

QUASI-ANALYTIC SOLUTIONS OF ANALYTIC ORDINARY DIFFERENTIAL EQUATIONS AND O-MINIMAL STRUCTURES

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ABSTRACT

It is well known that the non-spiraling leaves of real analytic foliations of codimension 1 all belong to the same o-minimal structure. Naturally, the question arises if the same statement is true for non-oscillating trajectories of real analytic vector fields.

We show, under certain assumptions, that such a trajectory generates an o-minimal and model complete structure together with the analytic functions. The proof uses the asymptotic theory of irregular singular ordinary differential equations in order to establish a quasi-analyticity result from which the main theorem follows.

As applications, we present an infinite family of o-minimal structures such that any two of them do not admit a common extension, and we construct a non-oscillating trajectory of a real analytic vector field in \mathbb{R}^5 that is not definable in any o-minimal extension of \mathbb{R} .

1. Introduction

Consider a system of ordinary differential equations having an irregular singular point:

$$x^{p+1} \frac{dy}{dx} = A(x, \mathbf{y}), \quad \mathbf{y} \in \mathbb{R}^r, \quad (1.1)$$

where A is real analytic in a neighborhood of $0 \in \mathbb{R}^{r+1}$ with $A(0, 0) = 0$ and the *Poincaré rank* p is greater or equal to one. In a series of articles [14], [4], [5], *non-oscillating solutions* of (1.1) have been studied; here a solution (vector) $H = (H_1, \dots, H_r) : (0, \varepsilon] \rightarrow \mathbb{R}^r$ of (1.1) tending to 0 as $x \rightarrow 0$ is called non-oscillating if for any function f real analytic in some neighborhood of $0 \in \mathbb{R}^{r+1}$, the function $x \mapsto f(x, H(x))$ either is identically 0 or has finitely many zeros.

One of the motivations of our work is the more general question whether the structure $\mathbb{R}_{an, H}$ generated by H_1, \dots, H_r and the restricted analytic functions is *o-minimal*, i.e. the sets definable in that structure have a finite number of connected components (with respect to the usual topology of \mathbb{R}^m). The definable sets of $\mathbb{R}_{an, H}$ are those that belong to the smallest family of subsets of \mathbb{R}^m , $m \in \mathbb{N}$, that contains the graphs of addition, multiplication, restrictions of analytic functions to cartesian products of compact intervals and H_1, \dots, H_r and that is closed under finite unions and intersections, cartesian products, complements and linear projections. It is easy to show that the graphs of the above functions $f(x, H(x))$, f real analytic in a neighborhood of 0, are definable, but more generally also the graph of any function obtained from restricted analytic functions and H_j , $j = 1, \dots, r$ by (repeated) compositions and resolution of implicit equations. The desired o-minimality of $\mathbb{R}_{an, H}$ thus would imply that they all have a finite number of zeros.

For bounded non-spiraling leaves of real analytic foliations of codimension 1, such a result has been shown (see [28], [15]): the structure generated by all such leaves is o-minimal.

In our main result (Theorem 2.5), we consider a system (1.1) such that a) the eigenvalues $\lambda_1, \dots, \lambda_r$ of $\frac{\partial A}{\partial \mathbf{y}}(0, 0)$ are nonzero and have distinct arguments; and b) the Stokes phenomenon of the formal solution \widehat{H} is non-trivial for at least one direction corresponding to each λ_j , $j = 1, \dots, r$. Observe that this implies the divergence of \widehat{H} . We consider an arbitrary solution $H : (0, \varepsilon] \rightarrow \mathbb{R}^r$ of such an equation having an asymptotic expansion $H(x) \sim \widehat{H}(x) = \sum_{n=1}^{\infty} h_n x^n$ as $x \rightarrow 0$ and prove that the above structure $\mathbb{R}_{an,H}$ generated by the analytic functions and H_1, \dots, H_r is o-minimal and model complete; a structure is called *model complete*, if the same family of subsets is generated without taking complements. Observe that our theorem can also be applied to C^∞ -solutions $H : [-\varepsilon, \varepsilon] \rightarrow \mathbb{R}^r$ of (1.1) if p is odd: it suffices to consider $\widehat{H} : (0, \varepsilon] \rightarrow \mathbb{R}^{2r}$ defined by $\widehat{H}(x) = (H(x), H(-x))$. Condition a) has to be replaced by the following one: the eigenvalues $\lambda_1, \dots, \lambda_r$ of $\frac{\partial A}{\partial \mathbf{y}}(0, 0)$ are nonzero and have arguments distinct mod $\pi\mathbb{Z}$.

The model completeness is linked to the problem of describing $\mathbb{R}_{an,H}$ in terms of H and analytic functions: here we show (Proposition 2.2) that any definable set in $\mathbb{R}_{an,H}$ can be obtained as finite union of linear projections of sets of the form $A = X \cap V_1 \cap V_2 \cdots \cap V_n$ where X is globally semi-analytic (see subsection 2.1) and $V_i = \{(x_1, \dots, x_m) / x_k = H_s(x_l)\}$ for some $1 \leq k, l \leq m$ and $1 \leq s \leq r$.

As applications of our result, we present a two-parameter family of solutions of some two-dimensional system (1.1) such that the o-minimal structures generated by each of them are mutually incompatible, i.e. any two of them do not admit a common o-minimal extension – this is very different from the theory of (bounded) non-spiraling leaves of analytic foliations: they *all* belong to one common o-minimal structure. Moreover, we construct a non-oscillating trajectory of a system (1.1) in \mathbb{R}^4 that cannot belong to any o-minimal structure.

Our main result is a consequence of two theorems. The first (Theorem 2.1) states that strong quasi-analyticity of H (see Definition 1) implies o-minimality and model completeness of the structure $\mathbb{R}_{an,H}$, the second (Theorem 2.4) states that our conditions a), b) imply strong analytic transcendence of the formal solution $\widehat{H}(x)$ of (1.1) which in turn yields that the above H is strongly quasi-analytic. Its proof is based on the theory of irregular singular points of ordinary differential equations in the complex domain.

The article is organized as follows: in section 2, we present precise definitions and theorems as well as examples and applications, section 3 is devoted to the proof of Theorem 2.1, section 4 to that of Theorem 2.4. An outline of these proofs can be found at the end of subsection 2.2.

2. Results

2.1. Preliminaries

Before stating the results, let us briefly recall the terminology of o-minimal structures (see [7] for an introduction). Consider a family $\mathcal{F} = (\mathcal{F}_m)_{m \in \mathbb{N}}$ of sets \mathcal{F}_m of functions $f : \mathbb{R}^m \rightarrow \mathbb{R}$. A subset of \mathbb{R}^m is said to be *definable in the structure*

$\mathbb{R}_{\mathcal{F}}$ generated by \mathcal{F} (or in the extension of the real field by \mathcal{F}) if it belongs to the smallest collection of subsets of \mathbb{R}^m , $m \in \mathbb{N}$, satisfying:

- S1) It is closed under cartesian products, finite unions and intersections, complements and under images of linear projections $\mathbb{R}^{m+1} \rightarrow \mathbb{R}^m$.
- S2) It contains the diagonals $\Delta_{ij} = \{(x_1, \dots, x_m) \in \mathbb{R}^m / x_i = x_j\}$ for $1 \leq i < j \leq m$.
- S3) It contains the point sets $\{a\}$ for $a \in \mathbb{R}$, as well as the graphs of addition and multiplication $+, \cdot : \mathbb{R}^2 \rightarrow \mathbb{R}$.
- S4) It contains the graph of all function in \mathcal{F} .

If some function of the family \mathcal{F} is defined on a proper subset $D \subset \mathbb{R}^m$, then it is understood without mentioning this, that the function is extended to \mathbb{R}^m by the value 0 in the above definition.

The structure $\mathbb{R}_{\mathcal{F}}$ is called *o-minimal* if every definable set has a finite number of connected components (with respect to the ordinary topology). It is called *model complete* if in the above definition the condition of closure under complements is superfluous.

For $\mathcal{F} = \emptyset$, the structure $\mathbb{R}_{\mathcal{F}} = \mathbb{R}_{alg}$ is o-minimal and model complete and its definable sets are the *semialgebraic sets*, i.e. finite unions of subsets of \mathbb{R}^m of the form $\{\mathbf{x} \in \mathbb{R}^m / P(\mathbf{x}) = 0, Q_1(\mathbf{x}) > 0, \dots, Q_s(\mathbf{x}) > 0\}$ where P, Q_1, \dots, Q_s are polynomials. Another classical example uses real analytic functions instead of polynomials. A subset $A \subset \mathbb{R}^n$ is called *semianalytic at a point* $a \in \mathbb{R}^n$ if there exists an open neighborhood U of a such that $A \cap U$ is a finite union of sets of the form $\{\mathbf{x} \in U / f(\mathbf{x}) = 0, g_1(\mathbf{x}) > 0, \dots, g_s(\mathbf{x}) > 0\}$, where f, g_1, \dots, g_s are analytic functions on U . The set $A \subset \mathbb{R}^n$ is called *global semianalytic* if $\rho_n(A) \subset [-1, 1]^n$ is semianalytic at every point, where $\rho_n(x_1, \dots, x_n) = (x_1/\sqrt{1+x_1^2}, \dots, x_n/\sqrt{1+x_n^2})$. Finally, a *global subanalytic set* of \mathbb{R}^m is the linear projection of a global semianalytic set $A \subset \mathbb{R}^n$ for some $n \geq m$. The collection of global subanalytic sets is precisely the collection of definable sets in the structure \mathbb{R}_{an} generated by the set \mathcal{F}_{an} of *restricted analytic functions*. These are the functions $f : \mathbb{R}^m \rightarrow \mathbb{R}$ for which there exists an analytic function f defined in a neighborhood of $[-1, 1]^m$ such that $\tilde{f} = f$ on $[-1, 1]^m$ and $\tilde{f} = 0$ outside $[-1, 1]^m$. It is known that \mathbb{R}_{an} is o-minimal and model complete: this result is a consequence of Gabrielov's Complement Theorem asserting that the complement of a global subanalytic set is again global subanalytic [9].

2.2. Statement of the results

Consider a system (1.1) of analytic ordinary differential equations and a solution (vector) $H = (H_1, \dots, H_r) : (0, \varepsilon] \rightarrow \mathbb{R}^r$ with $\lim_{x \rightarrow 0} H(x) = 0$ admitting an asymptotic expansion $H(x) \sim \widehat{H}(x) = \sum_{n=0}^{\infty} h_n x^n \in \mathbb{R}[[x]]^r$ as $\mathbb{R}_+ \ni x \rightarrow 0$, i.e. for any natural number $N \geq 0$, there are constants $C_N, \delta_N > 0$ such that

$$\| H(x) - \sum_{n=0}^N h_n x^n \| \leq C_N x^{N+1}, \quad 0 < x < \delta_N.$$

We prove below that $\widehat{H}(x)$ is a formal solution of (1.1) and that H extends to a \mathcal{C}^∞ function on $[0, \varepsilon]$ having $\widehat{H}(x)$ as its Taylor series at the origin (cf. subsection 3.1 for details).

As a matter of notation, if $\phi(x)$ is a formal power series or a \mathcal{C}^∞ function at the origin, let $J_k\phi(x) = \sum_{i=0}^k \frac{\phi^{(i)}(0)}{i!} x^i \in \mathbb{R}[x]$ denote the k -jet of ϕ as $\mathbb{R}^+ \ni x \rightarrow 0$ and let $T_k\phi(x)$ denote $(\phi(x) - J_k\phi(x))/x^k$. Denote by $\text{val } \phi$ the valuation of ϕ at the origin, i.e. the minimum of all l with $\phi^{(l)}(0) \neq 0$ and by $\text{deg } P$ the degree of a polynomial P .

DEFINITION 1. We say that a solution $H = (H_1, \dots, H_r)$ of (1.1) is *strongly quasi-analytic* if it tends to 0 at the origin, admits an asymptotic expansion $H(x) \sim \widehat{H}(x)$ as $x \rightarrow 0$ and satisfies the following condition:

(SQA) If $k \geq 0$, $n \geq 1$, an analytic function $f \in \mathbb{R}\{x, z_{11}, \dots, z_{rn}\}$ with $f(0) = 0$ and polynomials $P_1(x), \dots, P_n(x)$ with $\text{val } P_l > 0$ and $P_l^{(\text{val } P_l)}(0) > 0$ are given, then one has

$$f(x, \{T_k \widehat{H}_j(P_l(x))\}_{j,l}\}) \equiv 0 \implies f(x, \{T_k H_j(P_l(x))\}_{j,l}\}) \equiv 0.$$

Here, the word “strongly” reflects the fact that polynomials P_l are allowed as arguments of H ; this will be necessary in the sequel (see also Example 3 below). The condition on the derivatives of the P_l is imposed by the fact that H is defined only for $x > 0$.

The condition (SQA) not only excludes exponentially small solutions of (1) (if those exist as e.g. for $x^2 y' = y$), but also the possibility that exponentially small functions might be obtained as certain combinations of H_j and analytic functions.

THEOREM 2.1. *Suppose that $H : (0, \varepsilon] \rightarrow \mathbb{R}^r$ is a strongly quasi-analytic solution of the system (1.1). Then the structure $\mathbb{R}_{an,H} := \mathbb{R}_{\widehat{\mathcal{F}}_{an} \cup \{H\}}$ generated by the restricted analytic functions and H is o-minimal and model complete.*

We can be more explicit about the description of the definable sets in $\mathbb{R}_{an,H}$. Let us say that a set $A \subset \mathbb{R}^m$ is *H-semianalytic at a point* $a \in \mathbb{R}^m$ if there exists $\delta > 0$ such that $(A - a) \cap [-\delta, \delta]^m$ is a finite union of sets of the form $X \cap V_1 \cap \dots \cap V_n$, for $n \geq 0$, where X is semianalytic at every point of $[-\delta, \delta]^m$ and $V_i = \{(x_1, \dots, x_m) \in [-\delta, \delta]^m \mid 0 < x_l \leq \varepsilon, x_k = H_s(x_l)\}$ for some $1 \leq k, l \leq m$ and $1 \leq s \leq r$. The set A is said to be *global H-semianalytic* if $\rho_m(A)$ is *H-semianalytic* at any point of $[-1, 1]^m$, where $\rho_m(x_1, \dots, x_m) = (x_1/\sqrt{1+x_1^2}, \dots, x_m/\sqrt{1+x_m^2})$. Finally, a set $B \subset \mathbb{R}^m$ is said to be (*global*) *H-subanalytic* if there exists $n \geq m$ and some global *H-semianalytic* set $A \subset \mathbb{R}^n$ such that $B = \pi(A)$, where $\pi : \mathbb{R}^n \rightarrow \mathbb{R}^m$ is the projection onto the first m coordinates.

PROPOSITION 2.2. *Under the hypotheses of Theorem 2.1, the collection of H-subanalytic sets is stable with respect to taking complements and coincides with the collection of definable sets in $\mathbb{R}_{an,H}$.*

Property (SQA) is trivially satisfied for its sum if the formal series solution $\widehat{H}(x)$ converges. In this case, the sum of $\widehat{H}(x)$ is the only strongly quasi-analytic solution among those having $\widehat{H}(x)$ as an asymptotic expansion (irrespective of the fact that they may generate an o-minimal structure: $H(x) = \exp(-1/x)$, $x > 0$, or not: $H(x) = (H_1(x), H_2(x)) = (\exp(-1/x) \sin(1/x), \exp(-1/x) \cos(1/x))$, $x > 0$).

If $\widehat{H}(x)$ is *multisummable* in the positive real direction \mathbb{R}^+ in the sense of [20], then the restriction $H : (0, \varepsilon) \rightarrow \mathbb{R}$ of the multisum of \widehat{H} along \mathbb{R}^+ is strongly quasi-analytic. (We point out that, for this particular solution, the \mathfrak{o} -minimality of the structure $\mathbb{R}_{an, H}$ is a consequence of the results in [8]).

In the above examples only particular solutions are (or are shown to be) strongly quasi-analytic. We will state conditions on the system (1.1) which imply (SQA) for *any* solution $H : (0, \varepsilon] \rightarrow \mathbb{R}^r$ having an asymptotic expansion $H(x) \sim \widehat{H}(x)$ as $\mathbb{R}^+ \ni x \rightarrow 0$:

- We assume that the linear part $A_0 = \frac{\partial A}{\partial \mathbf{y}}(0)$ is non singular and that its eigenvalues $\{\lambda_1, \dots, \lambda_r\}$ satisfy:

$$\arg(\lambda_i) \not\equiv \arg(\lambda_j) \pmod{2\pi\mathbb{Z}} \quad \text{if } i \neq j. \quad (2.1)$$

- Condition (2.1) implies that the system has a unique formal power series solution $\widehat{H}(x) \in \mathbb{R}[[x]]^r$ (see for instance [27]). Moreover, $\widehat{H}(x)$ is *p*-summable (see subsection 4.1 for details) in each direction $d_\theta = \{z / \arg(z) = \theta\}$ in the complex plane except for the *singular directions*

$$\{d_{\theta_{l,j}} / p \theta_{l,j} = \arg(\lambda_j) + 2\pi l\}_{1 \leq j \leq r; 0 \leq l \leq p-1}.$$

To each $d_{\theta_{l,j}}$ corresponds a *Stokes coefficient* $c_{l,j} \in \mathbb{C}$ (see again subsection 4.1 for details). If $\widehat{H}(x)$ diverges, at least one of the Stokes coefficients must be nonzero; we assume that moreover

$$\forall j \in \{1, \dots, r\} \exists l = l(j) \in \{0, \dots, p-1\} \text{ with } c_{l,j} \neq 0. \quad (2.2)$$

Observe that $\widehat{H}(x)$ is not summable in the real positive direction and thus the results of [8] do not apply if $c_{l,j} \neq 0$ for some (l, j) with $\theta_{l,j} \equiv 0 \pmod{2\pi\mathbb{Z}}$.

