$

This is a consequence of the uniqueness of blowing-up, blowing-down and the fact that after reduction of singularities (even redundant), the system of indices given by $\mathcal{F}$ is obviously Lins-compatible.

A *system of exponents* $\Lambda = \{[\lambda_P]\}_{P \in E}$ on a pair $\mathcal{A} = (M \overset{\sigma}{\to} (\mathbb{C}^2, 0), S)$ is a collection of classes $[\lambda_P]$, where

$$\lambda_P : \{\text{branches at } P \text{ of } \ E \cup S^\sigma\} \to \mathbb{C}^*$$

under the equivalence $\lambda_P \sim \lambda'_P$ iff $\lambda_P = c\lambda'_P$, $c \in \mathbb{C}^*$.

Let $\pi : M' \to M$ be the blowing-up with center P and $\Lambda' = \{[\lambda'_Q]\}$ be a system of exponents on $\mathcal{A}'$. We define the *blowing-down* $\Lambda = \pi_* \Lambda'$ as follows. For any $Q \neq P$, we put $[\lambda_Q] = [\lambda'_{\pi^{-1}(Q)}]$. For each $Q' \in D = \pi^{-1}(P)$, select $\lambda'_{Q'} \in [\lambda'_{Q'}]$ with the property that $\lambda'_{Q'}(D) = 1$ and put, for any branch Γ of $E \cup S^\sigma$ at P,

$$\lambda_P(\Gamma) = \lambda'_{P'}(\Gamma'),$$

where Γ' is the strict transform of Γ and $P' = \Gamma' \cap D$.

In order to transform a system of exponents Λ by blowing-up, we must avoid the main resonance. More precisely, we say that Λ is *transformable at a point* $P \in E$ iff

$$\sum_{\Gamma} \lambda_P(\Gamma) \cdot m_P(\Gamma) \neq 0,$$

where Γ varies over the branches of $E \cup S^\sigma$ at P. In this case, we define the *transform* $\Lambda' = \pi^*\Lambda$ of Λ by the blowing-up $\pi : M' \to M$ with center P by putting

$$\lambda'_{Q'}(\pi^{-1}(P)) = \sum_{\Gamma} \lambda_P(\Gamma) \cdot m_P(\Gamma)$$

for any $Q' \in \pi^{-1}(P)$ and $\lambda'_{P'}(\Gamma') = \lambda_P(\Gamma)$ if $P' \in \pi^{-1}(E)$ is in the strict transform Γ' of a branch Γ of $E \cup S^\sigma$.

We say that Λ is *indefinitely transformable at* $P \in E$ iff Λ is transformable at P, its transform Λ^1 by the blowing-up π_1 at P is transformable at any point $P_1 \in \pi_1^{-1}(P)$, the transform of Λ^1 by the blowing up π_2 at any such point P_1 is transformable at any point $P_2 \in \pi_2^{-1}(P_1)$ and so on. Roughly speaking, Λ is indefinitely transformable if its transform is transformable at any infinitely near point of P. We say finally that Λ is *indefinitely transformable* if it is so at any point $P \in E \cup S^\sigma$.

Proposition 3 *Given a Lins-compatible system of indices $\mathcal{I}$ on $\mathcal{A}$, there is a unique indefinitely transformable system of exponents $\Lambda(\mathcal{I})$ such that for any $P \in E \cup S^\sigma$ and any branch Γ of $E \cup S^\sigma$ at P, we have that*

$$\lambda_P(\Gamma) \cdot I_P(\Gamma) + \sum_{\Gamma^* \neq \Gamma} \lambda_P(\Gamma^*) \cdot (\Gamma^*, \Gamma)_P = 0, \qquad (\star)$$

where Γ^* varies over all branches of $E \cup S^\sigma$ at P. Moreover, the system $\Lambda(\mathcal{I})$ satisfies $\pi^*(\Lambda(\mathcal{I})) = \Lambda(\pi^*\mathcal{I})$ for any blowing-up $\pi : M' \to M$.