THEOREM 2.3. *Consider a system of analytic ordinary differential equations (1.1) such that conditions (2.1) and (2.2) are fulfilled. Then any solution $H = (H_1, \dots, H_r) : (0, \varepsilon] \rightarrow \mathbb{R}^r$ of (1.1) having an asymptotic expansion $H(x) \sim \widehat{H}(x)$ as $\mathbb{R}^+ \ni x \rightarrow 0$ is strongly quasi-analytic.*

In fact, we will prove the following stronger result:

THEOREM 2.4. *Under the hypothesis of Theorem 2.3, the formal solution \widehat{H} is strongly analytically transcendental in the following sense:*

(SAT) *If $k \geq 0$, $n \geq 0$, an analytic function $f \in \mathbb{R}\{x, z_{11}, \dots, z_{rn}\}$ with $f(0) = 0$ and distinct polynomials $P_1(x), \dots, P_n(x)$ with $\deg P_l < (p+1)\text{val } P_l$ and $P_l^{(\text{val } P_l)}(0) > 0$ are given, then one has*

$$f(x, \{T_k \widehat{H}_j(P_l(x))\}_{j,l}) \equiv 0 \implies f \equiv 0.$$

We will prove below (cf. Lemma 4.1) that property (SAT) for the formal solution $\widehat{H}(x)$ of (1.1) implies property (SQA) for any solution $H(x)$ with $H(x) \sim \widehat{H}(x)$ as $\mathbb{R}^+ \ni x \rightarrow 0$. Combining Theorems 2.1 and 2.3, we can state our main result

THEOREM 2.5. *Let $x^{p+1}dy/dx = A(x, \mathbf{y})$ be a system of analytic ordinary differential equations satisfying the above conditions (2.1) and (2.2). Then, for any solution $H : (0, \varepsilon] \rightarrow \mathbb{R}^r$ admitting an asymptotic expansion $H(x) \sim \widehat{H}(x)$ as*

$\mathbb{R}^+ \ni x \rightarrow 0$, the structure $\mathbb{R}_{an,H}$ generated by the restricted analytic functions and H is o -minimal and model complete.

The proof of these results is organized as follows. In paragraph 3, we prove Theorem 2.1 and Proposition 2.2; the main idea is to apply a fundamental result of [22] to a certain class of functions corresponding to H . Let us give here a brief description of the principal arguments.

First, we show that we can suppose the solution H to be a C^∞ function on $[-\varepsilon, \varepsilon]$, by considering the ramification $x \mapsto H(x^2)$. Then we consider the smallest class $\mathcal{A} = (\mathcal{A}_m)_{m \geq 1}$ of germs (at the origin of \mathbb{R}^m) of C^∞ functions that contains the germs of analytic functions and those of the components H_j of H and that is stable under composition, division by monomials and under taking solutions of implicit equations. It is shown in [22], that the quasi-analyticity of the class \mathcal{A} suffices to prove a *Theorem of the Complement for \mathcal{A}* : the “subanalytic sets” defined by using elements of \mathcal{A} have a finite number of connected components and they form a family that is stable under taking complements. Theorem 2.1 and Proposition 2.2 follow by classical arguments.

The difficult part is to show that \mathcal{A} is a *quasi-analytic* class: if $\phi \in \mathcal{A}_m$ has a vanishing Taylor series at $0 \in \mathbb{R}^m$ then $\phi = 0$. Here, we first reduce the problem, by restricting to appropriate paths, to the quasi-analyticity of the subclass \mathcal{A}_1 of germs of functions of a single variable.

The quasi-analyticity of \mathcal{A}_1 is not an obvious consequence of hypothesis (SQA); \mathcal{A}_1 seems to be a very large class of functions, containing for instance compositions $H_j \circ H_l$ of components of H and implicitly defined functions. By means of the crucial Lemma 3.1 and a description of the functions of \mathcal{A} (Lemma 3.9), we get rid of this “double transcendence” and we can prove that \mathcal{A}_1 is the set of so-called *simple functions*: germs of functions of the form $x \mapsto f(x, \{T_k H_j(P_l(x))\}_{j,l})$ where f is analytic and P_l are polynomials. This reduction justifies our condition (SQA).

In paragraph 4, we prove Theorem 2.4; roughly, we proceed as follows: since $T_k H(x)$ satisfies a system like (1.1) as well, we can first suppose that $k = 0$. Consider the formal series $\widehat{F}(x) = f(x, \{\widehat{H}_j(P_l(x))\})$, where f and P_l are as in the hypothesis of the theorem. On a certain family V_j of sectors covering a disk punctured at $0 \in \mathbb{C}$, we construct functions $F_j : V_j \rightarrow \mathbb{C}$ having \widehat{F} as their common asymptotic expansion.

We show two results. On the one hand, if we suppose that $\widehat{F}(x)$ is identically zero, then, by Ramis-Sibuya’s Lemma and the Relative Watson’s Lemma, any F_j is exponentially flat of order strictly greater than $p\nu$ where p is the Poincaré rank of the system and ν is the minimum of the valuations of the polynomials P_l . On the other hand, assuming the condition (2.2) (and a preliminary normalization of the function f), we show that if f is not identically zero, then at least one of the differences $F_{j+1} - F_j$ is exponentially flat of order exactly $p\nu$. Thus, by the first result, f must vanish identically if $\widehat{F}(x) \equiv 0$.

2.3. Examples and applications

In the following examples and applications we discuss the conditions (2.1) and (2.2) and their relations with the properties (SQA) and (SAT).

EXAMPLE 1. It is not sufficient in Theorem 2.5 (and Theorem 2.3 as well) to assume only that $H(x) \rightarrow 0$ as $x \rightarrow 0$. In the example

$$x^2 y' = \left[\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} + \mu x \operatorname{id} \right] y + \begin{pmatrix} x \\ x \end{pmatrix}, \quad \mu > 0,$$

the eigenvalues of the leading matrix are $\pm i$ and hence the unique formal solution is 1-summable in the positive real direction; conditions (2.1) and (2.2) are also satisfied (compare examples 4, 5). Moreover, the system has a homogeneous solution of the form $x^\mu (\sin(1/x), \cos(1/x))^T$ tending to 0 as $x \rightarrow 0$, whose graph obviously cannot belong to any o-minimal structure and which *does not have an asymptotic expansion* in a power series as $x \rightarrow 0$. By superposition, the same is true for the above non-homogeneous system.

EXAMPLE 2. (**Plane Pfaffian curves**) A first consequence of our results is the model completeness of the structures generated by certain pfaffian curves. Consider a 1-form $\omega = a(x, y)dx + b(x, y)dy$ real analytic in some neighborhood of $0 \in \mathbb{R}^2$ with $a(0, 0) = b(0, 0) = 0$. A *pfaffian curve* defined by ω is the graph \mathcal{C} of a non-oscillating solution $H : (0, \varepsilon] \rightarrow \mathbb{R}$ tending to 0 as $x \rightarrow 0$ of the differential equation $b(x, y) \frac{dy}{dx} = -a(x, y)$. The o-minimality of the structure $\mathbb{R}_{an, H}$ is a particular case of the results proved in [28, 15].

Analytic changes of coordinates or blowups are obviously inessential for the question of model completeness. Therefore, by a classical theorem on the reduction of singularities [24], we can suppose that the origin is a simple singularity of ω and hence that H is of one of the following types:

- (i) H is analytic also at $x = 0$. In this case, the model completeness of $\mathbb{R}_{an, H}$ is proved in [10].
- (ii) $H(x) = x^\lambda$ for some irrational $\lambda > 0$. In this case, the model completeness is proved in [18].
- (iii) H is a solution of some *saddle-node equation*

$$x^{p+1} \frac{dy}{dx} = y + A_1(x, y), \quad p \geq 1, \quad (2.3)$$

where A_1 is analytic at the origin and $A_1(0) = \frac{\partial A_1}{\partial y}(0) = 0$. In this case, if the Taylor series $\widehat{H}(x)$ is divergent, the conditions of Theorem 2.5 are satisfied, because a formal solution $\widehat{H}(x)$ without a singular direction having non-trivial Stokes phenomenon would be convergent (see also [17]). Thus the structure $\mathbb{R}_{an, H}$ is model complete. If $\widehat{H}(x)$ converges, however, the model completeness is still an open question.

EXAMPLE 3. Assume that (1.1) has a formal power series solution $\widehat{H}(x) \in \mathbb{R}[[x]]^r$. Contrary to the scalar case, if $r > 1$ then divergence of $\widehat{H}(x)$ does not suffice for strong quasi-analyticity of solutions with asymptotic expansion $\widehat{H}(x)$. Even the condition that $\widehat{H}(x)$ is *analytically transcendental*, i.e. $f(x, \widehat{H}(x))$ is not identically zero for all non zero analytic function $f(x, \mathbf{y})$, does not imply strong quasi-analyticity (compare with (SAT)).

For example, let $E : x \mapsto E(x)$, $x > 0$, be a solution of the Euler equation $x^2 \frac{dy}{dx} = y - x$ and define the function $H : (0, \infty) \rightarrow \mathbb{R}^2$ by $H(x) = (E(x) +$

$\exp(-1/x), E(2x)$). It satisfies the system of differential equations

$$x^2 \frac{dy_1}{dx} = y_1 - x, \quad x^2 \frac{dy_2}{dx} = \frac{y_2}{2} - x. \quad (2.4)$$

Clearly, H does not satisfy condition (SQA). However, its asymptotic expansion $\widehat{H}(x) = (\widehat{E}(x), \widehat{E}(2x))$ diverges, $\widehat{E}(x) = \sum_{n=1}^{\infty} (n-1)!x^n$ being the *Euler series*. Moreover, since $\widehat{E}(x)$ satisfies the property (SAT) by Theorem 2.4 applied to the Euler equation, $\widehat{H}(x)$ is analytically transcendental.

Notice that the linear part of (2.4) has eigenvalues $1, 1/2$, so that condition (2.1) is not satisfied.

EXAMPLE 4. (Two dimensional systems with non-real eigenvalues)

Consider a system (1.1) with $r = 2$ such that the linear part $A_0 = \partial A / \partial \mathbf{y}(0)$ has two non real conjugate eigenvalues $\lambda_1, \lambda_2 = \overline{\lambda_1}$ (in particular, it satisfies condition (2.1)). Let $\widehat{H}(x) \in \mathbb{R}[[x]]^2$ be its formal power solution. The singular directions $d_{\theta_{l,j}}$ of $\widehat{H}(x)$ satisfy $\theta_{l,1} = -\theta_{k,2}$ if $l \equiv -k \pmod{p}$ and, since $\widehat{H}(x)$ has real coefficients, the corresponding Stokes coefficients $c_{l,1}, c_{k,2}$ are complex conjugate. Thus, condition (2.2) is equivalent to the existence of a non zero Stokes coefficient for some singular direction and hence to the divergence of the series $\widehat{H}(x)$. We can summarize the preceding discussion by stating the following

COROLLARY 2.6. *Consider a two dimensional system (1.1) such that the linear part has no real eigenvalues and suppose that its formal power series solution $\widehat{H}(x) \in \mathbb{R}[[x]]^2$ does not converge. Then $\widehat{H}(x)$ is strongly analytically transcendental (SAT). As a consequence, any solution $H : (0, \varepsilon] \rightarrow \mathbb{R}^2$ with $H(x) \sim \widehat{H}(x)$ as $\mathbb{R}^+ \ni x \rightarrow 0$ is strongly quasi-analytic and the structure $\mathbb{R}_{an,H}$ is o-minimal and model complete.*

EXAMPLE 5. (Non-oscillating solutions and o-minimal structures)

Two dimensional systems (1.1) with non real eigenvalues as in Corollary 2.6 were already treated in [5] in the context of analytic three dimensional vector fields. There, the authors study the whole family *IP* (“integral pencil”) of solutions $H : (0, \varepsilon] \rightarrow \mathbb{R}^2$ of (1.1) having $\widehat{H}(x)$ as the asymptotic expansion as $\mathbb{R}^+ \ni x \rightarrow 0$. They show that any element in *IP* is non-oscillating if and only if $\widehat{H}(x)$ is divergent. Moreover, in that case, $\widehat{H}(x)$ is analytically transcendental and any pair of solutions in *IP* are *asymptotically linked* in the sense that each one “turns around” the other.

Let us apply our results to an example considered in [5], generalizing the Euler equation to dimension 2:

$$\begin{cases} x^2 \frac{dy_1}{dx} = y_1 + y_2 - x, \\ x^2 \frac{dy_2}{dx} = y_2 - y_1. \end{cases} \quad (2.5)$$

The eigenvalues of its linear part are $1 \pm i$. It is easily seen that any solution $H(x) = (H_1(x), H_2(x))$ of (2.5) defined for $x > 0$ verifies $\lim_{x \rightarrow 0} H(x) = 0$ and that all such solutions have a common asymptotic expansion $\widehat{H}(x) \in \mathbb{R}[[x]]^2$ as $\mathbb{R}^+ \ni x \rightarrow 0$. Moreover, $\widehat{H}(x)$ is divergent; this follows from the fact that $\widehat{y}(x) = \widehat{H}_1(x) + i\widehat{H}_2(x)$ is the formal solution of a modification $x^2 y' = (1-i)y - x$ of Euler’s equation and hence $\widehat{y}(x) = \sum_{n=1}^{\infty} (n-1)! (1-i)^{-n} x^n$ diverges. By Corollary 2.6, the structure $\mathbb{R}_{an,H}$ is o-minimal and model complete for any such solution.

The most important feature of this example is the following: given two *different* solutions $H, G : (0, \varepsilon) \rightarrow \mathbb{R}^2$ of (2.5), the argument of the vector $x \mapsto H(x) - G(x)$ tends to infinity as $x \rightarrow 0$ (“asymptotic linking”). As a consequence, the two structures $\mathbb{R}_{an,H}, \mathbb{R}_{an,G}$ do not admit any common o-minimal extension. Thus, we obtain an explicit family of “mutually incompatible” o-minimal structures. Examples of incompatible pairs of o-minimal structures were already exhibited in [22].

It is natural to wonder if the o-minimality of these structures could arise from well-known results of pfaffian geometry. It is proved in [25] that the so-called *pfaffian closure* $\mathcal{S}_{\text{Pfaff}}$ of any o-minimal structure \mathcal{S} is o-minimal as well. Because of the previous remark, it is clear that at most one of the structures $\mathbb{R}_{an,H}$ could be a reduct of the pfaffian closure $\mathbb{R}_{an,\text{Pfaff}}$ of \mathbb{R}_{an} . We claim that it is actually the case for none of these structures.

Let us suppose indeed that for some solution H of example (2.5) the structure $\mathbb{R}_{an,H}$ is a reduct of $\mathbb{R}_{an,\text{Pfaff}}$. Then, for any other solution G , the pfaffian closure $\mathbb{R}_{an,G,\text{Pfaff}}$ (which is of course an extension of $\mathbb{R}_{an,\text{Pfaff}}$) would be a common o-minimal extension of $\mathbb{R}_{an,H}$ and $\mathbb{R}_{an,G}$, which is impossible.

Example (2.5) also provides an example of a non-oscillating solution of systems of analytic ordinary differential equations which is not definable in any o-minimal extension of \mathbb{R} :

PROPOSITION 2.7. *Let $H, G : (0, \varepsilon) \rightarrow \mathbb{R}^2$ be two different solutions of (2.5). Then, the function $H^*(x) = (H(x), G(2x)) : (0, \varepsilon) \rightarrow \mathbb{R}^4$ is a non-oscillating solution of a system of analytic ordinary differential equations of the form (1.1) such that the structure \mathbb{R}_{H^*} is not o-minimal.*

Proof. The functions $H(x)$ and $G(x)$ have the same asymptotic expansion $\widehat{H}(x)$ as $\mathbb{R}^+ \ni x \rightarrow 0$. Therefore the asymptotic expansion $\widehat{H}^*(x) = (\widehat{H}(x), \widehat{H}(2x))$ of $H^*(x)$ is analytically transcendental by Theorem 2.4. This implies that $H^*(x)$ is non-oscillating. On the other hand, the image of the curve $x \mapsto (H(x), G(x))$ in \mathbb{R}^4 is definable in the structure \mathbb{R}_{H^*} but it cuts infinitely many times the semialgebraic set $\{(v_1, v_2) \in \mathbb{R}^4 / v_1 - v_2 = 0\}$. \square

We finish with some applications of Proposition 2.7 to *Hardy fields*. The latter are, by definition, subfields of the ring of germs of differentiable real functions on semi-intervals $(0, \varepsilon)$ which are closed under the usual differentiation (see [23]). A non-oscillating solution $H(x)$ of a system (1.1) determines a Hardy field K_H consisting of the germs of quotients $f(x, H(x))/g(x, H(x))$ where f, g are analytic functions and $g(x, H(x)) \not\equiv 0$. Another example of a Hardy field is the field $K_{\mathcal{R}}$ of germs of functions of one variable definable in some o-minimal structure \mathcal{R} [14]. Proposition 2.7 gives an example of a *polynomially bounded* Hardy field which is not contained in the field $K_{\mathcal{R}}$ for any o-minimal structure \mathcal{R} . An example of this kind, but less explicit, is already exhibited in [2].

2.4. Open questions

We have three main questions:

1) Given a solution $H : (0, \varepsilon) \rightarrow \mathbb{R}$ of a scalar equation as (2.3) with a convergent asymptotic expansion at $x = 0$, is the structure $\mathbb{R}_{an, H}$ model complete? We conjecture that this is not always the case.

2) Does any non-oscillating solution $H : (0, \varepsilon) \rightarrow \mathbb{R}^r$ of a system of analytic ordinary differential equations (1.1) with $r = 2$ or $r = 3$ generate an o-minimal structure?

3) Which are the relations between analytic transcendence, quasianalyticity, the hypotheses (2.1) and (2.2) and (non linear) differential Galois theory?

3. Quasi-analyticity and model-completeness

This paragraph is devoted to the proof of Theorem 2.1.

3.1. Some generalities

Let $H = (H_1, \dots, H_r) : (0, \varepsilon] \rightarrow \mathbb{R}^r$ be a solution of a system of (real) analytic differential equations of the form (1.1) such that $\lim_{x \rightarrow 0} H(x) = 0$. The following general result, which will be essential in the sequel, might already be known. However, we have not found any reference.

LEMMA 3.1. *Suppose that $A(0, 0) = 0$. Then, for any $L > 0$ there exists a neighborhood V of $0 \in \mathbb{R}^{r+1}$, $\delta_L > 0$ and an analytic function $B : [-L, L] \times V \rightarrow \mathbb{R}^r$ such that*

$$H(x + x^{p+1}z) = B(z, x, H(x)), \quad \text{for } |z| \leq L \text{ and } 0 < x \leq \delta_L.$$

Observe that, conversely, any function H satisfying the above functional equation also solves a certain system (1.1) of differential equations, namely $x^{p+1}H' = \frac{\partial B}{\partial z}(0, x, H)$.

Proof. Let V_1 be a neighborhood of the origin in \mathbb{R}^{r+1} such that the function $A(x + x^{p+1}z, \mathbf{w})$ is defined for $(z, x, \mathbf{w}) \in (-L - 1, L + 1) \times V_1$ and $1 + x^p z$ does not vanish there. Consider the regular analytic differential equation with analytic parameter x :

$$(E_x) \quad \frac{d\mathbf{w}}{dz} = (1 + x^p z)^{-p-1} A(x + x^{p+1}z, \mathbf{w}).$$

Since $A(0, 0) = 0$, the solution of the initial value problem (E_0) , $w(0) = 0$ is the zero function $w : (-L - 1, L + 1) \rightarrow \mathbb{R}$, $z \mapsto 0$. By the theorem of analytic dependence upon parameters and initial values, there exists a neighborhood $0 \in V \subset V_1$ such that for $(x, w_0) \in V$ the solution of (E_x) , $w(0) = w_0$ exists on the interval $[-L, L]$. Moreover, the mapping $B : [-L, L] \times V \rightarrow \mathbb{R}^r$ associating (z, x, w_0) to $w(z)$, the value of this solution at some $z \in [-L, L]$, is analytic. Choose finally $\delta_L > 0$ such that $(x, H(x)) \in V$ for $0 < x \leq \delta_L$. Then $G_x(z) = B(z, x, H(x))$ is defined for $z \in [-L, L]$, $0 < x < \delta_L$ and denotes the solution of (E_x) , $G_x(0) = H(x)$ which is thus defined on the interval $[-L, L]$. For sufficiently small z , this solution can also be expressed by $G_x(z) = H(x + x^{p+1}z)$. The identity theorem for (real) analytic functions yields the theorem. \square

Suppose now that H has an asymptotic expansion

$$H(x) \sim \widehat{H}(x) = (\widehat{H}_1(x), \dots, \widehat{H}_r(x)) \in \mathbb{R}[[x]]^r$$

as $\mathbb{R}^+ \ni x \rightarrow 0$.

LEMMA 3.2. *The function H can be continued to a C^∞ function on $[0, \varepsilon]$ and its Taylor series at $x = 0$ is equal to $\widehat{H}(x)$.*

Proof. The function $x \mapsto A(x, H(x))$, $x > 0$ has $A(x, \widehat{H}(x))$ as its asymptotic expansion as $0 < x \rightarrow 0$. Hence, $x \mapsto H'(x) = A(x, H(x))/x^{p+1}$ admits the asymptotic expansion $A(x, \widehat{H}(x))/x^{p+1}$, which might contain negative powers of x . Since asymptotic expansions are compatible with integration, this cannot be the case and $A(x, \widehat{H}(x))$ is divisible by x^{p+1} . Moreover, we have the following equation for formal series: $\widehat{H}(x) = \int_0^x A(t, \widehat{H}(t))t^{-p-1}dt$ and hence $\widehat{H}(x)$ (formally) satisfies the system (1.1). This implies that $H'(x) \sim \widehat{H}'(x)$ as $\mathbb{R}^+ \ni x \rightarrow 0$. We conclude that H is of class \mathcal{C}^1 at $x = 0$ and that $\lim_{x \rightarrow 0^+} H'(x) = H'(0) = \widehat{H}'(0)$. The same argument for $H'(x)$, which is a solution of $x^{p+1}z' = \frac{\partial A}{\partial x}(x, H(x)) + \frac{\partial A}{\partial y}(x, H(x))z - (p+1)x^p z$, shows that H is of class \mathcal{C}^2 at $x = 0$. The proof can be completed by induction in this way. \square

3.2. Simple functions and the class \mathcal{A}

Lemma 3.2 allows to reduce the statement of Theorem 2.1 to \mathcal{C}^∞ solutions defined on intervals symmetric to 0. Indeed, consider $G : [-\varepsilon, \varepsilon] \rightarrow \mathbb{R}^r$ defined by $G(x) = H(x^2)$ if $x \neq 0$ and $G(0) = 0$. This is a \mathcal{C}^∞ function by Lemma 3.2 and it satisfies a system of ordinary differential equations of the form (1.1). Denote by $\tilde{H}, \tilde{G} : \mathbb{R} \mapsto \mathbb{R}^r$, respectively, the extensions of H resp. G that vanish outside their corresponding domains of definition. In a straightforward manner, it can be shown that $\mathbb{R}_{\tilde{F}_{an} \cup \{\tilde{G}\}}$ is o-minimal (respectively model complete) if and only if $\mathbb{R}_{\tilde{F}_{an} \cup \{\tilde{H}\}}$ is o-minimal (respectively model complete).

Moreover, if H satisfies the strong quasi-analyticity property (SQA) then G has the analogous property without the condition that the first non-zero coefficients $P_l^{(\text{val} P_l)}(0)$ of the polynomials P_l have to be positive.

We introduce some notation useful for the proofs (recall that $T_k \phi(x) = (\phi(x) - J_k \phi(x))/x^k$ for a \mathcal{C}^∞ function or a formal series ϕ , where $J_k \phi(x) \in \mathbb{R}[x]$ denotes its k -jet at 0).

DEFINITION 2. Let $H = (H_1, \dots, H_r) : [-\varepsilon, \varepsilon] \rightarrow \mathbb{R}^r$ be a \mathcal{C}^∞ function. A germ $\varphi(x)$ of a \mathcal{C}^∞ function at $0 \in \mathbb{R}$ will be called a *simple function* (relative to H) if there exists $n \geq 0$, an analytic function $f \in \mathbb{R}\{x, z_{11}, \dots, z_{rn}\}$, polynomials $P_j(x)$ with $P_j(0) = 0$ for $j = 1, \dots, n$ and an integer $k \geq 0$ such that

$$\varphi(x) = f(x, T_k H(P_1(x)), \dots, T_k H(P_n(x))). \quad (3.1)$$

The family of simple functions is an algebra denoted by \mathcal{S}_H (or simply by \mathcal{S} if H is obvious from the context).

Using the above considerations, Theorem 2.1 follows immediately from the following seemingly weaker statement:

THEOREM 3.3. *Let $H = (H_1, \dots, H_r) : [-\varepsilon, \varepsilon] \rightarrow \mathbb{R}^r$ be a \mathcal{C}^∞ solution of a system (1.1) of ordinary differential equations. Suppose that the algebra \mathcal{S} of simple functions relative to H is quasi-analytic; i.e. $\varphi = 0$ whenever the Taylor series of some $\varphi \in \mathcal{S}_H$ at $x = 0$ vanishes. Then the extension $\mathbb{R}_{an,H}$ is \mathcal{O} -minimal and model complete.*

Our proof of Theorem 3.3, which will be given in subsection 3.4, relies on a fundamental result of [22] about the model completeness of structures generated by certain quasi-analytic classes. Therefore we first study the quasi-analyticity of certain classes of functions defined using H .

Given a \mathcal{C}^∞ function $H = (H_1, \dots, H_r) : [-\varepsilon, \varepsilon] \rightarrow \mathbb{R}^r$, let \mathcal{A}_H be the smallest collection \mathcal{A}_H^m , $m \in \mathbb{N}$, of subalgebras of germs of \mathcal{C}^∞ functions at $0 \in \mathbb{R}^m$ satisfying the following conditions (we drop the subscript “ H ” for the sake of clarity):

- A1) The germs of analytic functions of m variables are elements of \mathcal{A}^m , for all m , and the germ of H_l at 0 is in \mathcal{A}^1 for $l = 1, \dots, r$.
- A2) (Stability by composition). If $\phi \in \mathcal{A}^m$ and $\phi_1, \dots, \phi_m \in \mathcal{A}^n$ with $\phi_1(0) = \dots = \phi_m(0) = 0$ then $\phi(\phi_1, \dots, \phi_m) \in \mathcal{A}^n$.
- A3) (Stability by implicit equations). If $\phi \in \mathcal{A}^{m+1}$ satisfies $\phi(0) = 0$ and $\partial\phi/\partial x_{m+1}(0) \neq 0$ then there exists $\bar{\phi} \in \mathcal{A}^m$ such that $\bar{\phi}(0) = 0$ and the germ $\phi(\mathbf{x}, \bar{\phi}(\mathbf{x})) \in \mathcal{A}^m$, with $\mathbf{x} = (x_1, \dots, x_m)$, is equal to zero.
- A4) (Stability by monomial division). If $\phi \in \mathcal{A}^m$ satisfies $\phi(0, \bar{\mathbf{x}}) \equiv 0$, where $\bar{\mathbf{x}} = (x_2, \dots, x_m) \in \mathbb{R}^{m-1}$ then there exists $\bar{\phi} \in \mathcal{A}^m$ such that $\phi(\mathbf{x}) = x_1 \bar{\phi}(\mathbf{x})$, where $\mathbf{x} = (x_1, \bar{\mathbf{x}})$.

REMARK 1. Properties A2-A4 imply that the algebras \mathcal{A}^m are stable by partial differentiation. In fact, if $\phi \in \mathcal{A}^m$, the germ $\phi_1(u, \mathbf{x}) = \phi(x_1 + u, \bar{\mathbf{x}}) - \phi(\mathbf{x})$ is in \mathcal{A}^{m+1} (here again $\mathbf{x} = (x_1, \bar{\mathbf{x}})$), and thus also $\bar{\phi}_1(u, \mathbf{x}) = \phi_1(u, \mathbf{x})/u$ defines an element of \mathcal{A}^{m+1} by A4. Thus, $\partial\phi/\partial x_1(\mathbf{x}) = \bar{\phi}_1(0, \mathbf{x})$ is in \mathcal{A}^m .

REMARK 2. Property A3 of the class \mathcal{A} can be generalized to systems of implicit equations:

Let $\Phi = (\Phi_1, \dots, \Phi_n) \in (\mathcal{A}^{m+n})^n$ such that $\Phi(0) = 0$ and $\partial\Phi/\partial\mathbf{y}(0)$ is an invertible matrix (with the notation (\mathbf{x}, \mathbf{y}) of the coordinates of \mathbb{R}^{m+n}). Then there exists $\phi = (\phi_1, \dots, \phi_n) \in (\mathcal{A}^m)^n$ such that $\Phi(\mathbf{x}, \phi(\mathbf{x})) \equiv 0$.

Proof. By induction over n . The case $n = 1$ is property A3. So suppose that $n > 1$. Denote $\partial\Phi/\partial\mathbf{y}(0) = (a_{ij})_{1 \leq i, j \leq n}$. Up to some permutation of the rows, we can suppose that its principal minors $\det((a_{ij})_{1 \leq i, j \leq k})$, $k = 1, \dots, n$, are all different from zero. By property A3, the solution ψ_1 of

$$\Phi_1(\mathbf{x}, \psi_1(\mathbf{x}, y_2, \dots, y_n), y_2, \dots, y_n) = 0$$

is in \mathcal{A}^{m+n-1} . For $j = 2, \dots, n$ consider $\bar{\Phi}_j(\mathbf{x}, \bar{\mathbf{y}}) = \Phi_j(\mathbf{x}, \psi_1(\mathbf{x}, \bar{\mathbf{y}}), \bar{\mathbf{y}})$ and $\bar{\Phi} = (\bar{\Phi}_2, \dots, \bar{\Phi}_n)$, where $\bar{\mathbf{y}} = (y_2, \dots, y_n)$. We have that $\bar{\Phi}_j \in \mathcal{A}^{m+n-1}$ and

$$\frac{\partial \bar{\Phi}_j}{\partial y_k}(0) = \frac{1}{a_{11}} \det \begin{pmatrix} a_{11} & a_{1k} \\ a_{j1} & a_{jk} \end{pmatrix},$$

so that $\bar{J} = \partial\bar{\Phi}/\partial\bar{\mathbf{y}}(0)$ is invertible. By the hypothesis of induction, the system $\bar{\Phi}(\mathbf{x}, \bar{\phi}(\mathbf{x})) = 0$ is solved by some $\bar{\phi} = (\bar{\phi}_2, \dots, \bar{\phi}_n) \in (\mathcal{A}^m)^{n-1}$ and completing the vector with $\phi_1(\mathbf{x}) = \psi_1(\mathbf{x}, \bar{\phi}(\mathbf{x})) \in \mathcal{A}^m$ we have solved $\Phi(\mathbf{x}, \phi(\mathbf{x})) \equiv 0$. \square

We pass from germs to actual functions by means of the following definition. A function $F : U \rightarrow \mathbb{R}$ on an open subset $U \subset \mathbb{R}^m$ is called an \mathcal{A} -analytic function if for any $a \in U$ there exists $\phi_a \in \mathcal{A}^m$ such that the germ of F at a is equal to the germ of $\mathbf{x} \mapsto \phi_a(\mathbf{x} - a)$. This definition is justified by the following:

LEMMA 3.4. *If $\phi \in \mathcal{A}^m$, there exists a representative of ϕ defined in some neighborhood of $0 \in \mathbb{R}^m$ that is an \mathcal{A} -analytic function.*

Proof. For any m , let $\tilde{\mathcal{A}}^m$ be the subset of \mathcal{A}^m consisting of all germs having the required property. One can verify easily that the collection $\tilde{\mathcal{A}} = (\tilde{\mathcal{A}}^m)_m$ verifies properties A1–A4. For example, for each $l = 1, \dots, r$, the germ at $0 \in \mathbb{R}$ of H_l belongs to $\tilde{\mathcal{A}}^1$: its representative H_l is analytic at any nonzero point $a \in (-\varepsilon, \varepsilon)$.

Thus, $\tilde{\mathcal{A}}^m = \mathcal{A}^m$ for any m . □

3.3. Quasi-analyticity of the class \mathcal{A}

Clearly, if $\mathcal{A} = (\mathcal{A}^m)$ is a family of algebras of germs of C^∞ functions that satisfies properties A1–A4 then the collection of simple functions with respect to H is a subalgebra of \mathcal{A}^1 . Our goal in this paragraph is to prove that if H is a solution of a system of analytic ordinary differential equations, then the quasi-analyticity property of \mathcal{S}_H is inherited by the entire class \mathcal{A}_H .

THEOREM 3.5 *Quasi-analyticity of \mathcal{A} . Under the hypothesis of Theorem 3.3, let $\mathcal{A} = \mathcal{A}_H$ be the smallest class of germs satisfying A1–A4. Then \mathcal{A} is a quasi-analytic class: for any $m \geq 1$, if the Taylor series of $\phi \in \mathcal{A}^m$ at 0 vanishes then $\phi = 0$.*

The proof is given in the remainder of this subsection and it is divided in two parts. In the first one, we deduce the general statement from the one for germs of a single variable. In the second part, we show the surprising fact that any element of \mathcal{A}^1 is a simple function.