Proof The result is local at each point $P \in M$ and thus we can assume that $M = (\mathbb{C}^2, 0)$, $S = \cup_{j=1}^r S_j$ and $\mathcal{I} = \{I_0\}$. Let us do induction on $n = n(\mathcal{A})$. Assume that $n = 0$: if $S = S_1$ has a single non singular branch, we put $\lambda_0(S_1) = 1$ (note that in this case $I_0(S_1) = 0$); if $S = S_1 \cup S_2$ define $\lambda_0(S_1) = 1$, $\lambda_0(S_2) = -1/I_0(S_2) = -I_0(S_1)$. Moreover, Λ is indefinitely transformable since by blowing-up at 0 the situation repeats. Consider now the case $n > 0$. Let $\pi : M \to (\mathbb{C}^2, 0)$ be the blowing-up of the origin and let $Q_1, Q_2, \ldots, Q_k$ be the points where $S^\pi = \cup_{j=1}^r S'_j$ cuts the divisor $D = \pi^{-1}(0)$. Denote $A_l = \{j : S'_j \cap D = Q_l\}$, for $l = 1, \ldots, k$. Let us localize $\mathcal{I}' = \pi^*\mathcal{I}$ to get systems of indices $\mathcal{I}'_l$ at Q_l and take the systems of exponents $\Lambda'_l = \{[\lambda'_{Q_l}]\}$ given by the induction hypothesis. Choose $\lambda'_{Q_l} \in [\lambda'_{Q_l}]$, $l = 1, \ldots, k$, such that $\lambda'_{Q_l}(D) = 1$ and define $\lambda_0 : \{S_j\}_{j=1}^r \to \mathbb{C}^*$ by

$$\lambda_0(S_j) = \lambda'_{Q_l}(S'_j) = \lambda_j \quad \text{if} \quad j \in A_l.$$

In view of the property $(\star)$ for Λ'_l, we get for any l

$$0 = \lambda'_{Q_l}(D) \cdot I_{Q_l}(D) + \sum_{j \in A_l} \lambda'_{Q_l}(S'_j) \cdot (D, S'_j)_{Q_l} =$$
$$= I'_{Q_l}(D) + \sum_{j \in A_l} \lambda_j m_0(S_j).$$

Taking the summation over all $l = 1, \dots, k$, we obtain that

$$0 = \sum_{l=1}^{k} I'_{Q_l}(D) + \sum_{j=1}^{r} \lambda_j m_0(S_j) = -1 + \sum_{j=1}^{r} \lambda_j m_0(S_j).$$

Since $\sum_{j=1}^{r} \lambda_j m_0(S_j) = 1$, the system $\Lambda = \{[\lambda_0]\}$ is transformable at the origin and it is indefinitely transformable because its blowing-up is, by construction, an indefinitely transformable system on $\mathcal{A}' = (M \xrightarrow{\pi} (\mathbb{C}^2, 0), S)$. Let us show $(\star)$ for Λ. Fix $j \in \{1, \dots, r\}$ and let $Q_l \in D$ such that $j \in A_l$. Then

$$\begin{aligned} &\lambda_j I_0(S_j) + \sum_{i \neq j} \lambda_i \cdot (S_i, S_j)_0 = \\ &= \lambda_j (I'_{Q_l}(S'_j) + m_0(S_j)^2) + \sum_{i \neq j} \lambda_i [m_0(S_i) \cdot m_0(S_j) + (S'_i, S'_j)_{Q_l}] \\ &= \lambda_j I'_{Q_l}(S'_j) + \sum_{i \neq j} \lambda_i (S'_i, S'_j)_{Q_l} + m_0(S_j) \sum_{i=1}^{r} \lambda_i m_0(S_i) \\ &= \lambda'_{Q_l}(S'_j) \cdot I'_{Q_l}(S'_j) + \sum_{i \in A_l} \lambda'_{Q_l}(S'_i) \cdot (S'_i, S'_j)_{Q_l} + \lambda'_{Q_l}(D) \cdot (S'_j, D)_{Q_l} = 0, \end{aligned} \tag{2}$$

the last equality by the property $(\star)$ for Λ'_l. Now we prove uniqueness of system Λ. Suppose that there exists another system of indices $\Delta = \{[\delta_0]\}$ indefinitely transformable and satisfying $(\star)$ at the origin. Denote by $\Delta' = \{[\delta'_Q]\}_{Q \in D}$ the blowing-up of Δ by π. Fix $\delta_0 \in [\delta_0]$ and consider for any $l = 1, \dots, k$ the element $\delta'_{Q_l} \in [\delta'_{Q_l}]$ defined by $\delta'_{Q_l}(S'_j) = \delta_0(S_j) = \delta_j$ for $j \in A_l$ and $\delta'_{Q_l}(D) = \sum_{i=1}^{r} \delta_i m_0(S_i)$. From the equation

$$\delta_j I_0(S_j) + \sum_{i \neq j} \delta_i \cdot (S_i, S_j)_0 = 0$$

for $j \in A_l$ and $l \in \{1, \dots, k\}$ we prove, exactly as in (2) replacing λ by δ, that the localized of $\pi^*(\Delta)$ at Q_l satisfies the property $(\star)$ with respect to the branches S'_j with $j \in A_l$, for any l. Consequently it is enough to prove the property $(\star)$ also for the branch D at Q_l and then we get that $\pi^*(\Delta) = \pi^*(\Lambda)$ and $\Delta = \Lambda$ by the induction hypothesis. From

$$\begin{aligned} \lambda_j I_{Q_l}(S'_j) + \textstyle\sum_{i \in A_l, i \neq j} \lambda_i (S'_i, S'_j)_{Q_l} + \lambda_{Q_l}(D)(D, S'_j)_{Q_l} = 0 \\ \delta_j I_{Q_l}(S'_j) + \textstyle\sum_{i \in A_l, i \neq j} \delta_i (S'_i, S'_j)_{Q_l} + \delta_{Q_l}(D)(D, S'_j)_{Q_l} = 0 \end{aligned}$$

we obtain

$$\begin{aligned} \delta_j \sum_{i \in A_l, i \neq j} \lambda_i (S'_i, S'_j)_{Q_l} + \delta_j \lambda_{Q_l}(D)(D, S'_j)_{Q_l} = \\ \lambda_j \sum_{i \in A_l, i \neq j} \delta_i (S'_i, S'_j)_{Q_l} + \lambda_j \delta_{Q_l}(D)(D, S'_j)_{Q_l}. \end{aligned}$$

Summing for all $j \in A_l$, we have that

$$\lambda_{Q_l}(D) \sum_{j\in A_l} \delta_j(D, S'_j)_{Q_l} = \delta_{Q_l}(D) \sum_{j\in A_l} \lambda_j(D, S'_j)_{Q_l}$$

Since $\lambda_{Q_l}(D) I_{Q_l}(D) + \sum_{j\in A_l} \lambda_j(D, S'_j)_{Q_l} = 0$ by $(\star)$ for $\pi^*(\Lambda)$ at Q_l, then we get

$$\delta_{Q_l}(D) I_{Q_l}(D) + \sum_{j\in A_l} \delta_j(D, S'_j)_{Q_l} = 0$$

after dividing by $\lambda_{Q_l}(D)$, which is the required equation.

Theorem 1 *Let $S = \cup_{j=1}^r S_j$ be an analytic curve at $0 \in \mathbb{C}^2$. Let $\mathcal{I} = \{I_0\}$ be a Lins-compatible system of indices on $\mathcal{A} = ((\mathbb{C}^2, 0), S)$ and $\Lambda = \Lambda(\mathcal{I}) = \{[\lambda_0]\}$. Take $\underline{\lambda} = (\lambda_1, \dots, \lambda_r)$ with $\lambda_j = \lambda_0(S_j)$. Then, for any $F = (f_1, \dots, f_r)$ with f_j a reduced equation of S_j, the logarithmic foliation $\mathcal{L} = \mathcal{L}_{\underline{\lambda},F}$ satisfies:*

a) $\mathcal{L}$ is a non dicritical CGC with curve of separatrices S and $I_0(\mathcal{L}, S_j) = I_0(S_j)$ for any j.
b) Let $\sigma : M \to (\mathbb{C}^2, 0)$ be any finite sequence of blowing-ups, then $\sigma^*\mathcal{I}$ gives the Lins Neto indices of $\sigma^*\mathcal{L}$ at all the separatrices.

Proof The property b) follows from a) in view of the Example 1. Let us prove a) by induction on $n = n(\mathcal{A})$.

Assume that $n = 0$. Then S has normal crossings at 0 and either $S = S_1$ or $S = S_1 \cup S_2$. In the first case, $\mathcal{L} = \{df_1 = 0\}$ is non singular, hence a non dicritical CGC and $I(\mathcal{L}, S_1) = 0$. Since 0 is a non singular point of $S \cup E = S_1$, we have also that $I_0(S_1) = 0$. In the second case, $\mathcal{L} = \{\lambda_1 f_2 df_1 + \lambda_2 f_1 df_2\}$ with f_1, f_2 a system of coordinates at 0. We know that $I_0(\mathcal{L}, S_1) = -\lambda_2/\lambda_1$, $I_0(\mathcal{L}, S_2) = -\lambda_1/\lambda_2$ and these values coincide with $I_0(S_1)$, $I_0(S_2)$ respectively, in view of the property $(\star)$ for Λ. Moreover, $-\lambda_2/\lambda_1 \notin \mathbb{Q}_{\geq 0}$ since $\mathcal{I}$ is Lins-compatible. Thus, 0 is a hyperbolic simple singularity of $\mathcal{L}$ and a) follows.