PROPOSITION 3.6. *If the algebra \mathcal{A}^1 of elements of \mathcal{A} of one variable is quasi-analytic then Theorem 3.5 follows.*

Proof. Let $\phi \in \mathcal{A}^m$ such that its Taylor series $\hat{\phi}(\mathbf{x}) \in \mathbb{R}[[\mathbf{x}]]$ vanishes. By Lemma 3.4, we can consider a representative F of ϕ which is an \mathcal{A} -analytic function in the open ball $B(0, \varepsilon)$, $\varepsilon > 0$. Given a unitary vector $a \in \mathbb{R}^m$, let $F_a : (-\varepsilon, \varepsilon) \rightarrow \mathbb{R}$ be defined by $F_a(t) = F(ta)$. Then the germ of F_a at $t = 0$ is an element of \mathcal{A}^1 whose Taylor series vanishes. Our assumption implies that the function F_a vanishes identically in some interval centered at $t = 0$. Let $\delta_a = \sup\{\delta \in (0, \varepsilon) / F_a(t) = 0 \text{ for } 0 \leq t \leq \delta\} > 0$. The proof is complete if we show that $\delta_a = \varepsilon$ for any a . Suppose, on the contrary, that $\delta_a < \varepsilon$ for a certain a . Then the function G_a defined by $G_a(t) = F_a(t + \delta_a)$, defined on $] -\delta_a, \varepsilon - \delta_a[$, vanishes for $t < 0$. Thus, its Taylor series at $t = 0$ must vanish. Since the germ of G_a at $t = 0$ belongs to \mathcal{A}^1 by choice of F , the function G_a vanishes in a neighborhood of 0 . This contradicts the definition of δ_a . □

PROPOSITION 3.7. *Under the hypothesis of Theorem 3.3, we have $\mathcal{A}^1 = \mathcal{S}_H$.*

As \mathcal{S}_H is assumed to be quasi-analytic in the hypothesis of Theorem 3.3 (which is also that of Theorem 3.5), propositions 3.6 and 3.7 prove Theorem 3.5. It remains to prove the above proposition.

This result is by no means trivial and actually quite surprising: the definition of the class \mathcal{A} permits solutions of implicit equations and compositions, so that we could expect that \mathcal{A}^1 contains much more complicated functions than the simple ones. The proof of Proposition 3.7 needs several preparatory lemmas and will be given at the end of this subsection. The hypothesis of Theorem 3.3 (i.e. the quasi-analyticity of the class \mathcal{S}_H of simple functions) will be assumed in the remainder of this subsection without explicitly mentioning it.

LEMMA 3.8. *The class of simple functions \mathcal{S}_H is closed by monomial division and by composition.*

Proof. Let $\varphi \in \mathcal{S}_H$ be written as $\varphi(x) = f(x, T_k H(P_1(x)), \dots, T_k H(P_n(x)))$ with f analytic and polynomials $P_j(x)$ such that $P_j(0) = 0$. Suppose that $\varphi(0) = 0$. Using the formula $T_k H(x) = x\mathbf{a}_{k+1} + xT_{k+1}H(x)$, where $\mathbf{a}_{k+1} \in \mathbb{R}^r$ is the $k+1$ -th coefficient of the Taylor series $\hat{H}(x)$ of H at 0, we can write the quotient $\varphi(x)/x$ in the form

$$\frac{1}{x}f(x, \{P_l(x)\mathbf{a}_{k+1} + P_l(x)T_{k+1}H(P_l(x))\}_{l=1, \dots, n}).$$

This last function is of the form $\tilde{f}(x, T_{k+1}H(P_1(x)), \dots, T_{k+1}H(P_n(x)))$ with some analytic \tilde{f} . This proves the closure of the simple functions by monomial division.

Consider now $\varphi_1, \varphi_2 \in \mathcal{S}_H$ such that $\varphi_2(0) = 0$ and let us prove that the composition $\varphi = \varphi_1 \circ \varphi_2$ is also a simple function. It suffices to consider the case that φ_1 is a component of H , say H_1 . Assume that $\varphi_2 \neq 0$ (otherwise there is nothing to prove). Since \mathcal{S}_H is a quasi-analytic class, there exists $d > 0$ and $c_d \neq 0$ such that we can write

$$\varphi_2(x) = c_d x^d + \dots + x^{(p+1)d} T_{(p+1)d} \varphi_2(x) = P(x) + P(x)^{p+1} \tilde{\varphi}_2(x),$$

where $P(x) = c_d x^d + \dots + c_{pd+d-1} x^{pd+d-1} = J_{pd+d-1} \varphi_2$ and $\tilde{\varphi}_2$ is still a simple function as we already proved that \mathcal{S}_H is closed under monomial division. By Lemma 3.1, we have $\varphi(x) = H_1(\varphi_2(x)) = B_1(x, H_1(P(x)), \tilde{\varphi}_2(x))$ with some analytic function B_1 . We conclude that $\varphi \in \mathcal{S}_H$. \square

It is often convenient to view a function f obtained by the composition of several functions f_i in the following way: consider the graph of f as the projection of a set essentially constructed using the graphs of the f_i and cartesian products (see for example [19]).

The elements of \mathcal{A} are obtained from H and analytic functions by repeated compositions, monomial divisions and solutions of implicit equations. In the same spirit, in order to replace the recursive definition of elements of \mathcal{A} , we describe them as projections of solutions of (big) systems of (simple) implicit equations in several variables. The individual equations can be analytic or in a new class of functions slightly extending H , the *divided differences*.

We define a divided difference (generated by H) as an element of the smallest subcollection $\mathcal{D} = (\mathcal{D}^m)$ of \mathcal{A} containing the germ of each component H_l of H and such that, if Δ is in \mathcal{D}^m , then the germ $\tilde{\Delta}$ defined by one of the following formulas also is in \mathcal{D} :

- D1) $\bar{\Delta}(\mathbf{x}) = \Delta(0, x_2, \dots, x_m)$.
 D2) $\bar{\Delta}(\mathbf{x}) = \Delta(\mathbf{x}) - \Delta(0)$ or $\bar{\Delta}(\mathbf{x}) = \Delta(\mathbf{x}) - \Delta(0, x_2, \dots, x_m)$.
 D3) If $\Delta(0, x_2, \dots, x_m) \equiv 0$ then $\bar{\Delta}(\mathbf{x}) = \Delta(\mathbf{x})/x_1$.
 D4) $\bar{\Delta}(\mathbf{x}) = \Delta(x_{\sigma(1)}, \dots, x_{\sigma(m)})$ for some permutation σ of $\{1, \dots, m\}$.
 D5) $\bar{\Delta}(\mathbf{x}, z) = \Delta(\mathbf{x})$.
 D6) $\bar{\Delta}(\mathbf{x}, z) = \Delta(x_1 + x_2(\alpha + z), x_2, \dots, x_m)$ for some $\alpha \in \mathbb{R}$.

LEMMA 3.9. *Let $\beta \in \mathcal{A}^m$, $m \geq 1$. Then there exists $n \in \mathbb{N}$ and a system of n implicit equations $\Phi(\mathbf{x}, \mathbf{y}) = (\Phi_1(\mathbf{x}, \mathbf{y}), \dots, \Phi_n(\mathbf{x}, \mathbf{y}))^T = 0$, $(\mathbf{x}, \mathbf{y}) \in \mathbb{R}^{m+n}$ with $\Phi(0) = 0$ and $\det(\partial\Phi/\partial\mathbf{y}(0)) \neq 0$ such that its solution $\mathbf{y} = \phi(\mathbf{x}) = (\phi_1(\mathbf{x}), \dots, \phi_n(\mathbf{x}))$ satisfies $\phi_n = \beta - \beta(0)$ and such that each component Φ_i is either analytic or of the form $\Phi_i(\mathbf{x}, \mathbf{y}) = y_{l_i} - \Delta_i(\mathbf{x}, \mathbf{y})$ with some $1 \leq l_i \leq n$ and some divided difference Δ_i generated by H .*

Proof. Consider, for any m , the subset $\tilde{\mathcal{A}}^m \subset \mathcal{A}^m$ of all germs satisfying the statement of the lemma. Let us show that the collection $\tilde{\mathcal{A}} = (\tilde{\mathcal{A}}^m)$ verifies properties A1–A4 of the definition of the class \mathcal{A} .

Obviously, $\tilde{\mathcal{A}}$ verifies A1.

Let $\beta_0 \in \mathcal{A}^m$ and $\beta_1, \dots, \beta_m \in \tilde{\mathcal{A}}^n$ with $\beta_j(0) = 0$, $j = 0, \dots, m$. We want to show that the composition $\beta = \beta_0(\beta_1, \dots, \beta_m) \in \mathcal{A}^n$.

Consider the system $\Phi^{(0)}(\mathbf{z}, \mathbf{y}^{(0)}) = 0$ of n_0 equations in \mathbb{R}^{m+n_0} such that its solution $\mathbf{y}^{(0)} = \phi^{(0)}(\mathbf{z}) = (\phi_1^{(0)}(\mathbf{z}), \dots, \phi_{n_0}^{(0)}(\mathbf{z}))$ defines $\beta_0 = \phi_{n_0}^{(0)}$ and consider, for $j = 1, \dots, m$, the system $\Phi^{(j)}(\mathbf{x}, \mathbf{y}^{(j)}) = 0$ of n_j equations in \mathbb{R}^{n+n_j} such that its solution $\mathbf{y}^{(j)} = \phi^{(j)}(\mathbf{x}) = (\phi_1^{(j)}(\mathbf{x}), \dots, \phi_{n_j}^{(j)}(\mathbf{x}))$ defines $\beta_j = \phi_{n_j}^{(j)}$. Put

$$\Phi(\mathbf{x}, \mathbf{y}) = (\Phi^{(1)}(\mathbf{x}, \mathbf{y}^{(1)}), \dots, \Phi^{(m)}(\mathbf{x}, \mathbf{y}^{(m)}), \Phi^{(0)}((y_{n_1}^{(1)}, \dots, y_{n_m}^{(m)}), \mathbf{y}^{(0)})),$$

with $\mathbf{y} = (\mathbf{y}^{(1)}, \dots, \mathbf{y}^{(m)}, \mathbf{y}^{(0)}) \in \mathbb{R}^{n_0 + \dots + n_m}$; here the column vectors have to be stacked on top of each other. We have $\Phi(0) = 0$ and the matrix $\partial\Phi/\partial\mathbf{y}(0)$ is lower block diagonal with diagonal blocks given by the matrices $\partial\Phi^{(j)}/\partial\mathbf{y}^{(j)}(0)$, thus invertible. If $\mathbf{y} = \phi(\mathbf{x})$ is the solution of the system $\Phi(\mathbf{x}, \mathbf{y}) = 0$ then $\phi = (\phi^{(1)}, \dots, \phi^{(m)}, \psi)$ with ψ satisfying $\Phi^{(0)}((\phi_{n_1}^{(1)}(\mathbf{x}), \dots, \phi_{n_m}^{(m)}(\mathbf{x})), \psi(\mathbf{x})) \equiv 0$. Hence, the last component of ψ is $\psi_{n_0} = \beta_0(\beta_1, \dots, \beta_m)$. On the other hand, it is obvious that each component Φ_i of Φ is either analytic or of the form $y_{l_i}^{(j(i))} - \Delta_i(\mathbf{x}, \mathbf{y})$ with some divided difference Δ_i . This proves A2 for $\tilde{\mathcal{A}}$.

Let $\beta(\mathbf{x}, x_{m+1}) \in \tilde{\mathcal{A}}^{m+1}$ with $\beta(0) = 0$, $\partial\beta/\partial x_{m+1}(0) \neq 0$ and consider the solution $\beta'(\mathbf{x})$ of $\beta(\mathbf{x}, \beta'(\mathbf{x})) \equiv 0$. Let $\Phi((\mathbf{x}, x_{m+1}), \mathbf{y}) = 0$ be the system of implicit equations corresponding to β and consider its solution $\mathbf{y} = \phi = (\phi_1, \dots, \phi_n)$ which satisfies $\phi_n = \beta$. Put $\mathbf{y}' = (\mathbf{y}, y_{n+1})$ with an additional variable y_{n+1} and let

$$\Phi'(\mathbf{x}, \mathbf{y}') = (\Phi((\mathbf{x}, y_{n+1}), \mathbf{y}))^T, y_n)^T.$$

We verify that $\Phi'(0) = 0$ and, using $\partial\beta/\partial x_{m+1}(0) \neq 0$, that the matrix

$$\frac{\partial\Phi'}{\partial\mathbf{y}'}(0) = \begin{pmatrix} \frac{\partial\Phi}{\partial\mathbf{y}}(0) & \mathbf{b} \\ \mathbf{e}_n & 0 \end{pmatrix}, \quad \mathbf{b} = \partial\Phi/\partial x_{m+1}(0), \quad \mathbf{e}_n = (0, \dots, 0, 1) \in \mathbb{R}^n$$

is invertible. If $\mathbf{y}' = \phi' = (\phi'_1, \dots, \phi'_{n+1})$ is the solution of the system $\Phi'(\mathbf{x}, \mathbf{y}') = 0$ then we have $\Phi((\mathbf{x}, \phi'_{n+1}(\mathbf{x})), (\phi'_1(\mathbf{x}), \dots, \phi'_n(\mathbf{x}))) \equiv 0$ and $\phi'_n(\mathbf{x}) \equiv 0$. The first equation implies that $\phi'_l(\mathbf{x}) = \phi_l(\mathbf{x}, \phi'_{n+1}(\mathbf{x}))$ for $l = 1, \dots, n$. The second equation

then yields

$$\beta(\mathbf{x}, \phi'_{n+1}(\mathbf{x})) = \phi_n(\mathbf{x}, \phi'_{n+1}(\mathbf{x})) \equiv 0$$

and thus, by definition, $\phi'_{n+1} = \beta'$. This proves A3 for $\tilde{\mathcal{A}}$.

Let $\beta \in \tilde{\mathcal{A}}^m$ such that $\beta(0, \bar{\mathbf{x}}) \equiv 0$ with $\bar{\mathbf{x}} = (x_2, \dots, x_m)$ and let $\beta' \in \mathcal{A}^m$ be given by $\beta'(\mathbf{x}) = \beta(\mathbf{x})/x_1 \in \mathcal{A}^m$. Let $\Phi(\mathbf{x}, \mathbf{y}) = 0$ be a system of n equations for β given by the lemma and $\mathbf{y} = \phi = (\phi_1, \dots, \phi_n)$ its solution. Put $\psi = (\psi_1, \dots, \psi_n)$ with $\psi_j(\mathbf{x}) = (\phi_j(\mathbf{x}) - \phi_j(0, \bar{\mathbf{x}}))/x_1 \in \mathcal{A}^m$. Consider the vector $\mathbf{b} = \psi(0) = (\partial\phi/\partial x_1(0))^T = -\frac{\partial\Phi}{\partial\mathbf{y}}(0)^{-1}(\partial\Phi/\partial x_1(0))$ and the system of $2n$ equations

$$\Phi'(\mathbf{x}, \mathbf{y}, \mathbf{y}') = (\Phi((0, \bar{\mathbf{x}}), \mathbf{y}), \frac{1}{x_1}[\Phi(\mathbf{x}, \mathbf{y} + x_1(\mathbf{b} + \mathbf{y}')) - \Phi((0, \bar{\mathbf{x}}), \mathbf{y})])^T.$$

Denoting $\Psi(\mathbf{x}, \mathbf{y}, \mathbf{y}') = \Phi(\mathbf{x}, \mathbf{y} + x_1(\mathbf{b} + \mathbf{y}'))$, we have $\Phi'(0) = (\Phi(0), \partial\Psi/\partial x_1(0))^T$ and this is the zero vector by the definition of \mathbf{b} . Furthermore, the matrix $\partial\Phi'/\partial(\mathbf{y}, \mathbf{y}')$ (0) is lower block triangular and its two diagonal blocks are equal to $\partial\Phi/\partial\mathbf{y}(0)$, hence invertible. Moreover, each component of Φ is either analytic or a divided difference generated by H in the variables $(\mathbf{x}, \mathbf{y}, \mathbf{y}')$. (Observe that this last part makes the introduction of the divided differences unavoidable.)

We check that $\phi'(\mathbf{x}) = (\phi(0, \bar{\mathbf{x}}), \psi(\mathbf{x}) - \mathbf{b})$ satisfies $\Phi'(\mathbf{x}, \phi'(\mathbf{x})) \equiv 0$. Its last component is

$$\psi_n(\mathbf{x}) - b_n = \frac{1}{x_1}(\phi_n(\mathbf{x}) - \phi_n(0, \bar{\mathbf{x}})) - b_n = \frac{1}{x_1}(\beta(\mathbf{x}) - \beta(0, \bar{\mathbf{x}})) - b_n = \beta'(\mathbf{x}) - \beta'(0),$$

and the proof is complete. \square

In the following three lemmas, we establish some useful relations between the divided differences and the simple functions.

LEMMA 3.10. *Let $\Delta(\mathbf{x}) \in \mathcal{D}^m$ be a divided difference in the variables $\mathbf{x} = (x_1, \dots, x_m)$. Then there exists natural numbers $n, N \geq 0$, polynomials $P_1(\mathbf{x}), \dots, P_n(\mathbf{x}), Q(\mathbf{x})$ with $Q(\mathbf{x}) \not\equiv 0$ and a function B analytic in a neighborhood of $0 \in \mathbb{R}^{m+nr(N+1)}$ such that*

$$Q(\mathbf{x})\Delta(\mathbf{x}) = B\left(\mathbf{x}, \{H^{(l)}(P_j(\mathbf{x}))\}_{0 \leq l \leq N, 1 \leq j \leq n}\right), \quad (3.2)$$

where $H^{(l)}$ denotes the l -th derivative of H with respect to x .