Assume that $n > 0$. Let $\pi : \widetilde{(\mathbb{C}^2, 0)} \to (\mathbb{C}^2, 0)$ be the blowing-up at the origin. Since Λ is transformable, $\mathcal{L}$ does not have the main resonance. Denote by $\Lambda' = \{[\lambda'_P] : P \in \pi^{-1}(0)\}$ the transform of Λ. In view of Proposition 1, we know that $D = \pi^{-1}(0)$ is invariant for $\mathcal{L}' = \pi^*\mathcal{L}$ and the singularities of $\mathcal{L}'$ over D are exactly the points $Q_1, \dots, Q_k$ where the strict transform $S^\pi = \cup_{j=1}^r S'_j$ cuts D. Let $A_l = \{j : S'_j \cap D = Q_l\}$. Choose $\lambda'_{Q_l} \in [\lambda'_{Q_l}]$ with

$$\lambda'_{Q_l}(D) = \sum_{j=1}^r \lambda_j m_0(S_j).$$

Hence $\lambda'_{Q_l}(S'_j) = \lambda_j$ for $j \in A_l$ by the definition of Λ'. Let $\mathcal{I}'_l$ be the localization of $\mathcal{I}'$ at Q_l. We know that $\Lambda(\mathcal{I}'_l) = \{[\lambda'_{Q_l}]\}$, in view of Proposition 3. In order to end the proof, by our induction hypothesis, let us show that $\mathcal{L}'_{Q_l}$ has a logarithmic expression associated to $S'_{(l)} = D \cup (\cup_{j\in A_l} S'_j)$ with λ'_{Q_l} as vector of exponents. In fact, we have that $\mathcal{L}'_{Q_l}$ is given by

$$d((xu_{(l)})^{\lambda'_{Q_l}(D)} \cdot \prod_{j\in A_l} {f'_j}^{\lambda_j}) = 0;$$

where $x = 0$ is a reduced equation of D at Q_l, f'_j is the strict transform of f_j at Q_l and $u_{(l)}$ is a unit given by

$$u_{(l)} = \prod_{j\notin A_l} {f'_j}^{\frac{\lambda_j}{\lambda'_{Q_l}(D)}}.$$

A consequence of the theorem above is the existence of a *logarithmic model* for a non dicritical CGC:

Corollary 1 ([9, 10]) *Given $\mathcal{F}$ a non dicritical CGC there exists a non dicritical logarithmic foliation $\mathcal{L}$ such that $Sep_0(\mathcal{F}) = Sep_0(\mathcal{L})$ and $\mathcal{I}(\mathcal{F}) = \mathcal{I}(\mathcal{L})$.*

3 Real logarithmic models

Let $\mathcal{F}$ be a singular foliation at $0 \in \mathbb{C}^2$. We say that $\mathcal{F}$ is a *real foliation* if it can be defined by a holomorphic 1-form $\omega = a(x, y)dx + b(x, y)dy$ such that the coefficients $a, b \in \mathbb{R}\{x, y\}$. In this case $\mathcal{F}$ is invariant by the complex conjugation $(x, y) \mapsto (\bar{x}, \bar{y})$. More generally, a real foliation $\mathcal{F}$ in a complex surface M with a self-conjugation $\rho : M \longrightarrow M$ ([13]) will be called a *real foliation* if $\rho^*\mathcal{F} = \mathcal{F}$. It induces a real analytic foliation $\mathcal{F}_\mathbb{R}$ over the real surface $M_\mathbb{R}$ consisting of the set of fixed points of ρ, also called *real points*. We have that

1. $P \in Sing(\mathcal{F})$ iff $\rho(P) \in Sing(\mathcal{F})$.
2. $\Gamma \in Sep_P(\mathcal{F})$ iff $\rho(\Gamma) \in Sep_{\rho(P)}(\mathcal{F})$.
3. $I_P(\mathcal{F}, \Gamma) = \overline{I_{\rho(P)}(\mathcal{F}, \rho(P))}$.

If P is a real point, a *real separatrix* of $\mathcal{F}$ at P is a ρ-invariant separatrix $\Gamma \in Sep_P(\mathcal{F})$ and thus $I_P(\mathcal{F}, \Gamma) \in \mathbb{R}$. This is equivalent to say that the real part $S_\mathbb{R} = S \cap M_\mathbb{R}$ is a real analytic curve through P (of topological dimension 1) invariant by $\mathcal{F}_\mathbb{R}$. Denote by $\mathbb{R}Sep_P(\mathcal{F})$ the set of real separatrices at $P \in M_\mathbb{R}$. A real foliation $\mathcal{F}$ is called *non real-dicritical* at 0 if $\mathbb{R}Sep_0(\mathcal{F})$ is finite.

Let $\sigma : M \longrightarrow (\mathbb{C}^2, 0)$ be a composition of blowing-ups at real points. Then the conjugation of $\mathbb{C}^2$ lifts to a self-conjugation ρ of M and the transform $\mathcal{F}'$ of $\mathcal{F}$ is a real foliation with respect to ρ. If $\mathcal{F}$ is non real-dicritical, then every component of the divisor $E = \sigma^{-1}(0)$ is invariant by $\mathcal{F}$. The morphism σ is called a *real reduction of singularities* of $\mathcal{F}$ if every singular real point of $\mathcal{F}'$ is a simple singularity. Also, the singular foliation $\mathcal{F}$ is called a *real generalized curve* (RGC for short) if no real point of M is a saddle-node singularity of $\mathcal{F}'$.

Examples of RGC foliations which are not CGG can be easily obtained (see [19]).

In this paragraph we want to construct real logarithmic models for real foliations. To be precise, we adopt the following definition:

Definition 2 Let $\mathcal{F}$ be a real foliation at $0 \in \mathbb{C}^2$ which is a real generalized curve and non real-dicritical. A *real logarithmic model* of $\mathcal{F}$ is a non dicritical logarithmic foliation $\mathcal{L}$ which is a real foliation and such that:

i) $\mathbb{R}Sep_0(\mathcal{F}) = \mathbb{R}Sep_0(\mathcal{L})$
ii) $\mathcal{F}$ and $\mathcal{L}$ have a common real reduction of singularities $\sigma : M \longrightarrow (\mathbb{C}^2, 0)$ such that if $\mathcal{F}'$ and $\mathcal{L}'$ are the transforms of $\mathcal{F}$ and $\mathcal{L}$ by σ, then $\mathcal{F}'$ and $\mathcal{L}'$ have the same real singular points.
iii) For any real point P in $\sigma^{-1}(0)$ and any real separatrix Γ of $\mathcal{F}'$ at P, we have $I_P(\mathcal{F}', \Gamma) = I_P(\mathcal{L}', \Gamma)$.

We remark that it is not true, in general, that the minimal reduction of the real separatrices gives a real reduction of singularities of a non real-dicritical RGC foliation, even in the case of a real CGC. For example, consider the singular foliation given by $\{d(y(x^2 + y^2)) = 0\}$. It has a non simple singularity at the origin and a single non singular real separatrix. Thus, a real logarithmic model must bear in mind also the non real separatrices $\{x + iy = 0\}$ and $\{x - iy = 0\}$. This example also shows that real logarithmic models are far from being unique: all foliations of the type $\{d(y(x^2 + \lambda y^2)) = 0\}$ with $\lambda \in \mathbb{R}_{>0}$ have the same real reduction of singularities with the same indices at the real points.

On the other hand, condition iii) implies that the Lins Neto indices of $\mathcal{F}$ and $\mathcal{L}$ along real separatrices at 0 coincide. However, this last property alone does not guarantee that $\mathcal{L}$ and $\mathcal{F}$ satisfy condition iii) since the indices at the real corners are not determined by the indices of the real separatrices at 0.

Theorem 2 *Let $\mathcal{F}$ be a real analytic foliation at $0 \in \mathbb{C}^2$ and suppose that $\mathcal{F}$ is a non real-dicritical real generalized curve. Then there exists a real logarithmic model $\mathcal{L}$ of $\mathcal{F}$.*

Proof Let $\sigma : M \longrightarrow (\mathbb{C}^2, 0)$ be a real reduction of singularities of $\mathcal{F}$ and $\rho : M \longrightarrow M$ be the self-conjugation that lifts the conjugation of $\mathbb{C}^2$. Consider the decomposition $E = E_1 \cup \ldots \cup E_n$ of the total divisor $E = \sigma^{-1}(0)$ into irreducible components and let us suppose that $n \geq 1$, i.e., that we make at least a blowing-up, even if the singularity of $\mathcal{F}$ is already simple at the origin. Since $\mathcal{F}$ is non real-dicritical, E_j is invariant for any j by the strict transform $\mathcal{F}'$ of $\mathcal{F}$. Moreover, since σ is a composition of blowing-ups at real points, the corners of E are all real points and the indices at these corners of the strict transform along the components of E are real numbers.

Let A be the set of points $P \in E \cap Sing(\mathcal{F}')$ such that P is not a corner of E and $Re(I_P(\mathcal{F}', E)) \neq 0$. We have obviously that A is invariant by ρ. Moreover, we prove that A is non-empty by using the following claim.

Claim: Given a real foliation $\mathcal{F}$ and a divisor E recursively constructed by blowing-up at real points, there exists a point P in E such that either P is not a corner and $\mathrm{Re}(I_P(\mathcal{F},E)) \neq 0$ or P is a corner and if D_1, D_2 are the components of E through P then $I_P(\mathcal{F},D_1)\cdot I_P(\mathcal{F},D_2) \neq 1$.