Proof. The components of H trivially satisfy (3.2). Moreover, if Δ is a divided difference which satisfies (3.2), then the divided difference $\bar{\Delta}$ obtained from Δ by one of the operations D3-D6 also does. Also, if (3.2) is true for Δ_1 and Δ_2 then it is true for the difference $\Delta_1 - \Delta_2$. Let us now consider $\bar{\Delta}(\bar{\mathbf{x}}) = \Delta(0, \bar{\mathbf{x}})$, where $\bar{\mathbf{x}} = (0, x_2, \dots, x_m)$. Here the problem is that $Q(0, \bar{\mathbf{x}})$ might vanish identically. In any case, since $Q(\mathbf{x}) \not\equiv 0$, we can consider the smallest integer $s \geq 0$ such that $\partial^s Q/\partial x_1^s(0, \bar{\mathbf{x}}) \not\equiv 0$. We differentiate the identity (3.2) s times and obtain some analytic function \bar{B} in $m - 1 + nr(N + s + 1)$ variables such that

$$\frac{\partial^s Q}{\partial x_1^s}(0, \bar{\mathbf{x}})\Delta(0, \bar{\mathbf{x}}) = \bar{B}\left(\bar{\mathbf{x}}, \{H^{(l)}(P_j(0, \bar{\mathbf{x}}))\}_{0 \leq l \leq N+s, 1 \leq j \leq n}\right).$$

\square

LEMMA 3.11. *Let $\Delta \in \mathcal{D}^m$ be a divided difference and $\varphi_1, \dots, \varphi_m \in \mathcal{S}_H$ be simple functions. Then the composition $\Delta(\varphi_1, \dots, \varphi_m)$ is also simple.*

Proof. The result is true for any component of $H = (H_1, \dots, H_r)$ by Lemma 3.8. Furthermore, if the statement is true for some Δ then it is true for any divided difference $\bar{\Delta}$ obtained by one of the formulas D1–D6, again by Lemma 3.8. \square

LEMMA 3.12. *Let $\Delta(\mathbf{x}) \in \mathcal{D}^m$ be a divided difference and denote by $\hat{\Delta}$ its Taylor series at $0 \in \mathbb{R}^m$. Let $\hat{\alpha}(x) = (\hat{\alpha}_1(x), \dots, \hat{\alpha}_m(x)) \in \mathbb{R}[[x]]^m$ be a vector of formal power series in a single variable with $\hat{\alpha}(0) = 0$. Then there are natural numbers $d_0, n_0 \geq 0$ such that, for any $d > d_0$, there exists $L \in \mathbb{N}$, an analytic function $C = C(x, \mathbf{z}, \mathbf{w})$ in a neighborhood of $0 \in \mathbb{R}^{1+m+r n_0}$ with $\partial\Delta/\partial\mathbf{x}(0) = \partial C/\partial\mathbf{z}(0, 0, 0)$ and polynomials $Q_1(x), \dots, Q_{n_0}(x) \in \mathbb{R}[[x]]$ with each $Q_j(0) = 0$ such that*

$$T_d \left(\hat{\Delta}(\hat{\alpha}(x)) \right) = C \left(x, T_d \hat{\alpha}(x), \{T_L \hat{H}(Q_j(x))\}_{1 \leq j \leq n_0} \right). \quad (3.3)$$

(Recall the notation $J_d\phi$ for the d -jet of ϕ at the origin and the definition $T_d\phi(x) = (\phi(x) - J_d\phi(x))/x^d$).

The numbers n_0, d_0 (and also C, Q_l, L) do not depend continuously (with respect to the topology of formal series) upon $\hat{\alpha}(x)$, as can be seen in the proof. This means that these objects can not be determined from a finite number of coefficients of $\hat{\alpha}(x)$.

It is a tempting idea to try to prove the lemma using recursion, i.e. the properties D1–D6; the problematic formula is D3. Any recursive proof we found, however, is no simpler than the subsequent technical one.

REMARK 3. Before beginning the proof of Lemma 3.12, note that the result is true (with $d_0 = n_0 = 0$) if we replace the divided difference Δ by an analytic function $D \in \mathbb{R}\{\mathbf{x}\}$: the analytic function $C(x, \mathbf{z}) = x^{-d}[D(J_d\hat{\alpha}(x) + x^d\mathbf{z}) - J_d(D(\hat{\alpha}(x)))]$ verifies the statement. Moreover, if the lemma holds for Δ_1 and Δ_2 then, choosing a suitable d_0 , it also holds for $\Delta_1 + \Delta_2$. Therefore, given any integer M , we can suppose that $J_M\Delta = 0$.

Proof. We use formula (3.2) of Lemma 3.10. There might be a problem if $Q(\hat{\alpha}(x))$ vanishes identically. In any case, since the polynomial $Q(\mathbf{x})$ is not identically zero, we can find some $\tau = (\tau_1, \dots, \tau_m) \in \mathbb{N}^m$ of minimal length $|\tau| = \tau_1 + \dots + \tau_m$ such that $\partial^\tau Q/\partial\mathbf{x}^\tau(\hat{\alpha}(x)) \neq 0$. For convenience, put $P_0(\mathbf{x}) = \partial^\tau Q/\partial\mathbf{x}^\tau(\mathbf{x})$ and denote $\hat{\beta}_j(x) = P_j(\hat{\alpha}(x))$ for $j = 0, \dots, n$. Up to some permutation, we can suppose that there exists $0 \leq n_0 \leq n$ such that $\hat{\beta}_j(x) \neq 0$ if and only if $j \in \{0, \dots, n_0\}$. Let $\nu_j = \text{val}(\hat{\beta}_j(x))$ for $j \in \{0, \dots, n_0\}$ and put $d_0 = \max(\nu_0, (p+1)\nu_1, \dots, (p+1)\nu_{n_0})$. Given $d > d_0$, consider $M > d + \nu_0 + |\tau|$. We can suppose, following Remark 3, that $J_M\Delta = 0$.

We consider the τ -th derivative of equation (3.2)

$$\sum_{\tau' \leq \tau} \binom{\tau}{\tau'} \frac{\partial^{\tau'} Q}{\partial\mathbf{x}^{\tau'}}(\mathbf{x}) \frac{\partial^{\tau-\tau'} \Delta}{\partial\mathbf{x}^{\tau-\tau'}}(\mathbf{x}) = \tilde{B} \left(\mathbf{x}, \{H^{(l)}(P_j(\mathbf{x}))\}_{0 \leq l \leq N+|\tau|, 1 \leq j \leq n} \right) \quad (3.4)$$

where \tilde{B} is some analytic function and by $\tau' \leq \tau$ we mean that, if $\tau' = (\tau'_1, \dots, \tau'_m)$ then $0 \leq \tau'_s \leq \tau_s$ for each s . Since $x^{p+1}H'(x) = A(x, H(x))$, we deduce by induction that $x^{(p+1)l}H^{(l)}(x)$ are analytic functions of $(x, H(x))$ for all $l \in \mathbb{N}$. This fact, together with the formula $H(x) = J_L H(x) + x^L T_L H(x)$ for any L , leads to the property that there exists $L \in \mathbb{N}$ and polynomials $R_{s,j}(\mathbf{x})$ with $\text{val}(R_{s,j}) \geq M - |\tau| + 1$

such that the right hand side of equation (3.4) is of the form

$$\tilde{B}(\mathbf{x}, \{H^{(l)}(P_j(\mathbf{x}))\}) = \overline{B}(\mathbf{x}, \{R_{sj}(\mathbf{x}) T_L H_s(P_j(\mathbf{x}))\}_{1 \leq s \leq r, 1 \leq j \leq n}), \quad (3.5)$$

with \overline{B} analytic. Inserting $\mathbf{x} = \hat{\alpha}(x)$ into equation (3.4) we obtain from the minimality of $|\tau|$:

$$\hat{\beta}_0(x) \Delta(\hat{\alpha}(x)) = \overline{B}(\hat{\alpha}(x), \{R_{sj}(\hat{\alpha}(x)) T_L H_s(P_j(\hat{\alpha}(x)))\}_{1 \leq s \leq r, 1 \leq j \leq n_0}, 0). \quad (3.6)$$

(Recall that for $j > n_0$ we have $P_j(\hat{\alpha}(x)) \equiv 0$). Write for $j = 0, \dots, n$

$$P_j(J_d \hat{\alpha}(x) + x^d \mathbf{z}) = J_d \hat{\beta}_j(x) + x^d D_{jd}(x, \mathbf{z})$$

where D_{jd} is some polynomial in the variables $(x, \mathbf{z}) \in \mathbb{R}^{1+m}$. By our choice of d_0 we can write for $j = 1, \dots, n_0$:

$$\begin{aligned} P_0(J_d \hat{\alpha}(x) + x^d \mathbf{z}) &= J_d \hat{\beta}_0(x) (1 + x^{d-\nu_0} \tilde{D}_{0d}(x, \mathbf{z})), \\ P_j(J_d \hat{\alpha}(x) + x^d \mathbf{z}) &= J_d \hat{\beta}_j(x) + (J_d \hat{\beta}_j(x))^{p+1} \tilde{D}_{jd}(x, \mathbf{z}), \end{aligned} \quad (3.7)$$

where \tilde{D}_{jd} are again analytic functions. Lemma 3.1 (applied to $T_L H(x) = (H(x) - J_L H(x))/x^L$ which satisfies a system of ordinary differential equations analogous to (1.1)) implies that for $s = 1, \dots, r$ and $j = 1, \dots, n_0$ there exists an analytic function C_{sjdL} such that

$$(T_L H_s)(P_j(J_d \hat{\alpha}(x) + x^d \mathbf{z})) = C_{sjdL}(x, \tilde{D}_{jd}(x, \mathbf{z}), (T_L H)(J_d \hat{\beta}_j(x)))$$

On the other hand, if $j > n_0$ then $J_d \hat{\beta}_j(x) \equiv 0$ and

$$D_{jd}(x, T_d \hat{\alpha}(x)) = x^{-d} (P_j(J_d \hat{\alpha}(x) + x^d T_d \hat{\alpha}(x))) = x^{-d} P_j(\hat{\alpha}(x)) \equiv 0. \quad (3.8)$$

We insert $\mathbf{x} = J_d \hat{\alpha}(x) + x^d \mathbf{z}$ into \overline{B} and obtain

$$\begin{aligned} \overline{B}(J_d \hat{\alpha}(x) + x^d \mathbf{z}, \{R_{sj}(J_d \hat{\alpha}(x) + x^d \mathbf{z})(T_L H_s)(P_j(J_d \hat{\alpha}(x) + x^d \mathbf{z}))\}) &= \\ \check{B}(x, \mathbf{z}, \{x^{M_0} (T_L H)(J_d \hat{\beta}_j(x))\}_{j=1}^{n_0}, \{x^{M_0} (T_L H)(x^d D_{jd}(x, \mathbf{z}))\}_{j=n_0+1}^n) & \end{aligned} \quad (3.9)$$

with $M_0 = M - |\tau| + 1$ and some analytic function \check{B} . Define for sufficiently small $\mathbf{w} \in \mathbb{R}^{r n_0}$

$$C(x, \mathbf{z}, \mathbf{w}) := \frac{\check{B}(x, \mathbf{z}, \{x^{M_0} \mathbf{w}\}, 0)}{x^d J_d \hat{\beta}_0(x) (1 + x^{d-\nu_0} \tilde{D}_{0d}(x, \mathbf{z}))}.$$

Let us first show that C is an analytic function. Since the denominator in the quotient above has valuation $d + \nu_0 < M_0 - 1$ it suffices to show that the numerator is divisible by x^{M_0-1} . We compute this numerator modulo x^{M_0-1} from (3.4), (3.5) and (3.9):

$$\check{B}(x, \mathbf{z}, 0, 0) \equiv \sum_{\tau' \leq \tau} \binom{\tau}{\tau'} \frac{\partial^{\tau'} Q}{\partial \mathbf{x}^{\tau'}}(J_d \hat{\alpha}(x) + x^d \mathbf{z}) \frac{\partial^{\tau-\tau'} \Delta}{\partial \mathbf{x}^{\tau-\tau'}}(J_d \hat{\alpha}(x) + x^d \mathbf{z}).$$

Since $J_M \Delta = 0$, we obtain

$$\text{val}_x \left(\frac{\partial^{\tau-\tau'} \Delta}{\partial \mathbf{x}^{\tau-\tau'}}(J_d \hat{\alpha}(x) + x^d \mathbf{z}) \right) \geq M - (|\tau| - |\tau'|) \geq M - |\tau| = M_0 - 1$$

and thus $\check{B}(x, \mathbf{z}, 0, 0) \equiv 0 \pmod{x^{M_0-1}}$. Set $Q_j(x) = J_d \hat{\beta}_j(x)$ for $j = 1, \dots, n_0$. We use equations (3.7), (3.8), (3.9) and (3.6) and that $\hat{\alpha}(x) = J_d \hat{\alpha}(x) + x^d T_d \hat{\alpha}(x)$ and

calculate:

$$\begin{aligned}
 x^d \hat{\beta}_0(x) C(x, T_d \hat{\alpha}(x), \{T_L \hat{H}(Q_j(x))\}_j) &= \\
 &= \tilde{B}(x, T_d \hat{\alpha}(x), \{x^{M_0} T_L \hat{H}(J_d \hat{\beta}_j(x))\}, 0) = \\
 &= \overline{B}(\hat{\alpha}(x), \{R_{s_j}(\hat{\alpha}(x)) (T_L \hat{H}_s)(P_j(\hat{\alpha}(x)))\}) = \\
 &= \hat{\beta}_0(x) \hat{\Delta}(\hat{\alpha}(x)).
 \end{aligned}$$

This immediately yields (3.3). Finally $\partial\Delta/\partial\mathbf{x}(0) = \partial C/\partial\mathbf{z}(0, 0, 0) = 0$ since $J_M \Delta = 0$ and the function C is divisible by x . \square

Proof of Proposition 3.7.- Let $\beta \in \mathcal{A}^1$ and suppose that $\beta(0) = 0$. By Lemma 3.9 there exists a system $\Phi(x, \mathbf{y}) = 0$ of n implicit equations in \mathcal{A} (here x denotes a single variable and $\mathbf{y} \in \mathbb{R}^n$) with $\partial\Phi/\partial\mathbf{y}(0)$ invertible, such that each row of Φ is either analytic or of the form $y_{l_i} - \Delta_i(x, \mathbf{y})$ for some $\Delta_i \in \mathcal{D}^{1+n}$ and such that its solution $\mathbf{y} = \phi = (\phi_1, \dots, \phi_n)$ satisfies $\phi_n = \beta$. By Remark 3, both kinds of rows satisfy the statement of Lemma 3.12 for the vector of formal series $\hat{\alpha}(x) = (x, \hat{\phi}(x)) \in \mathbb{R}[[x]]^{1+n}$, $\hat{\phi}$ being the Taylor series of ϕ at $x = 0$. Taking the maximum of the numbers d_0 for any of the rows and combining the corresponding polynomials Q_j , we see that there exists an analytic (vectorial) function $C(x, \mathbf{z}, \mathbf{w})$ and $d \in \mathbb{N}$ such that $\partial C/\partial\mathbf{z}(0) = \partial\Phi/\partial\mathbf{y}(0)$ is invertible and

$$C(x, T_d \hat{\phi}(x), \{T_L \hat{H}(Q_j(x))\}_j) = T_d(\hat{\Phi}(\hat{\alpha}(x))) \equiv 0. \quad (3.10)$$

The solution of the analytic equation $C(x, \mathbf{z}, \mathbf{w}) = 0$ with respect to \mathbf{z} is an analytic function $f(x, \mathbf{w})$. The germ $\varphi(x) = f(x, \{T_L \hat{H}(Q_j(x))\}_j)$ is a simple function and its Taylor series $\hat{\varphi}(x)$ satisfies $C(x, \hat{\varphi}(x), \{T_L \hat{H}(Q_j(x))\}_j) \equiv 0$. By the uniqueness of solutions of implicit equations, $\hat{\varphi}(x) = T_d \hat{\phi}(x)$. Now, define the function

$$\tilde{\varphi}(x) = J_d \phi(x) + x^d \varphi(x) = (\tilde{\varphi}_1(x), \dots, \tilde{\varphi}_n(x)).$$

It is a vector of simple functions whose Taylor series $\hat{\tilde{\varphi}}(x)$ coincides with $\hat{\phi}(x)$. Hence

$$\hat{\Phi}(x, \hat{\tilde{\varphi}}(x)) = \hat{\Phi}(x, \hat{\phi}(x)) \equiv 0.$$

On the other hand, $\hat{\Phi}(x, \hat{\tilde{\varphi}}(x))$ is the Taylor series of $\Phi(x, \tilde{\varphi}(x))$. This last vector belongs to $(\mathcal{S}_H)^n$ by Lemma 3.11. Due to the quasi-analyticity of the class \mathcal{S}_H , the germ of $x \mapsto \Phi(x, \tilde{\varphi}(x))$ is equal to 0 and thus $\phi = \tilde{\varphi} \in (\mathcal{S}_H)^n$. In particular $\phi_n = \beta \in \mathcal{S}_H$ is a simple function. \square

3.4. Theorem of the Complement for \mathcal{A}

As a consequence of the quasi-analyticity results of the preceding subsection, the theory developed in [22] allows to complete the proofs of Theorem 3.3 and Proposition 2.2.