The proof of the claim is completely analogous to the proof given in [7] concerning a property named $(*)$ there. It is broadly as follows. If E has a single component, there are no corners and then the index along E at one of the singularities appearing in E has non-zero real part since the sum of the indices along E must be equal to -1. Now, suppose that Q is a real point of some divisor E as in the hypothesis which has the required properties and let π_Q be the blowing-up at Q. We have to show that there is a point P in the exceptional divisor $D = \pi_Q^{-1}(Q)$ with the required properties. We analyze the two cases:

1) Q is not a corner and $\mathrm{Re}(I_Q(\mathcal{F},E)) \neq 0$. Let D_1 be the strict transform by π_Q of the component of E where Q belongs. Either there is a singular point of D which is not a corner having index along D with non-zero real part or, otherwise, the index at the corner $P = D \cap D_1$ with respect to D is equal to -1, since it is real. The index at P with respect to D_1 cannot be -1 since $\mathrm{Re}(I_Q(\mathcal{F},E)) \neq 0$ and we are done;
2) Q is a corner with $I_Q(\mathcal{F},D_1)I_Q(\mathcal{F},D_2) \neq 1$, where D_1, D_2 are the components through Q. If there are no non-corners in D for which the real part of the index along D is non-zero, we have two corners P_1, P_2 in D corresponding to the tangent lines of D_1, D_2 respectively with $I_{P_1}(\mathcal{F},D) + I_{P_2}(\mathcal{F},D) = -1$ since these indices are real. We conclude from the fact that $I_Q(\mathcal{F},D_1)I_Q(\mathcal{F},D_2) \neq 1$ that the same property holds either for P_1 or P_2.

We continue with the proof of the theorem. Consider the subset $A_1 = \{P \in A\ :\ Re(I_P(\mathcal{F}',E)) \in \mathbb{Q}_{>0}\}$. Note that A_1 is invariant by ρ and does not have real points. It is possible that A_1 is the empty set. Otherwise, we make the following construction for any pair of points $P, \rho(P) \in A_1$: let E_j be the (unique) component of E such that $P \in E_j$ and consider two different points $Q, Q' \in E_j$ close to P with $Q, Q' \notin A$, and put $B_P = B_{\rho(P)} = \{Q, Q', \rho(Q), \rho(Q')\}$. Consider the set $A' = (A \setminus A_1) \cup (\cup_{P\in A_1} B_P)$ written as

$$A' = \{P_1, \ldots, P_s, Q_1, \rho(Q_1), \ldots, Q_t, \rho(Q_t)\}$$

in such a way that P_j is real and $Q_j, \rho(Q_j)$ are not real. Each P_j is a simple hyperbolic singularity of $\mathcal{F}'$, since $\mathcal{F}$ is a RGC. Then, there exists a non singular real separatrix S_j^σ of $\mathcal{F}'$ at P_j transversal to E. For each pair $Q_j, \rho(Q_j)$, $j = 1, \ldots, t$, take non singular curves S^σ_{s+2j-1}, $S^\sigma_{s+2j} = \rho(S^\sigma_{s+2j-1})$ transversal to E at Q_j and at $\rho(Q_j)$, respectively. Then $S = \sigma(\cup_{j=1}^r S_j^\sigma) = \cup_{j=1}^r S_j$, $r = s + 2t$, is a complex curve at the origin. Consider equations $F = (h_1, \ldots, h_s, g_1, g_1^\rho, \ldots, g_t, g_t^\rho)$ of the branches S_j at the origin, where h_j is real over $\mathbb{R}^2$ and g_j, g_j^ρ are not real. Consider the pair $\mathcal{A} = (\sigma : M \to (\mathbb{C}^2, 0), S)$. Note that $n(\mathcal{A}) = 0$. Let us define a system of indices $\mathcal{I} = \{I_P\}_{P\in E}$ on $\mathcal{A}$ as follows.

1. For $j = 1, \ldots, s$, put $I_{P_j}(E) = I_{P_j}(\mathcal{F}', E)$ and $I_{P_j}(S_j^\sigma) = I_{P_j}(\mathcal{F}', S_j^\sigma)$.
2. For any $k \in \{1, \ldots, t\}$ such that $Q_k \in A \setminus A_1$, define $I_{Q_k}(E) = I_{\rho(Q_k)}(E) = Re(I_{Q_k}(\mathcal{F}', E))$ and $I_{Q_k}(S_{s+2k-1}^\sigma) = I_{\rho(Q_k)}(S_{s+2k}^\sigma) = 1/Re(I_{Q_k}(\mathcal{F}', E))$.
3. If $B_P = \{Q_{k_1}, Q_{k_2}, \rho(Q_{k_1}), \rho(Q_{k_2})\}$ for some $P \in A_1$, choose α_1, α_2 irrational numbers such that $\alpha_1 + \alpha_2 = Re(I_P(\mathcal{F}', E))$ and define $I_{Q_{k_j}}(E) = I_{\rho(Q_{k_j})}(E) = \alpha_j$ and $I_{Q_{k_j}}(S_{s+2k_j-1}^\sigma) = I_{\rho(Q_{k_j})}(S_{s+2k_j}^\sigma) = 1/\alpha_j$, $j = 1, 2$.

Finally, define $I_P = 0$ if $P \notin A'$. By construction, $\mathcal{I}$ is Lins-compatible. The logarithmic foliation $\mathcal{L} = \mathcal{L}_{\underline{\lambda}, F}$ associated to $\mathcal{I}$ in Theorem 1 is a real foliation and a real logarithmic model of $\mathcal{F}$.

Remark 2 The system of indices $\mathcal{I}$ constructed in the above proof satisfy also that the associated exponents can be chosen to be real and equal for the conjugated branches. We deduce that we can get a real logarithmic model of the type $\mathcal{L} = \{d(h_1^{\lambda_1} \cdots h_m^{\lambda_m}) = 0\}$, $m = s + t$, where h_i is real over $\mathbb{R}^2$ and $\lambda_i \in \mathbb{R}$.

4 Examples

Let $\mathcal{F}$ be a real foliation at $0 \in \mathbb{C}^2$ and suppose that $\mathcal{F}$ is a non dicritical CGC. Then it can be shown easily that there exist complex logarithmic models given by Corollary 1 which are also real logarithmic models. The following example shows that sometimes we can construct a simpler real logarithmic model without using the whole complex reduction of singularities.

Example 2 Let $\mathcal{F} = \mathcal{L}_{\underline{\lambda}, F}$ with $F = (y - x^2, y + ix^2, y - ix^2)$ and $\underline{\lambda} = (1, i, -i)$, $i = \sqrt{-1}$. Let $\sigma : M \longrightarrow (\mathbb{C}^2, 0)$ be the composition of two blowing-ups that gives the minimal reduction of singularities of the parabolas $S = S_1 \cup S_2 \cup S_3$, separatrices of $\mathcal{F}$. Then the strict transforms S_1', S_2' and S_3' cut the divisor $E = \sigma^{-1}(0)$ at three different simple singular points P_1, P_2 and P_3 where the indices of the transform $\mathcal{F}'$ are

$$I_{P_1}(\mathcal{F}', S_1') = -2,\ I_{P_2}(\mathcal{F}', S_2') = 2i,\ I_{P_3}(\mathcal{F}', S_3') = -2i.$$

The real part of the indices relative to S_2' and S_3' is zero. Hence $\mathcal{L} = \{d(y - x^2) = 0\}$ is a real logarithmic model of $\mathcal{F}$. This example also shows that not every real logarithmic foliation has a multivalued first integral of the type $h_1^{\lambda_1} \cdots h_m^{\lambda_m}$, with h_j real over real points and $\lambda_i \in \mathbb{R}$, although real logarithmic models having this property always exists as mentioned in Remark 2.

It is well known that the multiplicity of a non dicritical CGC at the origin is given by $m - 1$ where m is the multiplicity of its set of separatrices. Moreover, as it is shown in [11], two non dicritical CGC sharing a complex logarithmic model (equivalently with the same set of separatrices and with the same system of indices) of multiplicity ν are defined by 1-forms with the same ν-jet when we fix the coordinates. In the real case these properties are not satisfied.

Example 3 *Let* $\mathcal{F} = \{\omega = y^3 dx + (x^2 + y^2 - x^3)dy = 0\}$. *It is a RGC but not a CGC. It has a single non singular real separatrix* S_1 *at* 0. *The (non singular) foliation* $\mathcal{L} = \{df_1 = 0\}$ *is a real logarithmic model of* $\mathcal{F}$, *where* $f_1 = 0$ *is a real equation of* S_1.

5 Applications to center-focus foliations

Let us finish by discussing the relation between center-focus real foliations and existence of real separatrices.

Recall that a real analytic foliation $\mathcal{F}$ at $0 \in \mathbb{C}^2$ is said to be of *center-focus* type if there is a first returning map $P : \ell \to \ell$ of the real leaves of $\mathcal{F}$ on a (germ of a) half straight real line ℓ through 0, transversal to $\mathcal{F}$. This is equivalent (see [18, 1, 6, 12]) to any of the following properties:

a) there are no real leaves of $\mathcal{F}$ which accumulate to the origin with a well defined tangent (*characteristic orbits*).

b) $\mathcal{F}$ is non real-dicritical and if $\pi : M \longrightarrow (\mathbb{C}^2, 0)$ is a real resolution of singularities of $\mathcal{F}$ the real singular points of the transform $\mathcal{F}'$ are situated at corners of the divisor $E = \pi^{-1}(0)$ and are all of *saddle type* (with indices of opposite sign along the two separatrices).