Let $\mathcal{A} = \mathcal{A}_H$ be the smallest class of germs of C^∞ functions satisfying properties A1–A4. We define the \mathcal{A} -semianalytic and the \mathcal{A} -subanalytic sets in the usual way. Given $a = (a_1, \dots, a_m) \in \mathbb{R}^m$ and $\delta > 0$ denote $I_{a,\delta} = [a_1 - \delta, a_1 + \delta] \times \dots \times [a_m - \delta, a_m + \delta]$. A set $A \subset I_{a,\delta}$ is called \mathcal{A} -basic if it is of the form

$$\{\mathbf{x} \in I_{a,\delta} / F(\mathbf{x}) = 0, G_1(\mathbf{x}) > 0, \dots, G_n(\mathbf{x}) > 0\}$$

for some \mathcal{A} -analytic functions F, G_j in a neighborhood of $I_{a,\delta}$. A set $A \subset \mathbb{R}^m$ is said to be \mathcal{A} -semianalytic at the point $a \in \mathbb{R}^m$ if there exists $\delta > 0$ such that $A \cap I_{a,\delta}$

is a finite union of \mathcal{A} -basic sets. We say that $A \subset \mathbb{R}^m$ is \mathcal{A} -semianalytic if it is \mathcal{A} -semianalytic at any point of \mathbb{R}^m . In order to obtain finiteness of the number of connected components we restrict our considerations to *global \mathcal{A} -semianalytic sets* $A \subset \mathbb{R}^m$, defined by the condition that $\rho_m(A) \subset I^m$ is a \mathcal{A} -semianalytic set, where $\rho_m(x_1, \dots, x_m) = (x_1/\sqrt{1+x_1^2}, \dots, x_m/\sqrt{1+x_m^2})$. Finally, a set $B \subset \mathbb{R}^m$ is said to be (global) \mathcal{A} -subanalytic if there exists $n \geq m$ and a global \mathcal{A} -semianalytic set $A \subset \mathbb{R}^n$ such that $B = \pi(A)$, where $\pi : \mathbb{R}^n \rightarrow \mathbb{R}^m$ is the projection onto the first m coordinates. The following theorem is one of the main results of the article [22]:

THEOREM 3.13 *Theorem of the Complement for \mathcal{A} . Assume that \mathcal{A} is a quasi-analytic class satisfying A1–A4. Then, given any \mathcal{A} -subanalytic set $A \subset \mathbb{R}^m$, the complement $\mathbb{R}^m \setminus A$ is also \mathcal{A} -subanalytic and A has a finite number of connected components.*

For completeness, let us briefly recall the idea of the proof of Theorem 3.13. The quasi-analyticity of the class \mathcal{A} makes it possible to associate a finite *multiplicity* to any non zero element $\phi \in \mathcal{A}^m$. Using the properties A1–A4 of this class and a finite number of blowups and ramifications, we can write ϕ in a *normal form* $\phi(\mathbf{x}) = m(\mathbf{x})u(\mathbf{x})$, where $m(\mathbf{x})$ is a monomial in the coordinates \mathbf{x} and u is a unit in \mathcal{A}^m . Normalizing all the germs involved in the description of some \mathcal{A} -subanalytic set A , one proves the finiteness of the number of connected components of A , as well as the property that A has a dimension and that its boundary is included in some \mathcal{A} -subanalytic set of dimension strictly smaller than that of A . The property of the complement is a classical consequence of this property.

Proof of Theorem 3.3. By Theorem 3.5, we can assume that the Theorem of the Complement for \mathcal{A} is true. Denote by \mathcal{SA} the collection of global \mathcal{A} -subanalytic sets. It is routine to check that \mathcal{SA} is closed under finite unions and intersections, linear projections and cartesian products. Also, it contains the graph of any restricted analytic function as well as the graph of the extension \tilde{H} of the solution H . It is closed under taking complements by the first part of Theorem 3.13. Thus, any definable set in the structure $\mathbb{R}_{an,H}$ belongs to \mathcal{SA} and $\mathbb{R}_{an,H}$ is o-minimal by the second part of that theorem. On the other hand, denote by \mathcal{T} the smallest collection of sets satisfying S1–S4 for $\mathcal{F} = \tilde{\mathcal{F}}_{an} \cup \{\tilde{H}\}$ without the condition of taking complements and let us show that $\mathcal{SA} \subset \mathcal{T}$. This will prove that the elements of \mathcal{SA} are exactly the definable sets in $\mathbb{R}_{an,H}$ and that the structure $\mathbb{R}_{an,H}$ is model-complete. We only have to show that the \mathcal{A} -basic sets (at the origin, for instance) are contained in \mathcal{T} . Consider the collection $\tilde{\mathcal{A}} = (\tilde{\mathcal{A}}^m)$ of subalgebras of \mathcal{C}^∞ germs at $0 \in \mathbb{R}^m$ having a representative in some neighborhood of 0 whose graph is contained in \mathcal{T} . We show that $\mathcal{A} \subset \tilde{\mathcal{A}}$ by verifying that $\tilde{\mathcal{A}}$ satisfies properties A1–A4. This proves that any \mathcal{A} -basic set of the type $\{\mathbf{x} \in I_{0,\delta}/F(\mathbf{x}) = 0\}$ belongs to \mathcal{T} . On the other hand, any \mathcal{A} -basic set of the type $\{\mathbf{x} \in I_{0,\delta}/G(\mathbf{x}) > 0\} = \{\mathbf{x} \in I_{0,\delta}/\exists(y, z) \in \mathbb{R} \times \mathbb{R}^* \text{ with } G(\mathbf{x}) = y \text{ and } y = z^2\}$ is a projection of $(\text{graph } G \times \mathbb{R}) \cap X$ where X is some semialgebraic set. It also belongs to \mathcal{T} and the proof is complete.

Proof of Proposition 2.2.- Denote by \mathcal{SH} the collection of H -subanalytic sets and let us prove that $\mathcal{SH} = \mathcal{SA}$. Clearly $\mathcal{SH} \subset \mathcal{SA}$. On the other hand, it is easy to check that \mathcal{SH} is closed under finite unions and intersections, linear projections and cartesian products. We need only to prove that the \mathcal{A} -basic sets belong to \mathcal{SH} .

Consider the collection $\tilde{\mathcal{A}} = (\tilde{\mathcal{A}}^m)$ of subalgebras of \mathcal{C}^∞ germs at $0 \in \mathbb{R}^m$ having a representative in some neighborhood of 0 whose graph is H -subanalytic. The inclusion $\mathcal{A} \subset \tilde{\mathcal{A}}$ follows by verifying properties A1–A4 for $\tilde{\mathcal{A}}$ and we conclude as in the previous proof. \square

REMARK 4. It is interesting to note that the collection \mathcal{A}_H of algebras introduced below theorem 3.3 does *not* allow (in general) to factor regular elements as in the Weierstrass preparation theorem which holds e.g. for the algebras of germs of functions analytic at the origin or for those introduced in [8] in the context of summability. Thus we have to rely on a result [22] that uses desingularization techniques not needing Weierstrass preparation.

Precisely, consider any strictly increasing solution $H : [0, a] \rightarrow \mathbb{R}$, $a > 0$ of the equation $x^3 H' = (1 - x^2)H - x$. As its formal solution $\hat{H}(x) = \sum_{n \geq 0} 2^n n! x^{2n+1}$ is odd, H can be extended to an odd solution $H : [-a, a] \rightarrow \mathbb{R}$. Denote by \mathcal{A}^m , $m = 1, 2, \dots$ the algebras generated by the germs of functions analytic at the origin and the germ of H having the properties A1–A4 (see below theorem 3.3). As the formal solution \hat{H} is divergent, the hypotheses of Theorem 2.4 are satisfied. As discussed in example 2, if the Stokes coefficients vanished then there would be no singular direction and the formal series would have to converge. Thus \hat{H} is strongly analytically transcendental (SAT).

Consider now the inverse function $I : [0, H(a)] \rightarrow \mathbb{R}$ of H which by proposition 3.7 is a simple function (this can also be shown directly) and put $F(x, y) = I(y)^2 + x$. This clearly is a function of \mathcal{A}^2 and it is regular as $F(0, y) = I(y)^2 = y^2 + \mathcal{O}(y^3)$, but *it does not have a factor $y^2 - G_1(x)y + G_2(x)$ that is polynomial in y and belongs to \mathcal{A}^2* as we will show below.

First of all, the formal series $\hat{F}(x, y) = \hat{I}(y)^2 + x$ can be factored $\hat{F}(x, y) = (y^2 - \hat{G}_1(x)y + \hat{G}_2(x))\hat{U}(x, y)$ where $\hat{G}_1, \hat{G}_2, \hat{U}$ are formal series and $\hat{U}(x, y) = 1 + \dots$ is a unit. The zeros of \hat{F} are clearly $y = \hat{H}(\pm i x^{1/2}) = \pm \hat{H}(i x^{1/2})$ and thus $\hat{G}_1(x) = 0$, $\hat{G}_2(x) = -\hat{H}(i x^{1/2})^2$. Putting $t = i x^{1/2}$ this yields $\hat{G}_2(-t^2) = -\hat{H}(t)^2$. This implies that $\hat{G}_2(x)$ cannot be the Taylor series of a simple function $G_2 = f(x, \{T_k H(P_l(x)), l = 1, \dots, n\})$ as in (3.1) because an equation

$$f(-t^2, \{T_k \hat{H}(P_l(-t^2)), l = 1, \dots, n\}) = -\hat{H}(t)^2$$

would contradict the (SAT); as $\hat{H}(x)$ is odd, $T_k \hat{H}(P_l(-t^2))$ can be reduced to some $\pm T_k \hat{H}(\tilde{P}_l(t^2))$, where $\tilde{P}_l^{\text{(val } \tilde{P}_l)}(0) > 0$.

On the other hand, a factor $y^2 - G_1(x)y + G_2(x)$ of $F(x, y)$ in \mathcal{A}^2 would have to have coefficients G_1, G_2 in \mathcal{A}^1 , i.e. simple functions according to proposition 3.7. Therefore such a factor cannot exist.

4. Quasi-analyticity of the algebra of simple functions

This paragraph is devoted to the proof of Theorem 2.4. Let us first show how this result implies Theorem 2.3. This is an immediate consequence of the following

LEMMA 4.1. *Let $\hat{H}(x) \in \mathbb{R}[[x]]^r$ be a formal solution of a system of analytic ordinary differential equations of the form (1.1). If $\hat{H}(x)$ is strongly analytically transcendental then any actual solution $H : (0, \varepsilon] \rightarrow \mathbb{R}^r$ such that $H(x) \sim \hat{H}(x)$ as $\mathbb{R}^+ \ni x \rightarrow 0$ is strongly quasi-analytic.*

Proof. Let $f \in \mathbb{R}\{x, z_{11}, \dots, z_{rn}\}$ and $P_1(x), \dots, P_n(x) \in \mathbb{R}[x]$ be as in (SQA) and assume that $f(x, \{T_k \widehat{H}_j(P_l(x))\}_{j,l}) \equiv 0$. Let $\nu_l = \text{val} P_l > 0$ and denote by $Q_l(x) = J_{(p+1)\nu_l-1} P_l(x)$, a polynomial of valuation ν_l and degree $d_l < (p+1)\nu_l$. Up to some permutation, we can assume that the set $\{Q_l | l = 1, \dots, n\} = \{Q_l | l = 1, \dots, m\}$ with some $m \in \{1, \dots, n\}$ and *distinct* $Q_l, l = 1, \dots, m$. We can write

$$P_l(x) = Q_l(x) + Q_l(x)^{p+1} u_l(x)$$

where u_l is some analytic function. By Lemma 3.1 applied to $T_k H$ (which is a solution of a system of ordinary differential equations of the form (1.1)), there exists another analytic function $\tilde{f} \in \mathbb{R}\{x, z_{11}, \dots, z_{rm}\}$ such that

$$f(x, \{T_k H_j(P_l(x))\}_{1 \leq j \leq r, 1 \leq l \leq n}) = \tilde{f}(x, \{T_k H_j(Q_l(x))\}_{1 \leq j \leq r, 1 \leq l \leq m}).$$

Thus by assumption, $\tilde{f}(x, \{T_k \widehat{H}_j(Q_l(x))\}_{j,l}) \equiv 0$. By (SAT), we obtain $\tilde{f} \equiv 0$ and we conclude that $f(x, \{T_k H_j(P_l(x))\}_{j,l}) \equiv 0$. \square

4.1. Background on elementary theory of summation and statement of the result

Consider a system (1.1) of analytic ordinary differential equations. We assume that its linear part $A_0 = \partial A / \partial \mathbf{y}(0)$ has non zero eigenvalues $\lambda_1, \dots, \lambda_r$ satisfying the *distinct argument condition*:

$$(DA) \quad \arg(\lambda_i) \not\equiv \arg(\lambda_j) \pmod{2\pi\mathbb{Z}} \quad \text{if } i \neq j.$$

As we have already mentioned in paragraph 2.2, condition (DA) implies that there exists a unique formal power series solution $\mathbf{y} = \widehat{H}(x) = (\widehat{H}_1(x), \dots, \widehat{H}_r(x)) \in \mathbb{R}[[x]]^r$ with $\widehat{H}(0) = 0$. Its coefficients can easily be determined recursively due to the invertibility of A_0 .

REMARK 5. Moreover, although we do not use the result, there exists at least one actual solution $H = (H_1, \dots, H_r) : (0, \varepsilon] \rightarrow \mathbb{R}^r$ of (1.1) having the formal series $\widehat{H}(x)$ as its asymptotic expansion as $\mathbb{R}^+ \ni x \rightarrow 0$. When there are no eigenvalues on the imaginary axis, the existence of H is a consequence of the classical theory of center manifolds ([11, 6]). In the general situation, we can proceed as in the fundamental theorem of existence of complex valued solutions of (1.1) [27]. See also [1] for the case $r = 2$.

A general result about systems of ordinary differential equations in the complex domain asserts that formal solutions of systems (1.1) are multisummable under very weak conditions (see for instance [3]); furthermore it is known that the algebra of multisummable series is closed under composition. Instead of using these facts as a “black box”, we decided to present our result using only some relatively elementary facts about summability of divergent series. In our particular case where the linear part A_0 is non singular we only need the following result (we use the standard notation $S(\alpha, \beta; \rho)$, $\alpha, \beta \in \mathbb{R}$, $\rho > 0$ for the sector $\{z \in \mathbb{C} / |z| < \rho, \alpha < \arg(z) < \beta\}$ in the complex plane of opening $\beta - \alpha$):

LEMMA 4.2. [21] *The vector $\widehat{H}(z)$ of formal series is p -summable. More precisely, for any $\theta \in \mathbb{R}$, except for those which satisfy $p\theta \equiv \arg(\lambda_j) \pmod{2\pi\mathbb{Z}}$ for some eigenvalue λ_j of A_0 , and for any open sector $S_\theta = S(\theta - \frac{\pi}{2p} - \delta, \theta + \frac{\pi}{2p} + \delta; \rho)$ of*

opening slightly greater than π/p (with δ, ρ sufficiently small) there exists a unique holomorphic function $\tilde{H}_\theta = (\tilde{H}_{\theta,1}, \dots, \tilde{H}_{\theta,r}) : S_\theta \rightarrow \mathbb{C}^r$ such that:

- 1) \tilde{H}_θ satisfies the (complexified) system of equations (1.1) and
- 2) For any i , $\tilde{H}_{\theta,i}$ has the series $\hat{H}_i(z)$ as the asymptotic expansion on the sector S_θ . This asymptotic expansion is, furthermore, Gevrey of order $\kappa = 1/p$.

The function \tilde{H}_θ is called the p -sum of the series $\hat{H}(z)$ along the ray $d_\theta = \{z/\arg(z) = \theta\}$.

By definition (see [20]), a bounded holomorphic function $h : S \rightarrow \mathbb{C}$ on a sector $S = S(\alpha, \beta; \rho)$ has a Gevrey asymptotic expansion of order κ (or a κ -Gevrey asymptotic expansion) with right hand side $\hat{h}(z) = \sum_{k \geq 0} a_k z^k \in \mathbb{C}[[z]]$, and we write $h(z) \sim_\kappa \hat{h}(z)$, if for any $\eta > 0$ there are constants $K, A > 0$ such that

$$\left| h(z) - \sum_{k=0}^{N-1} a_k z^k \right| \leq K A^N \Gamma(N\kappa + 1) |z|^N,$$

for any $N \in \mathbb{N}$ and any $z \in S(\alpha + \eta, \beta - \eta; \rho - \eta)$. Observe, that $h(z) \sim_\kappa 0 + 0z + \dots$ if and only if $h(z)$ is exponentially small of order $1/\kappa$, i.e. for any $\eta > 0$, there is a positive constant a such that $|h(z) \exp(a|z|^{-1/\kappa})|$ is bounded on $S(\alpha + \eta, \beta - \eta; \rho - \eta)$. This follows easily from Stirling's Formula by choosing N as the integer closest to $|z|^{-1/\kappa}$.