If $\mathcal{F}$ is a RGC of center-focus type then any of its real logarithmic models is also of center-focus type, but we can not distinguish, using real logarithmic models, if the original foliation is a center (first returning map P equal to the identity) or a focus ($P \neq id$). For instance, the foliations

$$\mathcal{F}_1 = \{d((x^2 + y^2)(x + iay)^i (x - iay)^{-i}) = 0\}$$
$$\mathcal{F}_2 = \{d((x^2 + y^2)(x^2 + a^2 y^2)) = 0\}$$

are real logarithmic models one of each other, $\mathcal{F}_1$ having a focus and $\mathcal{F}_2$ having a center at the origin.

Evidently, center-focus foliations can not have real analytic separatrices, but the converse is not true. An explicit example is given for instance in [19]:

$$\mathcal{F} = \{(xy + x^3 + x^3 y)dx - (-y^2 + x^4)dy = 0\}.$$

For this foliation, all the characteristic orbits are (infinitely) tangent with a formal, non convergent (real) separatrix $\widehat{S}$ at 0.

In general, any formal non convergent real separatrix $\widehat{S}$ of a real analytic foliation $\mathcal{F}$ corresponds to a saddle-node in the real reduction of singularities (and thus such a foliation is not a RGC) and it has associated at least a characteristic orbit with the same tangent as $\widehat{S}$ (actually the same iterated tangents, see [6]).

Hence, a center-focus vector field can not have formal or convergent real separatrices. Our final example shows that the converse is not true, neither for real CGC: there can be characteristic orbits without formal or convergent real separatrix. The example is giving by using the construction in Theorem 1 of a logarithmic foliation with prescribed indices.

Example 4 *Let $\pi_1 : M_1 \to (\mathbb{C}^2, 0)$ be the blowing up at the origin and $\pi_2 : M \to M_1$ be the blowing up at a real point in the exceptional divisor $\pi_1^{-1}(0)$ of π_1. Let $\pi = \pi_1 \circ \pi_2$. The exceptional divisor E of π consists of two components $E = E_1 \cup E_2$, where E_2 is the exceptional divisor of π_2 and E_1 is the strict transform of $\pi_1^{-1}(0)$ by π_2, intersecting at a real point $P \in M$. Consider $P_i \in E_i$, $i = 1, 2$ non real points and S_i^π a (germ of a) non singular complex analytic curve through P_i transversal to E_i. Let $\rho(P_i)$, $\rho(S_i^\pi)$ be the images of P_i, S_i^π by the complex self-conjugation $\rho : M \to M$. Then $S = \pi(S_1^\pi) \cup \pi(S_2^\pi) \cup \pi(\rho(S_1^\pi)) \cup \pi(\rho(S_1^\pi))$ is a complex curve at the origin of $\mathbb{C}^2$ for which π is the minimal reduction of singularities. Consider the pair $\mathcal{A} = (M \xrightarrow{\pi} (\mathbb{C}^2, 0), S)$ and a system of indices $\mathcal{I}$ on $\mathcal{A}$ given as follows: let λ be an irrational positive number and put*

$$
\begin{array}{l}
I_{P_1}(E_1) = -(1 + \lambda/2), \;\; I_{P_1}(S_1^\pi) = 1/I_{P_1}(E_1) \\
I_{P_2}(E_2) = -(\lambda + 1)/2\lambda, \;\; I_{P_1}(S_2^\pi) = 1/I_{P_2}(E_2) \\
I_{\rho(P_1)}(E_1) = I_{P_1}(E_1), \;\; I_{\rho(P_1)}(S_1^\pi) = 1/I_{\rho(P_1)}(E_1) \\
I_{\rho(P_2)}(E_2) = I_{P_2}(E_2), \;\; I_{\rho(P_2)}(S_2^\pi) = 1/I_{\rho(P_2)}(E_2) \\
I_P(E_1) = \lambda, \; I_P(E_2) = 1/\lambda.
\end{array}
$$

A calculation shows that $\mathcal{I}$ is a Lins-compatible system of indices on $\mathcal{A}$. By Theorem 1, there exists a logarithmic non-dicritical foliation $\mathcal{F}$ with $Sep_0(\mathcal{F}) = S$ so that the strict transform $\mathcal{F}'$ by π has indices given by $\mathcal{I}$. On the other hand, the choice of the curves $S_i^\pi, \rho(S_i^\pi)$ and the system of indices implies that $\mathcal{F}$ is a real foliation and π is a real reduction of singularities of it. Finally, there is no real singular points of the transform $\mathcal{F}'$ except for the point $P = E_1 \cap E_2$ where $\mathcal{F}'$ has a focus with eigenvalues $1, \lambda$ and E_1, E_2 as the only real separatrices at P. We conclude that $\mathcal{F}$ is not a center-focus foliation.

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