There are more algorithmical definitions of p -summability (see also e.g. [12]). The function $\tilde{H}_\theta(z)$ can be expressed as p -Borel-Laplace sum of $\hat{H}(z)$ in the direction d_θ : The formal Borel transform of order p , i.e. the series $\mathbf{H}(t) = \sum_{m \geq 0} \frac{1}{\Gamma(1+m/p)} H_{m+1} t^m$ converges in some neighborhood of $t = 0$; it can be continued analytically to the ray $\arg t = \theta$ (again named $\mathbf{H}(t)$) and has at most exponential growth of order p ; finally \tilde{H}_θ is the Laplace transform of order p of $\mathbf{H}(t)$, i.e. $\tilde{H}_\theta(z) = \int_{d_\theta} e^{-t^p/z^p} \mathbf{H}(t) \frac{p t^{p-1}}{z^{p-1}} dt$.

The excluded angles $\theta \in \{\frac{1}{p}(\arg(\lambda_j) + 2\pi s) / j = 1, \dots, r, s \in \mathbb{Z}\}$ in the above lemma are called *singular* and the corresponding rays d_θ are called the *singular directions* (also called the *anti-Stokes directions*) of $\hat{H}(z)$. By the hypothesis (DA), there are exactly rp different singular directions modulo $2\pi\mathbb{Z}$. Denote them by d_{θ_l} , $l = 0, \dots, rp - 1$ with angles $\theta_l \in [0, 2\pi)$. Up to reordering the eigenvalues λ_j , we can choose the indices such that $0 \leq \theta_0 < \dots < \theta_{rp-1}$ and $p\theta_l \equiv \arg(\lambda_j) \pmod{2\pi\mathbb{Z}}$ if and only if $l \pm 1 \equiv j \pmod{r}$.

Two p -sums $\tilde{H}_\theta, \tilde{H}_{\theta'}$ of $\hat{H}(z)$ coincide in the intersection of their domains if there is no singular direction between d_θ and $d_{\theta'}$. We obtain, by analytic continuation, holomorphic functions

$$\tilde{H}_l : \tilde{S}_l = S(\theta_l - \frac{\pi}{2p} + \delta, \theta_{l+1} + \frac{\pi}{2p} - \delta; \rho) \rightarrow \mathbb{C}^r, \quad l = 0, \dots, rp - 1$$

where $\delta, \rho > 0$ are sufficiently small and where we put $\theta_{rp} = \theta_0 + 2\pi$. Mostly, we will use some smaller sectors $S_l = S(\theta_l - \varepsilon, \theta_{l+1} + \varepsilon; \rho) \subset \tilde{S}_l$ with some $\varepsilon > 0$, so that the family $\{S_l\}_l$ forms a *good covering* of the punctured disc $\dot{D}_\rho \subset \mathbb{C}$, i.e. $S_l \cap S_k \neq \emptyset$ if and only if $|l - k| \leq 1 \pmod{rp}$. In general, the functions \tilde{H}_l cannot be continued analytically or change their asymptotic behavior beyond the rays of angles $\theta_l - \frac{\pi}{2p}, \theta_{l+1} + \frac{\pi}{2p}$, $l = 0, \dots, rp - 1$, called the *Stokes directions* of $\hat{H}(z)$. The so called *Stokes phenomenon* of the series $\hat{H}(z)$ is the description of the behavior of the

difference $\Delta_l(z) = \tilde{H}_{l+1}(z) - \tilde{H}_l(z)$ of two such consecutive functions. This difference is defined in the intersection $\tilde{S}_{l,l+1} = \tilde{S}_l \cap \tilde{S}_{l+1} = S(\theta_{l+1} - \frac{\pi}{2p} + \delta, \theta_{l+1} + \frac{\pi}{2p} - \delta; \rho)$, a sector of opening slightly smaller than π/p . It satisfies a system of linear differential equations (in the complex domain)

$$z^{p+1} \frac{d\mathbf{y}}{dz} = B_l(z)\mathbf{y}, \quad (4.1)$$

where $B_l(z)$ is the matrix of holomorphic functions on $\tilde{S}_{l,l+1}$ defined by $B_l(z) = \overline{B}(z, \tilde{H}_l(z), \tilde{H}_{l+1})$ with $\overline{B}(z, \mathbf{y}_1, \mathbf{y}_2)$ analytic and satisfying

$$A(z, \mathbf{y}_2) - A(z, \mathbf{y}_1) = \overline{B}(z, \mathbf{y}_1, \mathbf{y}_2)(\mathbf{y}_2 - \mathbf{y}_1).$$

We see that all matrices $B_l(z)$ for $l = 0, \dots, rp - 1$ have the same $\frac{1}{p}$ -Gevrey asymptotic expansion on the sector $\tilde{S}_{l,l+1}$ equal to $\hat{B}(z) = \overline{B}(z, \hat{H}(z), \hat{H}(z)) = \partial A / \partial \mathbf{y}(z, \hat{H}(z))$. The initial term $\hat{B}(0) = A_0$ has distinct eigenvalues. In this situation, the classical theory of linear ordinary differential equations (see [27]) asserts that there exists a fundamental matrix solution of (4.1) of the form

$$Y_l(z) = G_l(z) \exp(Q(z)) z^J, \quad (4.2)$$

where:

- (1) $G_l(z)$ is a matrix of holomorphic functions on $\tilde{S}_{l,l+1}$. There exists a formal series $\hat{G}(z)$ such that, for every l , the matrix $G_l(z)$ has $\hat{G}(z)$ as its $\frac{1}{p}$ -Gevrey asymptotic expansion in $\tilde{S}_{l,l+1}$, moreover $\det(\hat{G}(0)) \neq 0$;
- (2) $J = \text{diag}(\alpha_1, \dots, \alpha_r)$ is a constant diagonal matrix and
- (3) $Q(z) = \text{diag}(q_1(z), \dots, q_r(z))$ is a diagonal matrix where $q_j(z) = -\frac{\lambda_j}{p} z^{-p} + \dots \in \mathbb{C}[z^{-1}]$ are polynomials in the variable z^{-1} of degree p without constant term.

The anti-Stokes or singular directions (respectively the Stokes directions) are precisely the rays where the initial term of some of the polynomials $q_j(z)$ is real negative (respectively purely imaginary). Denote the columns of the matrix $G_l(z)$ by $G_{lj}(z)$, $j = 1, \dots, r$. The particular solution Δ_l of (4.1) defined above (4.1) can be written as

$$\Delta_l(z) = \sum_j c_{lj} G_{lj}(z) e^{q_j(z)} z^{\alpha_j}, \quad (4.3)$$

where $c_l = (c_{l1}, \dots, c_{lr}) \in \mathbb{C}^r$ is some constant vector.

LEMMA 4.3. *Given $l \in \{0, \dots, rp - 1\}$ and $\mu = \mu(l) \in \{1, \dots, r\}$ defined by $l + 1 \equiv \mu \pmod{r}$, we have that $c_{lj} = 0$ for every $j \neq \mu$.*

Proof. This is a classical result. It is due to the fact that for every $j \neq \mu$, the function $\exp(q_j(z))$ is exponentially large on some ray in $\tilde{S}_{l,l+1}$ while Δ_l remains bounded on that sector. \square

The coefficient $\gamma_l = c_{l\mu(l)}$ is called the *Stokes multiplier* of the solution $\hat{H}(z)$ associated to the singular direction $d_{\theta_{l+1}}$.

Now we restate Theorem 2.4 in the following more precise form:

THEOREM 4.4. *Consider a system of (real) analytic ordinary differential equations (1.1) such that the linear part A_0 satisfies condition (DA). Assume, furthermore, that the Stokes multipliers of its unique formal solution $\widehat{H}(z) \in \mathbb{C}[[z]]^r$ satisfy the following condition: for any $\mu \in \{1, \dots, r\}$ there exists some $l \in \{0, \dots, pr - 1\}$ with $l + 1 \equiv \mu \pmod{r}$ such that $\gamma_l \neq 0$. Then, the series $\widehat{H}(z)$ is strongly analytically transcendental in the following sense:*

(SAT) *If $k \geq 0$, $n \geq 0$, an analytic function $f \in \mathbb{C}\{z, z_{11}, \dots, z_{rn}\}$ with $f(0) = 0$ and distinct real polynomials $P_1(z), \dots, P_n(z) \in \mathbb{R}[z]$ with $\deg P_l < (p + 1)\text{val } P_l$ and $P_l^{(\text{val } P_l)}(0) > 0$ are given, then one has*

$$f(z, \{T_k \widehat{H}_j(P_l(z))\}_{j,l}) \equiv 0 \implies f \equiv 0.$$

We will denote by (SD) (for *singularity of all directions* $\arg(\lambda_j)$) the condition on the Stokes multipliers in the above statement. In the proof of Theorem 4.4 we will need the following reformulations of two well known results of the theory of Gevrey series (see [16]).

LEMMA 4.5 Ramis-Sibuya. *Let $\kappa > 0$. Let $h_l : V_l \rightarrow \mathbb{C}$, $l = 0, \dots, N$ be a family of bounded holomorphic functions in sectors $V_l = S(\alpha_l, \beta_l; \rho)$ such that $\{V_0, \dots, V_N\}$ is a good covering of the punctured disc $\{z \in \mathbb{C} / 0 < |z| < \rho\}$ (i.e. the only two by two non-empty intersections are the consecutive ones $V_{l,l+1} = V_l \cap V_{l+1}$, where $V_{N+1} = V_0$). Then the differences $h_{l+1} - h_l$ are exponentially small of order κ on $V_{l,l+1}$ for $l = 0, \dots, N$ if and only if there exists a formal series $\widehat{h}(z) \in \mathbb{C}[[z]]$ such that $h_l(z) \sim_{1/\kappa} \widehat{h}(z)$ on V_l for each l .*

LEMMA 4.6 Relative Watson's Lemma. *Let $\kappa_2 > \kappa_1 > 0$ and $\beta > \alpha + \pi/\kappa_1$. Consider a good covering of $V = S(\alpha, \beta; \rho)$ by sectors $V_l = S(\alpha_l, \beta_l; \rho)$, $l = 1, \dots, N$. Suppose that we have holomorphic functions $h_l : V_l \rightarrow \mathbb{C}$ satisfying the following property: there are some constants $a, b > 0$ such that $|h_l(z) \exp(a|z|^{-\kappa_1})|$ and $|(h_{l+1}(z) - h_l(z)) \exp(b|z|^{-\kappa_2})|$ are bounded on V_l and $V_l \cap V_{l+1}$, respectively for $l = 0, \dots, N$. Then, h_l is exponentially small of order κ_2 on V_l for any l .*

4.2. Proof of Theorem 4.4

First, taking into account that $T_k \widehat{H}(z)$ is the unique formal solution of a system of equations analogous to (1.1), we can suppose without loss of generality that $k = 0$. Let $f \in \mathbb{C}\{z, z_{11}, \dots, z_{rn}\}$ and $P_j(z) \in \mathbb{R}[z]$ be given satisfying the hypothesis of (SAT). Consider the subset $\Lambda \subset \{1, \dots, r\} \times \{1, \dots, n\}$ of indices (i, j) such that f effectively depends on the variable z_{ij} , i.e. its derivative with respect to this variable does not vanish identically. We prove Theorem 4.4 by induction on the number of elements of Λ . Put $\widehat{K}(z) = (z, \{\widehat{H}_i(P_j(z))\}_{(i,j) \in \Lambda})$ in order to simplify the notation.

If Λ is empty then the result is trivial.

Suppose that $\text{card } \Lambda = c \geq 1$ and that the statement has been shown if $\text{card } \Lambda = c - 1$. In order to prove the result we assume that $f(\widehat{K}(z)) \equiv 0$ but $f \not\equiv 0$ and we deduce a contradiction.

First we deduce that for every $(k, l) \in \Lambda$, there exists some s such that $\partial^s f / \partial z_{kl}^s(\widehat{K}(z)) \neq 0$. In fact, if this property is not true, then we have that the series in the two variables

z, z_{kl}

$$f(z, z_{kl}, \{\widehat{H}_i(P_j(z))\}_{(i,j) \in \Lambda \setminus \{(k,l)\}}) = \sum_{m=0}^{\infty} \frac{1}{m!} \frac{\partial^m f}{\partial z_{kl}^m}(\widehat{K}(z)) (z_{kl} - \widehat{H}_k(P_l(z)))^m$$

is identically zero. Thus, any coefficient of this series in z_{kl} is identically zero. Writing f as a series $f(z, \mathbf{z}) = \sum_{m=0}^{\infty} f_m(z, \{z_{ij}\}_{(i,j) \neq (k,l)}) z_{kl}^m$ we conclude, by the hypothesis of induction on $\text{card}(\Lambda)$, that $f_m \equiv 0$ for all m and hence $f \equiv 0$, contrary to our assumption.

For any fixed $(i, j) \in \Lambda$, we can suppose without loss of generality that $\frac{\partial f}{\partial z_{ij}}(\widehat{K}(z)) \neq 0$. The couple (i, j) will be chosen below. This can be justified as follows: Suppose for simplicity that $(i, j) = (1, 1)$ and consider the smallest integer $s \geq 1$ such that $\partial^s f / \partial z_{11}^s(\widehat{K}(z)) \neq 0$. Denote $g = \partial^{s-1} f / \partial z_{11}^{s-1}$. This function satisfies $g(\widehat{K}(z)) \equiv 0$ and $\partial g / \partial z_{11}(\widehat{K}(z)) \neq 0$. Renaming g to f , we can assume the desired conditions for f .

Given $j \in \{1, \dots, n\}$, the vector series $\widehat{H}(P_j(z)) \in \mathbb{C}[[z]]^r$ is $\frac{1}{p\nu_j}$ -summable, ν_j being the valuation of the polynomial $P_j(z)$; this follows easily from Ramis-Sibuya's Lemma. Moreover, $\varphi \in \mathbb{R}$ is a singular angle for $\widehat{H}(P_j(z))$ if and only if $\theta = \nu_j \varphi$ is a singular angle for $\widehat{H}(z)$. The $\frac{1}{p\nu_j}$ -sum of $\widehat{H}(P_j(z))$ along a non singular direction $d_\varphi = \{z / \arg(z) = \varphi\}$ is given by $\widetilde{H}_{\nu_j \varphi}(P_j(z)) : \widetilde{V}_{j, \varphi} \rightarrow \mathbb{C}^r$, defined in some sector $\widetilde{V}_{j, \varphi}$ of opening greater than $\pi/p\nu_j$ bisected by d_φ (recall the definition of \widetilde{H}_θ in Lemma 4.2). Let Γ be the set of all singular directions of the series $\widehat{H}(P_j(z))$, $j = 1, \dots, n$ and denote its elements by d_{φ_l} , $l = 0, \dots, N$ with $0 \leq \varphi_0 < \dots < \varphi_N < 2\pi$. If $d_\varphi \notin \Gamma$ then the function

$$F_\varphi(z) = f(z, \{\widetilde{H}_{\nu_j \varphi, i}(P_j(z))\}_{(i,j) \in \Lambda})$$

is defined in the sector $\widetilde{V}_\varphi = \bigcap \widetilde{V}_{j, \varphi}$ of opening slightly greater than $\pi/p\bar{\nu}$ where $\bar{\nu} = \max(\nu_1, \dots, \nu_n)$. In a similar way as was explained before for the series $\widehat{H}(z)$, for any given angle φ in the interval $]\varphi_k, \varphi_{k+1}[$, $k = 0, \dots, N$ (with $\varphi_{N+1} = \varphi_0 + 2\pi$), F_φ can be continued analytically to a holomorphic function $\widetilde{F}_k : \widetilde{V}_k \rightarrow \mathbb{C}$ in a sector of the form $\widetilde{V}_k = S(\varphi_k - \frac{\pi}{2p\bar{\nu}} - \delta, \varphi_{k+1} + \frac{\pi}{2p\bar{\nu}} + \delta; \rho)$ for some $\delta, \rho > 0$ sufficiently small. We consider subsectors $V_k = S(\varphi_k - \varepsilon, \varphi_{k+1} + \varepsilon; \rho) \subset \widetilde{V}_k$, $k = 0, \dots, N$ which form a good covering of a punctured disc at $0 \in \mathbb{C}$ and the restrictions $F_k = \widetilde{F}_k|_{V_k} : V_k \rightarrow \mathbb{C}$. Let $\nu = \min(\nu_1, \dots, \nu_n)$ and let us show the two following properties to obtain the desired contradiction (from now on, any statement is valid, without mentioning this explicitly every time, if ε and ρ are considered sufficiently small in the definition of the sectors V_k):

- I) Each F_k is exponentially small on V_k of order $\kappa > p\nu$ on V_k .
- II) There is at least one $k_0 \in \{0, \dots, N\}$ such that the difference $F_{k_0+1} - F_{k_0}$ is exponentially small of order exactly (not larger than) $p\nu$ on the intersection $V_{k, k+1} = V_k \cap V_{k+1}$ (with $V_{N+1} = V_0$).

Proof of I). Consider new variables $\bar{\mathbf{z}} = \{\bar{z}_{ij}\}_{(i,j) \in \Lambda}$ and analytic functions $D_{ij}(z, \mathbf{z}, \bar{\mathbf{z}})$ satisfying

$$f(z, \bar{\mathbf{z}}) - f(z, \mathbf{z}) = \sum_{(i,j) \in \Lambda} D_{ij}(z, \mathbf{z}, \bar{\mathbf{z}})(\bar{z}_{ij} - z_{ij}). \quad (4.4)$$

For any $(i, j) \in \Lambda$ and any k denote by $h_{ij, k} : V_k \rightarrow \mathbb{C}$ the restriction of the function $\widetilde{H}_{\nu_j \varphi, i}(P_j(z))$ with some $\varphi \in (\varphi_k, \varphi_{k+1})$. Let $D_{ij, k} : V_{k, k+1} \rightarrow \mathbb{C}$ be the

function obtained by replacing the variables z_{uv} and \bar{z}_{uv} in $D_{ij}(z, \mathbf{z}, \bar{\mathbf{z}})$ by $h_{uv,k}(z)$ and $h_{uv,k+1}(z)$ respectively. This yields for $k = 0, \dots, N$ and $z \in V_{k,k+1}$

$$F_{k+1}(z) - F_k(z) = \sum_{(i,j) \in \Lambda} D_{ij,k}(z)(h_{ij,k+1}(z) - h_{ij,k}(z)). \quad (4.5)$$

Notice that $D_{ij,k}$ has on $V_{k,k+1}$ a Gevrey asymptotic expansion of order $1/p\nu$ with right hand side $\partial f / \partial z_{ij}(\widehat{K}(z))$. The factor $h_{ij,k+1} - h_{ij,k}$ is exponentially small of order greater or equal to $p\nu_j$ on $V_{k,k+1}$ by Ramis-Sibuya's Lemma, since $h_{ij,k}$ and $h_{ij,k+1}$ are two $\frac{1}{p\nu_j}$ -sums of the same series $\widehat{H}_i(P_j(z))$. Thus, $F_{k+1} - F_k$ is exponentially small of order greater or equal to $p\nu$ on $V_{k,k+1}$. Again by Ramis-Sibuya's Lemma, the functions $F_k : V_k \rightarrow \mathbb{C}$ have a (common) Gevrey asymptotic expansion of order $1/p\nu$, by construction equal to $f(z, \{\widehat{H}_i(P_j(z))\}) \equiv 0$. Hence F_k is exponentially small of order greater or equal to $p\nu$ on V_k .

Let us show that this order is strictly greater than $p\nu$. First, note that F_k is defined as F_φ for any φ in the interval $(\varphi_k, \varphi_{k+1})$ and that F_φ is defined at least in the sector \widetilde{V}_φ bisected by the ray d_φ and whose opening does not depend on φ . Hence, it suffices to prove

LEMMA 4.7. *If $d_\varphi \notin \Gamma$ then F_φ is exponentially small of order (strictly) greater than $p\nu$ in \widetilde{V}_φ .*

Proof. Fix $d_\varphi \notin \Gamma$ and consider the set of sectors $\{V_{k_1}, \dots, V_{k_2}\}$ that have a non empty intersection with the closed sector $\{z \mid |z| < \rho, |\arg(z) - \varphi| \leq \frac{\pi}{2p\nu}\}$. The union $V = V_{k_1} \cup \dots \cup V_{k_2}$ is a sector of opening strictly greater than $\pi/p\nu$ and contains d_φ . We can suppose that $\widetilde{V}_\varphi \subset V$. Moreover, we can suppose that for any $j \in \{1, \dots, n\}$ such that $\nu_j = \nu$, the holomorphic function $\widetilde{H}_{\nu,\varphi}(P_j(z))$ is defined in the whole sector V . Consider for any $k \in \{k_1, \dots, k_2\}$ the holomorphic function $\overline{F}_k : V_k \rightarrow \mathbb{C}$ defined by $\overline{F}_k(z) = f(z, \{\overline{h}_{ij,k}(z)\}_{(i,j) \in \Lambda})$ where $\overline{h}_{ij,k}$ is the suitable modification of the function $h_{ij,k}$ given by

$$\overline{h}_{ij,k}(z) = \begin{cases} \widetilde{H}_{\nu,\varphi,i}(P_j(z)), & \text{if } \nu_j = \nu; \\ h_{ij,k}(z), & \text{if } \nu_j > \nu. \end{cases}$$

Notice that the differences $\overline{h}_{ij,k+1} - \overline{h}_{ij,k}$ are either zero or equal to $h_{ij,k+1} - h_{ij,k}$ when $\nu_j > \nu$, so they are exponentially small of order strictly greater than $p\nu$ on $V_{k,k+1}$. Using an equation analogous to (4.5), the same is true for $\overline{F}_{k+1} - \overline{F}_k$. By the Relative Watson's Lemma, the functions $\overline{F}_k : V_k \rightarrow \mathbb{C}$ are also exponentially small of order greater than $p\nu$. Finally, we compare \overline{F}_k to F_φ using an equation similar to (4.5): there are bounded functions $\overline{D}_{ij,k}(z)$ in $\widetilde{V}_\varphi \cap V_k$ with

$$\overline{F}_k(z) - F_\varphi(z) = \sum_{(i,j) \in \Lambda} \overline{D}_{ij,k}(z)(\overline{h}_{ij,k}(z) - \widetilde{H}_{\nu,\varphi,i}(P_j(z))).$$

Hence, $\overline{F}_k - F_\varphi$ is exponentially small of order strictly greater than $p\nu$ on $\widetilde{V}_\varphi \cap V_k$ for any $k \in \{k_1, \dots, k_2\}$. The result follows. \square

Proof of II.- We want to estimate the exponential growth of the difference $F_{k+1}(z) - F_k(z)$, for some index k chosen below. We use equation (4.5). We first look at the second factor of each summand in that expression. We have for any $(i, j) \in \Lambda$ and

$z \in V_{k,k+1}$:

$$h_{ij,k+1}(z) - h_{ij,k}(z) = \widetilde{H}_{\nu_j \bar{\varphi}, i}(P_j(z)) - \widetilde{H}_{\nu_j \varphi, i}(P_j(z)), \quad (4.6)$$

where $\varphi < \varphi_{k+1} < \bar{\varphi}$ are arbitrarily chosen but sufficiently close to φ_{k+1} . Notice that if $\nu_j \varphi_{k+1}$ is not one of the singular angles $\{\theta_0, \dots, \theta_{rp-1}\}$ of the series $\widehat{H}(z)$ then $\widetilde{H}_{\nu_j \varphi}$ and $\widetilde{H}_{\nu_j \bar{\varphi}}$ coincide and hence $h_{ij,k+1} - h_{ij,k} \equiv 0$. Otherwise, if $\nu_j \varphi_{k+1} = \theta_{l(k,j)+1}$ for some $l(k,j) \in \{0, \dots, rp-1\}$ (recall that we set $\theta_{rp} = \theta_0 + 2\pi$) then we have $h_{ij,k+1}(z) - h_{ij,k}(z) = \Delta_{l(k,j), i}(P_j(z))$ (the function $\Delta_l = (\Delta_{l,1}, \dots, \Delta_{l,r}) = \widetilde{H}_{l+1} - \widetilde{H}_l$ being defined in the preceding paragraph). Denote by J_k the set of indices j such that there exists $l(k,j)$ with $\nu_j \varphi_{k+1} = \theta_{l(k,j)+1}$. Using equation (4.3), Lemma 4.3 and the definition of the Stokes multipliers γ_l we obtain for any k

$$F_{k+1}(z) - F_k(z) = \sum_{(i,j) \in \Lambda, j \in J_k} D_{ij,k}(z) \gamma_{l(k,j)} G_{l(k,j)\mu(k,j)}^i(P_j(z)) \exp(q_{\mu(k,j)}(P_j(z))) P_j(z)^{\alpha_{\mu(k,j)}}. \quad (4.7)$$

Here $\mu(k,j) \in \{1, \dots, r\}$ is defined by $\mu(k,j) \equiv l(k,j) + 1 \pmod{r}$ and $G_{l\mu}^i$ is the i -th entry of the column vector $G_{l\mu}$ of the matrix G_l in (4.2).

We compute the real part of $q_{\mu(k,j)}(P_j(z))$:

$$\operatorname{Re}(q_{\mu(k,j)}(P_j(z))) = -\frac{|\lambda_{\mu(k,j)}|}{pa_j^p |z|^{p\nu_j}} \cos(\arg(\lambda_{\mu(k,j)}) - p\nu_j \arg(z)) (1 + O(|z|)), \quad (4.8)$$

where here and in the sequel $a_j := P_j^{(\nu_j)}(0)/\nu_j! > 0$ with $\nu_j = \operatorname{val} P_j$ denotes the first non-zero coefficient of P_j . Notice that if z is in a small sector bisected by $d_{\varphi_{k+1}}$ then $p\nu_j \arg(z)$ is close to $\arg(\lambda_{\mu(k,j)}) \pmod{2\pi\mathbb{Z}}$. Hence, the exponential term in (4.7) satisfies for $z \in V_{k,k+1}$ and $\delta > 0$ sufficiently small:

$$|\exp(q_{\mu(k,l)}(P_j(z)))| = \exp(\operatorname{Re}(q_{\mu(k,l)}(P_j(z)))) = O\left(\exp\left(-\frac{|\lambda_{\mu(k,j)}| - \delta}{pa_j^p |z|^{p\nu_j}}\right)\right). \quad (4.9)$$

Now we describe how to choose a suitable index k_0 . Suppose for instance that $\nu = \nu_1$. Up to a permutation of the indices, we can suppose that P_1, \dots, P_{n_1} are the polynomials having valuation ν and minimal first nonzero coefficient $A := a_1 = \dots = a_{n_1}$ (i.e. if $j > n_1$ then either $\nu_j > \nu$ or $\nu_j = \nu$ and $a_j > A$). Consider the subset $I \subset \{0, \dots, N\}$ of all indices k for which the (unique) singular direction $d_{\varphi_{k+1}} \in \Gamma$ in $V_{k,k+1}$ is a singular direction of the series $\widehat{H}(P_1(z))$ (and hence of any $\widehat{H}(P_j(z))$ for $j = 1, \dots, n_1$). With the above notation, if $k \in I$ then $\{1, \dots, n_1\} \subset J_k$ and in this case $l(k) := l(k,1) = \dots = l(k, n_1)$ is such that $\theta_{l(k)+1} = \nu \varphi_{k+1}$. Define also $\mu(k) \in \{1, \dots, r\}$ by $\mu(k) \equiv l(k) + 1 \pmod{r}$ for any $k \in I$. Then equation (4.7) yields with (4.9) for $k \in I$

$$F_{k+1}(z) - F_k(z) = \gamma_{l(k)} \sum_{j=1}^{n_1} E_{k,j}(z) \exp(q_{\mu(k)}(P_j(z))) P_j(z)^{\alpha_{\mu(k)}} + O(\exp(-B|z|^{-p\nu})), \quad (4.10)$$

where B is a certain constant $> \frac{|\lambda_{\mu(k)}|}{pA^p}$ and, for simplicity, we have denoted

$$E_{k,j}(z) = \sum_{i / (i,j) \in \Lambda} D_{ij,k}(z) G_{l(k,j)\mu(k,j)}^i(P_j(z)).$$

LEMMA 4.8. *We can assume, without loss of generality, that there exists $\mu_0 \in \{1, \dots, r\}$ with the property that $E_{k,1}(z)$ has a non trivial asymptotic expansion on $V_{k,k+1}$ for any $k \in I$ with $\mu(k) = \mu_0$.*

Proof. The asymptotic expansion of $E_{k,1}$ for $k \in I$ is given by

$$\widehat{E}_{k,1}(z) = (\widehat{D}_{11,k}(z), \dots, \widehat{D}_{r1,k}(z)) \widehat{G}(P_j(z)) (0, \dots, \overset{(\mu(k))}{1}, \dots, 0)^T$$

where $\widehat{D}_{ij,k}(z) = \partial f / \partial z_{ij}(z, \widehat{K}(z)) \in \mathbb{C}[[z]]$ is the asymptotic expansion of $D_{ij,k}$. As discussed in the beginning of this section, we can assume that the series $\partial f / \partial z_{11}(z, \widehat{K}(z)) \in \mathbb{C}[[z]]$ is not identically zero, hence the vector $(\widehat{D}_{11,k}(z), \dots, \widehat{D}_{r1,k}(z))$ is not trivial. We now reach the desired conclusion by using the surjectivity of the function $\mu : I \rightarrow \{1, \dots, r\}$ and the fact that $\widehat{G}(z)$ is a nonsingular matrix. \square

Now we use the hypothesis (SD) of Theorem 4.4: choose $l_0 \in \{0, \dots, rp - 1\}$ with $l_0 + 1 \equiv \mu_0 \pmod{r}$ such that $\gamma_{l_0} \neq 0$ and consider $k_0 \in \{0, \dots, N\}$ such that $\theta_{l_0+1} = \nu \varphi_{k_0+1}$. Then $k_0 \in I$ and $l_0 = l(k_0)$, $\mu_0 = \mu(k_0)$. We are now in a position to show that $F_{k_0+1} - F_{k_0}$ is exponentially small of the exact order $p\nu$.

We use (4.10) for $k = k_0$. By Lemma 4.8, at least the coefficient $E_{k_0,1}$ has a non zero asymptotic expansion on V_{k_0,k_0+1} . We can suppose, without loss of generality, that this is true for any coefficient $E_{k_0,j}$ for $j = 1, \dots, n_1$. In fact, if some $E_{k_0,j}$ has a vanishing asymptotic expansion then $E_{k_0,j}$ is exponentially small of order $p\nu$ on V_{k_0,k_0+1} (since this expansion is Gevrey of order $1/p\nu$). This property, together with (4.9), permits us to include the corresponding summand $E_{k_0,j}(z) \exp(q_{\mu_0}(P_j(z))) P_j(z)^{\alpha_{\mu_0}}$ in the remainder term $O(\exp(-B|z|^{-p\nu}))$ in (4.10) (maybe the constant B has to be reduced, but it remains $> \frac{|\lambda_{\mu(k)}|}{pA^p}$).

We remark that we have not yet used the hypothesis of Theorem 4.4 that the degree d_j of the polynomial $P_j(x)$ satisfies $d_j < (p+1)\nu_j$. This hypothesis (together with the fact that the polynomial $q_{\mu_0}(z) \in \mathbb{R}[z^{-1}]$ is of degree p and without constant term) is essential in order to show the following property (its proof is straightforward and left to the reader):

The principal parts at $z = 0$ of the meromorphic series $q_{\mu_0}(P_j(z))$ for $j = 1, \dots, n_1$ are two by two different.

Thus, for any pair of distinct indices $j_1, j_2 \in \{1, \dots, n_1\}$, there exists a complex number $\beta_{j_1 j_2} \neq 0$ and an integer $s_{j_1 j_2} > 0$ such that

$$q_{\mu_0}(P_{j_1}(z)) - q_{\mu_0}(P_{j_2}(z)) = \beta_{j_1 j_2} z^{-s_{j_1 j_2}} (1 + O(|z|)).$$

We deduce that for almost all $\varphi \in \mathbb{R}$, except for those satisfying $\arg(\beta_{j_1 j_2}) - s_{j_1 j_2} \varphi - \frac{\pi}{2} \in \pi\mathbb{Z}$, the real part of $q_{\mu_0}(P_{j_1}(z)) - q_{\mu_0}(P_{j_2}(z))$ has a constant sign (positive or negative) along d_φ (for $|z|$ sufficiently small depending on φ). Up to a permutation of the indices, we can suppose that $\operatorname{Re}(q_{\mu_0}(P_1(z))) > \operatorname{Re}(q_{\mu_0}(P_j(z)))$ for $j > 1$ along infinitely many rays d_φ where φ is arbitrarily close to φ_{k_0+1} . Moreover, given such an angle φ , there exists $b = b(\varphi) > 0$ and $s = s(\varphi) > 0$ (independent of φ) such that

$$\operatorname{Re}(q_{\mu_0}(P_1(|z|e^{i\varphi}))) - \operatorname{Re}(q_{\mu_0}(P_j(|z|e^{i\varphi}))) > b|z|^{-s},$$

for $|z|$ small. Due to the fact that $E_{k_0,j}$ has a non trivial asymptotic expansion, this implies that the first term in the summation in (4.10) dominates the remaining

ones; i.e.

$$\lim_{z \rightarrow 0, z \in d_\varphi} \frac{|E_{k_0,j}(z) \exp(q_{\mu_0}(P_j(z))) P_j(z)^{\alpha_{\mu_0}}|}{|E_{k_0,1}(z) \exp(q_{\mu_0}(P_1(z))) P_1(z)^{\alpha_{\mu_0}}|} = 0,$$

for $j = 2, \dots, n_1$. It also dominates the term $O(-B|z|^{-p\nu})$ since $B > \frac{|\lambda_{\mu(k)}|}{pA^p}$. Thus we have

$$F_{k_0+1}(z) - F_{k_0}(z) = \gamma_{l_0} E_{k_0,1}(z) \exp(q_{\mu_0}(P_1(z))) P_1(z)^{\alpha_{\mu_0}} (1 + O(z))$$

as $z \rightarrow 0$ restricted to $z \in d_\varphi$ for some φ arbitrarily close to φ_{k_0+1} . Finally, by (4.8), the function $\gamma_{l_0} E_{k_0,1}(z) \exp(q_{\mu_0}(P_1(z))) P_1(z)^{\alpha_{\mu_0}}$ is exponentially small of exact order $p\nu$ and thus the same is true for $F_{k_0+1} - F_{k_0}$.

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